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RESPONSE STATISTICS AND VARIANCE FOR A SINGLE SEA SUBSYSTEM: THEORY AND EXPERIMENT

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ABSTRACT

This paper is concerned with the ensemble statistics of the response of a single SEA subsystem. Both the point-to-point transfer function and the kinetic energy of the response are considered, and a comparison is made between theory and experiment for the case of a plate loaded with randomly located masses. Very close agreement is found, thus validating the theoretical predictions that are applicable to any type of subsystem.

INTRODUCTION

The response of a complex built-up structure at high frequencies can be predicted by using Statistical Energy Analysis (SEA) [1]. With this approach, the structure is modeled as an assembly of "subsystems" and the aim is to compute the vibrational energy of each subsystem. The predicted energy is a "mean" energy in the sense that the energy is averaged in up to three ways: (i) over the spatial extent of the subsystem; (ii) over a specified frequency band; (iii) over an ensemble of random systems. Whereas (i) and (ii) are conceptually straight forward, the averaging involved in (iii) requires the notion that at high frequencies the detailed properties of the system are random - thus, for example, successive cars from a production line may have different response properties, a fact that is well known from measurements [2]. It is often argued that the response might be expected to be "ergodic", in the sense that a frequency average taken on a particular structure should yield the same as an ensemble average taken at a fixed frequency, so that frequency and ensemble averaging yield the same result. While this argument may hold for some situations it is not guaranteed to be valid in general, and in fact the concept of an ensemble of structures, rather than frequency averaging, lies at the heart of the fundamental assumptions that underpin SEA [1]. For many years efforts have been made to extend the "mean" energy predicted by SEA to higher order statistical quantities, such as the ensemble variance, but as yet there is no robust and accurate method available for either a built-up system or the simpler case of a single isolated subsystem. The present work is concerned with the latter problem, and earlier work in this area is summarized below.

The statistics of the response of a single subsystem under single point loading has been analyzed by Lyon [3] under the assumption that the natural frequencies form a Poisson point process. This work was extended by Davy [4] to the case of multiple drive points and a response that is averaged over a specified number of receiver points. It was recognized in reference [3] that the Poisson model has certain shortcomings, and this issue was addressed by Lobkis et al [5] who modified the work of Lyon [3] and Davy [4] to allow for non-Poisson natural frequencies. The model adopted in reference [5] assumes that the natural frequencies conform to the statistics associated with the eigenvalues of a certain class of random matrix (the Gaussian Orthogonal Ensemble [6]) - a fact that has been confirmed numerically and experimentally for many practical systems (see for example references [7,8]). The results reported in reference [5] break down for the case of rain-on-the-roof excitation, and this issue was considered in detail by Langley and Brown [9]. A simple formula was developed for the variance of the energy under a specified number of excitation points (with rain-on-the-roof corresponding to an infinite number of points), and this was compared with numerical simulations and a limited experimental program. One of the aims of the present work is to provide a more extensive experimental corroboration of the theoretical results for the energy statistics.

The literature discussed above is largely concerned with the ensemble variance of the energy response. In some of the earliest work on the statistics of the response of an uncertain system, Schroeder [10] considered the point-to-point transfer function of an acoustic volume. It was shown that at sufficiently high frequency the real and imaginary parts become uncorrelated Gaussian variables of equal variance. This leads to an exponential distribution for the modulus squared transfer function, and experimental evidence in room acoustics [11] has confirmed the accuracy of this result. In principle the same result should apply to a structural component such as a plate, although no experimental evidence to support this can be found in the literature. Furthermore, the Schroeder result concerns the reverberant part of the response, and the influence of the "direct field" may be more significant for a structural component than for an acoustic volume. In the present work the analysis of Schroeder is modified to allow for the presence of the direct field, and the results obtained are compared with experimental results for a mass loaded plate.

In what follows, the results for the energy statistics derived in reference [9] are summarized. The Schroeder theory [10] for the point-to-point transfer function is then extended to include the direct field. An experimental program designed to test the validity of both theoretical results is then described, and finally the results from the program are reported.

