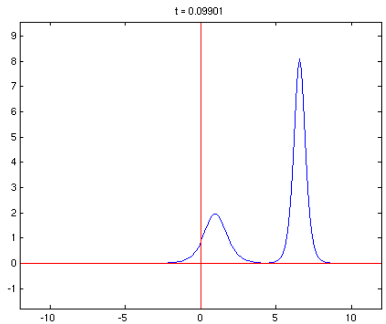
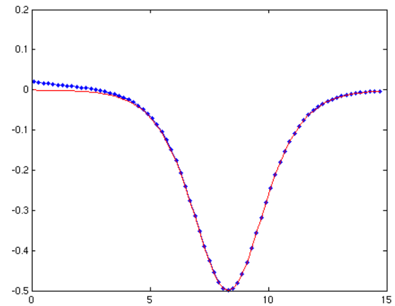
The KdV Equation

and the

Inverse Scattering Method (ISM)





Ryo Takei

APMA 935 Project, April 12, 2006

Introduction and History

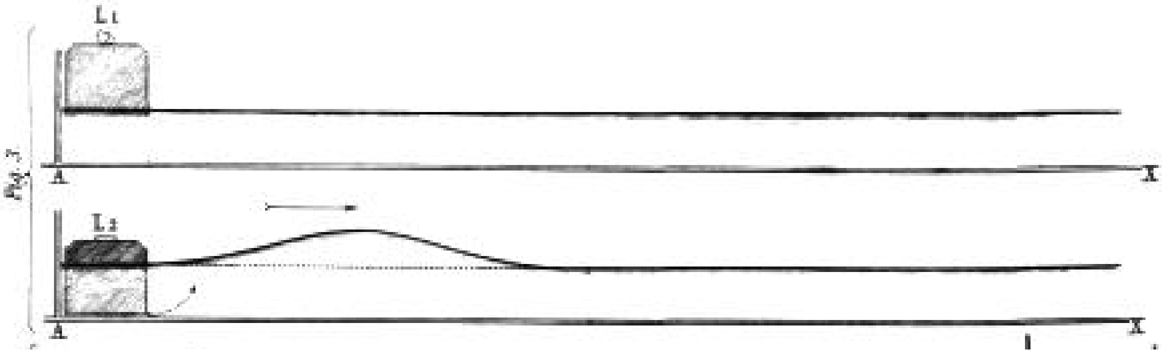
- The *Korteweg - de Vries* (KdV) Equation (long water waves of small amplitude):

$$\boxed{u_t - 6uu_x + u_{xxx} = 0 \qquad \text{(KdV)}}$$

- (KdV) known for *solitary wave* solutions, or *solitons*: localized, highly stable waves that, upon interaction, retains its identity.

- Solitons were discovered experimentally (Russell 1844), analytically (Korteweg & de Vries, 1895), then numerically (Zabusky & Kruskal 1965).

- J. Russell's sketch from his soliton experiment (*Report on Waves*, 1844):



- ISM for solving (KdV): Gardiner et.al. (1967).

Scattering Theory: Scattering

- Sturm-Liouville problem, given potential $u(x)$:

$$\psi_{xx} = (\lambda - u)\psi \qquad (u, u' \to 0 \text{ as } |x| \to \infty) \quad \text{(SL)}$$

- $\lambda < 0$: (discrete spectrum), assume the form:

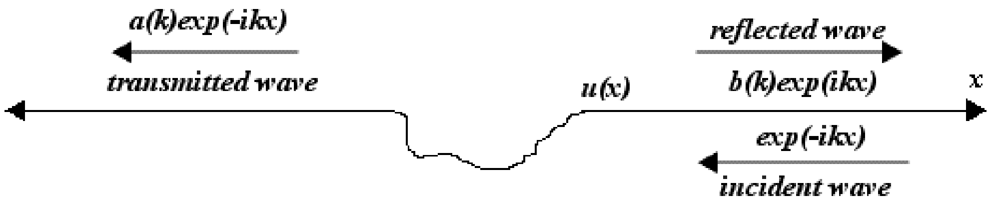
$$\psi_n(x) \sim c_n e^{-\kappa_n x} \qquad \text{as } x \to \infty,\ n = 1, ..., N$$

where $\kappa_n = \sqrt{-\lambda}$.

