## USING LINEAR ALGEBRA

# TO DISCOVER THE DEFINING IDENTITIES 

## FOR LIE AND JORDAN ALGEBRAS

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Let $F$ be any field. (For simplicity take $F=\mathbb{R}$, the field of real numbers). Let $A$ be a vector space over $F$. (For simplicity take $\operatorname{dim}_{F} A<\infty$, so that $A=\mathbb{R}^{n}$ for some positive integer $n)$. Let $p: A \times A \rightarrow A$ be a bilinear map, where $A \times A$ is the Cartesian product. We regard $p(X, Y)$ as the product of the elements $X, Y \in A$, and we write $X Y$ instead of $p(X, Y)$. With this notation the bilinearity of $p$ means that

$$
X(a Y+b Z)=a X Y+b X Z, \quad(a X+b Y) Z=a X Z+b Y Z
$$

for any scalars $a, b \in F$. So bilinearity of $p$ is equivalent to the distributivity of multiplication over addition. We call the pair $(A, p)$ an algebra over the field $F$.

Example: Take $A=M_{n}(\mathbb{R})$ (the $n \times n$ matrices with real entries) and take $p(X, Y)=X Y$ (the usual matrix product). This is an associative algebra because it satisfies the identity

$$
(X Y) Z=X(Y Z), \quad \text { for all } X, Y, Z \in A
$$

On any algebra $A$ we can define two new operations, the Lie bracket (Sophus Lie, Norway, 1842-1899)

$$
[X, Y]:=X Y-Y X
$$

and the Jordan product (Pascual Jordan, Germany, 19021980)

$$
X \circ Y:=\frac{1}{2}(X Y+Y X)
$$

(For the Jordan product we have to assume that char $F \neq 2$.)
It is clear from the definition that the Lie bracket satisfies the anticommutative identity

$$
[Y, X]=-[X, Y], \quad \text { for all } X, Y \in A
$$

(which is equivalent to $[X, X]=0$ if char $F \neq 2$ ), and that the Jordan product satisfies the commutative identity

$$
Y \circ X=X \circ Y, \quad \text { for all } X, Y \in A
$$

Both of these identities have degree 2 (each term is a product of 2 variables), whereas the associative identity has degree 3 .

How can we find identities of degree $d \geq 3$ satisfied by the

Lie bracket and the Jordan product in an associative algebra?
We can simplify this problem by considering only multilinear
identities: each term is a product of $d$ distinct variables.
For $d=3$ we use the variables $X, Y, Z$. There are 6 permutations of these variables:

$$
X Y Z, \quad X Z Y, \quad Y X Z, \quad Y Z X, \quad Z X Y, \quad Z Y X
$$

But we can't assume associativity, so we also need to take account of the 2 association types:

$$
(X Y) Z, \quad X(Y Z)
$$

This gives a total of $6 \cdot 2=12$ possible terms in a multilinear identity of degree 3 .

## TABLE ONE

The 12 possible terms in

## A MULTILINEAR NONASSOCIATIVE DEGREE-3 IDENTITY

$$
\begin{array}{llllll}
(X Y) Z, & (X Z) Y, & (Y X) Z, & (Y Z) X, & (Z X) Y, & (Z Y) X, \\
X(Y Z), & X(Z Y), & Y(X Z), & Y(Z X), & Z(X Y), & Z(Y X) .
\end{array}
$$

All 6 permutations are listed in the first association type, followed by all 6 permutations in the second association type.

These 12 terms form a basis of the vector space of possible multilinear nonassociative identities in degree 3 .

The Lie bracket satisfies anticommutativity, and the Jordan product satisfies commutativity. This reduces the number of terms to just 3 possibilities:

$$
[[X, Y], Z], \quad[[X, Z], Y], \quad[[Y, Z], X]
$$

for the Lie bracket, and

$$
(X \circ Y) \circ Z, \quad(X \circ Z) \circ Y, \quad(Y \circ Z) \circ X
$$

for the Jordan product. These 3 terms form a basis of the vector space of possible multilinear identities in degree 3 for a commutative or anticommutative algebra. (Note that all 3 terms are in the first association type.) Here are two examples showing how the other 9 terms can be reduced to these 3 :

$$
[Z,[Y, X]]=-[[Y, X], Z]=-(-[[X, Y], Z])=[[X, Y], Z]
$$

for the Lie bracket, and

$$
Z \circ(Y \circ X)=(Y \circ X) \circ Z=(X \circ Y) \circ Z
$$

for the Jordan product.

