# PRECALCULUS

A Study of Functions and Their Applications

## CHAPTER FIVE: TRIGONOMETRIC FUNCTIONS

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# 5.1 Two Ways of Defining Trigonometric Functions

In Section 2.5, you were introduced to periodic functions. This included an introduction to the two most commonly used periodic functions, sine and cosine. In this section, we will look at these functions in more detail and also define the tangent function. We will define these functions in two different ways, examine some of their properties, and look at ways to measure angles.

## Two Definitions of Sine and Cosine

The periodic functions sine and cosine were introduced in Section 2.5. Our definitions for these involved the unit circle. Recall that we defined $\sin x$ as the vertical position of a point on the unit circle where $x$ is the distance on the circle from the point $(1,0)$ *counter-clockwise* to the point. We also defined $\cos x$ as the horizontal position of a point on the unit circle. So the coordinates of this point are $(\cos x, \sin x)$. (See Figure 1.)

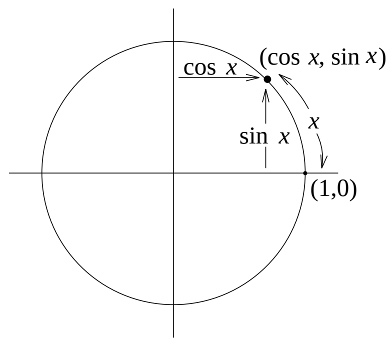



Figure 1: The coordinates of a point on the unit circle centered at the origin are $(\cos x, \sin x)$.

The sine and cosine functions can also be defined in terms of triangles. If you have a right triangle with angle $x$, then the definitions of sine and cosine are $\sin x = \frac{\text{opposite}}{\text{hypotenuse}}$ and $\cos x = \frac{\text{adjacent}}{\text{hypotenuse}}$. (See Figure 2.) Obviously, the two ways of defining these functions must be related

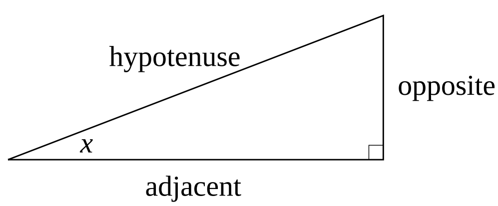



Figure 2: A right triangle showing $\sin x = \frac{\text{opposite}}{\text{hypotenuse}}$ and $\cos x = \frac{\text{adjacent}}{\text{hypotenuse}}$.

or they wouldn't be called the same thing. To see this relationship, look at Figure 3. Using the right triangle definitions, the length of the vertical side of the triangle is equal to $\sin x$ (since the length of the hypotenuse is one) and the length of the horizontal side of the triangle is equal to $\cos x$ (again, since the hypotenuse is equal to one). The lengths of these two lines also give the coordinates of the point $P$. So, the circle definitions and the right triangle definitions of sine and cosine seem to be the same. But are they?

Recall that, when thinking of $\sin x$ and $\cos x$ in terms of right triangles, the input is the angle. However, when thinking of $\sin x$ and $\cos x$ in terms of the unit circle, the input is the arc length. Figure 3 leads us to believe the two ways of defining these functions are the same until we realize that the inputs do not correspond.

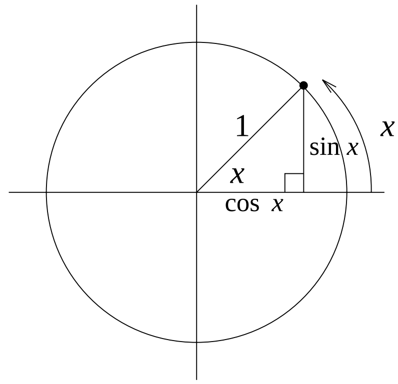



Figure 3: The sine and cosine functions for a right triangle in a unit circle.

The solution to this discrepancy is to have the angle equal to the arc length on the unit circle, thus forcing the inputs to be the same. This is exactly the purpose of using radians when working with $\sin x$ and $\cos x$ functions. The **radian** measure of an angle is the length of the arc on the unit circle intercepted by that angle. (See Figure 4.)

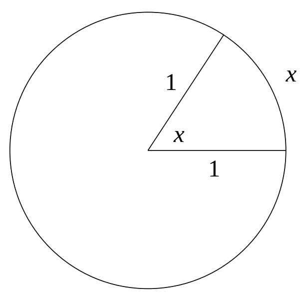



Figure 4: The measure of angle $x$ is $x$ radians.

Since the arc length of the unit circle is $2\pi \cdot 1 = 2\pi$, this means that the radian measure of the angle which intercepts the entire circle is also $2\pi$. The radian measure of the angle which intercepts half of the circle must be $\pi$, the radian measure of the angle which intercepts a quarter of the circle must be $\frac{\pi}{2}$, etc. As long as you are using radians for your input, the right triangle definitions and the unit circle definitions of sine and cosine give you the same answer for $0 \leq x \leq \frac{\pi}{2}$.

Why have two definitions for sine and cosine? The right triangle definitions are useful for solving problems involving right triangles. We do not have to think of these triangles as being in a circle and we are not limited to having the length of the hypotenuse equal to one. However, the right triangle definitions also have limitations. For these definitions, the input for sine or cosine must be one of the acute angles in the right triangle. That limits your input to being an angle between $0°$ and $90°$. This limitation also does not allow you to see the periodic nature of these functions. In the unit circle definition, we do not have to limit ourselves to a domain of $0° < x < 90°$ or, equivalently, $0 < x < \frac{\pi}{2}$. In fact, we do not even have to limit ourselves to a domain of $0 \leq x \leq 2\pi$ which is one revolution around the circle. If $x$ is greater than $2\pi$, the input just wraps around the circle more than once. If $x$ is negative, the input is the distance moving clockwise from $(1, 0)$ instead of counter-clockwise. (See Figure 5.)

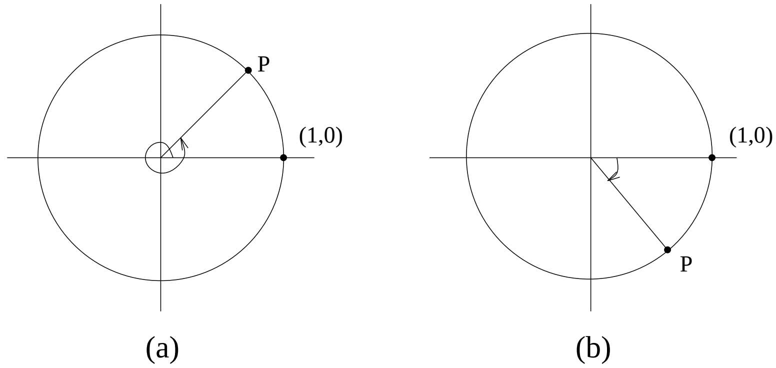
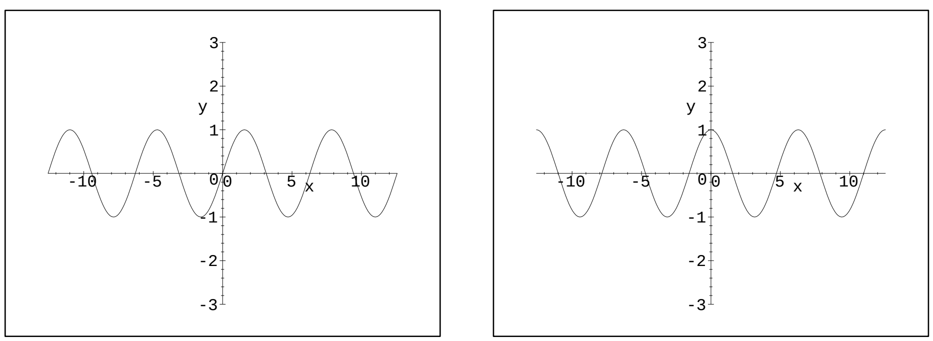



Figure 5: Angles can be greater than $2\pi$, as in shown in Figure 5(a), and negative, as shown in Figure 5(b).

So, using the unit circle definitions, we see that the domain of the sine and cosine functions is any real number and that the outputs will repeat every $2\pi$ radians. Therefore, these functions are periodic with periods of $2\pi$. The unit circle definitions also allow us to see that the range of the sine and cosine functions is $-1$ to 1. Since $\sin x$ and $\cos x$ are coordinates of a point on the unit circle, $-1 \le \sin x \le 1$ and $-1 \le \cos x \le 1$. The domain, range, and the periodic behavior of these functions can be seen in the graphs in Figure 6.

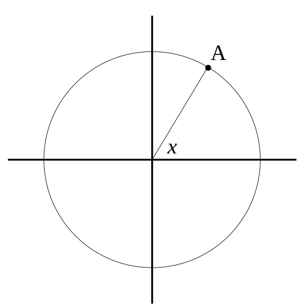



$y = \sin x$ $\qquad$ $y = \cos x$.

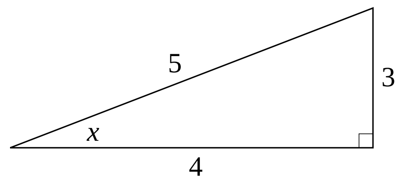
Figure 6: Graphs of $y = \sin x$ and $y = \cos x$.

### Reading Questions

**1.** The coordinates of point $A$ in the following figure are $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. What is $\cos x$?



**2.** Let $x$ represent the angle shown in the following triangle. What is $\sin x$?




**3.** Why is the input different when considering the circle definitions versus the triangle definitions of sine and cosine? How is this difference reconciled?

**4.** Why is it impossible to find an angle, $x$, such that $\cos x = 5$?

## Radians and Degrees

In addition to having two ways of defining trigonometric functions, there are also two ways of measuring angles. When you are using triangles, the input is typically given in degrees. However, when you are using circles or thinking of sine and cosine as periodic functions, the input is typically given in radians. Since, in both cases, the input is an angle, it is common to denote that angle with the Greek letter theta, $\theta$, instead of $x$. Writing $\cos\theta$ and $\sin\theta$ just reminds us that trigonometric functions are associated with angles.

Let's look at the connection between degrees and radians. Angles measured in degrees often have "nicer" numbers associated with them than angles measured in radians. This is because there are 360° in a circle and 360 has lots of factors such as 30, 45, 60, 90, and so on. However, radians are a more natural measurement of an angle since they are based on properties of the circle. Remember that when an angle in a unit circle is measured in radians, it is equal to the length of the arc that it intercepts. Therefore, an angle of one radian intercepts an arc of length one on a unit circle. (See Figure 7.)

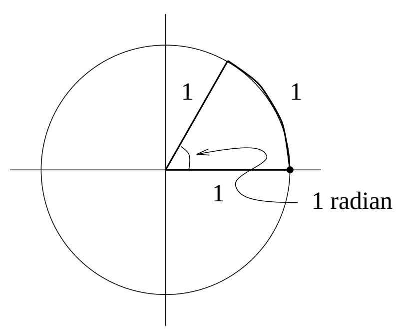



Figure 7: One radian is the measure of an angle that intercepts an arc of length one in a unit circle.

The use of radians can be expanded to include any circle. Figure 8 shows a one radian angle on a unit circle, a circle of radius 2, and a circle of radius $r$. Notice that as the radius of the circles increases, the length of the arc that is intercepted by an angle of one radian increases proportionally. On a circle of radius 2, an angle of one radian will intercept an arc that is 2 units long. On a circle of radius $r$, an angle of one radian will intercept an arc that is $r$ units long. In general, a one radian angle will intercept an arc that is the same length as the radius of the circle.

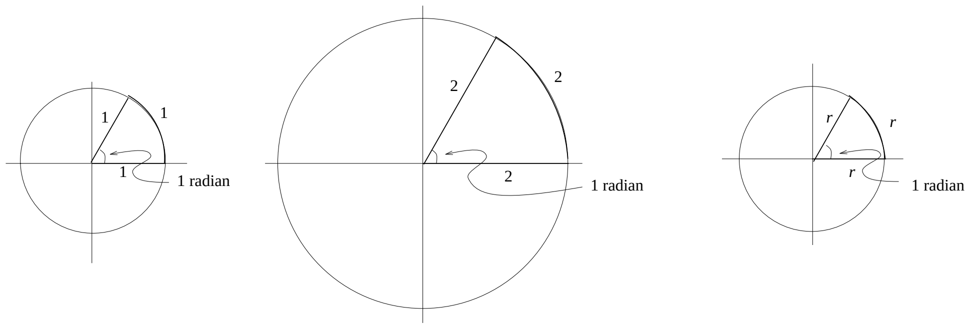



Figure 8: In a circle of radius $r$, a one radian angle intercepts an arc of length $r$.

Just as we can use radians to measure more than angles in a unit circle, we can also find the values of $\cos\theta$ and $\sin\theta$ for points on a circle of radius $r$. Figure 9 shows a right triangle drawn inside a circle of radius $r$. Since $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$, we have $\cos\theta = \frac{x}{r}$ which implies $x = r\cos\theta$. Similarly, we can show that $y = r\sin\theta$. So the coordinates of a point on a circle centered at the origin of radius $r$ will be $(r\cos\theta, r\sin\theta)$.

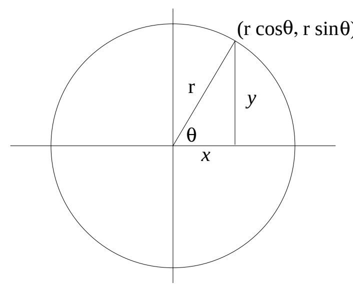



Figure 9: The coordinates of a point on a circle of radius $r$ are $(r\cos\theta, r\sin\theta)$.

Let's look at how to convert from radians to degrees. There are 360° in a circle. There are also $2\pi$ radians in a circle. Recall from Section 2.1 that most conversions of units of measurement can be represented by a linear function. Changing from degrees to radians is just converting units of measurement. For example, suppose you want to convert 90° to radians. Then

$$90^\circ \times \frac{2\pi \text{ radians}}{360^\circ} = 90^\circ \times \frac{\pi \text{ radians}}{180^\circ} = \frac{\pi}{2} \text{ radians}$$

In general,

$$\theta^\circ \times \frac{\pi \text{ radians}}{180^\circ} = \text{ measure of } \theta \text{ in radians.}$$

$$\theta \text{ radians} \times \frac{180^\circ}{\pi \text{ radians}} = \text{ measure of } \theta \text{ in degrees.}$$

**Example 1.** Convert 50° to radians.

**Solution:** To convert degrees to radians, we need to multiply by $\frac{\pi}{180^\circ}$.

$$50^\circ \cdot \frac{\pi}{180^\circ} = \frac{50\pi}{180} = \frac{5\pi}{18}\text{radians.}$$

■

**Example 2.** Convert $\frac{\pi}{6}$ radians to degrees.

**Solution:** To convert radians to degrees, we need to multiply by $\frac{180°}{\pi}$.

$$\frac{\pi}{6} \cdot \frac{180°}{\pi} = \frac{180°}{6} = 30°.$$

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Some angles, such as 30°, 45°, and 60°, occur frequently in applications. You should become accustomed to recognizing these common angles both in degrees and in radians. Figure 10 shows some of these common angles with both their degree measurement and their radian measurement.

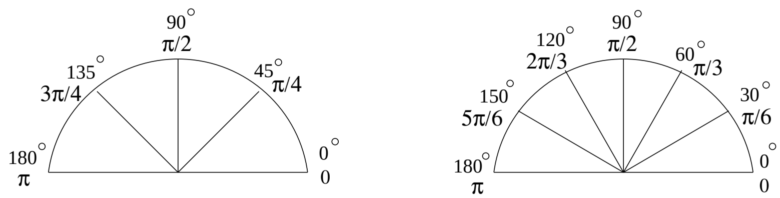
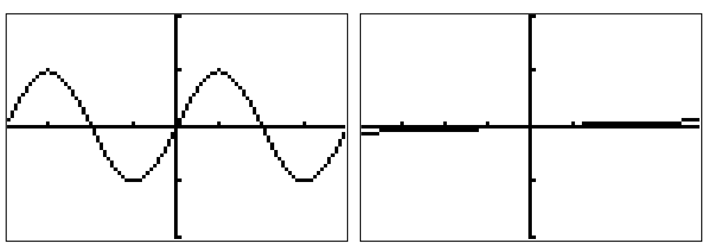



Figure 10: Angle measurements in degrees and radians for various angles.

**Technology Tip**

It is important to be sure that your calculator is interpreting angles with the same measurement units that you are. Many calculators allow you to select either degrees or radians using a menu that is accessed with a mode button. Having your calculator set in the wrong mode almost always gives you inaccurate results. For example, if the calculator is set in radians, then $\cos\frac{\pi}{2} = 0$. However, if the calculator is set in degrees, then $\cos\frac{\pi}{2}° \approx 0.99962$. The following graphs are of $y = \sin\theta$ for $-2\pi \le \theta \le 2\pi$. The graph on the left is when the calculator was in radian mode. The graph on the right is when the calculator was in degree mode.



$y = \sin\theta$ for $-2\pi \le \theta \le 2\pi$ $\qquad$ $y = \sin\theta$ for $-2\pi° \le \theta \le 2\pi°$

## Reading Questions

**5.** What is the arc length intercepted by a one radian angle in a circle of radius 5?

**6.** What is the $x$-coordinate of the point on a circle of radius 4 intercepted by an angle of $\pi$ radians?

**7.** Convert 45° to radians.

**8.** Convert $\frac{6\pi}{7}$ radians to degrees.

## The Tangent Function

Sine and cosine are not the only periodic functions associated with a triangle. Another commonly used trigonometric function is the tangent function. The **tangent** of $\theta$, $\tan\theta$, is defined as

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

While the tangent function is periodic, in many ways it behaves very differently than either sine or cosine. In a unit circle, sine and cosine are defined, respectively, as the $y$-coordinate and the $x$-coordinate of a point. Both of these are represented as *distances*. Tangent, however, is defined as the ratio of sine to cosine. This means that tangent is the ratio of a vertical distance to a horizontal distance. But such a ratio is the *slope* of a line passing through the origin. So, in a unit circle, $\tan\theta$ is the slope of the line which creates an angle of $\theta$ with the positive $x$-axis. (See Figure 11.)

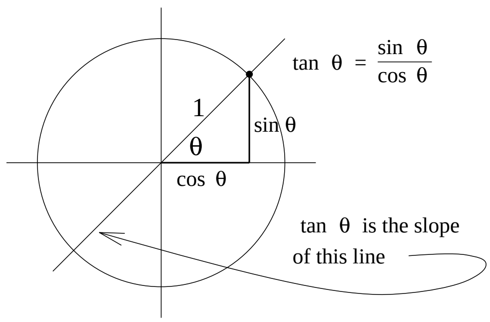



Figure 11: The unit circle definition of tangent is $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

There is also a right triangle definition of $\tan\theta$. Using our unit circle definition of tangent, our right triangle definitions of sine and cosine, and Figure 12 we have:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\text{opposite}}{\text{hypotenuse}}}{\frac{\text{adjacent}}{\text{hypotenuse}}} = \frac{\text{opposite}}{\text{adjacent}}.$$

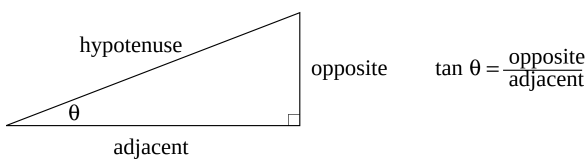



Figure 12: The right triangle definition of tangent is $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$.

Figure 13 shows a unit circle and a graph of $y = \tan\theta$. As the angle between the $x$-axis and a line passing through a point in the first quadrant increases, the slope of that line also increases. Starting at the $x$-axis, the slope is zero since you have a horizontal line. As the angle gets closer to $\frac{\pi}{2}$, the slope gets closer to infinity. This is because the line is becoming more and more similar to a vertical line. Also, $\cos\frac{\pi}{2} = 0$, so $\tan\frac{\pi}{2}$ is undefined. The slope of a line passing through a point on the unit circle in the second quadrant is negative since the line is decreasing. Therefore, the tangent function is negative in the second quadrant. As the angle gets closer to $\pi$, the slope is heading toward zero. Therefore, the tangent function will increase and at $\theta = \pi$, the slope will equal zero. As you can see in Figure 13, the lines in the third quadrant are the same as the lines in the first quadrant just as the lines in the fourth quadrant are the same as those in the second. Because of this, the graph of $y = \tan\theta$ has a period of $\pi$. Every $\pi$ units, the slopes

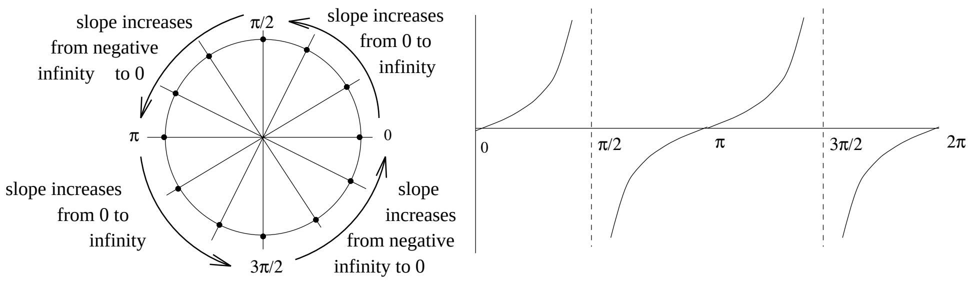



Figure 13: The tangent function is the slope of a line passing through the origin.

will repeat. The tangent function also has vertical asymptotes at $\frac{\pi}{2}$, $\frac{3\pi}{2}$, and every $\pi$ units after that. Symbolically, the tangent function has a vertical asymptote when $\theta = \frac{\pi}{2} + n\pi$ where $n$ is an integer. Notice that, unlike the sine and cosine, the range of the tangent function is not restricted. To summarize:

- $\tan\theta = \frac{\sin\theta}{\cos\theta}$ or, equivalently, $\tan\theta =$ slope of the line which creates an angle of $\theta$ with the positive $x$-axis.
- $y = \tan\theta$ is periodic with period $\pi$.
- The domain of $y = \tan\theta$ is all real numbers except for $\frac{\pi}{2} + n\pi$ where $n$ is an integer.
- The range of $y = \tan\theta$ is all real numbers.

### Historical Note[118]

The word *trigonometry* comes from the Greek word *trigon* meaning "triangle" and the Greek word *metron* meaning "a measure." So trigonometry is literally the measuring of triangles. The study of trigonometry goes back thousands of years. In 1858 a Scottish student of antiquities purchased a papyrus scroll that was written somewhere around 1800 BC. This scroll, now known as the Rhind papyrus (named after the man who purchased it) contains evidence that the ancient Egyptians knew many of the basics of trigonometry.

Even though there is additional evidence that the Babylonians also knew some trigonometry, it is Hippocrates, a Greek who lived in the second century BC, who is known as the "father of trigonometry." He has this distinction because he is the first person to publish a table of the ratios of arcs to chords for a series of angles. While Egyptian and Babylonian trigonometry seemed to be primarily related to triangles, the Greek tables were based on tables relating circles and their chords for use in astronomy.

One of the oldest trigonometric functions is the *gnomon* which is now called the *tangent.* The gnomon was originally used as a time-keeping device. A staff (called a gnomon by the Greeks) would be erected and its shadow would be measured. Not only would the length of the shadow depends on the time of the day, it would also depend on the time of the year since the shadow at noon is longest at the winter solstice and shortest at the summer solstice. The Greek gnomon and shadow functions became known in the Latin as the *umbra versa* (turned shadow) and the *umbra recta* (straight shadow). In the late 1500s, these functions became known as the tangent and the cotangent.

The modern-day approach to trigonometry using the unit circle as well as triangles came into being with Euler's work in the 1750's. Euler took an analytical approach to

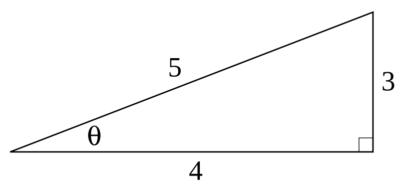
[118] Smith, D.E., *History of Mathematics, Volume II*, Dover Publications, Inc., 1953,, pp. 600–622.
Boyer, Carl B., A History of Mathematics, Princeton University Press, 1968, pp. 485–486.

mathematics. He unified the study of many different mathematical topics by looking at the abstract study of functions. For example, he considered the sine function to be the $y$-coordinate of a point on a unit circle rather than the length of a line segment used in finding the distance from the earth to the moon. It is this analytical approach which is now used in the study of trigonometry.

