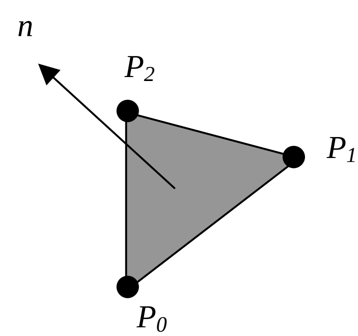
Answers to review questions

(March 1, 2001)

1. Suppose that you have a triangle in 3-space, defined by the three point $P_0, P_1, \textit{ and } P_2$. As a function of these three points, derive a formula for a normal vector $\vec{n}$ that is normal to this triangle. The normal vector should be directed so that from a viewer on the same side as the normal vector, the points appear in counterclockwise order.



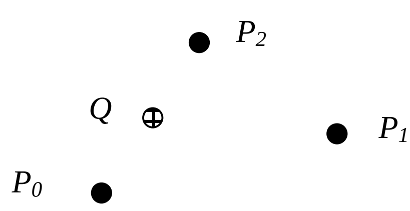
Answer: $\dfrac{(P_1 - P_0)\times(P_2 - P_1)}{|(P_1 - P_0)\times(P_2 - P_1)|}$ (with unit length)**.**

Problem (1)

2. Given a triple of noncollinear points $P_0, P_1, \textit{ and } P_2$ in the plane, any other point Q in the plane can be expressed uniquely as an affine combination of these three

$$Q = \omega_0 P_0 + \omega_1 P_1 + \omega_2 P_2 \qquad \text{(where } \omega_0 + \omega_1 + \omega_2 = 1\text{)}$$

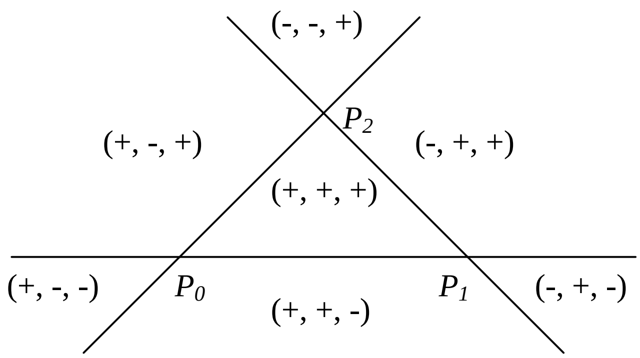
The triple $(\omega_0, \omega_1, \omega_2)$ of scalars is called the *barycentric coordinates* of Q relative to the P_i*'s* .
For example, in the figure below, Q coordinates are (0.5,0, 0.5).



(a) Give a drawing that illustrates the regions of the plane that are associated with each of the possible sign classes (+ or -) of the 3 barycentric coordinates. That is, label the region in which all 3 coordinates are positive with (+, +, +), label the region in which only the first coordinate is negative with (-, +, +). etc.

(b) Which sign class(es) are missing from the diagram? Why?

Answer: (a)



This can be derived by checking $(P_0 + P_1 + P_2)/3$, $(P_2 - P_0) + (P_1 - P_0)$ and $2P_0 - (P_0 + P_1 + P_2)/3$.

(b) (-, -, -) is missing from this diagram. Every point on this plane can be denoted by:

$$P_0 + u(P_2 - P_0) + v(P_1 - P_0)\text{, where } u, v \in (-\infty, +\infty).$$

It is easy to see that the coefficients of P_0, P_1, P_2 cannot be negative at the same time.

3. Given a triangle ΔPQR in 3-space, and given a line segment $\overline{AB}$ describe an algorithm to determine if the line segment intersects the triangle properly? (By proper intersection we mean that the interior of the line intersects the interior of the triangle.) Also describe how to compute the coordinates of the point of intersection.