STATISTICS OF THE RESPONSE ENERGY

For a proportionally damped system, the complex transfer function at frequency w between a unit force applied at location $\boldsymbol{x}_0$ and the response at location $\boldsymbol{x}$ can be written in the form

$$H(w,\mathbf{x}_0,\mathbf{x}) = \sum_n \frac{f_n(\mathbf{x}) f_n(\mathbf{x}_0)}{w_n^2 - w^2 + i h w w_n} , \tag{1}$$

where f_n is the nth mode shape (scaled to unit generalized mass), w_n is the nth natural frequency and h is the loss factor. The kinetic energy produced by the unit force can be written in the form

$$T(w) = \frac{w^2}{4R} \int_R r(\mathbf{x}) \left| H(w,\mathbf{x}_0,\mathbf{x}) \right|^2 \mathrm{d}\mathbf{x} , \tag{2}$$

where R is the area (or equivalent) of the subsystem and r is the density. It follows from Eqs. (1) and (2) that

$$T(w) = \sum_n \frac{w^2 a_n}{(w_n^2 - w^2)^2 + (h w w_n)^2} , \tag{3}$$

where mode shape orthogonality has been employed and

$$a_n = f_n^2(\mathbf{x}_0)/4R . \tag{4}$$

For different types of loading the definition of a_n will change, but Eq. (3) will remain valid. The statistics of the kinetic energy can be derived by considering the system natural frequencies to be a random point process with *cumulant* functions (related to the statistical moments) given by the Gaussian Orthogonal Ensemble. The derivations involved are lengthy, but it has been shown by Langley and Brown [9] that the resulting expressions for the mean energy and the relative variance have the following simple forms

$$\mu_T = \frac{\mathrm{E}[a_n]\pi\nu}{2\eta\omega}, \quad \rho_T^2 = \frac{\sigma_T^2}{\mu_T^2} = \left(\frac{\alpha - 1}{\pi m}\right) + \left(\frac{1}{\pi m}\right)^2, \tag{5,6}$$

where ν is the modal density of the subsystem and $m=\omega\eta\nu$ is the modal overlap factor. The term α is given by

$$\alpha = \frac{\mathrm{E}[a_n^2]}{\mathrm{E}[a_n]^2}. \tag{7}$$

For the case of N incoherent point loads, it can be shown that α has the form

$$\alpha = 1 + \frac{(K-1)}{N}, \qquad K = \frac{\mathrm{E}[\varphi_n^4]}{\mathrm{E}[\varphi_n^2]^2}. \tag{8,9}$$

If the mode shapes are Gaussian, then K=3. Simulations and experiments for a range of systems [5,9] suggest that K=2.6 is more appropriate. Eqs. (6,8) are compared with experimental results in what follows.

STATISTICS OF THE POINT-TO-POINT TRANSFER FUNCTION

The complex point-to-point transfer function $H=H_r+iH_i$ is now considered. In Schroeder's theory, it is assumed that the real and imaginary parts are Gaussian independent processes, with zero mean and same variance. Then, the probability density function (pdf) of $|H|$ is shown to be Rayleigh, the phase $\phi[H]$ is uniformly distributed and $|H|^2$ is exponential. The mean of the squared modulus is related to the variance σ_H^2 of the real and imaginary parts by $\mathrm{E}[|H|^2]=\mathrm{E}[H_r^2]+\mathrm{E}[H_i^2]=2\sigma_H^2$. The variance can then be estimated according to the energy balance for a single subsystem, $M\omega^2\mathrm{E}[|H|^2]=P_{inj}/\eta\omega$, where M is the total mass and P_{inj} is the injected power estimated by using the infinite system solution.