### Reading Questions

**9.** What is $\tan \frac{\pi}{4}$? [Hint: Recall that $\frac{\pi}{4} = 45°$ and think of your answer in terms of slope.]

**10.** Let $\theta$ represent the angle shown in the triangle shown below. What is $\tan \theta$?




**11.** Why does $\tan \theta$ have a vertical asymptote when $\theta = \frac{3\pi}{2}$?

**12.** Using a unit circle, draw a picture representing an angle where $\tan \theta = 5$.

## Determining Values for Sine, Cosine, and Tangent

You can use your calculator to determine approximate values for sine, cosine, and tangent. There are two things to remember when you do this. First of all, the answers that you get will most likely be approximations. For example, $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$. A calculator will give the decimal approximation as 0.8660254038. Secondly, make sure you have your calculator in the correct mode when determining a trigonometric function of an angle. If you are asked to find $\sin 10$, you need to know if this is $\sin 10°$ or $\sin$ (10 radians). Since there is no degree symbol, by default, we assume this is 10 radians. This gives $\sin 10 \approx -0.5440211109$.

**Example 3.** Determine decimal approximations of $\cos 4$ and $\cos 4°$.

**Solution:** Using a calculator in radian mode, we can determine that $\cos 4 \approx -0.6536$. Using a calculator in degree mode, we can determine that $\cos 4° \approx 0.9976$. ■

### Integer Multiples of $\frac{\pi}{2}$

Most of the time, we use a calculator to find approximate values for sine, cosine, or tangent. However, there are a few values which we can find exactly. We will start with the points that lie on either the $x$-axis or the $y$-axis. In the unit circle definitions of sine and cosine, sine is the vertical distance of a point on the unit circle and cosine is the horizontal distance. To be more precise, we should describe this distance as *directed distance*. Direction is important if we think of the coordinates of a point on the unit circle as $(\cos \theta, \sin \theta)$. For example, $\cos 0 = 1$ while $\cos \pi = -1$. (See Figure 14.)

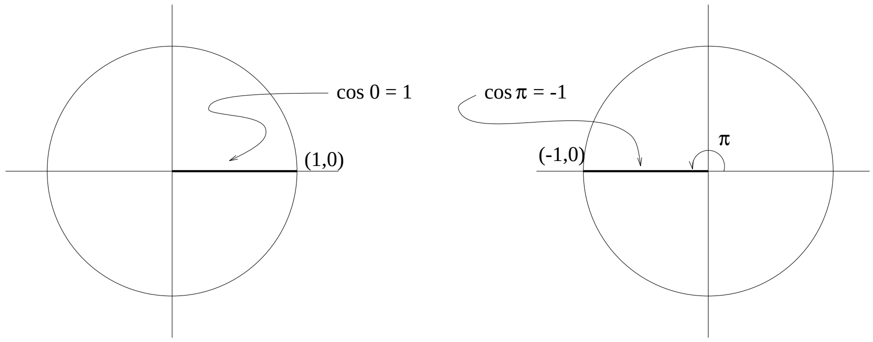



Figure 14: Directed distance is important as illustrated by $\cos 0 = 1$ and $\cos \pi = -1$.

**Example 4.** Determine the values of sine and cosine for $\theta = 0$, $\frac{\pi}{2}$, $\pi$, and $\frac{3\pi}{2}$.

**Solution:** Using Figure 15 as a guide and knowing that cosine is the $x$-coordinate of a point on the unit circle and sine is the $y$-coordinate, we can see that $\cos 0 = 1$, $\cos \frac{\pi}{2} = 0$, $\cos \pi = -1$, $\cos \frac{3\pi}{2} = 0$. We can also see that $\sin 0 = 0$, $\sin \frac{\pi}{2} = 1$, $\sin \pi = 0$, $\sin \frac{3\pi}{2} = -1$.

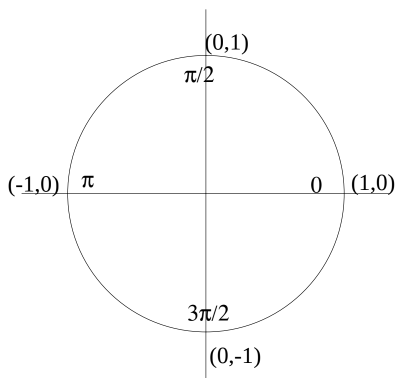



Figure 15

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It is also easy to determine values for the tangent function for $\theta = 0$, $\frac{\pi}{2}$, $\pi$, $\frac{3\pi}{2}$, since tangent is the ratio of sine and cosine or, equivalently, the slope of the line. Using Table 1 or Figure 15, we can see that $\tan 0 = \frac{0}{1} = 0$, $\tan \frac{\pi}{2}$ is undefined, $\tan \pi = \frac{0}{-1} = 0$, and $\tan \frac{3\pi}{2}$ is undefined.

**Integer Multiples of $\frac{\pi}{4}$ and $\frac{\pi}{6}$**

There are a few other angles for which we can find the exact values of sine, cosine, and tangent. These involve angles where it is possible to find the exact values of the sides of the triangles. In particular, because of their special properties, 45°- 45°- 90° and 30°- 60°- 90° triangles can be used to find the exact values of trigonometric functions for $\theta = 45°$ or $\frac{\pi}{4}$, $\theta = 30°$ or $\frac{\pi}{6}$, and $\theta = 60°$ or $\frac{\pi}{3}$.

For example, suppose we want to find $\sin \frac{\pi}{4}$. Knowing that sine is the $y$-coordinate of a point on the unit circle and $\frac{\pi}{4} = 45°$, we draw a 45°- 45°- 90° triangle inside the unit circle. (See Figure 16.) Using the Pythagorean theorem and Figure 16, we see that

$$a^2 + b^2 = 1^2.$$

However, $a = b$ since this is a 45° angle. Therefore, our equation is

$$b^2 + b^2 = 1.$$

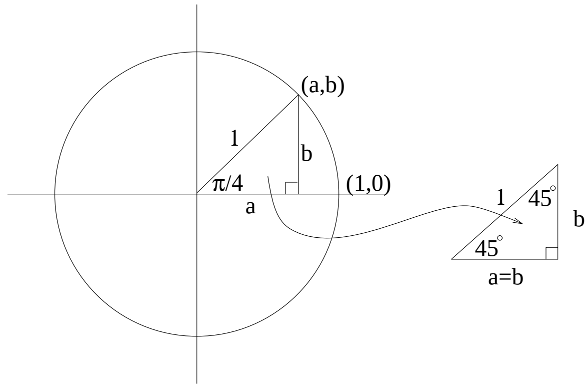



Figure 16: A 45°- 45°- 90° triangle inside the unit circle.

Solving this for $b$, we have

$$\begin{aligned} 2b^2 &= 1 \\ b^2 &= \frac{1}{2} \\ b &= \pm\sqrt{\frac{1}{2}} \\ b &= \pm\frac{\sqrt{2}}{2}. \end{aligned}$$

Since $b$ is positive, $b = \frac{\sqrt{2}}{2}$. Knowing $\sin\frac{\pi}{4}$ is the $y$-coordinate of our point,

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

The other triangle that we can use to find exact values for our trigonometric functions is a 30°- 60°- 90° triangle. For example, suppose we wanted to determine the value of $\cos\frac{\pi}{6}$. Knowing that cosine is the $x$-coordinate of a point on the unit circle and $\frac{\pi}{6} = 30°$, we draw a 30°- 60°- 90° triangle inside the unit circle. If we flip this triangle over the $x$-axis, a 60°- 60°- 60° triangle is formed. (See Figure 17.) A 60°- 60°- 60° triangle is equilateral and, therefore, has sides with

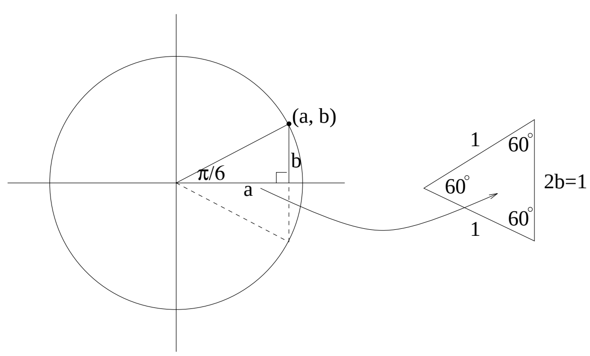



Figure 17: A 30°- 60°- 90° triangle inside the unit circle.

equal length. Since two of the sides are one unit, this implies that the third side is also one unit. So, $2b = 1$ or $b = \frac{1}{2}$. Using the Pythagorean theorem, we have

$$a^2 + \left(\frac{1}{2}\right)^2 = 1^2$$

$$\begin{aligned} a^2 + \frac{1}{4} &= 1 \\ a^2 &= \frac{3}{4} \\ a &= \pm\frac{\sqrt{3}}{2}. \end{aligned}$$

Since $a$ is positive, we know that $a = \frac{\sqrt{3}}{2}$. Knowing $\cos\frac{\pi}{6}$ is the $x$-coordinate of our point, we have

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

Remembering that the sides of a 30°- 60°- 90° triangle in a unit circle are $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, and 1, while the sides of a 45°- 45°- 90° triangle in a unit circle are $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{2}}{2}$, and 1, can help you determine the sine, cosine, and tangent of angles that are integer multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. (See Figure 18.) However, not all of these multiples will be in the first quadrant. For example, Figure 19 shows an

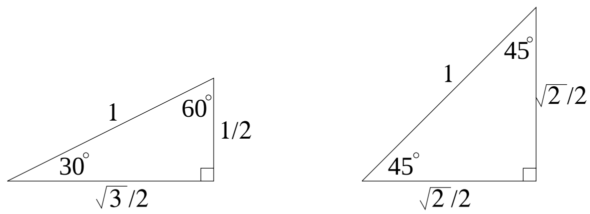



Figure 18: A 30°- 60°- 90° triangle and a 45°- 45°- 90° triangle.

angle of $\theta = \frac{2\pi}{3}$. To find the $x$-coordinate and the $y$-coordinate of the point on the unit circle (and hence $\cos\frac{2\pi}{3}$ and $\sin\frac{2\pi}{3}$), drop a perpendicular to the $x$-axis. Notice that this forms a triangle with

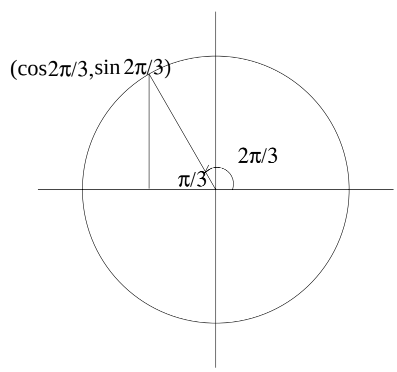



Figure 19: Finding $\cos\frac{2\pi}{3}$ and $\sin\frac{2\pi}{3}$ using the unit circle.

an angle of $\frac{\pi}{3}$. Knowing that $\cos\frac{\pi}{3} = \frac{1}{2}$ and that $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ almost gives us the coordinates of our point. However, since this point is in the second quadrant, the $x$-coordinate must be negative while the $y$-coordinate is positive. So $\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$.

Finding the exact values of the trigonometric functions for special angles involves not only knowing the triangles in Figure 18, but also remembering the signs of $\cos\theta$, $\sin\theta$, and $\tan\theta$ in each of the four quadrants. Since sine equals the $y$-coordinate, it will be positive in the first and second quadrants and negative in the other two. Since cosine equals the $x$-coordinate, it will be positive in the first and fourth quadrants and negative in the other two. Since tangent equals the slope (the ratio of sine and cosine), it will be positive in the first and third quadrant and negative in the other two. All of this is summarized in Figure 20.

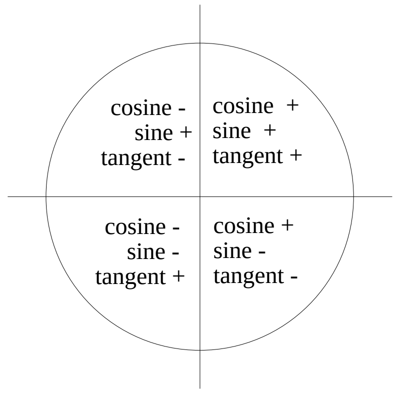



Figure 20: Summary of where the trigonometric functions are positive and negative.

**Example 5.** Determine $\sin \frac{5\pi}{4}$.

**Solution:** Since $\frac{5\pi}{4}$ is $\frac{\pi}{4}$ more than $\pi$, a 45°- 45°- 90° triangle is formed. (See Figure 21.) Since the vertical leg of this triangle is below the $x$-axis, its directed distance is negative. Therefore $\sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$. ■

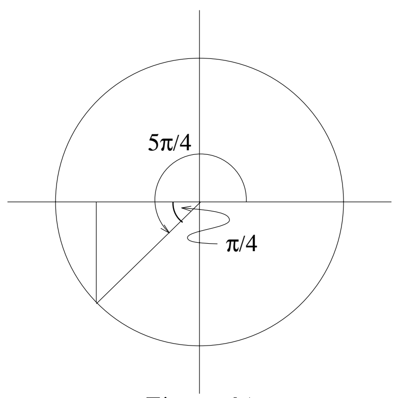



Figure 21

## Reading Questions

**13.** Determine a decimal approximation for $\cos 22°$ and $\cos 22$.

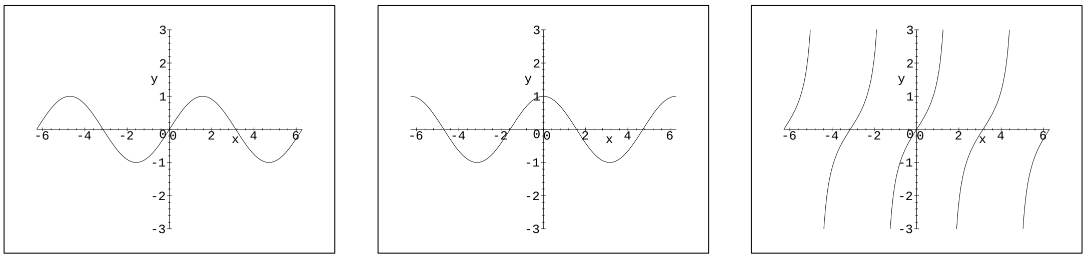
**14.** Which trigonometric function is positive in both the first and the third quadrants?

**15.** Find each of the following:

(a) $\cos \frac{3\pi}{2}$
(b) $\sin \frac{7\pi}{4}$
(c) $\tan \frac{4\pi}{3}$

## Identities Derived from the Graphs of Trigonometric Functions and the Unit Circle

We can derive some trigonometric identities just by looking at the graphs of trigonometric functions. Figure 22 shows the graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$. All of these graphs have some sort of symmetry. In Section 1.2, we categorized these types of symmetries as



$y = \sin x$ $y = \cos x$ $y = \tan x$

Figure 22: Graphs of three trigonometric functions with a domain of $-2\pi \leq x \leq 2\pi$.

*symmetric about the y-axis* and *symmetric about the origin*. You can see that the graph of the cosine function is symmetric about the $y$-axis since the part of the graph to the left of the $y$-axis is a reflection of the part to the right.

We also learned back in Section 1.2 that if a function, $f$, is symmetric about the $y$-axis then $f$ has the property $f(x) = f(-x)$. A function that has this property is called an *even function*. We can see that $\cos x$ is an even function by looking at the diagram of the unit circle in Figure 23. The input of $f(x) = \cos x$ is the distance on the circle from $(1, 0)$ to the point. If the $x$ is positive,

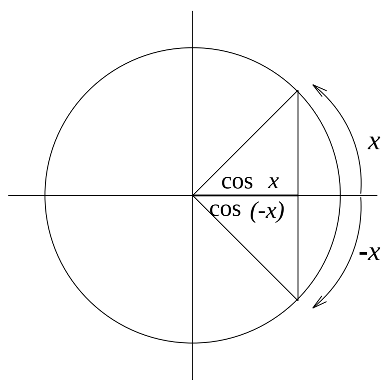



Figure 23: The same line segment represents $\cos x$ and $\cos(-x)$.

then the input is the distance moving *counter-clockwise* from the point $(1, 0)$. If $x$ is negative, then the input is the distance moving *clockwise* from the point $(1, 0)$. Regardless of whether $x$ is positive or negative, the output of $f(x) = \cos x$ is the horizontal coordinate of the point on the circle. For a given value of $x$, this horizontal position is the same for both $x$ and $-x$. (See Figure 23.) Therefore,

$$\cos x = \cos(-x).$$

Looking at the graphs in Figure 22, you can see that neither the sine function nor the tangent function are symmetric about the $y$-axis. These graphs are, however, symmetric about the origin. Remember that a graph is symmetric about the origin if, when rotated 180° with the origin being the center of rotation, it looks the same as it did before it was rotated.

If a function, $f$, is symmetric about the origin then $f$ has the property $f(x) = -f(-x)$. A function that has this property is called an *odd function*. We can see that $\sin x$ is an odd function by looking at the diagram of the unit circle in Figure 24. The input of $f(x) = \sin x$ is the same as

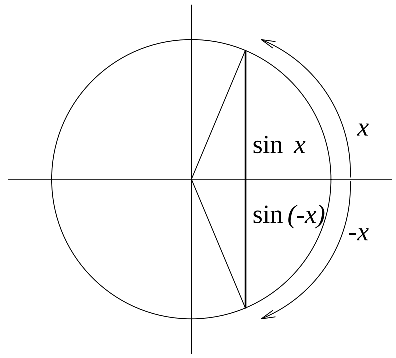



Figure 24: The line segment representing $\sin x$ is the same length as that representing $\sin(-x)$, but is below the $x$-axis so it will have the opposite sign.

the input for $f(x) = \cos x$. The output, however, is the vertical position of the point on the circle. For a given value of $x$, this vertical position is the same distance away from the $x$-axis regardless of whether $x$ is positive or negative. However, in one case the output is positive while in the other case the output is negative. (See Figure 24.) Therefore,

$$\sin x = -\sin(-x).$$

A similar situation occurs with the tangent function showing that

$$\tan x = -\tan(-x).$$

The unit circle definitions of sine and cosine also lead us to the most widely used trigonometric identity. Recall that $\cos x$ is the horizontal position of a point on the unit circle and $\sin x$ is the vertical position of a point on the unit circle. Using Figure 25 and the Pythagorean Theorem, we see that

$$\cos^2 x + \sin^2 x = 1.$$

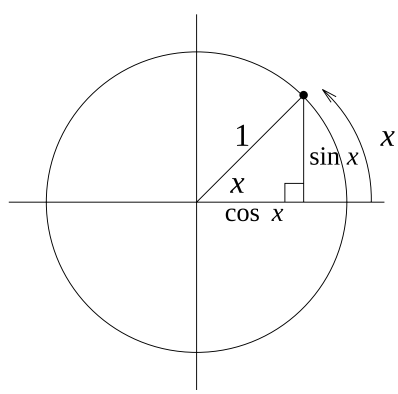



Figure 25: Deriving the identity $\cos^2 x + \sin^2 x = 1$.

Note: The notation $\cos^2 x$ means $(\cos x)^2$, i.e. find the cosine of an angle and then square your answer.

**Example 6.** If $\sin\theta = \frac{2}{3}$, find $\cos\theta$.

**Solution:** Using the identity $\cos^2\theta + \sin^2\theta = 1$ and substituting our value for $\sin\theta$ we have

$$\cos^2\theta + \left(\frac{2}{3}\right)^2 = 1.$$

Solving this for $\cos^2\theta$ we get

$$\cos^2\theta = \frac{5}{9}.$$

Therefore,

$$\cos\theta = \pm\frac{\sqrt{5}}{3}.$$

■

**Example 7.** If $\tan\theta = 2$, find $\sin\theta$ and $\cos\theta$.

**Solution:** Since $\tan\theta = \frac{\sin\theta}{\cos\theta} = 2$ then $\sin\theta = 2\cos\theta$. Using the identity $\cos^2\theta + \sin^2\theta = 1$ and substituting $2\cos\theta$ for $\sin\theta$ we have

$$\cos^2\theta + 4\cos^2\theta = 5\cos^2\theta = 1.$$

Solving this for $\cos^2\theta$ we get

$$\cos^2\theta = \frac{1}{5}.$$

Therefore,

$$\cos\theta = \pm\sqrt{\frac{1}{5}} = \pm\frac{\sqrt{5}}{5}.$$

Since $\sin\theta = 2\cos\theta$,

$$\sin\theta = \pm\frac{2\sqrt{5}}{5}.$$

Since $\tan\theta$ is positive, both $\sin\theta$ and $\cos\theta$ are positive or both $\sin\theta$ and $\cos\theta$ are negative. So, either $\cos\theta = \frac{\sqrt{5}}{5}$ and $\sin\theta = \frac{2\sqrt{5}}{5}$ or $\cos\theta = -\frac{\sqrt{5}}{5}$ and $\sin\theta = -\frac{2\sqrt{5}}{5}$. ■

### Reading Questions

**16.** If $\cos\theta = 0.34$, what is $\cos(-\theta)$?

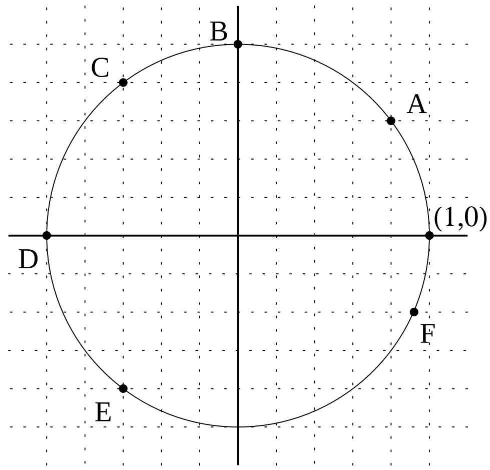
**17.** Explain why $y = \tan x$ is an odd function.

**18.** Suppose $\sin\theta = \frac{\sqrt{15}}{4}$. Use the identity $\cos^2\theta + \sin^2\theta = 1$ to find $\cos\theta$.

## Exercises

**1.** Why do we use radians rather than degrees when looking at the unit circle definitions of $\cos\theta$ and $\sin\theta$?

**2.** Use the following figure to approximate the values of the sine, cosine, and tangent functions at the indicated points.




**3.** Convert the following angle measurements from degrees to radians.

(a) $45°$
(b) $36°$
(c) $-60°$
(d) $-95°$
(e) $145°$
(f) $315°$

**4.** Convert the following angle measurements from radians to degrees.

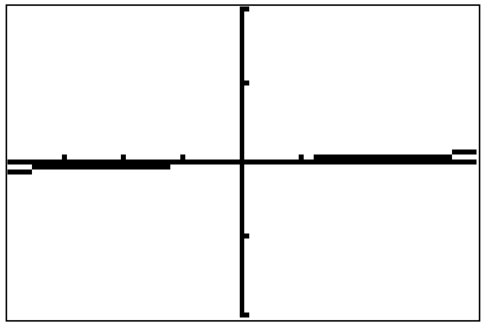
(a) $3\pi$
(b) $\frac{5\pi}{6}$
(c) $\frac{2\pi}{9}$
(d) $\frac{4\pi}{5}$
(e) $1.5$
(f) $6$

**5.** Use your calculator to find decimal approximations for each of the following.

(a) $\cos 42°$
(b) $\sin \frac{\pi}{7}$
(c) $\tan 132°$
(d) $\tan 132$

**6.** With the calculator in degree mode rather than radian mode, we made the following calculator graph of $y = \sin\theta$ for $-2\pi \le \theta \le 2\pi$. The graph increases very little in this viewing window. Explain why the calculator produced such a graph.

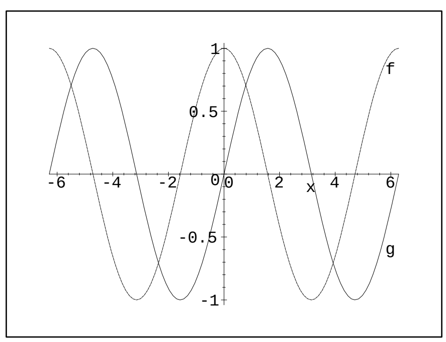


**7.** Using the right-triangle definition, we say that $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$. Explain why this is equivalent to saying that $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

**8.** (a) Using the definition that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, explain why $\tan\theta$ has a vertical asymptote for $\theta = \frac{\pi}{2}$.

(b) Using the definition that $\tan\theta$ is the slope of the line which makes an angle of $\theta$ with the positive $x$-axis, explain why $\tan\theta$ has a vertical asymptote for $\theta = \frac{\pi}{2}$.