Answer: Algorithm to determine if $\overline{AB}$ intersects ΔPQR:

(1) Compute the implicit equation of the plane determined by ΔPQR.

$$H: \quad \hat{n} \cdot \vec{X} + d = 0\text{, where } \hat{n} \text{ is the normal and } \vec{X} = (x, y, z)'$$

(Use the normal of ΔPQR as described in *problem 1*)

(2) Compute the parametric equation of line determined by $\overline{AB}$.

$$L: \quad \vec{X}(t) = (1 - t) \cdot \vec{A} + t \cdot \vec{B} \qquad t \in (-\infty, +\infty).$$

(We are considering the equation of the entire line, not just the line segment)

(3) Compute the intersection coordinates by substituting the parametric line equation into the plane equation:

$$\hat{n} \cdot (1 - t) \cdot \vec{A} + \hat{n} \cdot t \cdot \vec{B} + d = 0$$

Therefore,

$$t_0 = \frac{\hat{n} \cdot \vec{A} + d}{\hat{n} \cdot (\vec{A} - \vec{B})}$$

If $t_0 < 0$ or $t_0 > 1$, no intersection is returned. Otherwise, calculate the intersection point:

$$(x_0, y_0, z_0) = \vec{X}(t_0)$$

(4) Determine if (x_0, y_0, z_0) lies inside ΔPQR. This can be easily done by determining *barycentric coordinates of* (x_0, y_0, z_0) for ΔPQR (described in *problem 2*).

4. Using the vector representation of a quaternion as a generalized complex number

$$q=(s,\vec{u})=s+u_x i+u_y j+u_z k.$$

where the basis values *i*, *j*, and *k* satisfy the following multiplication rules:

$$i^2=j^2=k^2=-1 \qquad ij=k,\ jk=i, ki=j \qquad ji=-k, kj=-i, ik=-j.$$

Prove that

$$q_1 q_2=(s_1 s_2-(\vec{u}_1\cdot\vec{u}_2), s_1\vec{u}_2+s_2\vec{u}_1+\vec{u}_1\times\vec{u}_2)$$

(Hint: Express q_1 *and* q_2 using the above form, multiply everything out, and collect terms that share common basis values. Be very careful in your algebra, or it will take forever to get it right.)

Answer: Assume that $q_1=s_1+(u_{1x},u_{1y},u_{1z})\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}$, and $q_2=s_2+(u_{2x},u_{2y},u_{2z})\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}$.

Thus,

$$q_1q_2=s_1s_2+s_1(u_{2x},u_{2y},u_{2z})\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}+s_2(u_{1x},u_{1y},u_{1z})\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}+(u_{1x},u_{1y},u_{1z})\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}\cdot(u_{2x},u_{2y},u_{2z})\cdot\begin{pmatrix}i\\j\\k\end{pmatrix},$$

where $(u_{1x},u_{1y},u_{1z})\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}\cdot(u_{2x},u_{2y},u_{2z})\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}=(u_{1x},u_{1y},u_{1z})\cdot\begin{pmatrix}u_{2x}i & u_{2y}i & u_{2z}i\\ u_{2x}j & u_{2y}j & u_{2z}j\\ u_{2x}k & u_{2y}k & u_{2z}k\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}$

$$=(u_{1x},u_{1y},u_{1z})\cdot\begin{pmatrix}u_{2x}i^2 & u_{2y}ij & u_{2z}ik\\ u_{2x}ji & u_{2y}j^2 & u_{2z}jk\\ u_{2x}ki & u_{2y}kj & u_{2z}k^2\end{pmatrix}=(u_{1x},u_{1y},u_{1z})\cdot\begin{pmatrix}-u_{2x} & u_{2y}k & -u_{2z}j\\ -u_{2x}k & -u_{2y} & u_{2z}i\\ u_{2x}j & -u_{2y}i & -u_{2z}\end{pmatrix}$$

$$=-(u_{1x}u_{2x}+u_{1y}u_{2y}+u_{1z}u_{2z})+(u_{1y}u_{2z}-u_{1z}u_{2y}, u_{1z}u_{2x}-u_{1x}u_{2z}, u_{1x}u_{2y}-u_{1y}u_{2x})\begin{pmatrix}i\\j\\k\end{pmatrix}$$

$$=-(\vec{u}_1\cdot\vec{u}_2)+(\vec{u}_1\times\vec{u}_2).\begin{pmatrix}i\\j\\k\end{pmatrix}$$

So we have:

$$q_1q_2 = s_1s_2 - (\vec{u}_1 \cdot \vec{u}_2) + s_1\vec{u}_2 \cdot \begin{pmatrix} i \\ j \\ k \end{pmatrix} + s_2\vec{u}_1 \cdot \begin{pmatrix} i \\ j \\ k \end{pmatrix} + (\vec{u}_1 \times \vec{u}_2) \cdot \begin{pmatrix} i \\ j \\ k \end{pmatrix}$$

$$= s_1s_2 - (\vec{u}_1 \cdot \vec{u}_2), \quad s_1\vec{u}_2 + s_2\vec{u}_1 + (\vec{u}_1 \times \vec{u}_2)$$