In what follows, Schroeder's theory is improved by considering a deterministic part to the transfer function, with the result that the real and imaginary parts have a non-zero mean. The transfer function is split into deterministic and random parts, $H=H^{det}+H^{rand}$, and only the random part follows the Schroeder statistics. The pdf of the real or imaginary part is then written

$$\mathrm{p}_{H_{r,i}}(s) = \frac{1}{\sqrt{2\pi}\,\sigma_H}\exp\left[-\frac{(s-H_{r,i}^{\det})^2}{2\sigma_H^2}\right], \tag{10}$$

where s is the range variable related to H_r or H_i. The following change of variables can now be made

$$\begin{cases} H_r^{\mathrm{rand}} = |H|\cos\varphi - H_r^{\det}, \\ H_i^{\mathrm{rand}} = |H|\sin\varphi - H_i^{\det}, \end{cases} \tag{11}$$

so that $\mathrm{p}_{|H|,\varphi}(s_m, s_p) = \mathrm{p}_{H_r^{\mathrm{rand}},H_i^{\mathrm{rand}}}(s_r, s_i)\det(\mathbf{J})$ where **J** is the Jacobian matrix of the transformation. In this expression, s_r, s_i, s_m and s_p are the range variables respectively related to H_r, H_i, $|H|$, $\phi[H]$. It can be shown that $\det(\mathbf{J})=s_m$. Then, using the assumption that the real and imaginary parts of the random component are statistically independent, the joint pdf of the modulus and phase is obtained as

$$\mathrm{p}_{|H|,\phi}(\sigma_m,\sigma_p)=\frac{\sigma_m}{2\pi\sigma_H^2}\exp\left[-\frac{\sigma_m^2+|H^{\det}|^2-2\sigma_m|H^{\det}|\cos(\sigma_p-\phi^{\det})}{2\sigma_H^2}\right], \tag{12}$$

where $\phi^{\det}$ denotes the phase of the deterministic field. Consider the ratio $\Gamma^2=|H^{\det}|^2/2\sigma_H^2$. Based on Schroeder's statistics of the random part, it is known that $2\sigma_H^2=E[|H^{\mathrm{rand}}|^2]$, so that Γ is a measure of the relative importance of the deterministic and random components. The pdf's of $|H|$ and ϕ are found by integrating the joint distribution respectively over the possible values of ϕ ([-π, π]) and the possible values of $|H|$ ([0, ∞]). The software Mathematica for symbolic calculations gives

$$\mathrm{p}_{|H|}(\sigma_m)=\int_{-\pi}^{\pi}\mathrm{p}_{|H|,\phi}(\sigma_m,\sigma_p)d\sigma_p=\frac{\sigma_m}{\sigma_H^2}\mathrm{I}_0\left[2\Gamma\frac{\sigma_m}{\sigma_H}\right]\exp\left[-\left(\frac{\sigma_m^2}{2\sigma_H^2}+\Gamma^2\right)\right], \tag{13}$$

$$\mathrm{p}_{\phi}(\sigma_p)=\int_0^{\infty}\mathrm{p}_{|H|,\phi}(\sigma_m,\sigma_p)d\sigma_m=\frac{\mathrm{e}^{-\Gamma^2}}{2\pi}+\frac{\exp\left\{-\Gamma^2\left[1-\cos^2(\sigma_p-\phi^{\det})\right]\right\}}{2\sqrt{\pi}}\Gamma\cos(\sigma_p-\phi^{\det})\left(1+\mathrm{erf}\left[\Gamma\cos(\sigma_p-\phi^{\det})\right]\right) \tag{14}$$

where I_0 is the modified Bessel function of the first kind and erf is the error function. Notice that Γ is spatially dependent, and hence so are the pdf's of phase and modulus. In both cases, if the ratio Γ is small (meaning that the deterministic field is negligible), then the distributions from Schroeder's theory are recovered: $\mathrm{p}_{|H|}(\sigma_m)=(\sigma_m/\sigma_H^2)\exp(-\sigma_m^2/2\sigma_H^2)$ (Rayleigh) and $\mathrm{p}_{\phi}(\sigma_p)=1/2\pi$ (uniform). The maximum of the distribution of the phase occurs at $\phi^{\det}$, which is exactly given by the phase of the mean, $\phi(\mathrm{E}[H])=\tan^{-1}[H_i^{\det}/H_r^{\det}]$. The variance of the phase about this value only depends on the parameter Γ. In what follows Eqs. (10,13,14) are used to derive the mean and variance of the related quantities, and the resulting predictions are compared with experimental results.