**9.** The following graphs are of $y = \cos\theta$ and $y = \sin\theta$.



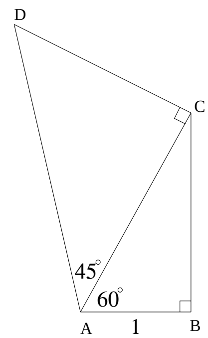

(a) Which graph is which?

(b) Describe the points on the graphs where $\tan\theta = 1$.

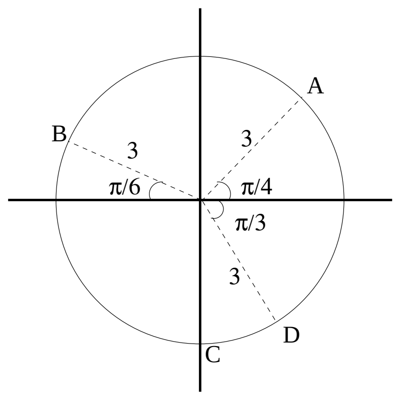
**10.** Complete the following table by filling in the missing entries.

| $\theta$ (in radians) | $\theta$ (in degrees) | $\cos\theta$ | $\sin\theta$ | $\tan\theta$ |
|---|---|---|---|---|
| $\frac{\pi}{6}$ | | | | |
| | $45°$ | | | |
| $\frac{\pi}{3}$ | | | | |
| | | $-\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | |
| | $330°$ | | | |
| | | | $\frac{\sqrt{2}}{2}$ | $-1$ |

**11.** In the following figure, the length of side AB is one unit. The measurement of $\angle CAB$ is $60°$ and the measurement of $\angle DAC$ is $45°$. Find the exact measurement of the length of side AD.




**12.** The circle below has a radius of 3 units. Give the $x$ and $y$ coordinates of each point marked on the circle.




**13.** Let $A$ be a point in the first quadrant such that the slope of the line connecting $A$ to the origin is $\sqrt{3}$ and the distance from $A$ to the origin is 5 units. Find the coordinates of point $A$.

**14.** Let $B$ be a point in the second quadrant such that the slope of the line connecting $B$ to the origin is $-1$ and the distance from $B$ to the origin is 4 units. Find the coordinates of point $B$.

**15.** Let $\theta$ be an angle in the first quadrant such that $\sin\theta = \frac{4}{5}$. Find the exact values of each of the following:

(a) $\cos\theta$

(b) $\tan\theta$

(c) $\sin(-\theta)$

**16.** Let $\theta$ be an angle in the first quadrant such that $\tan\theta = 3$. Find the exact values of each of the following:

(a) $\cos\theta$

(b) $\sin\theta$

(c) $\tan(\theta + \pi)$

**17.** Let $\theta$ be an angle in the first quadrant such that $\cos\theta = \frac{1}{6}$. Find the exact values of each of the following:

(a) $\sin\theta$

(b) $\tan\theta$

(c) $\cos(\theta+\pi)$

**18.** Let $\theta$ be an angle in the first quadrant such that $\cos\theta = b$.

(a) What are the restrictions, if any, on the value of $b$?

(b) Represent each of the following in terms of $b$.

i. $\sin\theta$
ii. $\tan\theta$
iii. $\cos(-\theta)$
iv. $\cos(\theta+2\pi)$

**19.** Let $\theta$ be an angle in the first quadrant such that $\tan\theta = c$.

(a) What are the restrictions, if any, on the value of $c$?

(b) Represent each of the following in terms of $c$.

i. $\sin\theta$
ii. $\cos\theta$
iii. $\tan(-\theta)$
iv. $\tan(\theta+\pi)$

**20.** (a) Explain, using the unit circle definition of the sine function, why $\sin\theta$ is an odd function.

(b) Explain why you can't use the right triangle definition of the sine function to show $\sin\theta$ is an odd function.

**21.** What's wrong with the following logic?

> Using the equation for a circle, I know that the points on the circle, $(x, y)$, satisfy the relationship $x^2 + y^2 = r^2$ where $r$ is the radius of the circle. I also know that the coordinates on a circle are represented by trigonometric functions. So the $x$-coordinate is equal to $\cos\theta$ and the $y$-coordinate is equal to $\sin\theta$. Using a circle of radius 3, I can conclude that $\cos^2\theta + \sin^2\theta = 9$.

**22.** Each of the following statements is false. Rewrite them so they become true statements.

(a) The domain of $y = \sin\theta$ is $-1 \le \theta \le 1$.

(b) The tangent function is defined for all real numbers.

(c) $\tan\frac{\pi}{3} = 1$.

(d) $\sin\theta + \cos\theta = 1$.

(e) $y = \sin\theta$ is a periodic function with period $\pi$.

(f) $y = \cos\theta$ is an odd function.

(g) $\sin\theta = \sin(-\theta)$.

**23.** What is the difference between $\sin^2\theta$ and $\sin\theta^2$?

## Investigations

### Investigation 1: A Stitch in Time.

Part of the process of making a quilt involves piecing shapes together. One of the most commonly used shapes is a triangle. Different types of material are often used with these pieces to create intricate patterns. After the pattern is designed, the quilter must determine the sizes of the various pieces. To allow the pieces to be sewn together, a seam allowance must be added to each edge of each piece. In quilting, this seam allowance is usually $\frac{1}{4}$ inch. This must be very precise. If the piece is cut incorrectly, the quilt pieces will not fit together as they should.

**1.** Isosceles right triangles are frequently used by quilters. The first thing a quilter determines is the size of the finished shape. Then the quilter determines the actual size of the piece that needs to be cut, allowing for the seam. Figure 1 shows an isosceles right triangle with

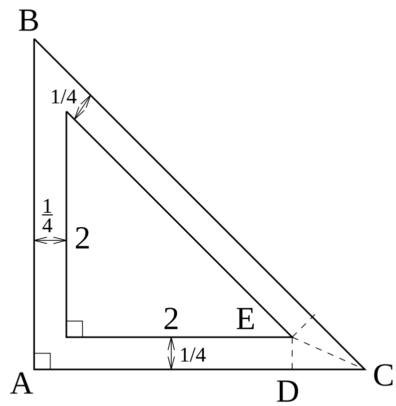



Figure 1

legs 2 inches long and the $\frac{1}{4}$ inch seam allowance added to all three sides.

(a) Find $AC$ by doing the following.

i. Explain why $AC = \frac{1}{4} + 2 + DC$.

ii. Use $\triangle DEC$ and an appropriate trigonometric function to approximate $DC$.

iii. What is $AC$ to the nearest $\frac{1}{16}$ inch?

(b) Find the length of the legs of the triangle that should be cut out if the finished triangle were to have legs which measure 3 inches instead of 2 inches. How does the amount you added to each leg of this triangle compare to the amount you added in question 1(a)?

(c) Find the length of the legs of the triangle that should be cut out if your finished piece were $n$ inches long. Justify your answer.

**2.** In a different quilt, the design calls for equilateral triangles with 2 inch sides. (See Figure 2.) The quilter must still add the $\frac{1}{4}$ inch seam allowance. What length should the sides of this triangle be? Can you use the same procedure that you used in question 1(a)? Why or why not?

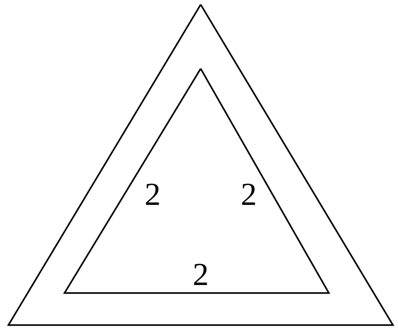



Figure 2

## Investigation 2: Trigonometric Functions Shown as Line Segments with a Unit Circle.

As you have seen, sine and cosine can be defined as lengths of line segments in a unit circle. This can be seen in Figure 1. In this investigation, we will see ways to define other trigonometric functions as line segments.

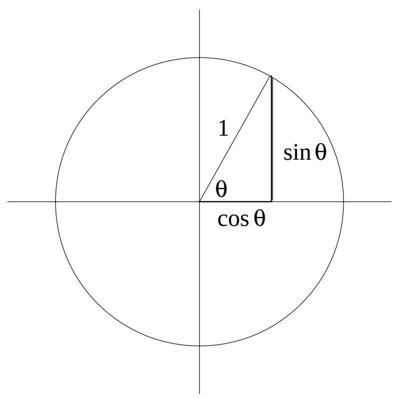



Figure 1

**1.** In this section, we described the tangent function as the slope of a line passing through the origin. However, we can also define the tangent function as the length of a line segment if the picture is drawn correctly. Figure 2 shows a right triangle in a unit circle. Explain why the length of segment $AB$ is $\tan\theta$.

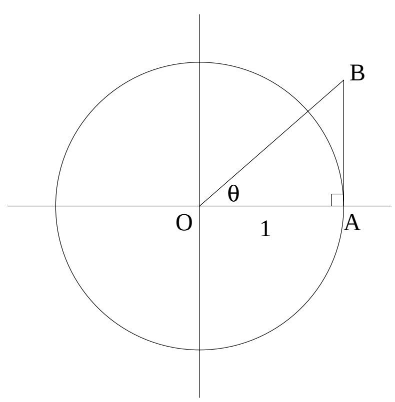



Figure 2

**2.** There are three other trigonometric functions that are commonly used. They are the cosecant function, written $\csc\theta$, the secant function, written $\sec\theta$, and the cotangent function, written $\cot\theta$. These three functions are defined as follows:

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}.$$

(a) Figure 3 shows the right triangle $OCD$ in a unit circle. Explain why the length of segment $OC$ is $\csc\theta$.

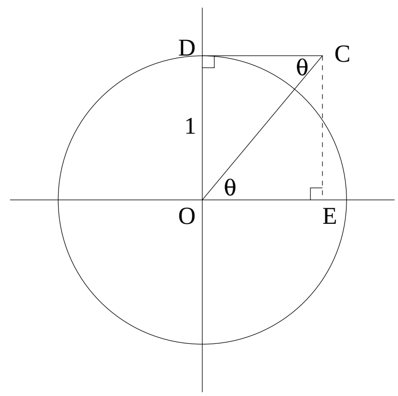



Figure 3

(b) The key to finding a line segment that has the length of the desired trigonometric function is to construct a right triangle where the denominator in the defining ratio is 1. With that in mind, draw a right triangle in a unit circle in such a way that one of its lengths is $\sec\theta$. Explain which segment is $\sec\theta$ and why. (Hint: Your drawing should either look like Figure 2 or Figure 3.)

(c) Draw a right triangle in a unit circle in such a way that one of its lengths is $\cot\theta$. Explain which segment is $\cot\theta$ and why.

**3.** Three other trigonometric functions that are much less commonly used are the versed sine function, written versin $\theta$, the coversine function, written cvs $\theta$, and the external secant function, written exsec $\theta$. These three functions are defined as follows:

$$\text{versin } \theta = 1 - \cos\theta, \quad \text{cvs } \theta = 1 - \sin\theta, \quad \text{exsec } \theta = \sec\theta - 1.$$

(a) Make a drawing similar to Figure 1. Identify the line segment in your drawing whose length is versin $\theta$. Explain why your segment is versin $\theta$.

(b) Make another drawing similar to Figure 1. By drawing an additional line segment in Figure 1, a line segment whose length is cvs $\theta$ is formed. Identify that segment and explain why it is cvs $\theta$.

(c) Draw a right triangle in a unit circle in such a way that a segment is formed whose length is exsec $\theta$. Explain which segment is exsec $\theta$ and why.

# 5.2 Arc Length and Area

In Section 5.1, we looked closer at periodic functions associated with circles and triangles – sine, cosine, and tangent. In this section, we'll spend some more time looking at circles and triangles. We'll see how to calculate the arc length of a piece of a circle, find the area of a sector of a circle, and look at angular velocity. We'll also look at area of triangles and several applications.

## Arc Length

Suppose you wanted to construct a patio on the back of your house shaped like the one shown in Figure 1. The patio will be built of concrete and the perimeter will be edged with brick. To

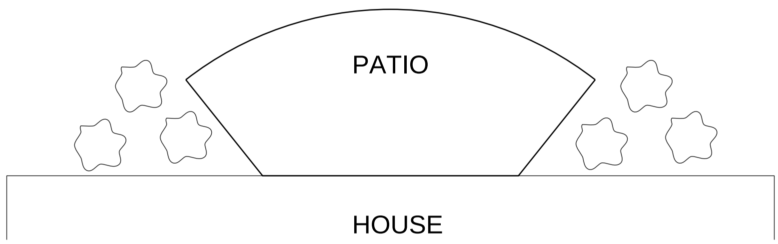



Figure 1: A patio built of concrete and edged with brick.

purchase the correct amount of both concrete and brick, you need to determine the area and the perimeter of the patio.

To start this process, a more detailed drawing is made. This is shown in Figure 2. The patio is a portion of a sector of a circle. A **sector** of a circle is the region bounded by two radii and a portion of the circumference. The sector shown in Figure 2 has a central angle of $60°$.

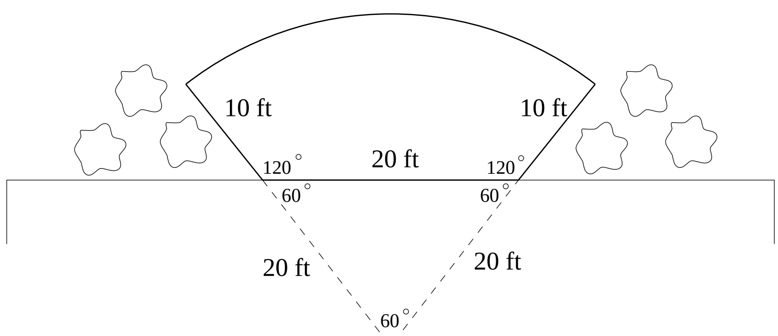



Figure 2: A detailed drawing of the patio showing how it is related to a sector with a central angle of $60°$.

Our first task will be determining the perimeter of the patio. The lengths of the straight lines are shown as 10 feet, 20 feet, and 10 feet. However, we also need to know the length of the arc. To do this, we will first develop the formula for finding the arc length of a portion of the circumference of a circle.

The Greek letter $\pi$ is defined to be the ratio of the circumference of any circle to its diameter. That is,

$$\pi = \frac{C}{d},$$

where $C$ is the circumference and $d$ is the diameter. The number $\pi \approx 3.14159$ is an irrational number so its exact value cannot be represented by a fraction. From the definition of $\pi$, we see that $C = \pi d$ which gives us a formula for computing the circumference of a circle. Since $d = 2r$ where $r$ is the radius of a circle, we also have the formula $C = 2\pi r$.

**Historical Note**[119]

The number $\pi$ is something that has intrigued and fascinated people for centuries. William Jones first used the symbol $\pi$ to represent the ratio of the circumference of a circle to its diameter in 1706, but the symbol was not generally accepted until Euler adopted it in 1737 and later used it in his book, *Analysis*. The estimation of the value of $\pi$, however, goes back much further. Around 1700 B.C., for example, the Egyptians used $\frac{256}{81} \approx 3.16$ as the value of $\pi$. Throughout the centuries, closer and closer approximations have been calculated. Today, computers have calculated $\pi$ to more than two billion digits. According to the *Guinness Book of Records*, Hideaki Tomoyori from Yokdiang, Japan recited the first 40,000 of those digits from memory in 1987. It took him 17 hours and 21 minutes.

To find a formula for a portion of the circumference or arc length of the circle, recall that in Section 5.1 we saw that a one radian angle intercepts an arc of length $r$ in a circle of radius $r$. (See Figure 3.) Assume the radius of your circle is $r$. If a one radian angle intercepts an arc of

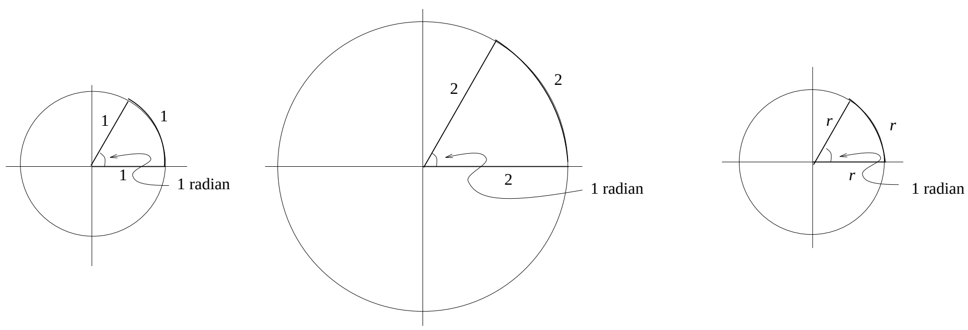



Figure 3: In a circle of radius $r$, a one radian angle intercepts an arc of length $r$.

length $r$, than a 2 radian angle intercepts an arc of length $r + r = 2r$, a 3 radian angle intercepts an arc of length $r + r + r = 3r$, and so forth. This is illustrated in Figure 4. From this you can see

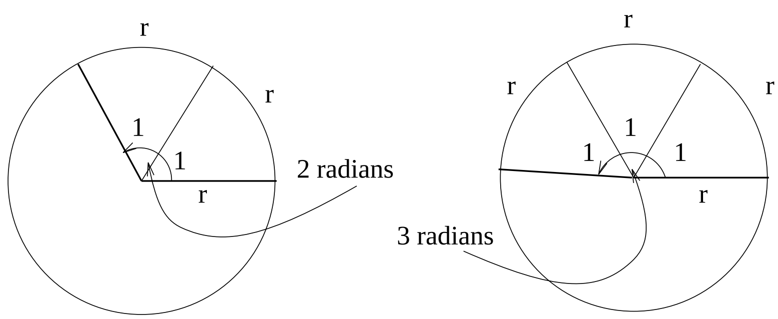



Figure 4: In a circle of radius $r$, a 2 radian angle intercepts an arc of length $2r$ and a 3 radian angle intercepts an arc of length $3r$.

that the length of the arc is the measure of the angle (in radians) times the radius of the circle.

[119] Ball, W.W. Rouse, and H.S.M. Coxeter, *Mathematical Recreations and Essays*, 13th ed., Dover Publications, Inc., 1987, pp 347-359.
McFarlan, Donald,ed., *The Guinness Book of Records 1991*, Guinness Publishing Ltd., 1990, pp 67,78.

- The length of an arc of a circle of radius $r$ that is intercepted by a central angle of $\theta$ radians is $s = r\theta$. (See Figure 5.)

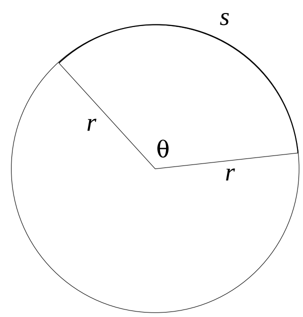



Figure 5: $s = r\theta$.

Notice that it is important that the measure of the angle is in *radians*. Radians define the angle in a unit circle in terms of the arc length intercepted by that angle. It is the fact that radians connect the measurement of the angle to the length of the arc that allows us to derive such a simple formula for the arc length of a circle. Degrees have no such connection. If you are given the measure of an angle in degrees, you must convert it to radians before finding the arc length. Notice also that the formula for arc length gives us the same formula for the circumference of a circle that we derived earlier from the definition of $\pi$. Since the full circle angle is $2\pi$, the circumference of a circle of radius $r$ is $C = r \cdot 2\pi = 2\pi r$.

Now back to our patio example. In Figure 2, we saw that the curved side of the patio is part of the circumference of a circle. To determine the arc length, we need to know the radius of the circle and the measure of the central angle in radians. In the picture, the central angle is given as $60°$. Converting $60°$ to radians, we have $60° \cdot \frac{\pi}{180°} = \frac{\pi}{3}$ radians. We can see that the radius of the circle is $20 + 10 = 30$ feet. Therefore, the length of the arc in the patio is $30 \cdot \frac{\pi}{3} = 10\pi \approx 31.4$ feet. So the perimeter of the patio is approximately $10 + 20 + 10 + 31.4 = 71.4$ feet which is about 71 feet 5 inches. With this information, we can determine how many bricks we need to go around the perimeter of our patio.

**Example 1.** A clock is constructed with a 6 inch second hand. Find the distance the tip of the second hand travels when it goes from the 1 on the face of the clock to the 5.

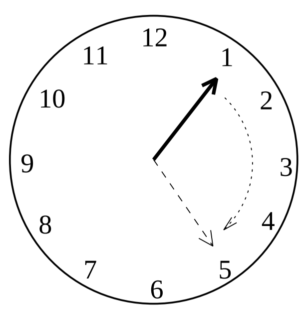
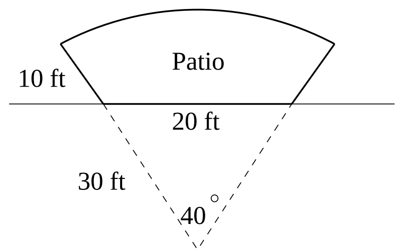



Figure 6

**Solution:** Going from 1 to 5 on the face of a clock is traveling $\frac{4}{12} = \frac{1}{3}$ of the way around. Therefore the second hand will move through an angle of $\frac{1}{3} \cdot 2\pi = \frac{2\pi}{3}$ radians. Since the hand is 6 inches long, its tip will travel $6 \cdot \frac{2\pi}{3} = 4\pi \approx 12.6$ inches. ∎

### Reading Questions

1. Suppose you have a circle of radius 5. Find the arc length intercepted by each of the following angles.
   (a) $\frac{\pi}{2}$
   (b) $\frac{2\pi}{3}$
   (c) $45°$
2. Why does the measure of your angle have to be in radians before you can find arc length?
3. Find the perimeter of the patio shown in the following figure.




## Area of a Sector

Let's return to building our patio. We found the perimeter so we could find the number of bricks we need. However, we still need to determine the area of the patio in order to calculate how much concrete we need. To do this, we need to find the area of the sector shown in Figure 2 and subtract the area of the equilateral triangle that is not part of the patio. We'll start by deriving the formula for the area of a sector.

To find the formula for a sector of a circle, look at Figure 7. If the central angle is $\theta = 2\pi$,

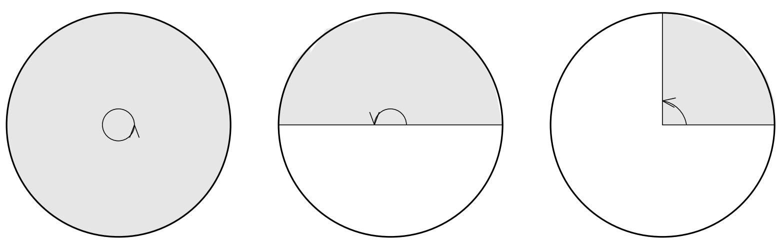



Figure 7: The ratio of the angle to $2\pi$ equals the ratio of the shaded area to the area of the entire circle.

then the shaded area is the entire circle. If the central angle is $\theta = \pi$ (half the full circle angle), then the shaded area is half the area of the circle. If the central angle is $\theta = \frac{\pi}{2}$ (one-fourth the full circle angle), then the shaded area is one-fourth of the area of the circle. In general, the ratio of the angle to the full circle angle of $\theta = 2\pi$ is equal to the ratio of the shaded area to the area of the entire circle. That is,

$$\frac{\theta}{2\pi} = \frac{A}{\pi r^2}.$$

Solving this for $A$, we have the following:

- The area of a sector of a circle of radius $r$ that is intercepted by a central angle of $\theta$ radians is $A = \frac{1}{2}r^2\theta$. (See Figure 8.)

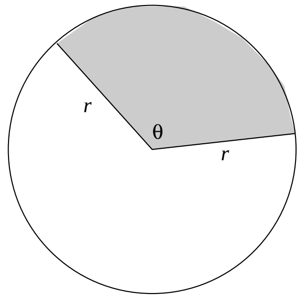



Figure 8: The area of a sector of a circle of radius $r$ that is intercepted by a central angle of $\theta$ radians is $A = \frac{1}{2}r^2\theta$.

This formula, similar to the arc length formula, assumes the measure of your angle is in radians. If your angle is given in degrees, be sure to convert it to radians before finding the area of the sector.

Now back to our patio example. Figure 9 shows that the patio is part of a sector of a circle. We need to find the area of this sector and subtract the area of the equilateral triangle.