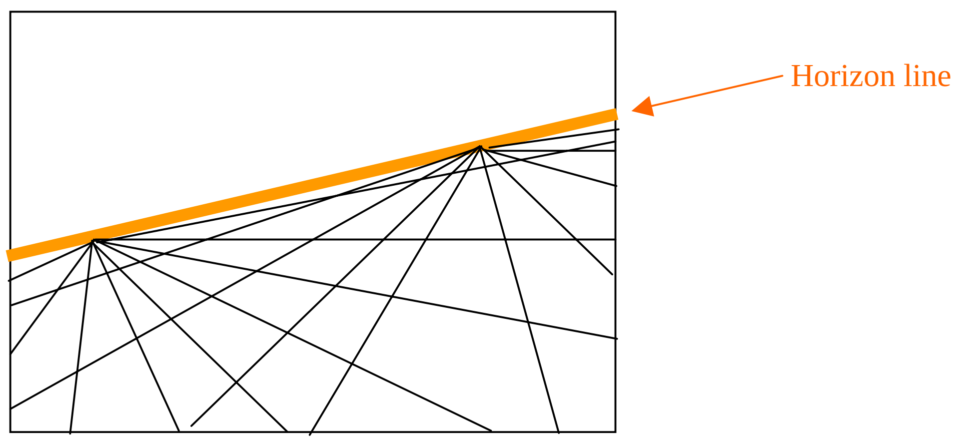
5. Consider the plane H in 3-space, defined by the following equation

$$H : ax + by + cz + d = 0,$$

for some nonzero constants a, *b*, *c*, and *d*. Suppose that we project this plane using our standard projective transformation

$$P(x, y, z) = (x/(-z), y/(-z), -1),$$

which projects the points onto the projection plane *z=-1*. Points at infinity on plane *H* are mapped to a horizon line on the projection plane. As a function of *a, b, c*, and *d*, derive the line equation (on the *x, y*-projection plane) for the horizon line.



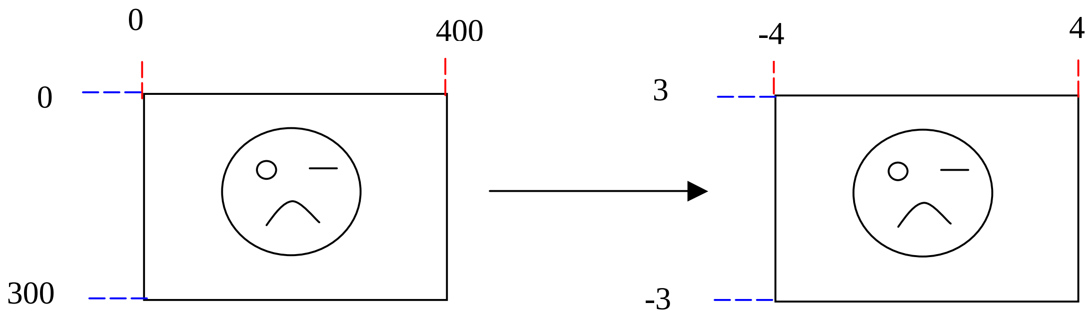
Answer: Consider another plane H' that is parallel to H and contains the origin (0, 0, 0). It's easy to obtain the equation of H':

$$H' : ax + by + cz = 0,$$

So the horizon line is the intersection of *H'* and projection plane. That is:

Horizon line: $\begin{cases} ax + by + cz = 0 \\ z = -1 \end{cases}$ or $\begin{cases} ax + by = c \\ z = -1 \end{cases}$

6. Give an affine transformation that maps the 400 x300 viewport shown below left to the 8 x 6 window shown on the right (Note the order reversal along the y-axis). You may express your transformation either in equation form or as a 3 x 3 homogeneous matrix.



Answer: The final affine transformation is a combination of several transformations:

Scale: $\begin{bmatrix} \frac{1}{50} & 0 & 0 \\ 0 & \frac{1}{50} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. *Reflection*: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. *Translation*: $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

Scale — due to the different size of rectangles.
Reflection — due to the axis direction flip
Translation — due to the shift of the origin

So the final transformation is:

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{50} & 0 & 0 \\ 0 & \frac{1}{50} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{50} & 0 & -4 \\ 0 & -\frac{1}{50} & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

7. Consider the tetrahedron whose vertices are A = (1,1,1) B = (0,1,1), C = (0,0,1), and D = (0,0,0).

(a). What are the normals of the four triangles that bound the tetrahedron? Remember that normals point to the outside of a closed object and that they are unit length.