EXPERIMENTAL INVESTIGATION

Experimental validations have been performed by using the plate shown in Fig. 1. The samples of the ensemble were produced by randomly moving a set of masses over the surface of the plate. A total of 19 configurations were tested over the frequency range [25 Hz, 1250 Hz]. The plate was excited by an impact hammer at 5 fixed locations and the acceleration was measured at the excitation point and also at 12 other points scattered over the plate. The damping of the plate has been estimated by measuring the injected power P_{inj} (with the force sensor of the hammer and the collocated accelerometer), and kinetic energy T (with the accelerometers on the plate) and by applying the power balance $P_{inj}=2\eta\omega T$. The damping loss factor was found to be approximately constant, at around η=1.39%. The modal overlap was then estimated from the formula $\mu=n\eta\omega$, with the modal density n being calculated (according to $n=S(\rho h/D)^{1/2}/4\pi$, where S is the surface of the plate, ρ its density, h its thickness, and D its bending stiffness).

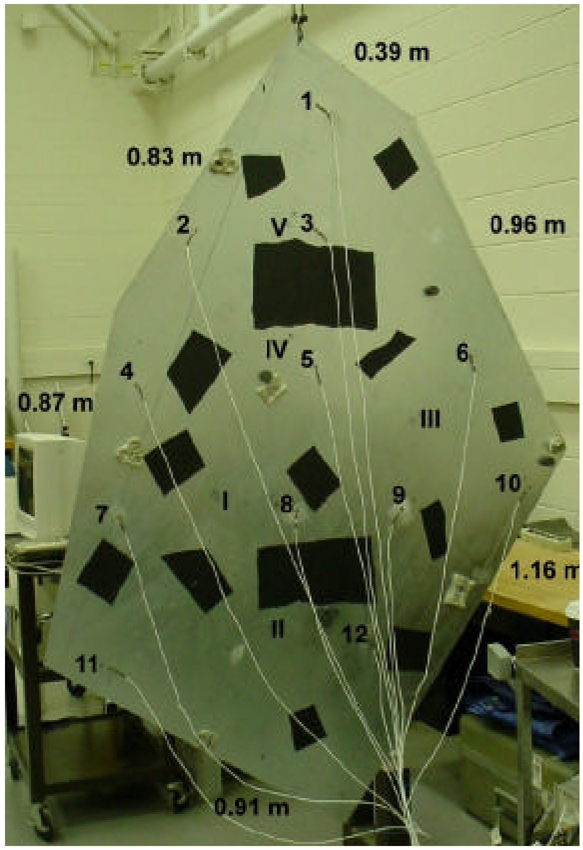

Fig. 1: ***Experimental setup. The Roman numbers indicate the 5 excitation points, the Arabic numbers indicate the 12 observation points. The added masses are taped or magnetically attached to the plate. The black patches are the damping treatment.***

RESULTS

The kinetic energy of the plate was measured for each of the 19 samples and for each of the 5 force locations. The measured results for the mean and relative variance versus the estimated modal overlap are shown in Fig. 2, together with the appropriate theoretical predictions. The mean energy was calculated by using standard SEA, while the relative variance was calculated by using Eqs. (6) and (8) with K=2.6, and N=1.