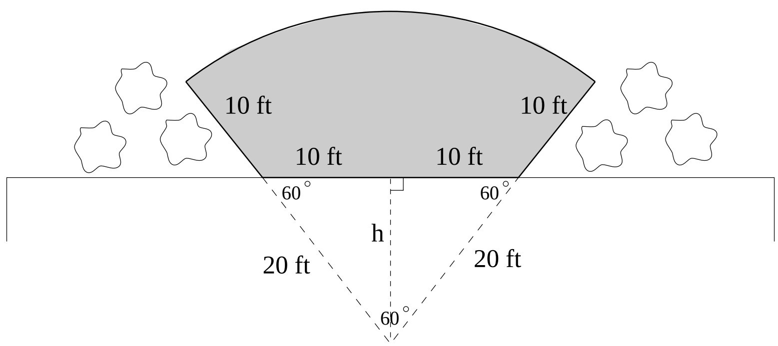



Figure 9: Finding the area of our patio involves finding the area of the sector and subtracting the area of the equilateral triangle.

To find the area of the sector, we convert the $60°$ central angle to $\frac{\pi}{3}$ radians. The radius is 30 feet, giving us the area of the sector as $\frac{1}{2} \cdot 30^2 \cdot \frac{\pi}{3} = 150\pi \approx 471$ square feet. To determine the area of the equilateral triangle, we need to know its height, $h$. The segment labeled $h$ in Figure 9 divides the equilateral triangle into two $30°$- $60°$- $90°$ triangles. In Section 5.1, we saw that $\sin 60° = \frac{\sqrt{3}}{2}$. We also know, from Figure 9 and the right triangle definition of sine, that $\sin 60° = \frac{h}{20}$. So $h = \sin 60° \cdot 20 = \frac{20\sqrt{3}}{2} = 10\sqrt{3}$. The area of the equilateral triangle is $\frac{1}{2}bh = \frac{1}{2} \cdot 20 \cdot 10\sqrt{3} = 100\sqrt{3} \approx 173$ square feet. Therefore, the area of the patio is approximately $471 - 173 = 298$ square feet.

**Example 2.** In a circle of radius 6 inches, find the area of a sector whose central angle is $100°$.

**Solution:** We first must convert $100°$ to radians. Doing this, we get $100° \cdot \frac{\pi}{180°} = \frac{5\pi}{9}$ radians. The area of the sector is $A = \frac{1}{2} \cdot 6^2 \cdot \frac{5\pi}{9} = 10\pi$ square inches. ■

#### Reading Questions

**4.** Suppose you have a circle of radius 5. Find the area of the sector that is intercepted by each of the following angles.

(a) $\frac{\pi}{2}$

(b) $\frac{2\pi}{3}$

(c) $45°$

**5.** What is the relationship between the area of a sector of a circle and the area of the entire circle?

## Area of a Triangle

You've seen the formula for the area of a triangle, $A = \frac{1}{2}bh$, where $b$ is the base and $h$ is the height. From this, we can derive another formula for the area of the triangle in terms of two of the sides and one of the angles. Figure 10 shows a triangle with the height, two sides, and one of the angles labeled.

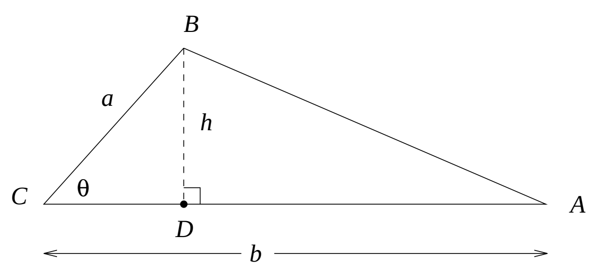



Figure 10: A triangle with the height, two sides, and one of the angles labeled.

Notice that $\triangle BDC$ is a right triangle. This means, using the right triangle definition of sine, that $\sin\theta = \frac{h}{a}$. Solving this for $h$ gives $h = a\sin\theta$. Substituting this into the standard formula for the area of a triangle leads us to $A = \frac{1}{2}b(a\sin\theta)$. This gives us the following formula for determining the area of a triangle.

- The area of a triangle is $A = \frac{1}{2}ab\sin\theta$ where $\theta$ is the angle between the sides whose lengths are $a$ and $b$.

**Example 3.** Find the area of the triangle in Figure 11. Angles are measured to the nearest degree and sides to the nearest tenth of a centimeter.

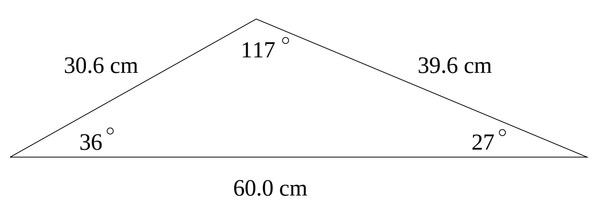



Figure 11

**Solution:** If we let $a = 30.6$ cm and $b = 60.0$ cm, then $\theta = 36°$. Using these and the formula $A = \frac{1}{2}ab\sin\theta$, we have $A = 0.5 \cdot 30.6 \cdot 60.0 \cdot \sin 36° \approx 540 \text{ cm}^2$. ■

In Example 3, we could have chosen *any* two sides as long as the angle we choose was between them. For example, if we let $a = 30.6$ cm and $b = 39.6$ cm, then $\theta = 117°$. In this case, $A = 0.5 \cdot 30.6 \cdot 39.6 \cdot \sin 117° \approx 540 \text{ cm}^2$ which is the same answer we got when using 36° as our angle. It doesn't matter which side you choose for $a$ and which side you choose for $b$ as long as $\theta$ is the angle between the two sides.

**Example 4.** Find the area of the shaded region in Figure 12 that is formed by the difference of the area of the sector and the triangle.

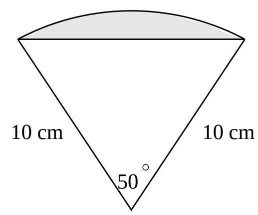



Figure 12

**Solution:** To find the difference between the area of the sector and the area of the triangle we need to subtract the two areas. This gives us

$$\text{Area of shaded region} = \frac{1}{2}r^2\theta - \frac{1}{2}ab\sin\theta.$$

Since $a = b = r = 10$ and $\theta = 50° = \frac{5\pi}{18}$ radians, we have

$$\begin{aligned}\text{Area of shaded region} &= \frac{1}{2}\cdot 10^2 \cdot \frac{5\pi}{18} - \frac{1}{2}\cdot 10 \cdot 10 \cdot \sin\frac{5\pi}{18} \\ &\approx 5.33 \text{ square centimeters.}\end{aligned}$$

■

**Example 5.** The baseball field shown in Figure 13 consists of a sector and two triangles. Find the area of the baseball field.

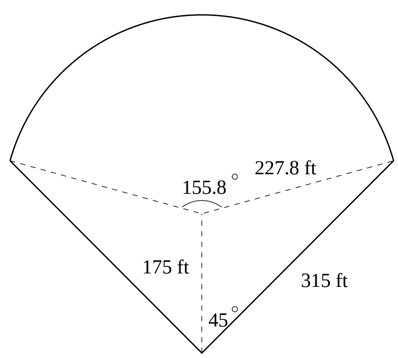



Figure 13

**Solution:** Since 155.8° is $\frac{155.8\pi}{180}$ radians, the area of the sector is

$$\frac{1}{2}\cdot\frac{155.8\pi}{180}\cdot 227.8^2 \approx 70,554 \text{ square feet.}$$

The area of each triangle is

$$\frac{1}{2}\cdot 315 \cdot 175 \cdot \sin 45° \approx 19,490 \text{ square feet.}$$

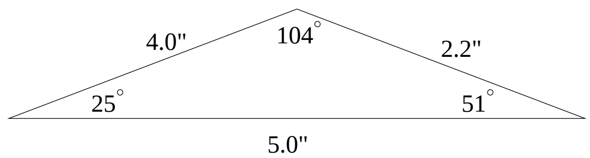
Therefore, the total area of the baseball field is approximately

$$70,554 + 2(19,490) = 109,534 \text{ square feet.}$$

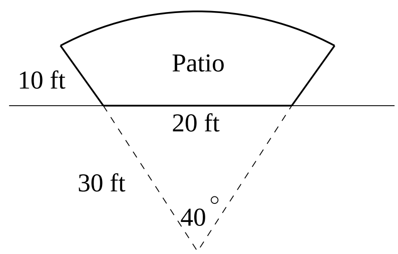
■

### Reading Questions

**6.** Find the area of the following triangle.




**7.** Find the area of the patio in the following figure.




## Angular Velocity

Riding on a merry-go-round at a playground can feel very different for someone riding near the center compared with someone riding near the edge. While both people are turning at the same rate and thus have the same angular velocity, both do not have the same linear velocity.

Suppose a merry-go-round is revolving at 6 revolutions per minute. This is the merry-go-round's angular velocity. **Angular velocity**, which we will denote by using the lower case Greek letter omega, $\omega$, is angular distance divided by time. **Angular distance** is some measure of the angle through which the radius of the circle moves. Angular distance can be given in degrees, radians, or revolutions. The most common unit used is revolutions since, most of the time, the angular distance will be more than one full circle. If a point is moving around the circumference of a circle at a constant angular velocity, then its angular velocity is given by

$$\omega = \frac{\theta}{t},$$

where $\theta$ is the angular measure through which the radius of the circle moved and $t$ is the time required for the radius to move that distance. For example, an angular velocity of 6 revolutions per minute means that the radius of the circle is moving six times completely around the circle in one minute.

We can use the arc length formula that we described earlier to relate angular velocity with linear velocity. **Linear velocity** is linear distance divided by time. This is what you typically think of when you think of velocity and is described by the formula $r = \frac{d}{t}$ where $d$ is distance and $t$ is time. If a point is moving around the circumference of a circle at a constant velocity, then its linear velocity is given by

$$v = \frac{s}{t},$$

where $s$ is the distance traveled around the circle (or arc length) and $t$ is the time required to go that distance. Recall that the arc length formula tells us that $s = r\theta$. Since the angular measure must be in radians to use this formula, we will assume that the angular velocity is in radians per unit time. Dividing both sides of $s = r\theta$ by $t$ gives us

$$\frac{s}{t} = \frac{r\theta}{t}.$$

Since we know $v = \frac{s}{t}$ and $\omega = \frac{\theta}{t}$, we obtain the following:

- If a circle of radius $r$ is rotating at $\omega$ radians per unit time, the linear velocity of a point on the circle is given by $v = r\omega$.

Let's return to our merry-go-round. Suppose there are two people on a merry-go-round that is spinning at 6 revolutions per minute. One person is 3 feet from the center and one is 6 feet from the center. They will both be revolving at the same angular velocity, but the one on the outside will have a greater linear velocity. To find the linear velocity of each, we first convert the angular velocity from revolutions per minute to radians per minute. Since there are $2\pi$ radians in one revolution, there are $6 \cdot 2\pi = 12\pi$ radians in 6 revolutions. Therefore, the merry-go-round is revolving at a rate of $12\pi$ radians per minute. The person 3 feet from the center is traveling at a rate of

$$3 \cdot 12\pi = 36\pi \approx 113 \text{ feet per minute.}$$

The person on 6 feet from the center is traveling at a rate of

$$6 \cdot 12\pi = 72\pi \approx 226 \text{ feet per minute.}$$

Note that the person 6 feet from the center is going twice as fast as the person 3 feet from the center.

**Example 6.** A 26 inch diameter bicycle tire is rotating at 3 revolutions per second. Determine how fast the bicycle is traveling in miles per hour.

**Solution:** The velocity of the bicycle is the same as the linear velocity of a point on the outside of the bicycle tire. To determine this velocity, we first need to convert 3 revolutions into radians and a 26 inch diameter to a radius. Since 1 revolution is $2\pi$ radians, 3 revolutions is $3 \cdot 2\pi = 6\pi$ radians. The radius of a circle is half the diameter so the radius is $\frac{1}{2} \cdot 26 = 13$ inches. Therefore, the linear velocity of the bicycle is $6\pi \cdot 13 = 78\pi$ inches per second. To convert $78\pi$ inches per second to miles per hour we do the following.

$$\frac{78\pi \text{ inches}}{1 \text{ second}} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} \cdot \frac{1 \text{ mile}}{5280 \text{ feet}} \cdot \frac{3600 \text{ seconds}}{1 \text{ hour}} \approx 13.9 \text{ miles per hour.}$$

Therefore, a bicycle that has 26 inch diameter tires rotating at 3 revolutions per second will be traveling approximately 13.9 miles per hour. ■

## Reading Questions

**8.** How is angular velocity different than linear velocity?

**9.** If a velocity is given as 30 radians per second, is this an angular velocity or a linear velocity?

**10.** A 24 inch diameter bicycle tire is rotating at 2 revolutions per second. Determine how fast the bicycle is traveling in miles per hour.

## Exercises

**1.** Each of the following represents either a linear measurement (such as a perimeter) or an area measurement for a common geometric figure. Determine if the formula is finding a linear measurement or an area measurement and then describe the geometric figure associated with the formula.

(a) $2w + 21$

(b) $\frac{1}{2}bh$

(c) $s^2$

(d) $2\pi r$

(e) $r\theta$

(f) $\frac{1}{2}ab\sin\theta$

(g) $lw$

(h) $\pi r^2$

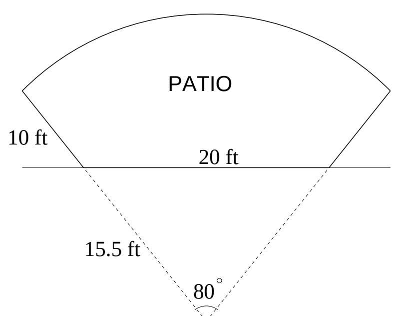
(i) $4s$

(j) $\frac{1}{2}r^2\theta$

**2.** Complete the following table by filling in the missing answers associated with a sector of a circle.

| | Sector $A$ | Sector $B$ | Sector $C$ | Sector $D$ | Sector $E$ |
|---|---|---|---|---|---|
| Length of the Radius | 2 | 7 | | $\frac{1}{4}$ | |
| Measure of the Central Angle | $\frac{\pi}{3}$ | | 2.5 | | $45°$ |
| Length of the Intercepted Arc | | $\frac{21\pi}{8}$ | 6.25 | $\frac{\pi}{4}$ | |
| Area of the Sector | | | | | $\frac{9\pi}{8}$ |

**3.** Find the perimeter and the area of the patio shown in the following figure.



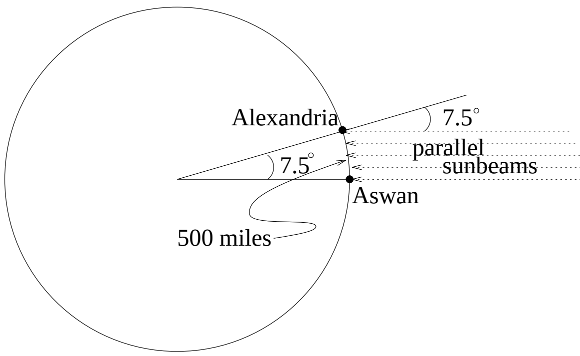

**4.** Glasgow, Scotland is at a latitude of approximately 56° north and a longitude of approximately 4° west. Madrid, Spain is at a latitude of approximately 40° north and a longitude of approximately 4° west. This means that the distance from Glasgow to Madrid is the length an arc of a circle with a central angle of 16° and a radius of 3960 miles (the radius of the Earth). Approximately how far is it from Glasgow to Madrid?

**5.** The Earth has a radius of about 3960 miles and has 24 time zones. What is the distance between time zones, at the equator, assuming they are equally spaced?

**6.** A nautical mile is one minute ($\frac{1}{60}$ of a degree) of the circumference of the Earth. The radius of the Earth is about 3960 (statute) miles. How far is one nautical mile in terms of (statute) miles?

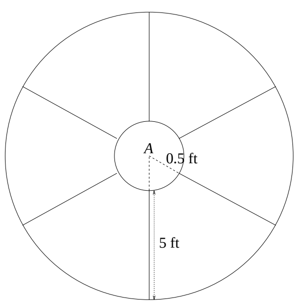
**7.** One of the classic problems in mathematics is how Eratosthenes calculated the circumference of the Earth in approximately 200 BC. He knew that on a certain day of the year the sun was directly overhead of Aswan, Egypt because there was no shadow from the sun as it shown into a well. At noon on that same day, he measured the shadow of the sun in Alexandria, Egypt which is 500 miles north of Aswan. From this, he concluded that the sun was 7.5° south of vertical in Alexandria. Using the figure below, answer the following questions.



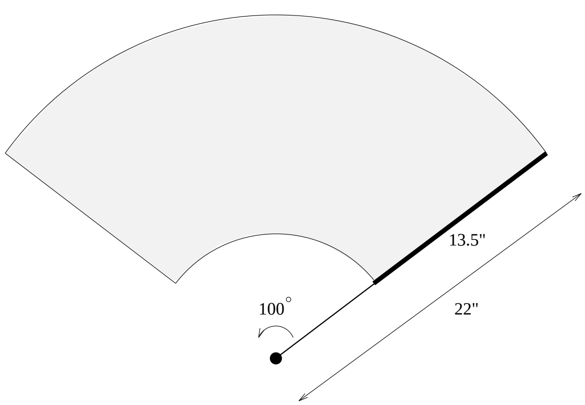

(a) There are two angles labeled 7.5° in the figure. Explain why if one is 7.5° the other must also be 7.5°.

(b) Calculate the radius and the circumference of the Earth.

**8.** The following figure shows a circular flower garden with a birdbath in the center. The radius of the circular base of the birdbath is 0.5 feet and the distance from the base of the birdbath to the edge of the garden is 5 feet. Assume that each plot is the same size. Determine the measurement of angle A and the area of each plot in the flower garden.




**9.** The back windshield wiper on a minivan is on an arm that is 22 inches long. The wiper blade is 13.5 inches long and is located on the outer portion of the wiper arm. The arm moves through an angle of 100'. (See the following figure.) What is the area of the back windshield that is wiped when the windshield wiper is operating?



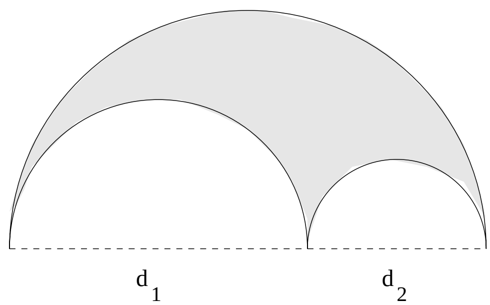
**10.** According to *The Rule Book* by the Diagram Group, the field on which a discus is thrown in a track and field competition must be a sector of a circle with a central angle of 40°. The radius of this can vary. Assume that the radius is 250 feet. What is the area of this sector?

**11.** A softball field is a sector with a 90° central angle and a radius of 225 feet.

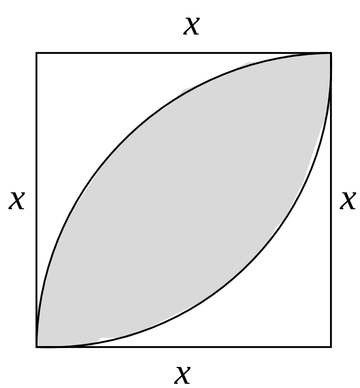
(a) What is the area of the softball field?

(b) What is the perimeter of the softball field?

**12.** An arbelos is the region bounded by a large semicircle and two smaller adjacent semicircles such that the sum of the two smaller diameters equals the large diameter. The placement of the semicircles is shown in the following figure.[120] Archimedes showed that the area of the arbelos is $A = \frac{\pi}{4} d_1 d_2$ where $d_1$ and $d_2$ are the diameters of the two smaller semicircles. Derive Archimedes' formula.

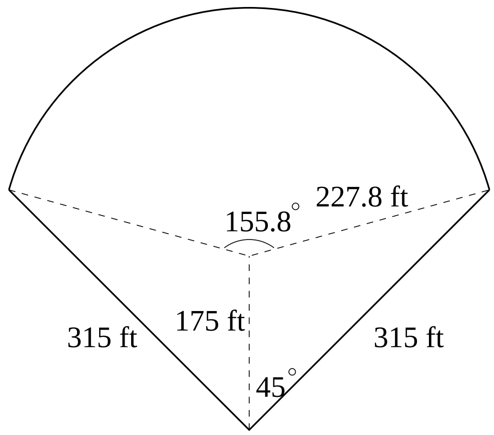



**13.** In the following figure, the arcs are portions of a circle whose center is one of the corners of the square. Find the area of the shaded region in terms of $x$.

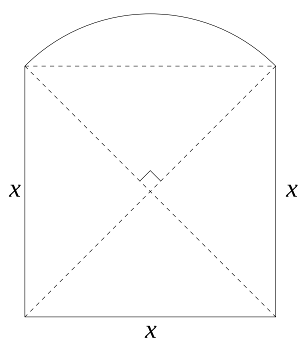



[120]The word arbelos in Greek means cobbler's knife. Apparently, a cobbler's knife looks something like the shape of an arbelos.

**14.** The drawing of the baseball field shown below consists of a sector and two triangles. Find the perimeter and area of the baseball field.

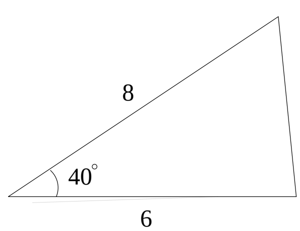



**15.** An arc is placed on top of a square with sides of length $x$. The arc is a portion of a circle whose center is also the center of the square. (See the following figure.) Find the area of the entire region in terms of $x$.

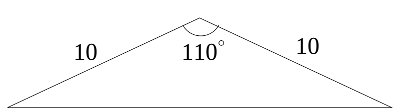



**16.** Find the area of the following triangles.

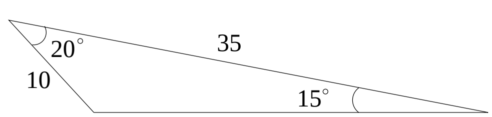
(a)



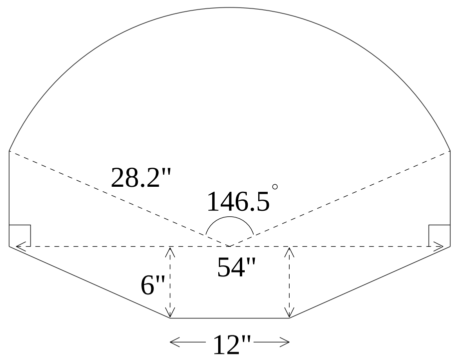

(b)




(c)



**17.** Find the area of the basketball backboard shown in the following figure.




**18.** Two people are on a merry-go-round. Person $A$ is sitting 1.5 feet from the center. How far from the center should person $B$ sit if person B wants to go three times as fast as person $A$?

**19.** One of the world's most famous clocks, commonly known as Big Ben[121], is in St. Stephen's Clock Tower at the House of Parliament in London, England. The minute hand of this clock is 14 feet long.

(a) How far does the end of the minute hand travel in ten minutes? Round your answer to the nearest inch.

(b) What is the angular velocity of the minute hand in radians per second?

(c) What is the linear velocity of a point on the end of the minute hand in inches per second?

(d) What is the linear velocity of a point 7 feet from the end of the minute hand in inches per second?

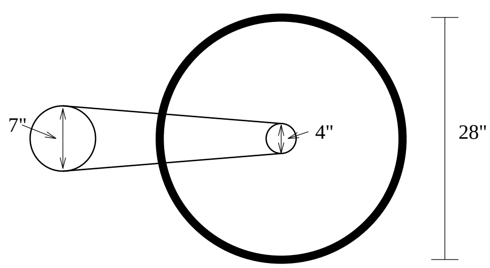
**20.** A phonograph record is 12 inches in diameter. It revolves on the record player at $33\frac{1}{3}$ revolutions per minute.

(a) What is the angular velocity of the record in radians per minute?

(b) What is the linear velocity of a point on the outside edge of the record in inches per minute?

**21.** The world's tallest Ferris wheel, the Millennium Wheel, is being built in London, England. The wheel, is 500 feet in diameter and has 60 enclosed capsules each of which can carry up to 16 people. The wheel takes 20 minutes to make one revolution. At what linear velocity, in feet per second, does each capsule on the Millennium Wheel move?

**22.** When a ten-speed bike is in first gear, the chain goes around a 7-inch diameter sprocket and then back to a 4-inch diameter sprocket. The outer diameter of the tire is 28 inches. (See the following figure.) Suppose the rider pedals the bike at a rate of 2 revolutions per second. This means the 7-inch sprocket will rotate at 2 revolutions per second.




[121] While the clock is commonly referred to as Big Ben, Big Ben is actually the name of the bell that chimes the hours.