(b). Assume that the eyepoint is at the point (-1,-1,-1). For each triangle (ABC, ABD, ACD, or BCD) tell whether it is frontfacing, backfacing, or edge on, with respect to the eye position.

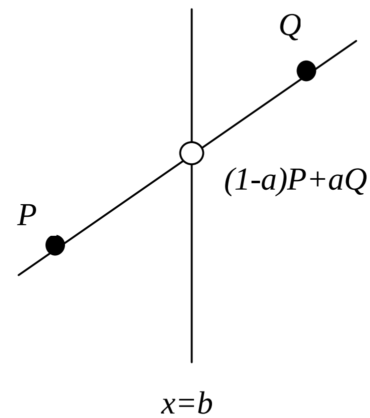
Answer: (a) Use vector cross product as in *problem 1* to compute the normals of the triangular faces. The unit normals of *BCD*, *ABD*, *ABC*, and *ACD* are respectively:

$$(-1,0,0), \qquad (0,\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}), \qquad (0,0,1), \qquad (0,-1,0)$$

(b) Take the center point of each triangle, and consider the vector from the center to the eyepoint. Compute the dot product of this vector with the normal of each triangle. Answer the question based on the sign of this dot production: positive (frontface), negative (backface), and zero (edge on).

Faces *ACD* and *BCD* are front facing, while the other two are back facing.

8. You are given a vertical line $x=b$ and a pair of points P and Q in the plane. As a function of b and the coordinates of P and Q, compute the affine combination of P and Q that lies on this vertical line.



problem (8)

Answer: Suppose $\vec{P} = (p_x, p_y)$ and $\vec{Q} = (q_x, q_y)$.

The affine combination of P and Q must be the intersection of line PQ and the vertical line. Denote the intersection point as $\vec{M} = (m_x, m_y)$. Suppose

$$\vec{M} = (1-a)\vec{P} + a\vec{Q}.$$

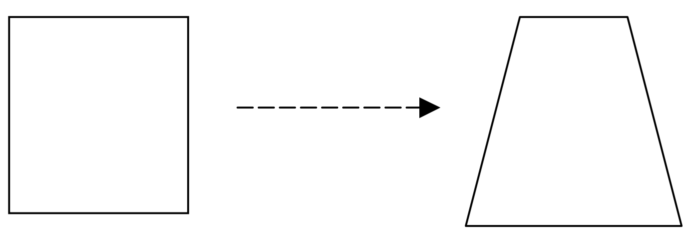
Thus,

$$(1-a)p_x + q_x = m_x = b.$$

Then we have: $a = \dfrac{b - p_x}{q_x - p_x}$

So, $\vec{M} = (1 - \dfrac{b-p_x}{q_x - p_x})\vec{P} + \dfrac{b-p_x}{q_x-p_x}\vec{Q}$.

9. Consider the transformation that maps a square into the trapezoid (shown below). Is this transformation *affine*? Explain.



Answer: This is not affine transformation because affine transformation maps parallel lines to parallel lines.

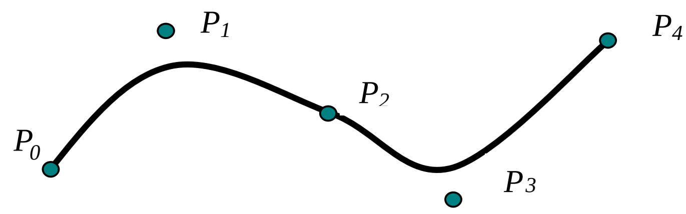
10. Consider the line $0 = 2x - 3y + 5$ in the 2-dimensional projective plane. What are the homogeneous coordinates of the (infinite) endpoints of this line?

Answer: (3, 2, 0).

11. Consider the parabola $y = x^2$ in the projective plane. Consider the extensions of the parabola out to infinity. What are the homogeneous coordinates of these points at infinity?

Answer: (0, 1, 0).

12. Suppose that we join two Bezier curves of degree 2 end-to-end, using the control points sequence $\langle P_0, P_1, P_2 \rangle$, and $\langle P_2, P_3, P_4 \rangle$, respectively. Exactly what conditions must be satisfied by these five points for the combined curve to have C^1 parametric continuity at the point at which they are joined. Prove your answer carefully by showing the continuity of the derivatives at this point.