We can expand each of the 3 terms into a sum in the original associative algebra. For the Lie bracket we get

$$
\begin{aligned}
{[[X, Y], Z] } & =(X Y-Y X) Z-Z(X Y-Y X) \\
& =X Y Z-Y X Z-Z X Y+Z Y X, \\
{[[X, Z], Y] } & =(X Z-Z X) Y-Y(X Z-Z X) \\
& =X Z Y-Z X Y-Y X Z+Y Z X, \\
{[[Y, Z], X] } & =(Y Z-Z Y) X-X(Y Z-Z Y) \\
& =Y Z X-Z Y X-X Y Z+X Z Y .
\end{aligned}
$$

We can store this information in the expansion matrix: the $6 \times 3$ matrix in which the rows are labelled by the 6 associative terms and the columns are labelled by the 3 anticommutative terms. The $i, j$ entry of the matrix gives the coefficient of the $i$-th associative term in the expansion of the $j$-th anticommutative term.

## TABLE TWO

## THE EXPANSION MATRIX FOR

## THE LIE BRACKET IN DEGREE 3

$$
[[X, Y], Z] \quad[[X, Z], Y] \quad[[Y, Z], X]
$$

| $X Y Z$ | 1 | 0 | -1 |
| :---: | :---: | :---: | :---: |
| $X Z Y$ | 0 | 1 | 1 |
| $Y X Z$ | -1 | -1 | 0 |
| $Y Z X$ | 0 | 1 | 1 |
| $Z X Y$ | -1 | -1 | 0 |
| $Z Y X$ | 1 | 0 | -1 |

Any multilinear degree-3 identity satisfied by the Lie bracket
has the form

$$
a[[X, Y], Z]+b[[X, Z], Y]+c[[Y, Z], X]=0,
$$

where $a, b, c \in F$. If we replace each anticommutative term with its expansion into associative terms, then we obtain

$$
\begin{aligned}
& a(X Y Z-Y X Z-Z X Y+Z Y X) \\
+ & b(X Z Y-Z X Y-Y X Z+Y Z X) \\
+ & c(Y Z X-Z Y X-X Y Z+X Z Y)=0 .
\end{aligned}
$$

If we group together common terms, then we obtain

$$
\begin{aligned}
& (a-c) X Y Z+(b+c) X Z Y+(-a-b) Y X Z \\
+ & (b+c) Y Z X+(-a-b) Z X Y+(a-c) Z Y X=0 .
\end{aligned}
$$

Since every coefficient must be zero, we obtain a system of 6 linear equations in 3 variables. The coefficient matrix of this system is the expansion matrix. So the nullspace of the expansion matrix consists of the identities we want.

Here is the expansion matrix $E$ and its row canonical form (reduced row echelon form, Gauss-Jordan form) $E^{R}$ :

$$
E=\left(\begin{array}{rrr}
1 & 0 & -1 \\
0 & 1 & 1 \\
-1 & -1 & 0 \\
0 & 1 & 1 \\
-1 & -1 & 0 \\
1 & 0 & -1
\end{array}\right) \quad E^{R}=\left(\begin{array}{rrr}
1 & 0 & -1 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

From $E^{R}$ we see that the nullspace has basis $(1,-1,1)$; this corresponds to the identity

$$
[[X, Y], Z]-[[X, Z], Y]+[[Y, Z], X]=0 .
$$

Using anticommutativity of the Lie bracket (and commutativity of addition) we can rewrite this as the cyclic sum

$$
[[X, Y], Z]+[[Y, Z], X]+[[Z, X], Y]=0
$$

which is called the Jacobi identity. The variety of Lie algebras is defined to be the class of algebras which satisfy the anticommutative and Jacobi identities.