- $\lambda > 0$: (continuous spectrum), assume:

$$\widehat{\psi}(x,k) \sim \begin{cases} e^{-ikx} + b(k)e^{ikx} & \text{as } x \to +\infty \\ a(k)e^{-ikx} & \text{as } x \to -\infty \end{cases}$$

where $k = \sqrt{\lambda}$. (see schematic picture below)



- $S(u) := \{(\kappa_n), (c_n), b(k)\}$: *scattering data* for $u(x)$.

Scattering Theory: Inverse Scattering

- In fact, $S(u)$ is sufficient to uniquely recover potential $u(x)$!
- Inverse Scattering Procedure:
  Given $S(u) = \{(\kappa_n), (c_n), b(k)\}$,

  1. Let $F(X) := \sum_{n=1}^{N} c_n^2 e^{-\kappa_n X} + \frac{1}{2\pi} \int_{-\infty}^{\infty} b(k) e^{ikX} dk$.
  2. Solve $K(x, z)$ in the *Marchenko Equation*:

  $$K(x,z) + F(x+z) + \int_x^{\infty} K(x,y)F(y+z)dy = 0 \tag{ME}$$

  3. Compute $u(x) = -2\frac{d}{dx}K(x,x)$.

- Remark: difficulty is in solving the Fredholm-type integral (ME).
- Conclusion: we have an equivalence between $S(u)$ and u in (SL). ie. One can be derived from the other via scattering / invserse scattering.

Lax Pair for the KdV

- Consider operators L, M:

$$\begin{aligned} L &= -\frac{\partial}{\partial x} + u(x,t) \\ M &= -4\frac{\partial^3}{\partial x^3} + 3u(x,t)\frac{\partial}{\partial x} + 3\frac{\partial}{\partial x}u(x,t) + A(t) \end{aligned}$$

(L, M is called a *Lax Pair*.)

- If $u(x,t)$ solves the (KdV),

$$\begin{aligned} \{L_t + (LM - ML)\}u &= u_t - 6uu_x + u_{xxx} \\ &= 0 \qquad \text{(LE)} \end{aligned}$$

- **Theorem** (Lax):
If (LE) *holds for* $u(x,t)$ *and* $L\psi = \lambda\psi$, *then*

$$\lambda_t = 0 \qquad \textit{and} \qquad \psi_t = M\psi.$$

The Inverse Scattering Method for KdV: Schematic Diagram

KdV: $u_t - 6uu_x + u_{xxx} = 0$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{IV: } u(x,0) \arrow[r, "1.\ L\psi = \lambda\psi \text{ at } x \to \pm\infty", "(\text{Scattering at } x \to \pm\infty)"'] & \text{Scattering Data } S(u(x,0)) \arrow[d, "2.\ \text{Evolve } S \text{ at } x \to \pm\infty \text{ by } \psi_t = M\psi"] \\
\text{Solution: } u(x,t) & S(u(x,t)) \arrow[l, "3.\ \text{Inverse Scattering Procedure}"']
\end{tikzcd}$$

- Remark 1: M is independent of u, u_x at $x \to \pm\infty$.
- Remark 2: similar to Fourier method of solving linear PDEs.

The Inverse Scattering Method for KdV: Overview

- Given IV u(x,0), solve the KdV via ISM:

1. Solve the eigenvalue problem $\psi_{xx} + (u(x,0) - \lambda)\psi = 0$, to get scattering data $S(u(x,0)) = \{(\kappa_n(t=0)), (c_n(t=0)), b(k,t=0)\}$. **(using (SL):** $L\psi = \lambda\psi$**)**

2. Evolve scattering data $S(u(x,0)) \to S(u(x,t))$:

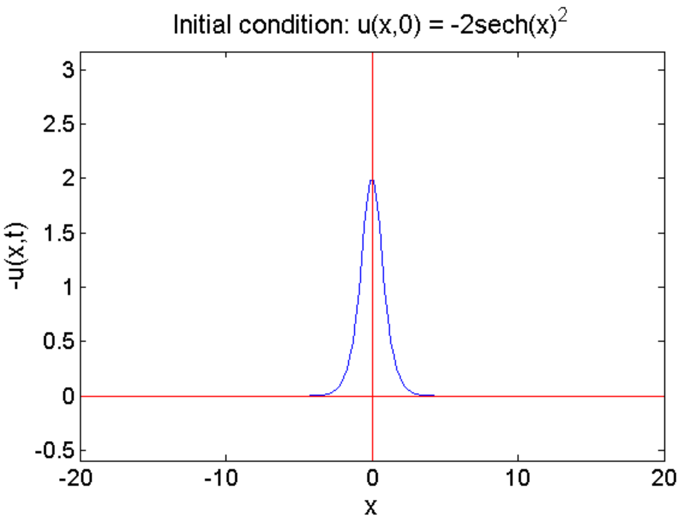
$$\begin{aligned} \kappa_n &= \text{constant}, && (1) \\ c_n(t) &= c_n(0)e^{4\kappa_n^3 t}, && (2) \\ b(k;t) &= b(k,0)e^{8ik^3 t}. && (3) \end{aligned}$$