(a) The linear velocity of the chain is the same as the linear velocity of a point on the 7-inch sprocket. What is the linear velocity of the chain?

(b) At what angular velocity will the 4-inch sprocket, and hence the wheel, be moving?

(c) How fast will the bike be traveling?

(d) When the same ten-speed bike is in tenth gear, the chain goes around an 8inch diameter sprocket and then back to a 2-inch diameter sprocket. If the rider continues to pedal the bike at a rate of 2 revolutions per second, how fast will the bike be traveling?

## Investigations

### Investigation 1: Broken Wheel.

If you are given an arc of a circle, you can determine both the radius of the circle and the radian measure of the angle forming the arc. This means that, given a piece of a circle, you can reconstruct the entire circle. Suppose an archeologist found the portion of the wheel shown in Figure 1 and is interested in reconstructing the dimensions of the wheel.

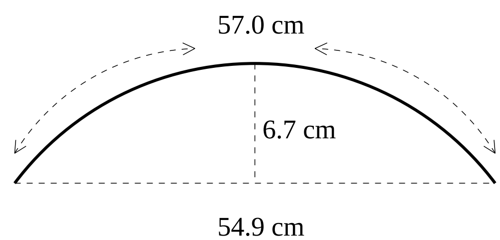



Figure 1

**1.** We'll start by deriving the general formula for computing the dimensions of the circle, given an arc. Figure 2(a) shows an arc that has a horizontal distance of $x$, a vertical distance of $y$, and an arc length of $s$. Notice that it would be easy to find measurements for $x$, $y$, and $s$ if this was a physical object found by an archeologist. In Figure 2(b), we have taken the arc from Figure 2(a) and drawn in lines representing the radius, r, of the original circle. Notice that $\triangle ADQ$ and $\triangle BDQ$ are both right triangles.

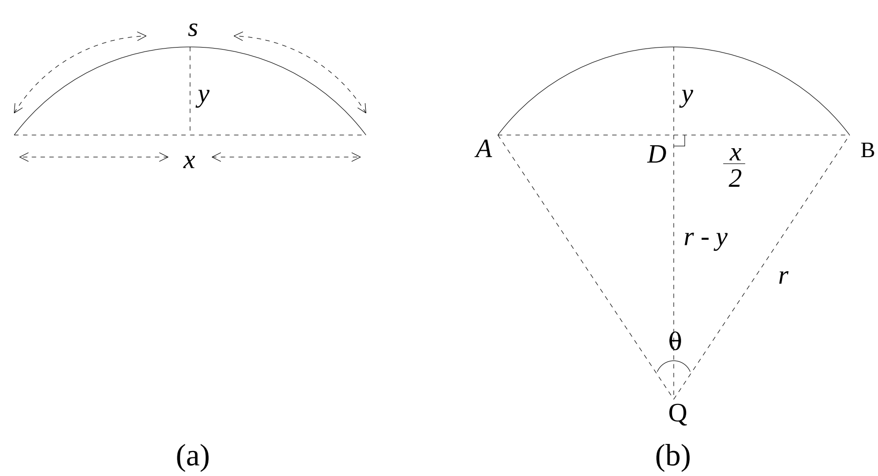



Figure 2

(a) Using the Pythagorean Theorem, derive a function whose inputs are $x$ and $y$ and whose output is $r$.

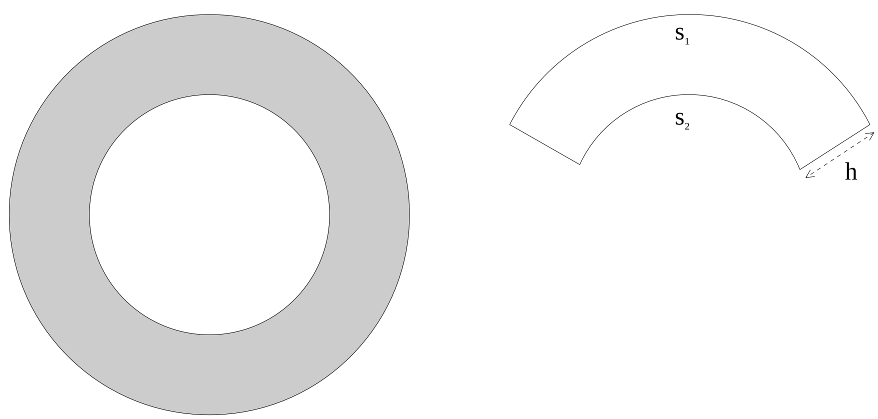
(b) Using the arc length formula and your function from question 1(a), derive a function whose inputs are $x$, $y$, and $s$ and whose output is $\theta$.

(c) Using the portion of a wheel shown in Figure 1, determine the radius of the wheel. The archeologist is also interested in determining what portion of the wheel is represented by the arc. Describe this in terms of the angle $\theta$ formed by drawing two radii from the end points of the arc to the center of the wheel. Give your answer in degrees.

(d) Write a short description of this arc and the reconstructed wheel from the viewpoint of an archeologist.

### Investigation 2: Area of a Sector of an Annulus.

An annulus is the region between two concentric circles. (See Figure l(a).) A sector of an annulus is much like a sector of a circle. Figure l(b) shows a sector of an annulus. The formula for the area of a sector of an annulus is $A = \frac{1}{2}h(s_1 + s_2)$ where $s_1$, and $s_2$ are the two arc lengths and h is the width of the annulus. In this investigation we will derive the formula for the area of a sector of an annulus and compare this to the area formula for a trapezoid.




Annulus
(a)

Sector of an Annulus
(b)

Figure 1

A sector of an annulus is similar to a "curved" trapezoid. Their area formulas are also similar. Figure 2 shows a trapezoid. Show that the area of a trapezoid is $A = \frac{1}{2}h(a + b)$. (Hint: Draw in appropriate lines to divide the trapezoid into either two triangles or two triangles and a rectangle.)

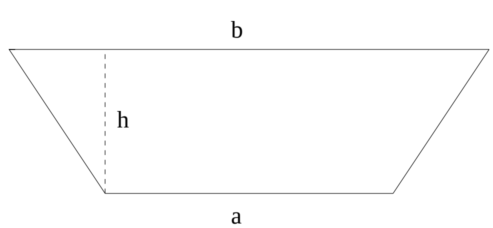



Figure 2

**2.** We want to derive the formula for the area of a sector of an annulus. Figure 3 shows such a sector where $s_1$ is the arc length of the outer circle, $s_2$ is the arc length of the inner circle, $r_1$, is the radius of the outer circle, $r_2$ is the radius of the inner circle, and $h$ is the width of the annulus. Note that $h = r_1 - r_2$

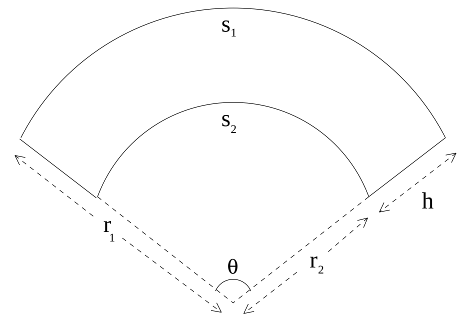



Figure 3

(a) Show that the area of the annulus can be written as $A = \frac{1}{2}\theta(r_1^2 - r_2^2)$.

(b) Modifying your formula from part (a), show that $A = \frac{1}{2}h(s_1 + s_2)$. (Hint: Factor the difference of two squares.)

## Investigation 3: Fence Posts.

Suppose you have two circles "bound together" similar to what would happen if you tied two fence posts together with wire. In this investigation, we will derive a formula for computing the length of the curve (or wire) "wrapped" around the two circles.

**1.** Figure 1 shows a cross-sectional view of a circular post of radius 9 inches and a circular post of radius 3 inches that are bound tightly together with wire. How long is the wire? (Hint: Begin by finding the lengths of the sides of $\triangle KOQ$.)

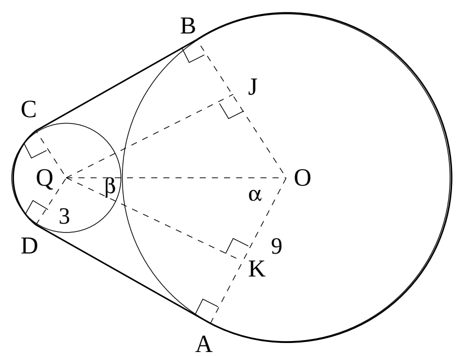



Figure 1

**2.** In general, show that $\gamma + \theta = 2\pi$. (See Figure 2.)

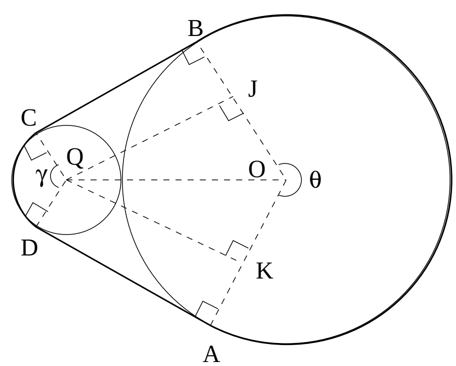



Figure 2

**3.** Find a formula for computing the length of the wire if the radius of the large circle is $r_1$, and the radius of the small circle is $r_2$. This function should contain an inverse cosine function. (See Figure 3).

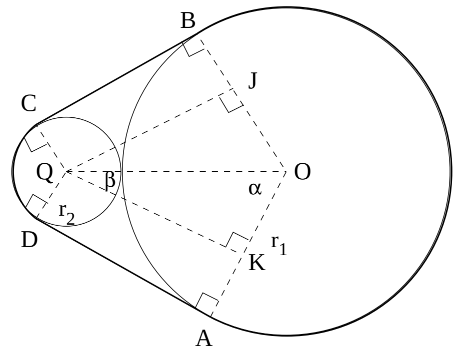



Figure 3

# 5.3 Transformations of Trigonometric Functions

In Chapter 3, we looked at transformations of functions. In particular, Sections 3.1 and 3.2 dealt with what happens when we change the input or output of a function by a constant. In this section, we will revisit that material while concentrating on trigonometric functions. We will see how to determine the amplitude, midline, period, and horizontal shift of either a sine or a cosine function. We will also revisit the material in Section 3.5 on inverse functions. In particular, we will explore the inverse sine, cosine, and tangent functions and see how they can be used to solve trigonometric equations.

## Output and Input Transformations

### Output Changes

In Section 3.1, we looked at the impact of changing the output of a function by a constant. Since output is represented on the vertical axis, changes in output cause vertical changes in the graph. These changes are summarized below and in Figure 1. For each statement, $y = f(x)$ is a function and $c$ is a positive number.

- The graph of $y = f(x) + c$ is the graph of $y = f(x)$ shifted up $c$ units. (See Figure 1(a).)
- The graph of $y = f(x) - c$ is the graph of $y = f(x)$ shifted down $c$ units. (See Figure 1(b).)
- When $c > 1$, the graph of $y = c \cdot f(x)$ is the graph of $y = f(x)$ vertically stretched away from the $x$-axis by a factor of $c$. (See Figure 1(c).)
- When $0 < c < 1$, the graph of $y = c \cdot f(x)$ is the graph of $y = f(x)$ vertically stretched towards the $x$-axis by a factor of $c$. (See Figure 1(d).) [Note: This type of stretch is often referred to as a compression.]
- The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ across the $x$-axis. (See Figure 1(e).)

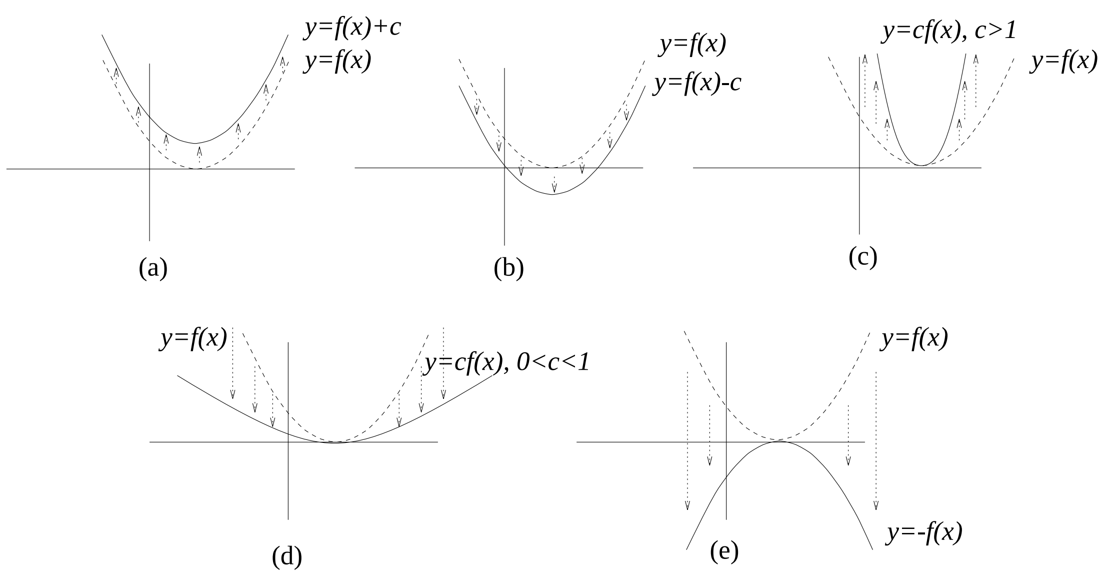



Figure 1: Output transformations for a function, $y = f(x)$, where $c > 0$.

**Example 1.** The graph of $y = f(x)$ in shown in Figure 2. Sketch the graph of $y = -2f(x) - 1$.

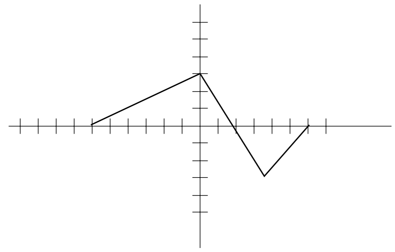


Figure 2

**Solution:** The graph of $y = 2f(x)$, shown in Figure 3(a), is the graph of $y = f(x)$ stretched away from the $x$-axis by a factor of 2. The graph of $y = -2f(x)$, shown in Figure 3(b), is the graph of $y = 2f(x)$ reflected across the $x$-axis. The graph of $y = -2f(x) - 1$, shown in Figure 3(c), is the graph of $y = -2f(x)$ shifted down 1 unit. ■

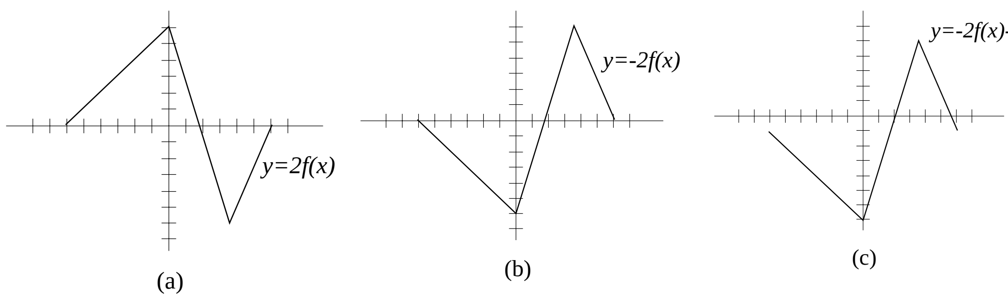



Figure 3

## Input Changes

In Section 3.2, we looked at the impact of changing the input of a function by a constant. Since input is represented on the horizontal axis, changes in input cause horizontal changes in the graph. These changes are summarized below and in Figure 4. For each statement, $y = f(x)$ is a function and $c$ is a positive number.

- The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted to the left $c$ units. (See Figure 4(a).)
- The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted to the right $c$ units. (See Figure 4(b).)
- When $c > 1$, the graph of $y = f(cx)$ is the graph of $y = f(x)$ horizontally compressed towards the $y$-axis by a factor of $c$. (See Figure 4(c).)
- When $0 < c < 1$, the graph of $y = f(cx)$ is the graph of $y = f(x)$ horizontally compressed away from the $y$-axis by a factor of $c$. (See Figure 4(d).) [Note: This type of compression is often referred to as a stretch.]
- The graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ across the $y$-axis. (See Figure 4(e).)

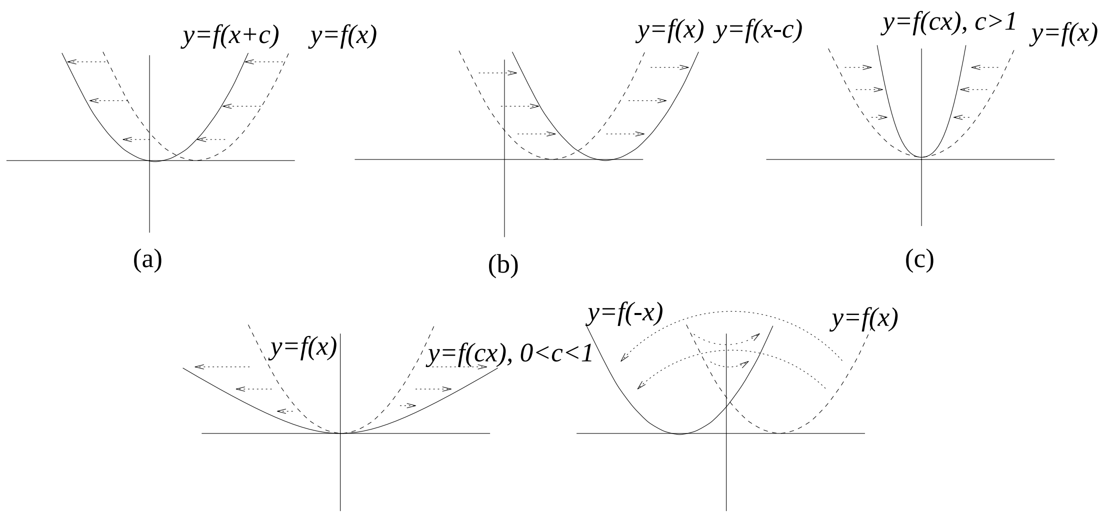
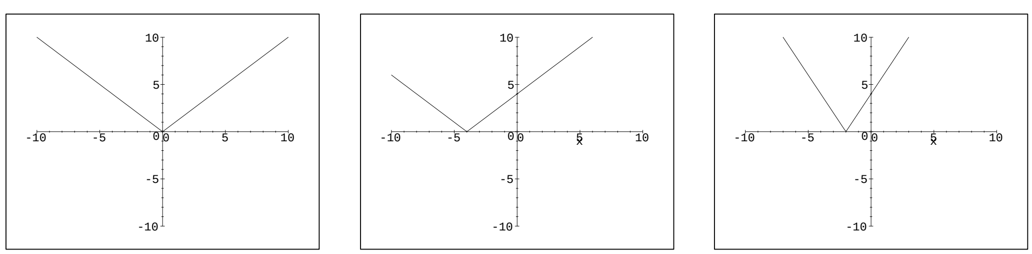



Figure 4: Input transformations for a function, $f$, where $c > 0$.

One thing we didn't look at in Section 3.2 is what happens if you want to both horizontally compress and horizontally shift a function. Given $f$, what is the graph of $g(x) = f(ax + b)$? The answer is that the graph of $g$ is the graph of $f$ first shifted to the left $b$ units and then compressed by a factor of $a$. Notice that this order may be opposite to the way you thought it should work.[122] Let's look at an example. Figure 5(a) is the graph of $f(x) = |x|$, Figure 5(b) is the graph of $g(x) = |x+4|$, and Figure 5(c) is the graph of $h(x) = |2x+4|$. Notice that the graph of $h$ started with the graph of $f$, shifted it four units to the left, and then compressed it horizontally by a factor of 2.



(a) $f(x) = |x|$ (b) $g(x) = |x+4|$ (c) $h(x) = |2x+4|$

Figure 5: The graph in Figure 5(b) is the graph in Figure 5(a) shifted left 4 units. The graph of Figure 5(c) is the graph in Figure 5(a), first shifted 4 units and then horizontally compressed towards the $y$-axis by a factor of 2.

[122] With inputs, a lot of things are opposite to the way you may think they should be. For example, $y = f(x + a)$ may seem like it should shift to the right, but it really shifts to the left.

**Example 2.** The graph of $y = f(x)$ is shown in Figure 6. Sketch a graph of $y = f(-2x - 4)$.

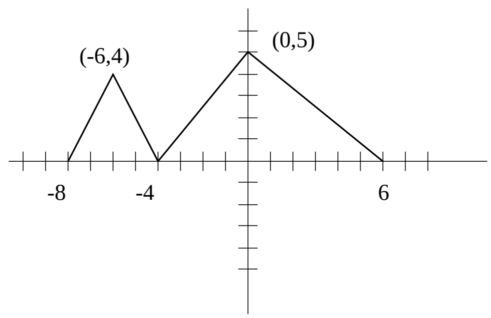



Figure 6

**Solution:** The graph of $y = f(x - 4)$, shown in Figure 7(a), is the graph of $y = f(x)$ shifted horizontally 4 units to the right. The graph of $y = f(2x + 4)$, shown in Figure 7(b), is the graph of $y = f(x - 4)$ horizontally compressed towards the $y$-axis by a factor of 2. The graph of $y = f(-2x - 4)$, shown in Figure 7(c), is the graph of $y = f(2x - 4)$ reflected across the $y$-axis. ■

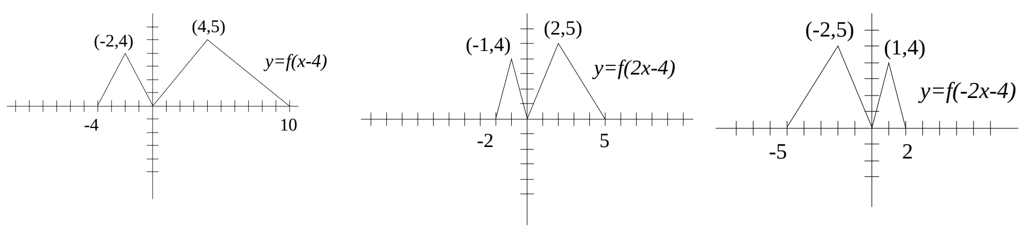
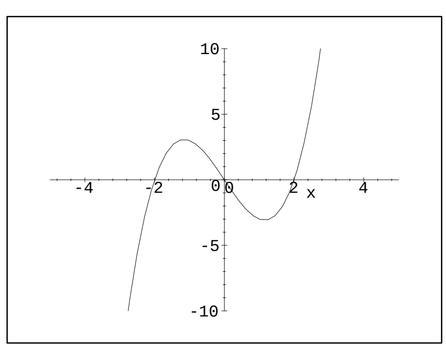



Figure 7

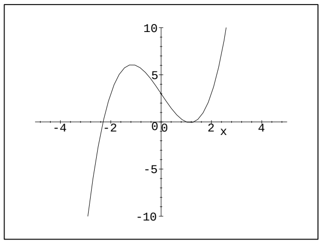
## Reading Questions

**1.** The following is a graph of $f$.

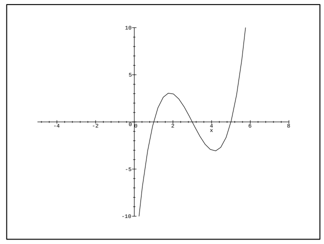



Match these equations to the graphs of transformations of $f$.

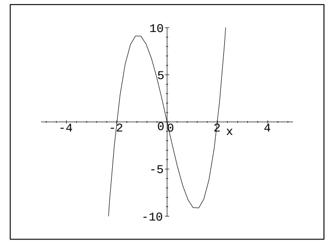
$$y = 3f(x), \quad y = f(x) + 3, \quad y = f(3x), \quad y = f(x - 3), \quad y = f(x + 3)$$



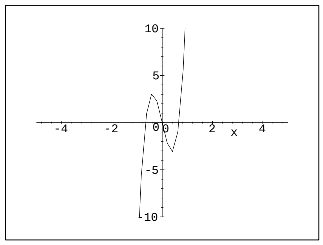

(a)



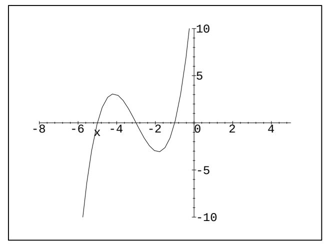
(b)




(c)

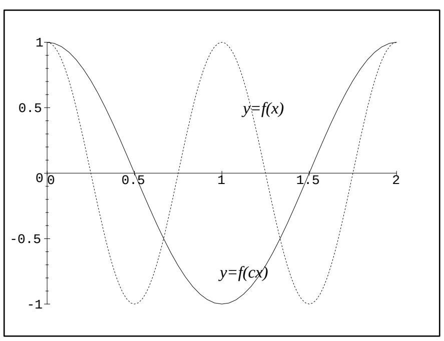



(d)




(e)

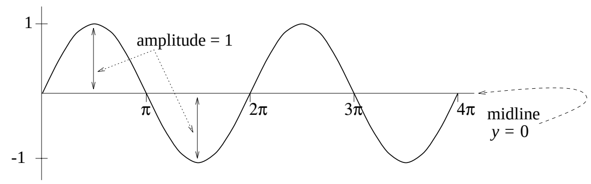
**2.** The following figure shows the graphs of both $y = f(x)$ and $y = f(cx)$. Is $c > 1$ or is $0 < c < 1$? Briefly justify your answer.