Answer: The Bezier curve of degree 2 determined by P_0, P_1, P_2 is:

$$\vec{f}(t): \qquad (1-t)^2\vec{P}_0 + 2(1-t)t\vec{P}_1 + t^2\vec{P}_2 \qquad t \in [0,1].$$

The first derivative of $\vec{f}(t)$ at P_2 (or t = 1) is:

$$\vec{f}\,'(t)\big|_{t=1} = 2(\vec{P}_2 - \vec{P}_1).$$

Similarly, the Bezier curve of degree 2 determined by P_2, P_3, P_4 is:

$$\vec{g}(t): \qquad (1-t)^2\vec{P}_2 + 2(1-t)t\vec{P}_3 + t^2\vec{P}_4 \qquad t \in [0,1].$$

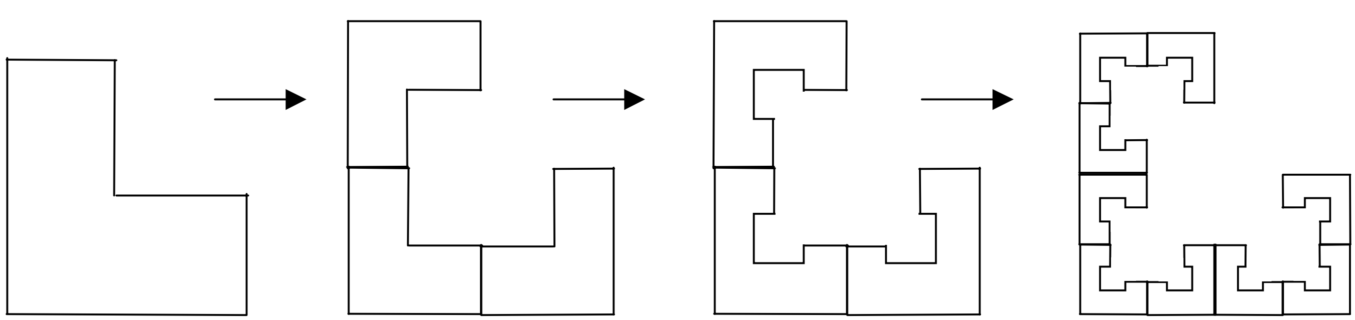
The first derivative of $\vec{g}(t)$ at P_2 (or t = 0) is:

$$\vec{g}\,'(t)\big|_{t=0} = 2(\vec{P}_3 - \vec{P}_2).$$

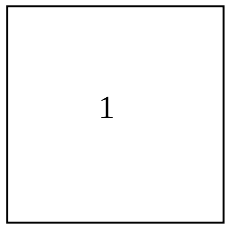
Therefore, the condition for the combined curve to have C^1 parametric continuity at jointed points is:

$\vec{g}\,'(t)\big|_{t=0} = c \cdot \vec{f}\,'(t)\big|_{t=1}$ or $(\vec{P}_3 - \vec{P}_2) = c \cdot (\vec{P}_2 - \vec{P}_1)$, which means that P_1, P_2, P_3 are collinear.

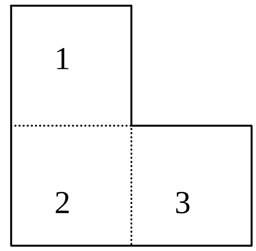
13. Consider the limit of the sequence shown in the figure below. What is the fractal dimension of the final object?



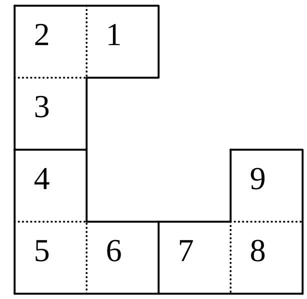
Answer:



0th iteration



1st iteration



2nd iteration

Take 1 square (with size 1) to cover the entire graph in the 0th iteration;

3 squares (with size $\frac{1}{2}$) to cover the entire graph in the 1st iteration;

9 squares (with size $\frac{1}{4}$) to cover the entire graph in the 2nd iteration;

………

3^k squares (with size $\frac{1}{2^k}$) to cover the entire graph in the k^{th} iteration;

Therefore the fractal dimension is:

$$\lim_{k\to\infty}\frac{\ln 3^k}{\ln 2^k} = \frac{\ln 3}{\ln 2} \cong 1.58.$$