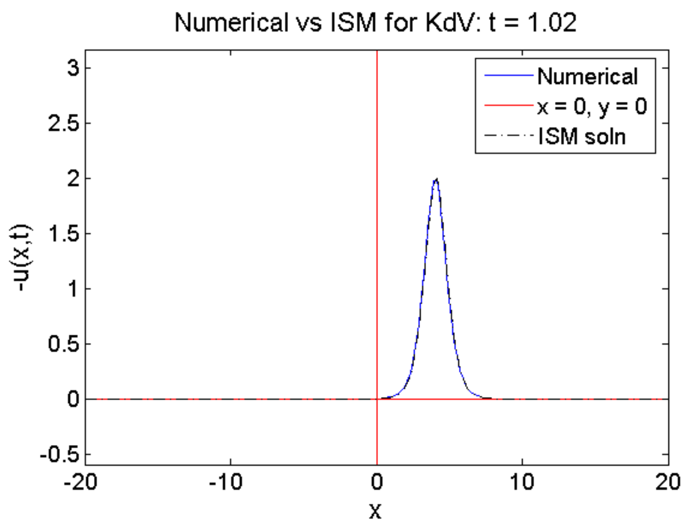
((1) is precisely $\lambda_t = 0$ **and (2), (3) are derived from** $\psi_t = M\psi$**)**

3. Define $F(X)$ from $S(u(x,t))$, and solve (ME) for $K(x,z)$ and subsequently for $u(x,t)$, via **Inverse scattering procedure**.

Example 1: single - soliton solution

- case $u(x,0) = -2\text{sech}^2(x)$
- obtain $\psi_1 = -\text{sech}(x)$ (solution to legendre's equation) and $b \equiv 0$ (reflection-less).
- Solution (by ISM): $u(x,t) = -2\text{sech}^2(x-4t)$.
- Remark: Agrees with solution obtained by traveling wave approach.





Example 2: two - soliton solution

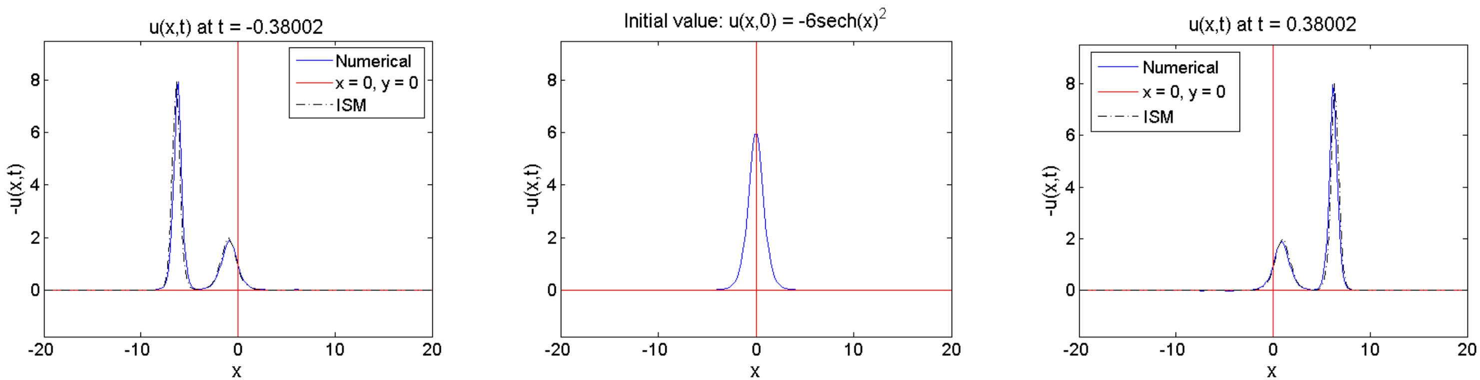
- case $u(x,0) = -6\text{sech}^2(x)$, where N is a positive integer.
- two discrete eigenvalues $\kappa_{1,2} = 1$ and 2, and $b \equiv 0$ (reflectionless).
- solution by ISM:

$$u(x,t) = -12\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{(3\cosh(x - 28t) + \cosh(3x - 36t))^2}$$