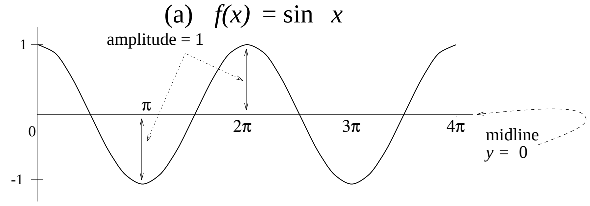

## Changes in Output for Sine and Cosine Functions: Midline and Amplitude

NOTE: Throughout this section, we are always using radians as the input for the trigonometric functions. This is because we are concentrating on graphical behavior. The familiar graphs of $y = \sin x$ and $y = \cos x$ are created from the unit circle definitions (rather than the triangle definitions). Typically, when using unit circle definitions, we assume the input is in radians.

Changing the output by a constant will either vertically shift or vertically stretch the function. For cosine and sine functions, these vertical changes impact the midline and the amplitude of the graph. The **midline** of a sine or a cosine function is the horizontal line that is halfway between the function's maximum and minimum outputs. If $M$ is the maximum output and $m$ is the minimum output, the midline is $y = \frac{M+m}{2}$. For example, the function $f(x) = \sin x$ has a midline of $y = 0$ since the maximum output of $f$ is 1 and the minimum output is $-1$ and $\frac{1+(-1)}{2} = 0$. (See Figure 8(a).) The **amplitude** of a sine or cosine function is the distance from the midline to the maximum (or minimum) output which is equal to half the difference between the maximum and minimum outputs. If $M$ is the maximum output and $m$ is the minimum output, the amplitude is equal to $\frac{M-m}{2}$. The amplitude is always a positive number since $M > m$. The function $f(x) = \sin x$ has an amplitude of 1 since the maximum output of $f$ is 1 and the minimum output is $-1$ and $\frac{1-(-1)}{2} = 1$. (See Figure 8(a).) Since the maximum and minimum outputs for $g(x) = \cos x$ are also $M = 1$ and $m = -1$, the midline for $g(x) = \cos x$ is also $y = 0$ and the amplitude is also 1. (See Figure 8(b).)




(a) $f(x) = \sin x$




(b) $g(x) = \cos x$

Figure 8: For $f(x) = \sin x$ and $g(x) = \cos x$, the amplitude is 1 and the midline is $y = 0$.

Adding to or subtracting from the output of a sine or a cosine function moves the function vertically, thus changing its midline. Multiplying the output of a sine or a cosine function stretches the function vertically, thus changing its amplitude. The midline and amplitude of a sine or cosine function can easily be determined by looking at the symbolic form. For example, suppose $h(x) = 3\sin x + 2.5$. Since $h$ is a vertical stretch of $f(x) = \sin x$ by a factor of 3 and $f$ has an amplitude of 1, $h$ has an amplitude of $3 \cdot 1 = 3$. Also, because $h$ is shifted up 2.5 units from $f(x) = \sin x$ and $f$ has a midline of $y = 0$, $h$ has a midline of $y = 2.5 + 0 = 2.5$. The amplitude and midline of $h(x) = 3\sin x + 2.5$ can also be determined algebraically. We know

$$\begin{aligned} -1 \leq &\quad \sin x &\leq 1 \\ -3 \leq &\quad 3\sin x &\leq 3 \\ -0.5 \leq &\quad 3\sin x + 2.5 &\leq 5.5. \end{aligned}$$

So the maximum value for $h(x)$ is $M = 5.5$ while the minimum value is $m = -0.5$. Therefore, the amplitude is $\frac{M-m}{2} = \frac{5.5-(-0.5)}{2} = \frac{6}{2} = 3$ while the midline is $\frac{M+m}{2} = \frac{5.5+(-0.5)}{2} = \frac{5}{2} = 2.5$. A graph of this function is shown in Figure 9. Notice that the amplitude, 3, and the midline, $y = 2.5$, are both numbers appearing in the equation for $h(x) = 3\sin x + 2.5$. The amplitude is the vertical stretch and the midline is the vertical shift. This will always be true for any sine or cosine function.

In general, for $f(x) = A\sin x + D$ or $g(x) = A\cos x + D$:

- The amplitude is $|A|$. If $A$ is negative, the graph will be reflected over the $x$-axis.
- The midline is $y = D$.

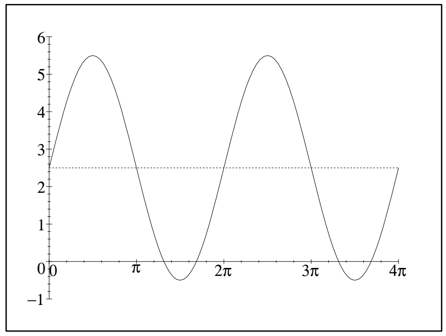



Figure 9: A graph of $h(x) = 3\sin x + 2.5$.

**Example 3.** Find an equation for the cosine function shown in Figure 10.

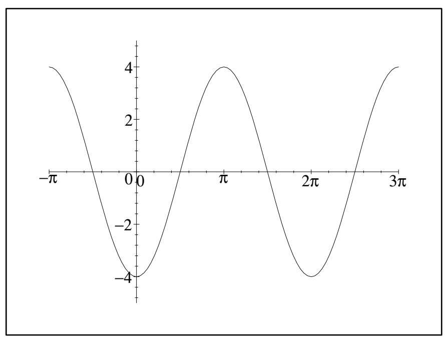
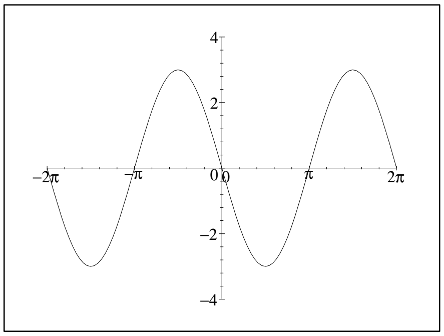



Figure 10

**Solution:** The maximum for the function shown in Figure 10 is 4 and the minimum is $-4$. So, the amplitude is $\frac{4-(-4)}{2} = 4$ and the midline is $y = \frac{4+(-4)}{2} = 0$. Our function is $h(x) = A\cos x + D$ where $|A| = 4$ and $D = 0$. Notice that the graph in Figure 10 has a $y$-intercept of $(0, -4)$ while the graph of $g(x) = \cos x$ has a $y$-intercept of $(0, 1)$. (See Figure 8(b).) So the graph in Figure 10 has been reflected over the $x$-axis. Hence, $A = -4$. Therefore, our equation is $h(x) = -4\cos x + 0 = -4\cos x$. ■

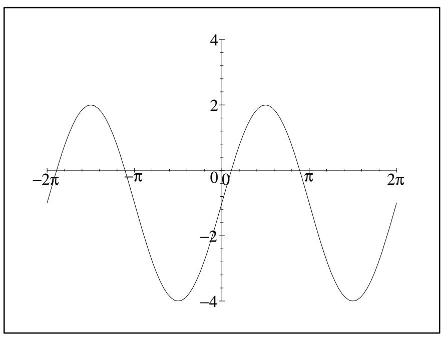
## Reading Questions

**3.** The following is a graph of $y = A\sin x + D$. Is $A$ positive or negative? Briefly justify your answer.




**4.** Let $f(x) = 2\cos x + \pi$. What is the amplitude of $f$? What is the midline of $f$?

**5.** Find an equation for the sine function, $y = A\sin x + D$, shown in the following graph.




# Changes in Input for Sine and Cosine Functions: Period and Horizontal Shift

## Period

Recall from Section 2.5 that a **periodic function** is one that gives the same output for inputs a fixed distance apart. Symbolically, this says that if $f$ is a periodic function, then for some constant $p$, $f(x) = f(x+p)$ for all $x$. The **period** of this function is the smallest value of $p$ for which this relationship is true. The functions $y = \sin x$ and $y = \cos x$ have a period of $2\pi$. (See Figure 11.) This means that $\sin x = \sin(x+2\pi)$ and $\cos x = \cos(x+2\pi)$ for all $x$. For $y = \sin x$ and $y = \cos x$,

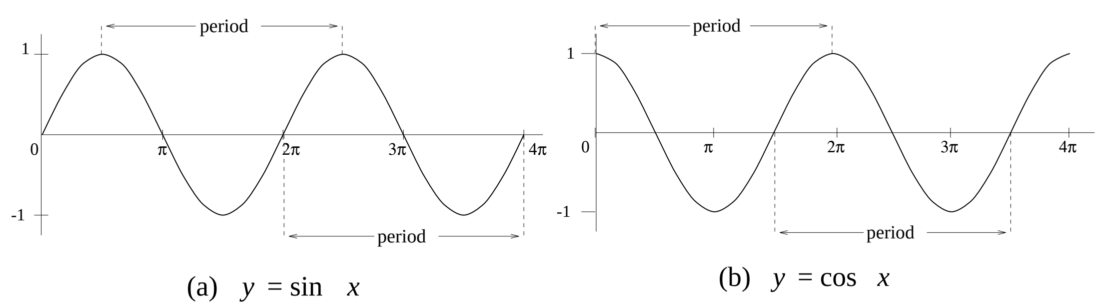



Figure 11: The period of $y = \sin x$ and $y = \cos x$ is $2\pi$.

inputs that are $2\pi$ units apart will have the same output. Notice that, for $p$ to be the period, the relationship $f(x) = f(x+p)$ has to work for *all* $x$. For example, $f(x) = \sin x$ is zero when $x$ is a multiple of $\pi$. Looking at graph 11(a), we see that $\sin 0 = \sin \pi = \sin(2\pi) = \sin(4\pi) = 0$. At first glance, this might lead you to conclude erroneously that the period of $f(x) = \sin x$ is $\pi$. However, $\sin(x + \pi) \neq \sin x$ for *all* values of $x$. In particular, $1 = \sin\left(\frac{\pi}{2}\right) \neq \sin\left(\frac{\pi}{2} + \pi\right) = \sin\left(\frac{3\pi}{2}\right) = -1$. So be careful when finding the period of a function by looking at points with the same output.

Since multiplying the input of a function by a constant compresses it horizontally towards the $y$-axis, multiplying the input of a trigonometric function by a constant will change the period. If you compress a function horizontally, you are changing the frequency at which the outputs will repeat. We can see this horizontal compression symbolically. For example, if $g(x) = \cos x$, then the period of $g$ is $2\pi$. This means that the graph of $g$ will complete one period between $x = 0$ and $x = 2\pi$. Now suppose $h(x) = \cos(3x)$. Because $h$ is compressed horizontally by a factor of 3, the outputs will repeat 3 times as often. This means that the graph of $h(x) = \cos(3x)$ will complete three periods between $x = 0$ and $x = 2\pi$. So the period of $h$ is $\frac{1}{3}$ of $2\pi$ or $\frac{2\pi}{3}$. The behavior of the sine function is similar. In general, the period of $f(x) = \sin(Bx)$ and $g(x) = \cos(Bx)$ is $\frac{2\pi}{B}$. Notice that the horizontal compression, $B$, is equal to $\frac{2\pi}{\text{period}}$.

**Example 4.** Determine the amplitude, midline, and period for $f(x) = 3\sin(\pi x)$.

**Solution:** Starting with the function $y = \sin x$, the function $f(x) = 3\sin(\pi x)$ has been vertically stretched by a factor of 3 and horizontally compressed by a factor of $\pi$. It has not been shifted either vertically or horizontally. So the amplitude of $f$ is 3 times the amplitude of $y = \sin x$ and the period of $h$ is $\frac{1}{\pi}$ times the period of $y = \sin x$. This means that the amplitude of $f(x) = 3\sin(\pi x)$ is 3 and the period is $\frac{2\pi}{\pi} = 2$. Since there is no vertical shift, the midline of $f$ is the same as the midline for $y = \sin x$ which is $y = 0$. ■

**Example 5.** Figure 12 is a graph of a cosine function which has been vertically stretched and horizontally "compressed." Using the amplitude and period, find the equation of the function.

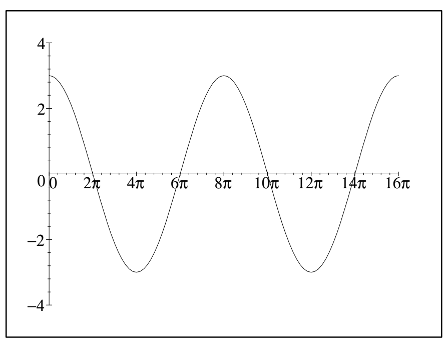



Figure 12

**Solution:** The amplitude is 3 and the period is $8\pi$. This means that there has been a vertical stretch of 3 and a horizontal "compression" of $\frac{1}{4}$ (since $\frac{2\pi}{8\pi} = \frac{1}{4}$). Therefore, the equation is $f(x) = 3\cos\left(\frac{x}{4}\right)$. ■

### Horizontal Shift

Now that we've looked extensively at what happens when you multiply the input of $y = \sin x$ or $y = \cos x$ by a constant, let's see what happens when we *add* a constant to the input. Recall that adding a constant to the input horizontally shifts the graph of the function. Look back at the graphs in Figure 11. Notice that the graphs of $y = \sin x$ and $y = \cos x$ will be identical if we appropriately shift them horizontally. If the graph of $y = \sin x$ were shifted to the left $\frac{\pi}{2}$ units, it would be the same as the graph of $y = \cos x$. Also, if the graph of $y = \cos x$ were shifted to the right $\frac{\pi}{2}$ units, it would be the same as the graph of $y = \sin x$. The observations give us the following two identities:

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x.$$
$$\cos\left(x - \frac{\pi}{2}\right) = \sin x.$$

When determining a horizontal shift given the graph of a sine or cosine function, focus on "familiar" points. For example, you know that the maximum values of $y = \sin x$ occur for $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$. You know that the maximum values of $y = \cos x$ occur at $x = 0, 2\pi, 4\pi, \ldots$. Be careful about concentrating on points where a sine or cosine function is zero. At some zeros, the function is changing from positive to negative while at other zeroes the function is changing from negative to positive. When using zeros to determine a horizontal shift, be sure to match them appropriately.

**Example 6.** Figure 13 is the graph of a sine function which has been shifted horizontally. Find the equation of the function.

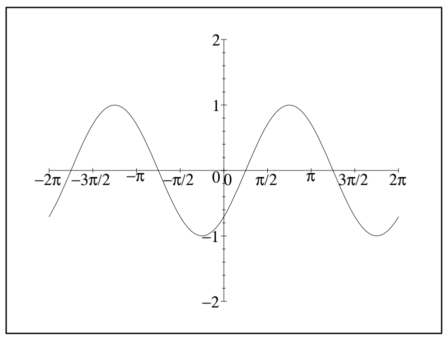


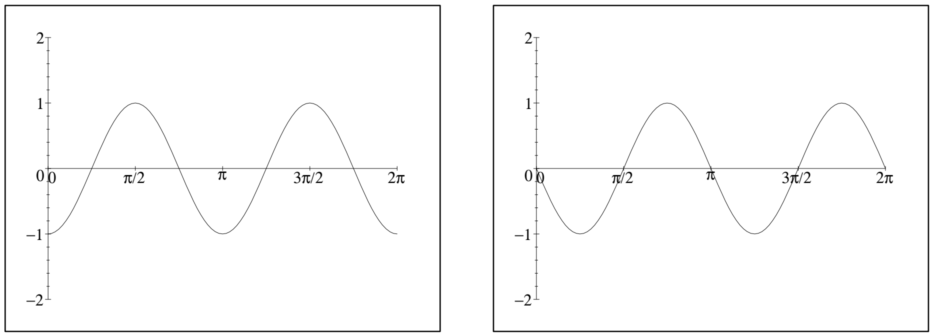

Figure 13

**Solution:** The graph in Figure 13 has a zero changing from negative to positive when $x = \frac{\pi}{4}$. The function $y = \sin x$ has a zero changing from negative to positive when $x = 0$. This means that $y = \sin x$ has been shifted $\frac{\pi}{4}$ units to the right. Alternatively, we can notice that the graph in Figure 13 appears to have a maximum value when $x = \frac{3\pi}{4}$ while the graph of $y = \sin x$ has a maximum when $x = \frac{\pi}{2}$. Again, this tells us that the graph in Figure 13 has been shifted $\frac{\pi}{4}$ units to the right. So the equation of the function shown in Figure 13 is $f(x) = \sin\left(x - \frac{\pi}{4}\right)$. ■

The equation $f(x) = \sin\left(x - \frac{\pi}{4}\right)$ is not the only equation that represents the graph in Figure 13. Since a sine function is periodic, we could have also used $f(x) = \sin\left(x - \frac{\pi}{4} + 2\pi\right) = \sin\left(x + \frac{7\pi}{4}\right)$ as the equation. If Example 6 had not stated that we were looking for the equation of a sine function, we could have written the equation for Figure 13 as a cosine function. Since $\cos x = \sin\left(x - \frac{\pi}{2}\right)$, we know that $f(x) = \sin\left(x - \frac{\pi}{4}\right) = \cos\left(x - \frac{\pi}{2} - \frac{\pi}{4}\right) = \cos\left(x - \frac{3\pi}{4}\right)$. Remember that, when finding equations for periodic functions, answers are not unique!

### Combining Changes in the Period with a Horizontal Shift

Combining a change in the period with a horizontal shift can be tricky. We mentioned earlier in this section that the graph of $f(x) = f(ax+b)$ is the graph of $f$, first shifted $b$ units horizontally and then horizontally compressed by a factor of $a$. Consider the two functions $f(x) = \sin\left(2x - \frac{\pi}{2}\right)$ and $g(x) = \sin\left[2\left(x - \frac{\pi}{2}\right)\right]$. For $f(x) = \sin\left(2x - \frac{\pi}{2}\right)$, the graph of $y = \sin x$ has first been shifted to the right $\frac{\pi}{2}$ units and then horizontally compressed by a factor of 2. For $g(x) = \sin\left[2\left(x - \frac{\pi}{2}\right)\right]$, the graph of $y = \sin x$ has first been horizontally compressed by a factor of 2 and then shifted to the right $\frac{\pi}{2}$ units. Both $f$ and $g$ are sine functions whose period has been changed to $\pi$ and whose graph has been shifted to the right $\frac{\pi}{2}$ units, but these two transformations have been done in the opposite order. The graphs of $f$ and $g$ are shown in Figure 14.




(a) $f(x) = \sin\left(2x - \frac{\pi}{2}\right)$ (b) $g(x) = \sin\left[2\left(x - \frac{\pi}{2}\right)\right]$

Figure 14: Two sine functions that have been shifted $\frac{\pi}{2}$ units to the right and have a period of $\pi$. Figure 14(a) shows the function with the shift first and the change in period second. Figure 14(b) shows the function with the change in period first and the shift second.

Let's take a closer look at these two functions. In particular, notice that $g(x) = \sin\left[2\left(x - \frac{\pi}{2}\right)\right] = \sin(2x - \pi)$. We can think of $g$ as compressing the period by a factor of 2 and then shifting $\frac{\pi}{2}$ units to the right (as we did earlier) or we can think of $g$ as shifting $\pi$ units to the right and then compressing the period by a factor of 2. Similarly, notice that $f(x) = \sin\left(2x - \frac{\pi}{2}\right) = \sin\left[2\left(x - \frac{\pi}{4}\right)\right]$. We can think of $f$ as shifting $\frac{\pi}{2}$ units to the right and then compressing the period by a factor of 2 (as we did earlier) or we can think of $f$ as compressing the period by a factor of 2 and then shifting $\frac{\pi}{4}$ units to the right. Which is the better way to think about this type of transformation? Should we typically write functions as $y = \sin(Bx + E)$ or as $y = \sin[B(x + C)]$? Look carefully at the graphs in Figure 14. Using Figure 14(a), it is fairly easy to see the period is $\pi$ and there is a shift of $\frac{\pi}{4}$ units to the right.[123] Using Figure 14(b), it is fairly easy to see that the period is $\pi$ and there is a shift of $\frac{\pi}{2}$ units to the right. The numbers 2 and $\frac{\pi}{4}$ for $f$ and the numbers 2 and $\frac{\pi}{2}$ for $g$ occur when the formulas are in the form $y = \sin[B(x + C)]$. This form gives a clearer connection between the graph and the equation. So even though algebraically it's simpler to multiply through and remove the parenthesis, we will typically write sine and cosine functions in the form $y = \sin[B(x + C)]$ or $y = \cos[B(x + C)]$.

In general, for $f(x) = \sin[B(x + C)]$ or $g(x) = \cos[B(x + C)]$:

- The period is $\dfrac{2\pi}{B}$. Also, $B = \dfrac{2\pi}{\text{period}}$.
- The horizontal shift is $C$ units to the left if $C$ is positive.
- The horizontal shift is $C$ units to the right if $C$ is negative.

**Example 7.** Determine the equation of the function shown in Figure 15. Write your in solution two ways: first as $f(x) = \sin[B(x + C)]$ and then as $f(x) = \cos[B(x + C)]$.

**Solution:** From the graph in Figure 15, we can see that the function has a period of 4. Therefore, $B = \frac{2\pi}{4} = \frac{\pi}{2}$. The function $y = \sin x$ has a zero at $x = 0$ where the function is changing from negative to positive. The graph in Figure 15 does the same. Therefore, this function has no horizontal shift from $y = \sin x$ so $C = 0$. The equation of this function is $f(x) = \sin\left(\frac{\pi x}{2}\right)$.

[123] When looking for a horizontal shift combined with a horizontal compression, it is best to use points that use to be at $x = 0$ rather than concentrating on maximums or minimums. This is because a horizontal compression changes the location of all points except for $x = 0$.

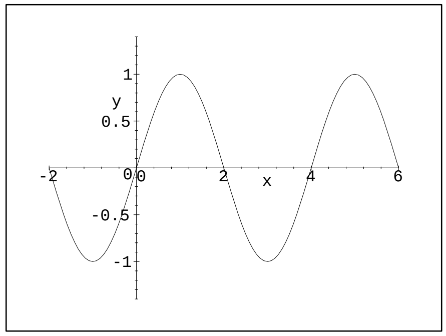


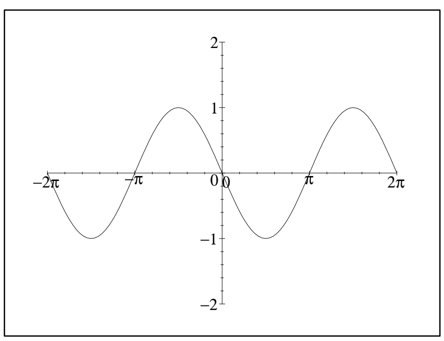

Figure 15

For the cosine function, we again have a period of 4 which means $B = \frac{\pi}{2}$. The function $y = \cos x$ has a maximum at $x = 0$ while the graph in Figure 15 has a maximum at $x = 1$. Therefore, this is a cosine graph which has been shifted 1 unit to the right so $C = -1$. The equation of this function is $g(x) = \cos\left[\frac{\pi}{2}(x-1)\right]$. Notice once again that the equation of a periodic function is not unique! ■

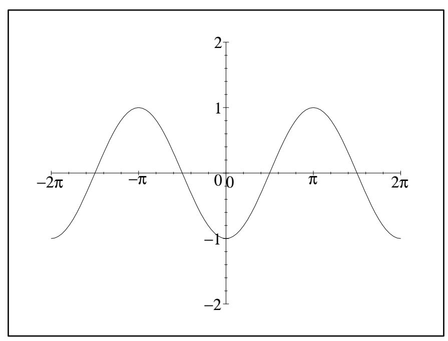
## Reading Questions

**6.** Match the following equations to the graphs.

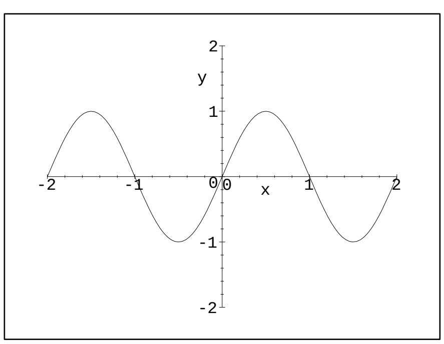
$$y = \sin(\pi x), \quad y = \sin(x - \pi), \quad y = \cos(\pi x), \quad y = \cos(x - \pi)$$



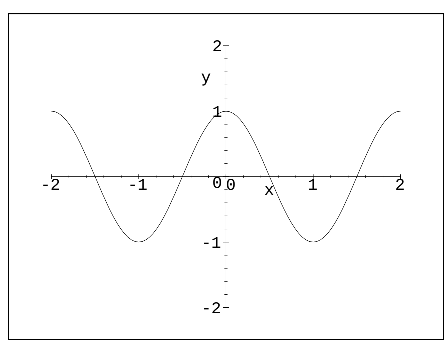

(a)




(b)




(c)




(d)

**7.** If a cosine function has been horizontally compressed by a factor of 10, what is its period? What is its equation?