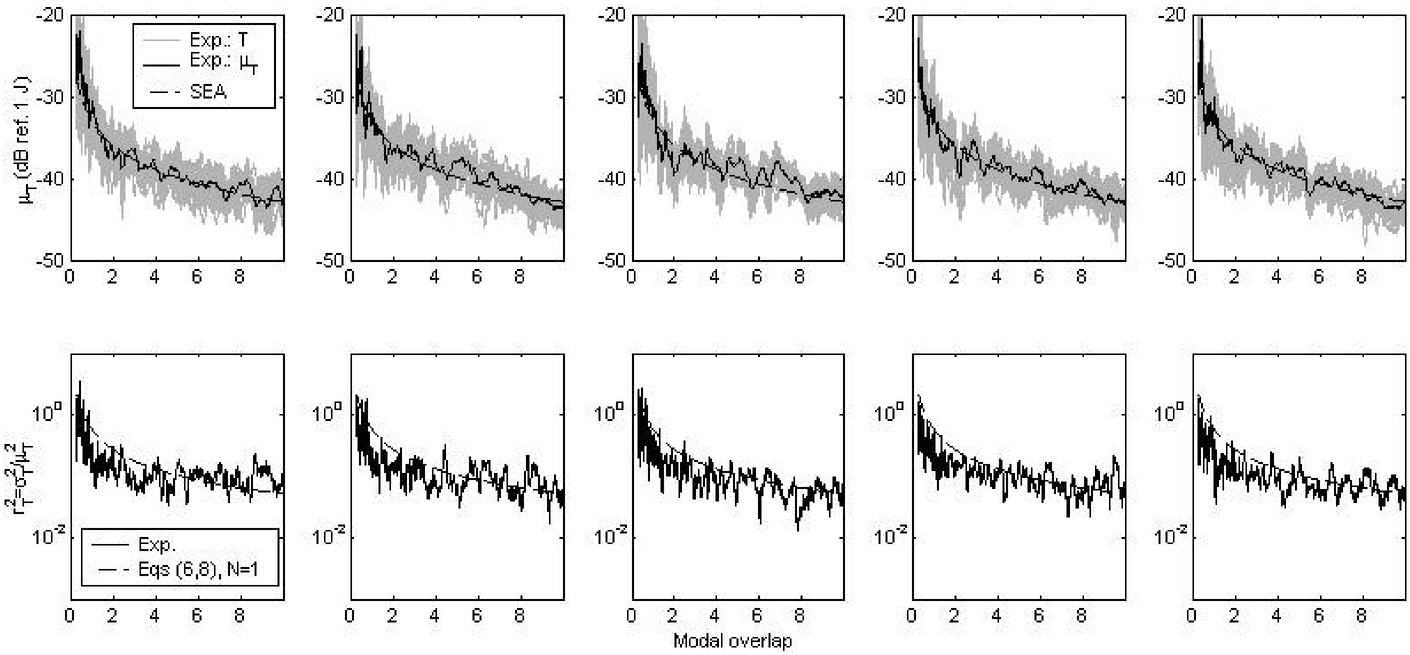

Fig. 2: ***Mean and relative variance of the total energy of the plate versus the modal overlap, with an ensemble of 19 plates, and for 5 excitation points. The dashed black lines are the theoretical predictions.***

The mean is accurately predicted, as should be the case since the damping loss factor used in the SEA calculation was estimated from the experimental data. The relative variance is also well predicted despite the small number of samples in the ensemble - theoretically the measured curve should be a smooth function of modal overlap (assuming that the structure has high statistical overlap); the fluctuations seen in Fig. 2 are due to the limited ensemble size.

The response to 5 incoherent simultaneous point forces can be found by summing the measured energy levels associated with each of the 5 point force locations. The corresponding mean and relative variance of the energy are shown in Fig. 3. Three theoretical estimates are shown for the variance: rain-on-the-roof forcing for which Eq. (6) reduces to $1/(\pi m)^2$; five forcing points (N=5 and K=2.6); and a single forcing point (N=1 and K=2.6). The 5 point prediction yields very good agreement with the experimental results. It is interesting to note that the predicted variance is very different for rain-on-the-roof forcing and five point forces. In SEA terms, five incoherent forces might be expected to give a good approximation to rain-on-the-roof excitation - the present analysis demonstrates that while this may be true for the mean response, it is certainly not true for the variance.