It is easy to check directly that the Lie bracket satisfies the Jacobi identity. The advantage of using linear algebra is that it shows that the Jacobi identity is the only multilinear identity in degree 3 satisfied by the Lie bracket.

An important theorem in the theory of Lie algebras, the Poincaré-Birkhoff-Witt Theorem (or PBW Theorem for short), implies that every identity of any degree satisfied by the

Lie bracket follows from the anticommutative identity and the Jacobi identity. So for the Lie bracket, our search for identities is now complete.

If we now consider the Jordan product then we get the expansions

$$
\begin{aligned}
(X \circ Y) \circ Z & =\frac{1}{4}(X Y+Y X) Z+\frac{1}{4} Z(X Y+Y X) \\
& =\frac{1}{4} X Y Z+\frac{1}{4} Y X Z+\frac{1}{4} Z X Y+\frac{1}{4} Z Y X, \\
(X \circ Z) \circ Y & =\frac{1}{4}(X Z+Z X) Y+\frac{1}{4} Y(X Z+Z X) \\
& =\frac{1}{4} X Z Y+\frac{1}{4} Z X Y+\frac{1}{4} Y X Z+\frac{1}{4} Y Z X, \\
(Y \circ Z) \circ X & =\frac{1}{4}(Y Z+Z Y) X+\frac{1}{4} X(Y Z+Z Y) \\
& =\frac{1}{4} Y Z X+\frac{1}{4} Z Y X+\frac{1}{4} X Y Z+\frac{1}{4} X Z Y .
\end{aligned}
$$

In this case the expansion matrix $E$ and its row canonical form $E^{R}$ are

$$
E=\frac{1}{4}\left(\begin{array}{ccc}
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{array}\right) \quad E^{R}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

The nullspace is $\{\mathcal{O}\}$ : the Jordan product does not satisfy any multilinear identity in degree 3 .

So for the Jordan product we have to look at identities in degree 4 ! There are 24 permutations of 4 letters:
$W X Y Z, W X Z Y, W Y X Z, W Y Z X, W Z X Y, W Z Y X$,
$X W Y Z, X W Z Y, X Y W Z, X Y Z W, X Z Y W, X Z W Y$,
$Y W X Z, Y W Z X, Y X W Z, Y X Z W, Y Z W X, Y Z X W$,
$Z W X Y, Z W Y X, Z X W Y, Z X Y W, Z Y W X, Z Y X W$.

There are 5 association types for a product of 4 letters:

$$
\begin{gathered}
((W X) Y) Z, \quad(W(X Y)) Z, \quad(W X)(Y Z) \\
W((X Y) Z), \quad W(X(Y Z))
\end{gathered}
$$

For a commutative product, these 5 types reduce to just 2 types:

$$
((W X) Y) Z, \quad(W X)(Y Z)
$$

We obtain a total of 15 possible multilinear degree- 4 terms.
There are 12 in the first association type:

$$
\begin{array}{lll}
((W \circ X) \circ Y) \circ Z, & ((W \circ X) \circ Z) \circ Y, & ((W \circ Y) \circ X) \circ Z, \\
((W \circ Y) \circ Z) \circ X, & ((W \circ Z) \circ X) \circ Y, & ((W \circ Z) \circ Y) \circ X, \\
((X \circ Y) \circ W) \circ Z, & ((X \circ Y) \circ Z) \circ W, & ((X \circ Z) \circ W) \circ Y, \\
((X \circ Z) \circ Y) \circ W, & ((Y \circ Z) \circ W) \circ X, & ((Y \circ Z) \circ X) \circ W ;
\end{array}
$$

and 3 more in the second association type:

$$
(W \circ X) \circ(Y \circ Z), \quad(W \circ Y) \circ(X \circ Z), \quad(W \circ Z) \circ(X \circ Y) .
$$