**8.** True or False:

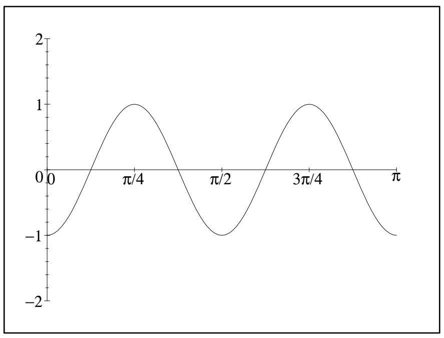
(a) $\sin\left(x + \frac{\pi}{2}\right) = \cos x$

(b) $\cos\left(x + \frac{\pi}{2}\right) = \sin x$.

(c) The period of $y = \sin(2x)$ is 2.

(d) The horizontal shift of $y = \cos(x - 3)$ is 3 units to the right.

**9.** The equation for the following graph can be written in the form $f(x) = \cos[B(x + C)]$. Find the value of $B$ and $C$.




## Putting it All Together

The general formulas for the sine and cosine functions are:

$$f(x) = A\sin[B(x + C)] + D \qquad \text{and} \qquad g(x) = A\cos[B(x + C)] + D$$

where

- The amplitude is $|A|$.
- The period is $\frac{2\pi}{B}$.
- The horizontal shift is $C$.
- The equation of the midline is $y = D$.

By combining input and output changes, we can find the equation for any sine or cosine function. More importantly, we can find equations to model real world periodic behavior.

**Example 8.** The graph of $f$ is shown in Figure 16. Determine an equation for $f$.

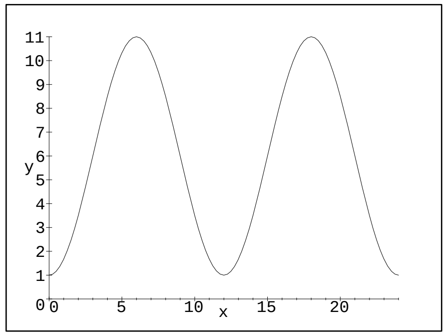



Figure 16

**Solution:** We can use either a sine or cosine function for the equation for $f$. We'll choose a cosine function since this graph is similar to $y = \cos x$, reflected over its midline, with no horizontal shift. The maximum value is 11 and the minimum value is 1, so the amplitude is $\frac{11-1}{2} = 5$. The equation of the midline is $y = \frac{11+1}{2} = 6$. Since, for $x = 0$, the output is below the midline, we know that $A = -5$ and $D = 6$. We chose a cosine function so we could ignore a horizontal shift. Therefore, $C = 0$. Finally, to find $B$, we need to look at the period. Looking at the graph, the period is 12 so $B = \frac{2\pi}{12} = \frac{\pi}{6}$. Therefore the equation of the function shown in Figure 16 is $f(x) = -5\cos\left(\frac{\pi}{6}x\right) + 6$. ■

**Example 9.** The graph of $g$ is shown in Figure 17. Determine an equation for $g$.

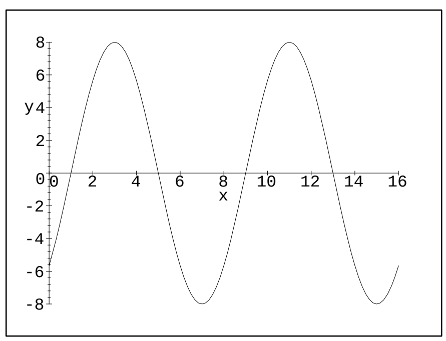



Figure 17

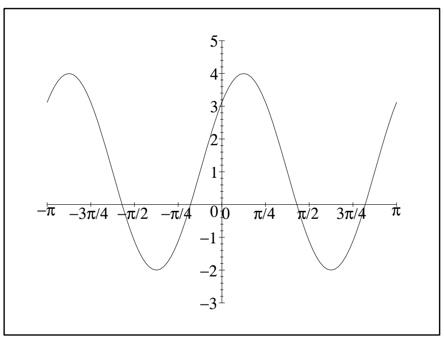
**Solution:** The graph shown in Figure 17 can be modeled by either a sine or a cosine function. Both will require a horizontal shift. We will choose to model it with a sine function. The maximum value is 8 and the minimum value is $-8$. These means the amplitude is $\frac{8-(-8)}{2} = 8$ and the midline is $y = \frac{8+(-8)}{2} = 0$. Since we're assuming this is a sine function that has been shifted to the right, the graph of $g$ is not reflected over it's midline. So $A = 8$ and $D = 0$. To find $B$, we determine that the period is also 8. So $B = \frac{2\pi}{8} = \frac{\pi}{4}$. The graph of $y = \sin x$ has a zero where $y = \sin x$ is increasing when $x = 0$. The graph in Figure 17 crosses the midline ($y = 0$) and is increasing when $x = 1$. So $g$ is a sine function shifted 1 unit to the right. Therefore $C = -1$. The equation of a function that will fit this graph is $g(x) = 8\sin\left[\frac{\pi}{4}(x-1)\right]$. ■

**Example 10.** The world's tallest Ferris wheel, the Millenium Wheel, is being built in London, England. The wheel is 500 feet in diameter and has 60 enclosed capsules each of which can carry up to 16 people. The wheel takes 20 minutes to make one revolution. Assume that the capsule starts at the bottom of the Ferris wheel and this height is 0. Find a function whose input is time (in minutes) and whose output is height of the capsule (in feet).

**Solution:** Since this is a periodic function where the minimum height occurs at $t = 0$, a cosine function reflected over the $x$-axis is a good choice for our modeling function. The maximum height height of the wheel is 500 feet and the minimum height is 0. Therefore, the amplitude is $\frac{500-0}{2} = 250$ and the midline is $y = \frac{500+0}{2} = 250$. The function is reflected over its midline (since it starts at the minimum rather than the maximum value) so $A = -250$ and $D = 250$. To find the period, we use the fact that it takes 20 minutes to make one revolution. So $B = \frac{2\pi}{20} = \frac{\pi}{10}$. We do not need a horizontal shift since the function starts at the minimum value. Therefore, a function that models the height of the capsule, in feet, is $h(t) = -250\cos\left(\frac{\pi t}{10}\right) + 250$, where $t$ is the time in minutes. ■

### Reading Questions

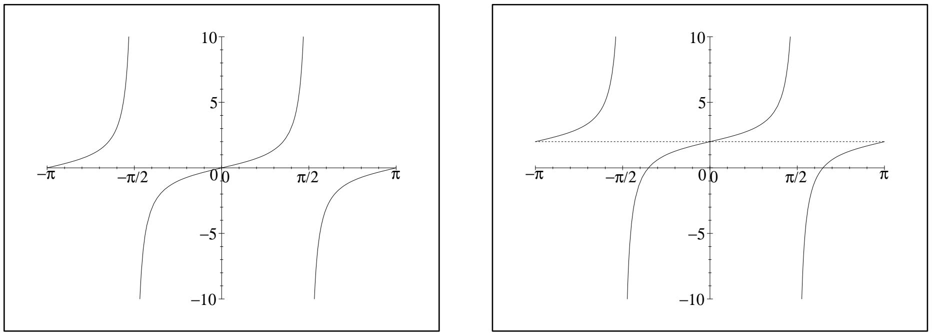
**10.** Determine an equation for the function shown in the following figure.




**11.** Suppose you have a kitchen clock with a 4-inch minute hand. Find an equation to model the height of the tip of the minute hand as it travels around the clock. Your input should be time (in minutes) and your output should be height from the line connecting the 3 and the 9 on the clock (in inches). Assume the minute hand starts at the 12 o'clock position.

## Transformations of the Tangent Function

The tangent function is a periodic function, like sine and cosine, but it is different in many ways. While sine and cosine functions have maximum and minimum outputs, the tangent function does not. Because of this, amplitude is not defined for the tangent function. We can still vertically stretch tangent functions (as we can for any function), but, without maximums and minimums, it is difficult to see this stretch when looking at the graph. A midline is also not defined for the tangent function. However, we can still vertically shift tangent functions. Since we can't focus on a maximum or minimum point when determining the vertical shift, we will instead focus on the point where the tangent function changes concavity. For $f(x) = \tan x$, the graph changes from concave down to concave up at the points where $y = 0$. (See Figure 18(a).) In the graph of $g(x) = \tan x + 2$, $g$ changes from concave down to concave up at the points where $y = 2$. (See Figure 18(b).)




(a) $f(x) = \tan x$ (b) $g(x) = \tan x + 2$

Figure 18: The graph of $f(x) = \tan x$ changes concavity at points where $y = 0$. The graph of $y = \tan x + 2$ changes concavity at the points where $y = 2$.

A horizontal shift can be seen using these same points. For $f(x) = \tan x$, the graph changes concavity when $x = n\pi$ where $n$ is an integer. (See Figure 18(a).) For $h(x) = \tan(x - 1)$, the graph changes concavity when $x = n\pi + 1$ where $n$ is an integer. (See Figure 19). This graph has been shifted 1 unit to the right.

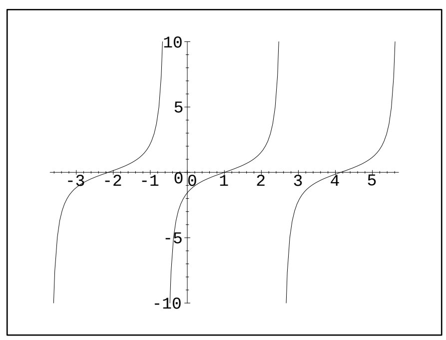



Figure 19: The graph of $h(x) = \tan(x - 1)$ changes concavity at the point $(1, 0)$. It is a tangent function which has been shifted to the right one unit.

A horizontal compression can be seen by observing the change in period. The period of $f(x) = \tan x$ is $\pi$. (See Figure 18(a).) The period of $k(x) = \tan(2x)$ is $\frac{\pi}{2}$ since this has been compressed by a factor of 2. The easiest way to see the period is to look at the distance between vertical asymptotes (or the distance between points where the graph changes concavity).

In general, for $y = A\tan[B(x + C)] + D$,

- The vertical stretch is $|A|$. If the graph changes from concave down to concave up between the vertical asymptotes, $A$ is positive. If the graph changes from concave up to concave down between the vertical asymptotes, $A$ is negative. (The value of $A$ is difficult to determine from a graph.)

- The vertical shift is $D$. The function will change concavity at the points where the output is $D$.

- The period is $\frac{\pi}{B}$. Notice that $B = \frac{\pi}{\text{period}}$.

- The horizontal shift is $C$. One of the points where the function changes concavity will be $(C, D)$.

**Example 11.** Determine the equation of the function shown in Figure 20.

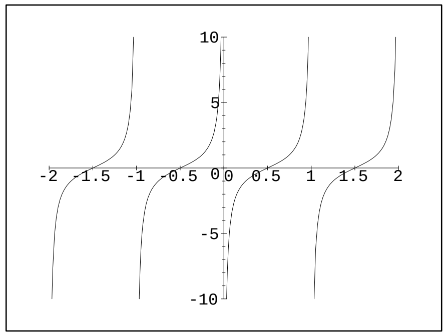



Figure 20

**Solution:** The graph shown in Figure 20 is similar to the graph of $y = \tan x$. The points where the graph changes concavity are at $y = 0$ so the function has not been shifted vertically and $D = 0$. One of the points where the concavity changes is $(0, 0.5)$ so the function has been shifted to the right 0.5 units and $C = -0.5$. The period has also changed. The period of the function shown in Figure 20 is 1, so $B = \frac{\pi}{1} = \pi$. Therefore, the equation of this function is $f(x) = \tan[\pi(x - 0.5)]$. ■

**Example 12.** Determine the equation of the function shown in Figure 21.

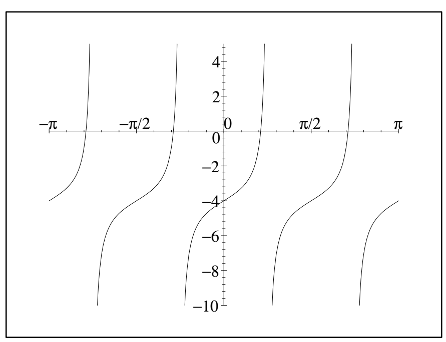



Figure 21

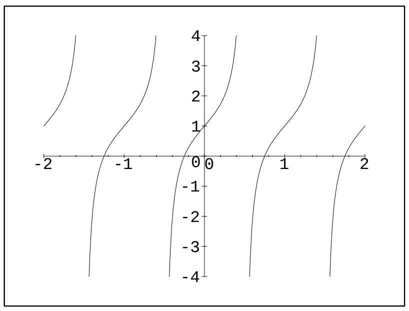
**Solution:** The graph shown in Figure 21 is similar to $y = \tan x$. However, the points where it changes concavity are at $y = -4$. So there is a vertical shift of $-4$ units, meaning that $D = -4$. One of the points where it changes concavity is at $(0, -4)$. So there is no horizontal shift, meaning that $C = 0$. The period of this function is $\frac{\pi}{2}$, so $B = \pi / \frac{\pi}{2} = 2$. Therefore, the equation is $f(x) = \tan(2x) - 4$. ■

**Reading Questions**

**12.** Why doesn't the tangent function have an amplitude or a midline?

**13.** For $y = A\tan[B(x + C)] + D$, how do you tell, looking at the graph, if $A$ is positive or negative?

**14.** Determine the equation of the tangent function shown in the following figure.




## Inverse Trigonometric Functions

Inverse trigonometric functions are frequently used to solve trigonometric equations. Recall from Section 3.5 that inverse trigonometric functions have a restricted domain. This is because, for the inverse of a function to exist, the function must pass the "horizontal line test" and not have repeated outputs. By the very nature of being periodic, trigonometric functions do not satisfy this condition unless the domain is restricted to avoid repeated outputs. We define the inverse of the sine and cosine functions as follows:

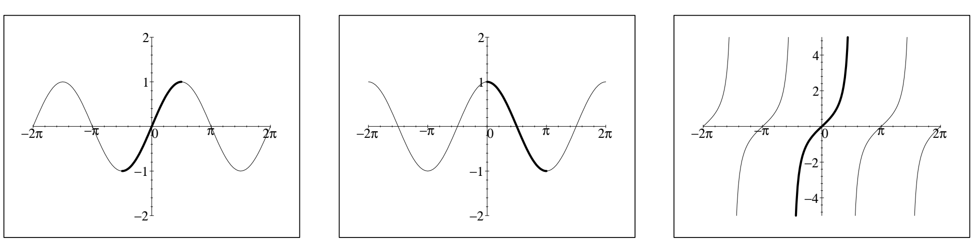
$$y = \sin^{-1} x \quad \text{if and only if} \quad x = \sin y \text{ and } \frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$$

$$y = \cos^{-1} x \quad \text{if and only if} \quad x = \cos y \text{ and } 0 \leq y \leq \pi$$

The inverse tangent function is defined with the same restricted domain as the sine function. Its inverse is defined as follows:

$$y = \tan^{-1} x \quad \text{if and only if} \quad x = \tan y \text{ and } \frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$$

Figure 22 shows the restricted domains of $y = \sin x$, $y = \cos x$, and $y = \tan x$ needed so that the inverse functions can be defined.



(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = \tan x$

Figure 22: The bold portions of these graphs show the restricted domains of the sine, cosine, and tangent functions so their inverse functions can be defined.

We will illustrate how inverse trigonometric functions can be used to solve equations by finding a solution to

$$2 = 3\cos\left(\frac{\pi}{2}x\right).$$

To begin, we divide both sides of this equation by 3 to obtain

$$\frac{2}{3} = \cos\left(\frac{\pi}{2}x\right).$$

Applying the inverse cosine function we obtain

$$\cos^{-1}\left(\frac{2}{3}\right) = \frac{\pi}{2}x.$$

Solving this for $x$ we get

$$\frac{2}{\pi}\cos^{-1}\left(\frac{2}{3}\right) = x.$$

Using a calculator to find a decimal approximation, we see that $x \approx 0.535$. [Note: There are an infinite number of solutions to $2 = 3\cos\left(\frac{\pi}{2}x\right)$, but the way inverse cosine is defined, it only gives us one solution. You will learn how to find other solutions in Chapter 6.]

**Example 13.** Solve $1 = 2\tan[\pi(x-2)] + 3$ for $x$.

**Solution:** We first must solve $1 = 2\tan[\pi(x-2)] + 3$ for $\tan[\pi(x-2)]$. Doing this we get

$$\begin{aligned} 1 &= 2\tan[\pi(x-2)] + 3 \\ -2 &= 2\tan[\pi(x-2)] \\ -1 &= \tan[\pi(x-2)]. \end{aligned}$$

Now we apply the inverse tangent function and solve for $x$.

$$\begin{aligned} -1 &= \tan[\pi(x-2)]. \\ \tan^{-1}(-1) &= \pi(x-2) \\ \frac{\tan^{-1}(-1)}{\pi} &= x-2 \\ \frac{\tan^{-1}(-1)}{\pi} + 2 &= x \end{aligned}$$

In Section 5.1, we used a $45° - 45° - 90°$ triangle to find that $\tan 45° = \tan\left(\frac{\pi}{4}\right) = 1$. Since $\tan x$ is an odd function, $\tan\left(-\frac{\pi}{4}\right) = -1$. So $\tan^{-1}(-1) = -\frac{\pi}{4}$. Using this, we find that $x = \frac{-\frac{\pi}{4}}{\pi} + 2 = -\frac{1}{4} + 2 = 1.75$. ■

**Example 14.** Solve $3 = \sin[\pi(x+1)]$ for $x$.

**Solution:** This equation does not have a solution because the maximum output of $y = \sin[\pi(x+1)]$ is 1, so $\sin[\pi(x+1)]$ will never equal 3. If we didn't notice this and just began to solve this equation, we encounter a problem. To try to solve this, we would apply the inverse sine function and solve for $x$.

$$\begin{aligned} \sin^{-1}(3) &= \pi(x+1) \\ \frac{\sin^{-1}(3)}{\pi} &= x+1 \\ \frac{\sin^{-1}(3)}{\pi} - 1 &= x \end{aligned}$$

In trying to find a decimal approximation for this, we get an error message on our calculator. This is because the domain of the inverse sine function is the range of the sine function, $-1 \le x \le 1$, so $\sin^{-1}(3)$ is not defined. Therefore, our equation has no solution. ■

### Reading Questions

**15.** Why must the domains of trigonometric functions be limited in order for their inverse functions to be defined?

**16.** Find a solution to $3 = \cos(x + 1) + 4$.

## Exercises

**1.** Indicate whether each of the following are True or False. If false, correct the statement so that it becomes a true statement.

(a) The graph of $y = f(x+3)$ is the graph of $y = f(x)$ shifted to the right 3 units.

(b) The graph of $y = f(2x+3)$ is the graph of $y = f(x)$ first horizontally compressed towards the $y$-axis by 2 units and then shifted to the left 3 units.

(c) The period of $y = \sin x$ is $\pi$ units.

(d) The amplitude of $y = 3\cos(2x) - 1$ is 3.

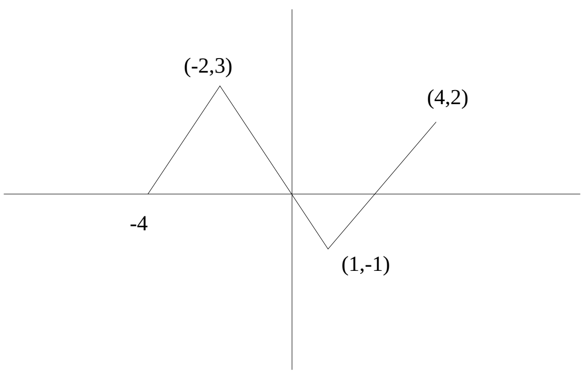
(e) The midline of $y = \sin(2x) - 1$ is $y = 2$.

(f) The period of $y = 3\sin(2x)$ is 2.

(g) The function $y = \tan x$ is undefined for $x = n\pi$ where $n$ is an integer.

(h) The domain of $y = \sin^{-1} x$ is $-1 \le x \le 1$.

**2.** The following graph is of $y = f(x)$.




Match the following functions to their graphs.

(a) $y = f(x-3)$

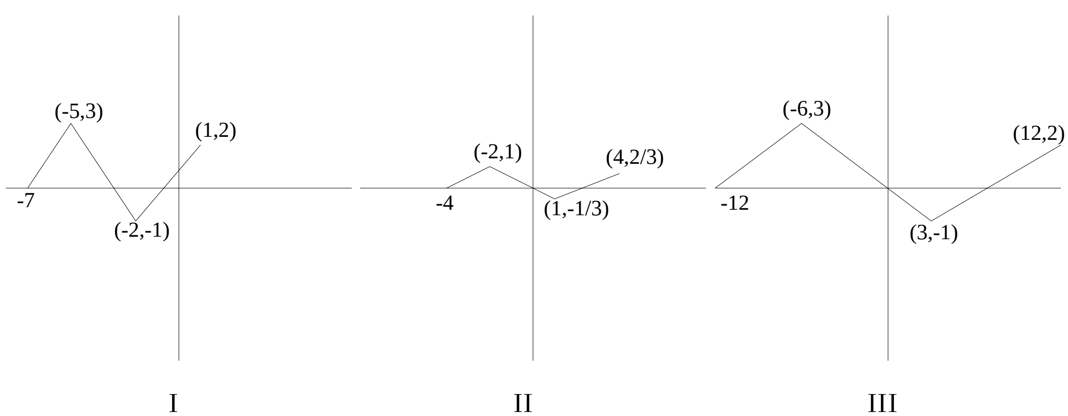
(b) $y = f(x+3)$

(c) $y = 3f(x)$

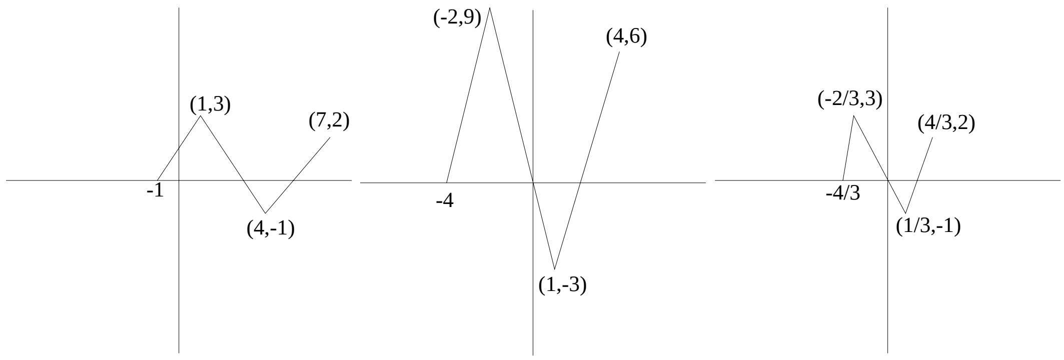
(d) $y = \frac{1}{3}f(x)$

(e) $y = f(3x)$

(f) $y = f(\frac{1}{3}x)$





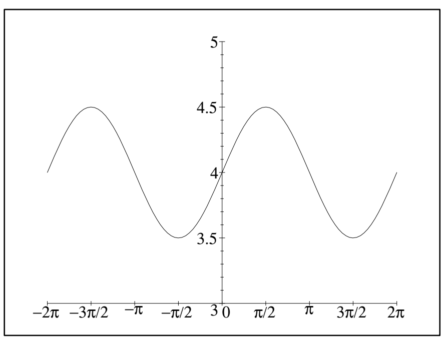

IV V VI

**3.** Give the amplitude and the midline for each of the following functions.

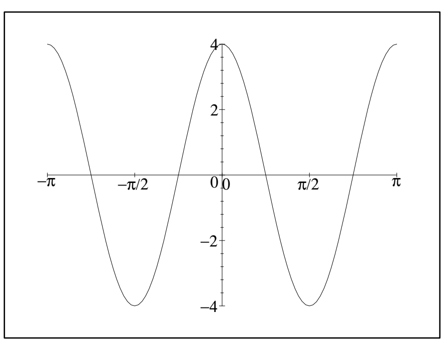
(a) $y = 3\sin(2x) + 4$

(b) $y = -3\cos x - \pi$

(c)




(d)




(e) A cosine function whose maximum is 10 and whose midline is $y = -2$.