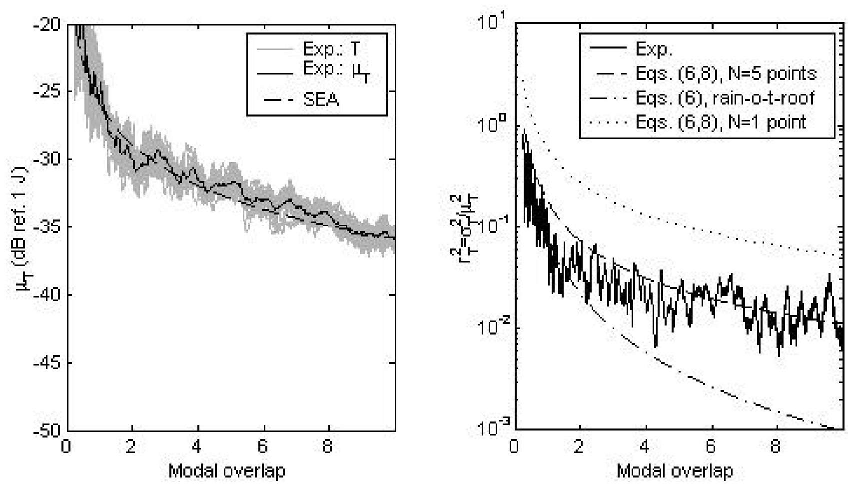

Fig. 3: ***Mean and relative variance of the total energy versus modal overlap, with an ensemble of 19 plates, for 5 points forcing. The dashed lines are the theoretical predictions.***

The statistics of the point-to-point transfer function has also been investigated using the experimental data. The mean and variance of the real and imaginary parts for the transfer function between point V and point 1, are shown in Fig. 4. The distance between the points is 0.31 m. The theoretical result from Eq. (10) is also shown in the figure (Schroeder's theory gives a zero mean and the same variance prediction). In Eq. (10), the deterministic part of the field was taken to be the direct field, $H^{det} = H^{dir}$.

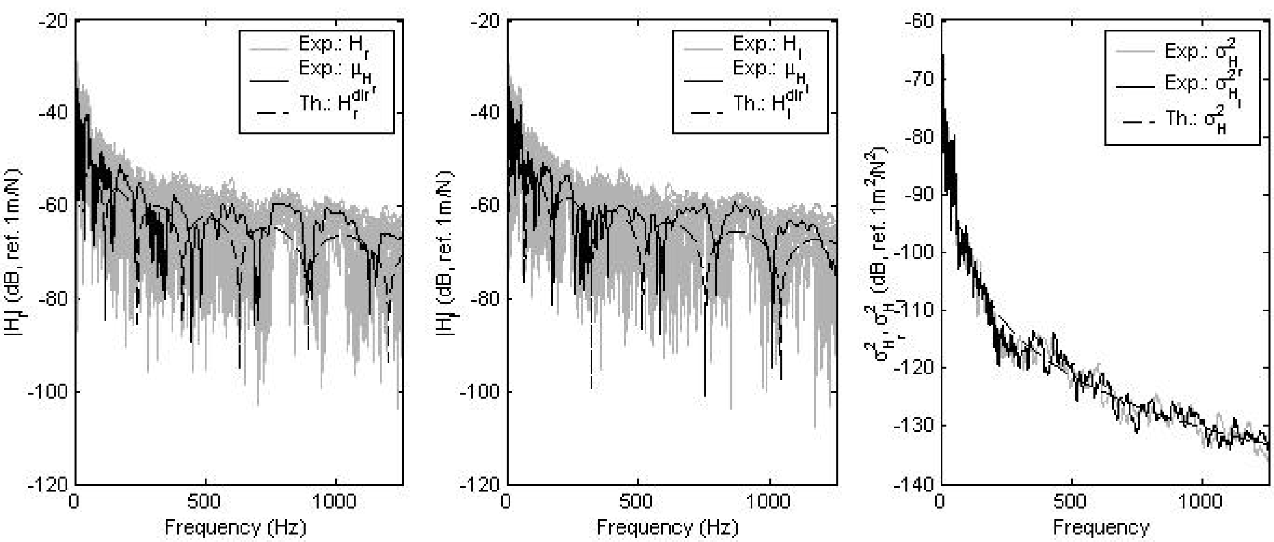

Fig. 4: ***Mean and variance of the real and imaginary parts of frequency response functions. The grey lines are the experimental results for each sample; the solid black line is the mean over all samples; the dashed black line is the present theory.***

The mean of the real and imaginary parts are shown to be non zero, and agree quite well with the direct field. However, even when averaging over all samples, the first reflections at the edges of the plate seem to remain coherent (do not averaged out to zero), so that the deterministic part of the field might be slightly different from the direct field (this is especially obvious for transfer functions between points close to the boundaries). The predicted variance based on the power balance is close to the experimental one. This is to be expected given that damping levels have been estimated from the same measurements and also by using the power balance method. The assumption that real and imaginary parts have the same variance proves to be valid.