The expansion matrix therefore has 24 rows and 15 columns.
The $i, j$ entry is the coefficient of the $i$-th associative term in the expansion of the $j$-th commutative term. This matrix can be worked out by hand, but it's more interesting to write a ...
... Maple procedure to compute the multilinear degree-4 identities for the Jordan product:
\# notation: ( $1,2,3,4$ ) $=(W, X, Y, Z)$
\# generate the associative terms s4 := combinat[permute] ([1,2,3,4]):
\# generate the terms in association type 1:
\# ((1.2).3). 4
type1 := []:
for $t$ in $s 4$ do
if $\mathrm{t}[1]<\mathrm{t}[2]$ then type1 := [op(type1),t] fi
od:
\# generate the terms in association type 2:
\# (1.2).(3.4)
type2 := []:
for $t$ in $s 4$ do
if ( $\mathrm{t}[1]<\mathrm{t}[2]$ ) and ( $\mathrm{t}[3]<\mathrm{t}[4]$ )
and $(\min (t[1], t[2])<\min (t[3], t[4]))$
then type2 := [op(type2),t] fi
od:
\# initialize the expansion matrix
expmat := matrix $(24,15)$ :
for i to 24 do for j to 15 do
expmat $[i, j]:=0$
od od:
\# expand the 15 commutative terms and store the \# coefficients in expmat

```
col := 0:
for t in type1 do
    col := col+1:
    e := [
        [t[1],t[2],t[3],t[4]], [t[2],t[1],t[3],t[4]],
        [t[3],t[1],t[2],t[4]], [t[3],t[2],t[1],t[4]],
        [t[4],t[1],t[2],t[3]], [t[4],t[2],t[1],t[3]],
        [t[4],t[3],t[1],t[2]], [t[4],t[3],t[2],t[1]] ]:
    for p in e do
        row := 1: while s4[row]<>p do row := row+1 od:
        expmat[row,col] := 1
    od
```

od:
for $t$ in type2 do
col := col+1:
e := [
[t[1], t[2], t[3], t[4]], [t[2], t[1], t[3], t[4]],
[t[1], t[2], t[4], t[3]], [t[2], t[1], t[4], t[3]],
$[\mathrm{t}[3], \mathrm{t}[4], \mathrm{t}[1], \mathrm{t}[2]],[\mathrm{t}[3], \mathrm{t}[4], \mathrm{t}[2], \mathrm{t}[1]]$,
[t[4], t[3], t[1], t[2]], [t[4], t[3], t[2], t[1]] ]:
for $p$ in e do
row := 1: while s4[row]<>p do row := row+1 od:
expmat[row,col] := 1
od
od:
\# compute the row canonical form of expmat rcfexpmat $:=$ linalg[rref] (expmat):
\# compute a basis for the nullspace of expmat nulexpmat := linalg[nullspace] (expmat):

TABLE THREE
The expansion matrix for
the Jordan product in Degree 4
$\frac{1}{8}\left(\begin{array}{lllllllllllllll}1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0\end{array}\right)$

Computing the row canonical form one step at a time requires 84 row operations:

| W (\%, 1, 7,-1); | addrow (\%, 1, 13, -1) ; |
| :---: | :---: |
| drow (\%, 1, 15,-1); | addrow (\%, 1, 19, -1) ; |
| drow (\%, 1,21,-1); | addrow (\%, 1, 23, -1) ; |
| drow (\%, 1, 24, -1) ; | addrow (\%, 2, 8,-1) ; |
| drow (\%, 2,13,-1); | addrow (\%, 2,15,-1) ; |
| ddrow (\%, 2,17,-1); | addrow (\%, 2,18,-1) ; |
| ddrow (\%, 2, 19,-1); | addrow (\%, 2, 21, -1) ; |
| ddrow (\%, 3, 7,-1) ; | addrow (\%, 3, 9,-1) ; |
| ddrow (\%, 3, 13, -1) ; | addrow (\%, 3, 20, -1) ; |
| addrow (\%, 3,21,-1) ; | addrow (\%, 3, 22, -1) ; |
| ddrow (\%, 3,23,-1); | addrow (\%, 4, 7,-1) ; |
| drow (\%, 4, 9,-1) ; | addrow (\%, 4,11,-1) ; |
| addrow (\%, 4,12,-1) ; | addrow (\%, 4,14,-1) ; |
| ddrow (\%, 4,20,-1); | addrow (\%, 4, 23, -1) ; |
| ddrow (\%, 5, 8,-1) ; | addrow (\%, 5, 11, -1) ; |
| drow (\%, 5,14,-1); | addrow (\%, 5, 15, -1) ; |
| addrow (\%, 5, 16, -1) ; | addrow (\%, 5,17,-1) ; |
| addrow (\%, 5, 19, -1) ; | addrow (\%, 6, 8,-1) ; |
| drow (\%, 6, 9,-1); | addrow (\%, 6,10,-1) ; |
| addrow (\%, 6, 11, -1) ; | addrow (\%, 6, 14, -1) ; |
| addrow (\%, 6,17,-1) ; | addrow (\%, 6, 20, -1) ; |
| ulrow (\%, 7, -1/2); | addrow (\%, 7, 1,-1); |
| addrow (\%, 7, 3,-1); | addrow (\%, 7,13, 2); |
| addrow (\%, 7,21, 2); | addrow (\%, 7, 23, 2) ; |
| lrow (\%, 8, -1/2); | addrow (\%, 8, 5, -1) ; |
| addrow (\%, 8, 6,-1) ; | addrow (\%, 8, 7,-1) ; |
| ddrow (\%, 8, 9, 2); | addrow (\%, 8,11, 2); |
| addrow (\%, 8,14, 2); | addrow (\%, 8,15, 2); |

addrow (\%, 8,17, 2);
addrow (\%, 8,20, 2);
addrow (\%, 9, 2,-1);
addrow (\%, 9, 7, 1);
addrow (\%, 9,11,-2);
addrow (\%, 9, 20,-2);
mulrow (\%,10, -1/2);
addrow (\%,10, 2,-1);
addrow (\%,10,19, 2);
mulrow (\%,11, 1/2);
addrow (\%,11, 4,-1);
addrow (\%,11, 8,-1);
addrow (\%,11,10, 1);
addrow (\%,11,15, 2);
addrow (\%, 8,19, 2);
mulrow (\%, 9, 1/2);
addrow (\%, 9, 6, 1);
addrow (\%, 9, 8,-1);
addrow (\%, 9,14,-2);
swaprow (\%,10,13) ;
addrow (\%,10, 1,-1);
addrow (\%,10,15, 2);
addrow (\%,10,21, 2);
addrow (\%,11, 2,-2);
addrow (\%,11, 6, 1);
addrow (\%,11, 9, 1);
addrow (\%,11,14,-2);
addrow (\%,11,19, 2);

TABLE FOUR

The Row canonical form of The Expansion matrix

$$
\left(\begin{array}{rrrrrrrrrrrrrr}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

The four row vectors of the following matrix form a basis of the nullspace of the expansion matrix:

$$
\left[\begin{array}{lllllllllllllll}
0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & -1 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

In terms of the 15 commutative terms these row vectors correspond to the following four identities:

$$
\begin{aligned}
& -((W \circ X) \circ Z) \circ Y-((W \circ Y) \circ Z) \circ X-((X \circ Y) \circ Z) \circ W \\
& \quad+(W \circ X) \circ(Y \circ Z)+(W \circ Y) \circ(X \circ Z)+(W \circ Z) \circ(X \circ Y)=0 \\
& \quad-((W \circ X) \circ Z) \circ Y-((W \circ Y) \circ Z) \circ X+((X \circ Y) \circ W) \circ Z \\
& \quad-((X \circ Y) \circ Z) \circ W+((X \circ Z) \circ Y) \circ W+((Y \circ Z) \circ W) \circ X=0 \\
& \quad-((W \circ X) \circ Z) \circ Y+((W \circ Y) \circ X) \circ Z-((W \circ Y) \circ Z) \circ X \\
& \quad+((W \circ Z) \circ X) \circ Y-((X \circ Y) \circ Z) \circ W+((Y \circ Z) \circ X) \circ W=0 \\
& \quad((W \circ X) \circ Y) \circ Z-((W \circ X) \circ Z) \circ Y-((W \circ Y) \circ Z) \circ X \\
& \quad+((W \circ Z) \circ Y) \circ X-((X \circ Y) \circ Z) \circ W+((X \circ Z) \circ W) \circ Y=0
\end{aligned}
$$