- Asymptotically, the above solution is:

$$u(x,t) \sim -8\text{sech}^2(2(x-16t) \mp \frac{1}{2}\log 3) - 2\text{sech}^2((x-4t) \pm \frac{1}{2}\log 3) \qquad \text{as } t \to \pm\infty$$

ie. interaction of solitons is nonlinear.



Example 3: Negative delta function initial condition

- If $u(x,0) = -\alpha\delta(x)$, cannot be solved analytically $\Longrightarrow$ Try numerical approach.

- Strategy I: Solve KdV numerically for $f_n(x)$ s.t. $f_n \to -\alpha\delta$.
  - tested with initial gaussians of area 500 ($\alpha = 500$) with increasing σ.
  - See top plots next slide.

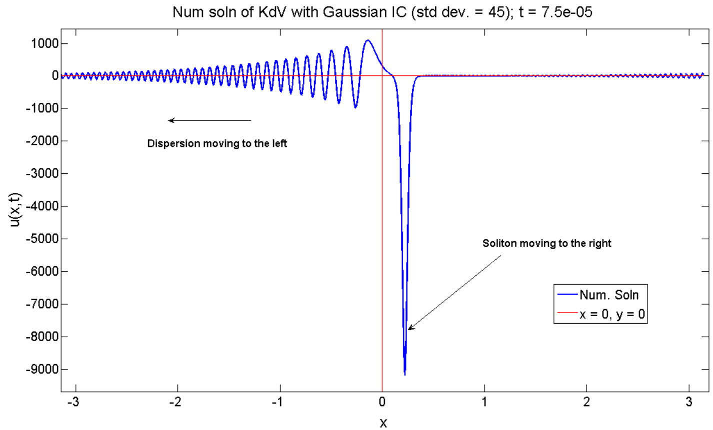
- Strategy II: Solve (ME) numerically; an ISM approach.
  - in (ME), $F(X;t)$ becomes:

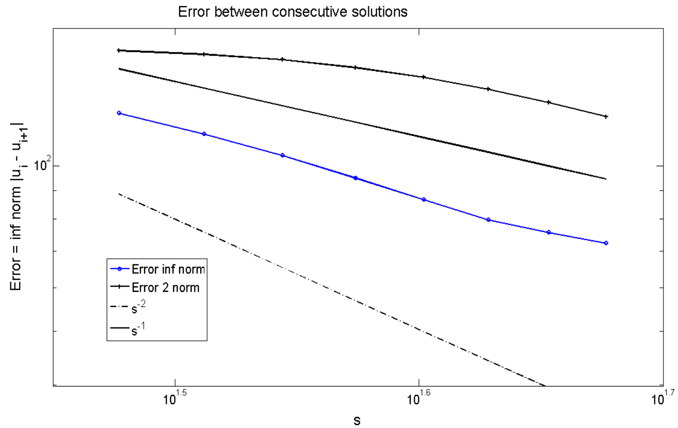
$$F(X;t) = \kappa_1 e^{8\kappa_1^3 t - \kappa_1 X} - \frac{\alpha}{2\pi} \int_{-\infty}^{\infty} \frac{e^{8ik^3t + ikX}}{2ik + \alpha} dk$$

  - integral in $F(X;t)$ solves a first order ODE with Airy inhomo. part.
  - thus numerically evaluate F, then solve (ME) by approximating the integral by the trapezoid rule.
  - See bottom plots next slide for the case $\alpha = 1$.

- Conclusion: soliton to the right, with (Airy-like) dispersion to the left.

(left) Spectral code for KdV with initial gaussian; (right) Convergence of error between consecutive increasing standard deviation. Here, $u(x,0) = -500\delta(x)$.





(left) Numerical solution to Marchenko solver; (right) convergence of error between consecutive increasing N, mesh points. Here $u(x,0) = -\delta(x)$.