For the same frf's, the mean and variance of the modulus, as well as the mean of the phase are shown in Fig. 5. Experimental data and theoretical results from both the present approach and Schroeder's statistics are shown in comparison.

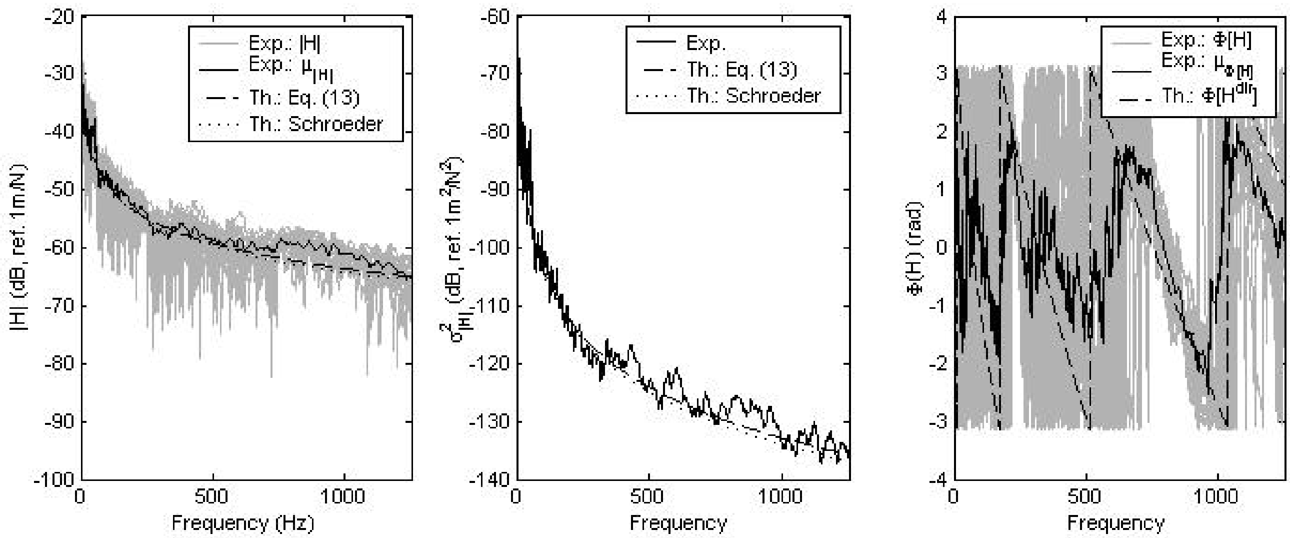

Fig. 5: ***Mean and variance of modulus and phase of the frf. The grey lines are the experimental results for each sample; the solid black line is the mean over all samples; the dashed black line is the present theory; the dot-dashed black line is Schroeder's result.***

The difference between Eqs (13,14) and the standard Schroeder's theory depends on the relative importance of the direct and reverberant fields. For the transfer functions considered here, the points are remote enough for the direct field to be much smaller than the reverberant field. As a result, Schroeder's theory yields a good prediction of the statistics of the modulus |*H*|. The trend followed by the mean of the phase is well predicted by the phase of the direct field.

CONCLUSIONS

Two theoretical results for the statistics of the response of a single SEA subsystem have been compared with experimental measurements: (i) the statistics of the response energy described by Eqs. (6,8), and (ii) the statistics of the point-to-point transfer function described by Eqs. (10,13,14). In each case good agreement with measurements has been obtained. It has been shown that while 5 drive points may represent a good approximation to rain-on-the-roof forcing so far as the mean energy is concerned, the same is not true with regard to the energy variance. The challenge now is to extend these results to built-up systems.

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