Note that the first of these identities is the only one that involves terms in both association types. If we set $W=X=Y$ in this identity we get

$$
-3((W \circ W) \circ Z) \circ W+3(W \circ W) \circ(Z \circ W)=0
$$

If char $F \neq 3$ then we get

$$
((W \circ W) \circ Z) \circ W=(W \circ W) \circ(Z \circ W),
$$

which is called the Jordan identity. It's not hard to show (using the procedure called linearizing an identity) that the Jordan identity implies the identity we started with, so these two identities are equivalent. In fact the Jordan identity implies all of the four identities in the nullspace of the expansion matrix. (Exercise!) The variety of Jordan algebras is defined to be the class of algebras which satisfy the commutative and Jordan identities.

It is easy to check directly that the Jordan product satisfies the Jordan identity. The advantage of using linear algebra is that it shows that the Jordan identity is (essentially) the only multilinear identity in degree 4 satisfied by the Jordan product.

At this point the theories of Lie and Jordan algebras diverge dramatically. There are identities of higher degree satisfied by the Jordan product that do not follow from the commutative and Jordan identities. These identities are called special identities (or s-identities for short). The simplest s-identity is the Glennie identity in degree 8; it was discovered in 1966:

$$
\begin{aligned}
& 2\{\{Y\{X Z X\} Y\} Z(X \circ Y)\}-\{Y\{X\{Z(X \circ Y) Z\} X\} Y\} \\
&-2\{(X \circ Y) Z\{X\{Y Z Y\} X\}\}+\{X\{Y\{Z(X \circ Y) Z\} Y\} X\}=0 .
\end{aligned}
$$

Here we have used the Jordan triple product which is defined by

$$
\{X Y Z\}:=(X \circ Y) \circ Z+(Y \circ Z) \circ X-(Z \circ X) \circ Y .
$$

Efim Zelmanov won the Fields Medal at the International Congress of Mathematicians in Zurich in 1994 for his work on the Burnside Problem in group theory. Before that he had solved some of the most important open problems in the theory of Jordan algebras. In particular he proved that Glennie's identity generates all s-identities in the following sense: if $G$ is the T-ideal generated by the Glennie identity in the free Jordan algebra $F J(X)$ on the set $X$ (where $X$ has at least 3 elements), then the ideal $S(X)$ of all s-identities is quasi-invertible modulo $G$ (and its homogeneous components are nil modulo $G$ ).

This is Theorem 6.7 in the paper Zelmanov's Prime Theorem for Quadratic Jordan Algebras by Kevin McCrimmon (Journal of Algebra 76 (1982) 297-326). Roughly speaking, this means that all other s-identities can be obtained by substituting into the Glennie identity, generating an ideal, extracting $n$-th roots, and summing up.

A good introductory reference on Lie and Jordan algebras is Basic Algebra I by Nathan Jacobson; see Chapter 7 (Algebras over a field), especially section 7.5 (Non-associative algebras. Lie and Jordan algebras).

For a comprehensive introduction to the theory of Lie algebras, see Lie Algebras by Nathan Jacobson, or Introduction to Lie Algebras and Representation Theory by James Humphreys.

For a comprehensive introduction to the theory of Jordan algebras, see Structure and Representations of Jordan Algebras by Nathan Jacobson, or the forthcoming book by Kevin McCrimmon.

For more information about general nonassociative algebras (especially Jordan and alternative algebras) see An Introduction to Nonassociative Algebras by Richard Schafer, or Rings that are Nearly Associative by Zhevlakov, Slinko, Shestakov and Shirshov.