(f) A sine function vertically stretched by a factor of $\pi$ and then shifted down 1 unit.

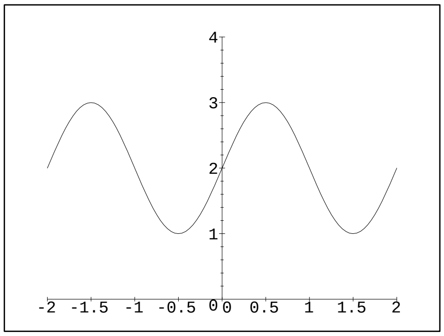
**4.** Give the period for each of the following functions.

(a) $y = 2\cos(x + \pi) - 1$

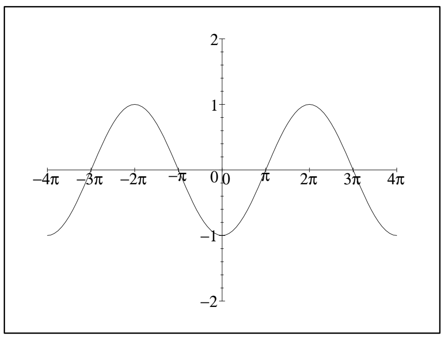
(b) $y = -\sin(3x)$

(c) $y = 3\tan[4(x - \frac{\pi}{4})]$

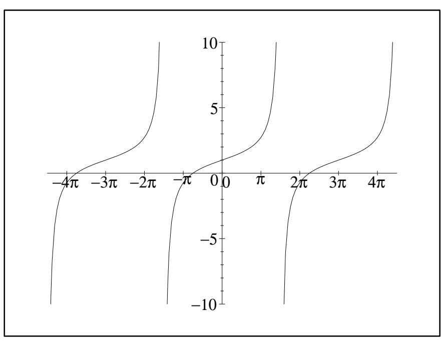
(d)




(e)




(f)




(g) A sine function which has been vertically stretched by a factor of 4, horizontally compressed by a factor of $\frac{1}{2}$, and then reflected over the $y$-axis.

(h) A cosine function which has a maximum at $x = 0$ and at $x = 3$ and no maximums for $0 < x < 3$.

(i) A tangent function which changes concavity at $(1, 1)$. The nearest point to the right where it has a vertical asymptote is $x = 2$.

**5.** (a) Write the equation of a cosine function with a period of $\frac{\pi}{2}$ that has been shifted 1 unit to the left.

(b) Rewrite your answer to part(a) as a sine function.

**6.** (a) Rewrite $y = 2\cos(3x - 1) + 5$ as a sine function.

(b) Rewrite $y = A\cos[B(x + C)] + D$ as a sine function.

(c) Rewrite $y = -3\sin(\pi x + 2) - \frac{\pi}{4}$ as a cosine function.

(d) Rewrite $y = A'\sin(B'x + C') + D'$ as a cosine function.

**7.** Match the following functions to their graphs.

(a) $y = \sin x$

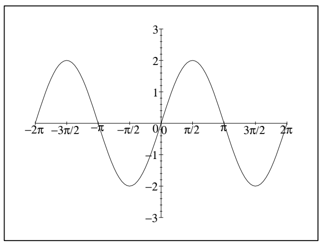
(b) $y = \sin(2x)$

(c) $y = 2\sin x$

(d) $y = \sin(x+2)$

(e) $y = \sin(x-2)$

(f) $y = \sin x + 2$



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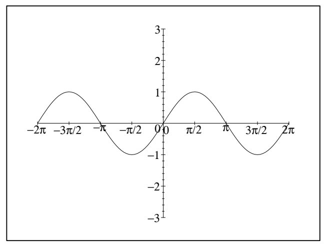

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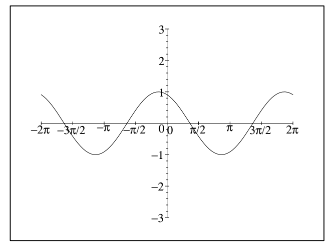
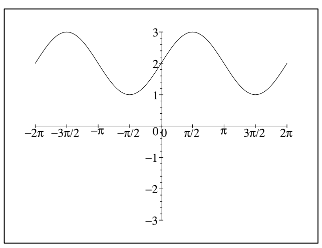
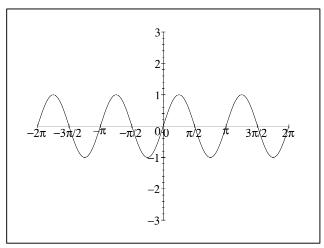


III



IV



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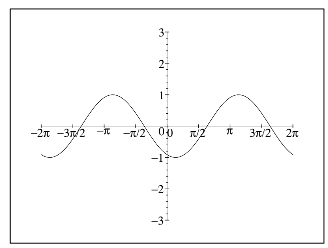


VI

**8.** Match the following functions to their graphs.

(a) $y = 3\cos x$

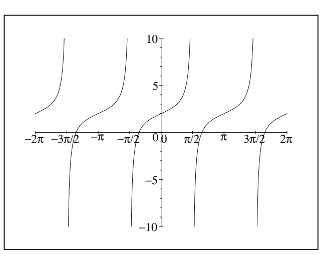
(b) $y = \cos(3x)$

(c) $y = \cos(\frac{1}{3}x)$

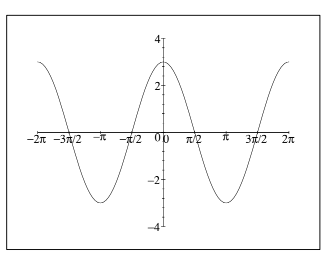
(d) $y = \tan x + 2$

(e) $y = \tan(2x)$

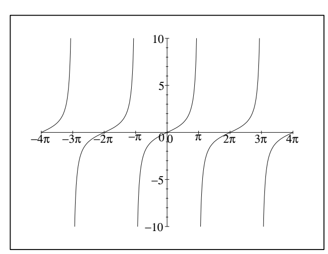
(f) $y = \tan(0.5x)$



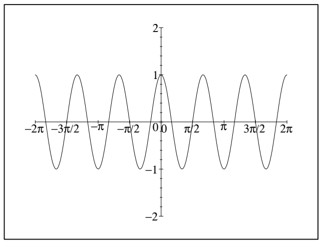
I



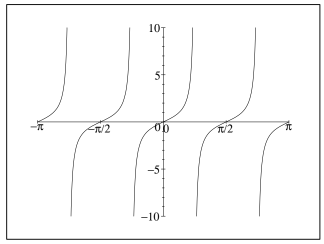
II



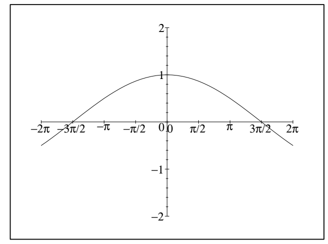
III




IV



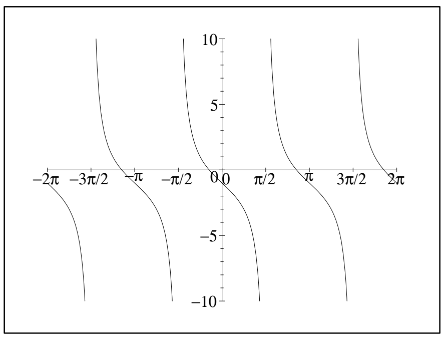

V



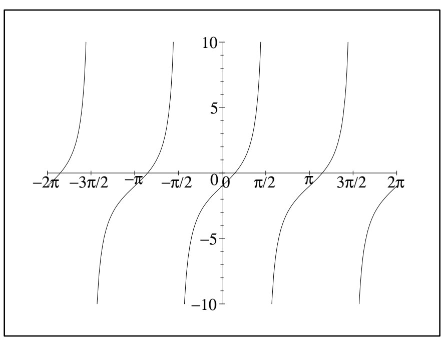

VI

**9.** The following graphs are functions of the form $y = A \tan x + D$. Determine if $A$ and $D$ are positive or negative. Justify your answer.

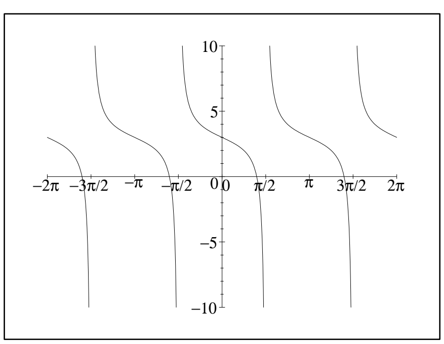
(a)




(b)

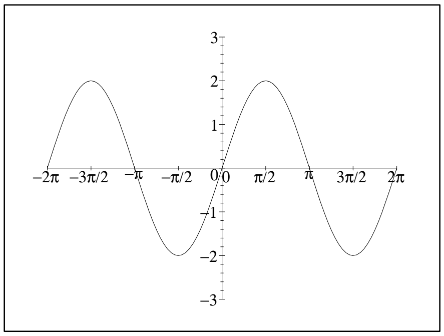



(c)

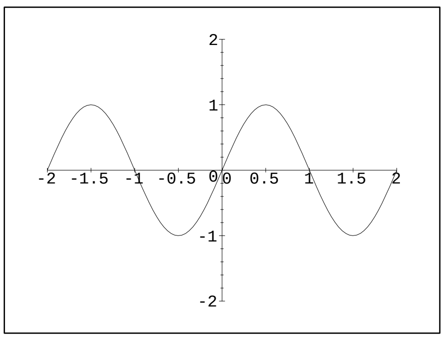


**10.** The following graphs are functions of the form $y = A\sin[B(x+C)] + D$. Find $A, B, C$ and $D$.

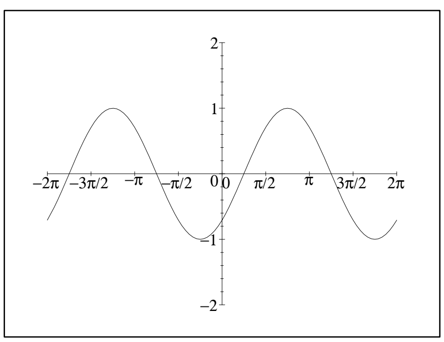
(a)



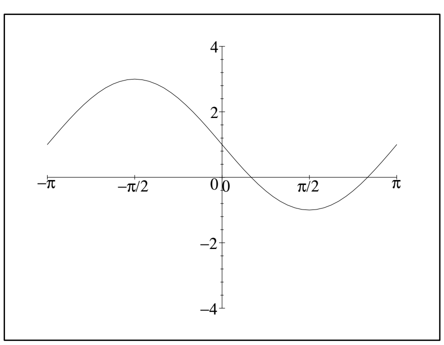

(b)



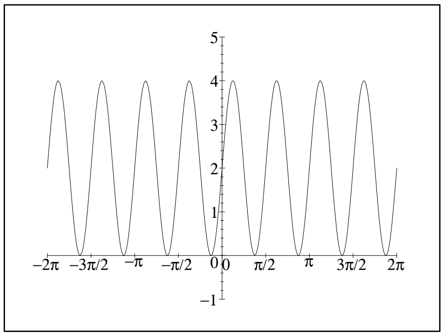

(c)



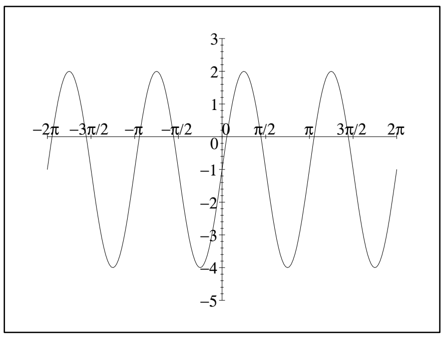

(d)



(e)

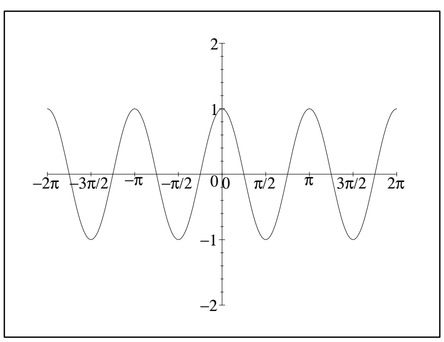



(f)

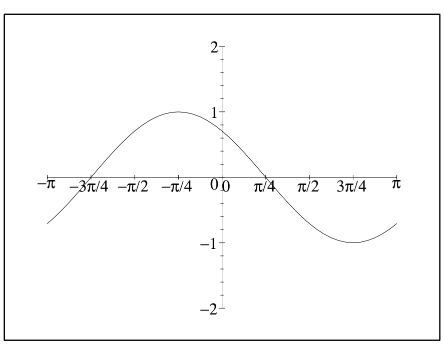



**11.** The following graphs are functions of the form $y = A\cos[B(x + C)] + D$. Find $A, B, C$ and $D$.

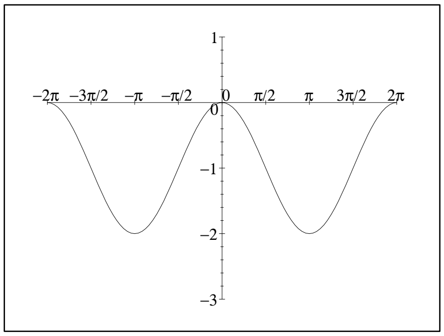
(a)



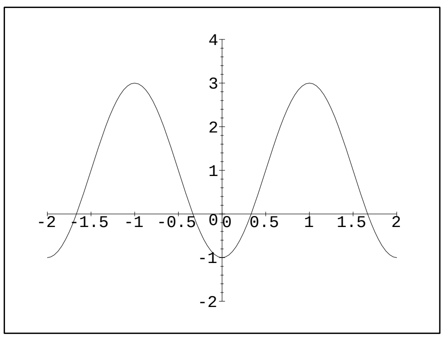

(b)



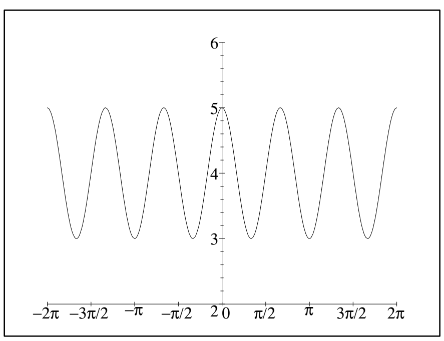
(c)


1
0
−1
−2
−3
−2π −3π/2 −π −π/2 0 π/2 π 3π/2 2π

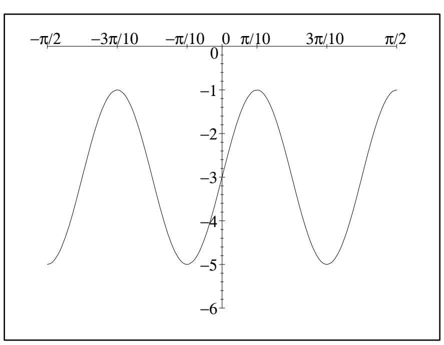
(d)


4
3
2
1
0
-1
-2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2

(e)

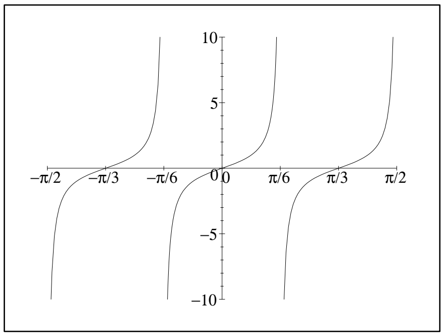

6
5
4
3
2
−2π −3π/2 −π −π/2 0 π/2 π 3π/2 2π

(f)

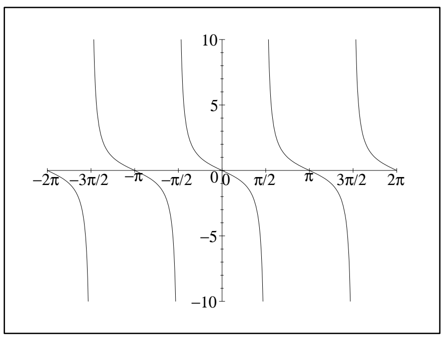

−π/2 −3π/10 −π/10 0 π/10 3π/10 π/2
0
−1
−2
−3
−4
−5
−6

**12.** Find the equations for the functions given in the following graphs.

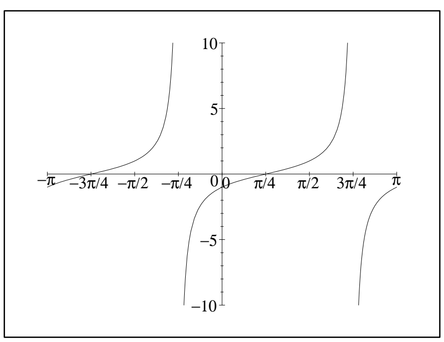
(a)



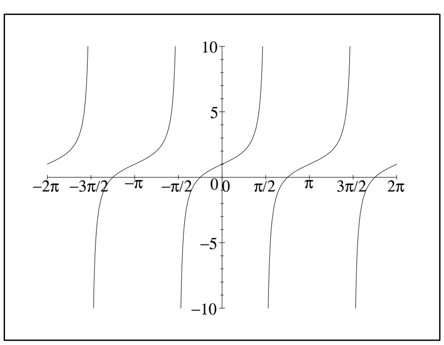

(b)




(c)




(d)



**13.** Correct each of the following statements.

(a) The domain of $y = \sin^{-1} x$ is $-1 \le x \le 1$ and the range is all real numbers.

(b) The range of $y = \tan^{-1} x$ is $0 \le y \le 2\pi$.

(c) The range of $y = \cos^{-1} x$ is $-\pi \le x \le \pi$.

(d) $\cos^{-1} \frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

(e) $\sin^{-1}(3)$ is positive.

(f) $\tan^{-1} 1 = \frac{\pi}{2}$.

**14.** Solve the following equations. Give exact answers where possible. Otherwise, round answers to two decimal places.

(a) $\frac{1}{2} = \sin\left(\frac{\pi}{4}x\right)$

(b) $4 = \tan\left(\frac{x}{3}\right)$

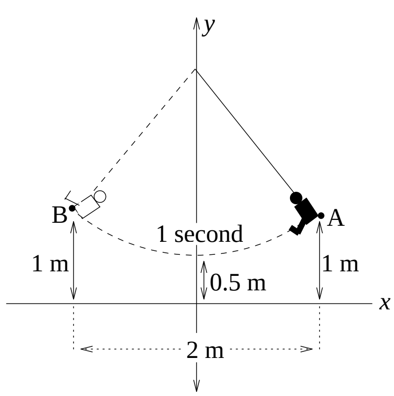
(c) $1 = 3\cos(4x - 3)$

(d) $1 = \cos\left(\frac{\pi}{3}x\right)$

(e) $0 = \sin[\pi(x - 6)]$

(f) $5 = \tan(x - \frac{\pi}{4}) - 4$.

**15.** The following figure shows a child's swing. As the child swings forward, she starts 1 meter above the ground (the highest position), then is at 0.5 meters above the ground (the lowest position), then is back up to 1 meter when she is at point $B$. Since she went from the highest position to the lowest and back to the highest, this completes one period for the vertical position function. It takes one second for her to swing from point $A$ to point $B$. Let $s(t)$ be the function whose input is time (in seconds) and whose output is vertical distance above the ground (in meters).




(a) Explain why $s(t)$ can be modeled by a cosine function.

(b) What is the amplitude for $s$?

(c) What is the period for $s$?

(d) Write an equation for $s$. Check your answer by making sure it gives you the points $(0, 1)$, $(0.5, 0.5)$, and $(1, 1)$.

**16.** Common household electricity is known as alternating current (AC). In the United States this current continually alternates between a maximum value of $110\sqrt{2}$ volts and a minimum value of $-110\sqrt{2}$ volts, at a rate of 60 times per second. Write the symbolic representation for a cosine function that models common household alternating current, where input is time in seconds.

**17.** Sound waves are periodic. A "pure" tone can be represented by a sine or cosine function. The period of the function is the reciprocal of the frequency of the note, i.e. frequency $= \frac{1}{\text{period}}$. The following table shows the frequency of the notes from middle $C$ to the $C$ one octave higher in hertz (or cycles per second). When you play the same note one octave lower, the frequency is half as much. When you play the same note one octave higher, the frequency is twice as much. Using this information and the table, determine what note is represented by the following functions.

| Note | $C$ | $C\sharp$ | $D$ | $E\flat$ | $E$ | $F$ | $F\sharp$ | $G$ | $A\flat$ | $A$ | $B\flat$ | $B$ | $C$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Frequency | 262 | 277 | 294 | 311 | 330 | 349 | 370 | 392 | 415 | 440 | 466 | 494 | 524 |

(a) $y = \sin(880\pi x)$

(b) $y = \cos(466\pi x)$

(c) $y = \sin(494\pi x)$

(d) $y = \cos(1568\pi x)$

**18.** The worlds tallest Ferris wheel is in Osaka, Japan. It measures 112.5 meters from the ground to it's apex and the wheel has a diameter of 100 meters (there is another Ferris wheel that has the same diameter in Otsu, but the wheel in Osaka is 4 meters higher). One revolution of this wheel takes about 15 minutes.[124] Assuming a point on the rim of the wheel starts at its minimum height, write a symbolic representation for the height of that point where time is the input (in minutes) and height above the ground is the output (in meters).

## Investigations

### Investigation 1: Aliasing.

Using calculators or computers is an easy way to quickly see a graph of a function. However, the picture on the screen is never completely accurate. Most of the time, the graph conveys enough information for you to correctly analyze the function. Unfortunately, this isn't always the case. One of these situations is when you are graphing periodic functions where the calculator or computer is not choosing a good set of inputs. This can cause you to misinterpret the period of the function and, therefore, think you have one periodic function when the graph is really representing a different periodic function. In physics, this situation is known as "aliasing." One place this causes problems is when the graphs are representing sound waves.

**1.** To explore the effect of "aliasing", we'll begin by using the function $f(x) = \cos x$.

(a) Complete Table 1 for the function $f(x) = \cos x$. Be sure your calculator is set in radians.

| $x$ | 0 | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\cos x$ | | | | | | | | | | | |

Table 1

(b) On graph paper (not on a calculator) sketch the graph that you get by plotting the points on Table 1. Make each square on your graph paper equal to one unit to that there are 6 horizontal squares between each point you plot. Connect the points with a smooth curve.

(c) What do you know about the period of $f(x) = \cos x$? What does the graph from part(b) imply about the period of $f(x) = \cos x$?

[124] Japan Information Network *Monthly News, Aug 1997: World's Tallest Ferris Wheel in Osaka,* http://www.jinjapan.org/kidsweb/news/97-8/wheel.html, May 27, 1998.

(d) Complete Table 2 and plot the points on the graph you used in part (b). Connect the points with a smooth curve. How do the two graphs compare?

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\cos x$ | | | | | | | | | | | | |

Table 2

(e) Why do the points in Table 1 give you an inaccurate graph of $f(x) = \cos x$?

**2.** To show the effect of aliasing on a calculator, we will look at the function $f(x) = \cos x$, using various dimensions for the viewing window.

(a) Have your calculator graph $f$ for $0 \leq x \leq 10\pi$. Sketch the graph you see on your calculator screen. By looking at this graph, estimate the period of $f$.

(b) Have your calculator graph $f$ for $0 \leq x \leq 50\pi$. Sketch the graph you see on your calculator screen. By looking at this graph, estimate the period of $g$.

(c) Have your calculator graph $f$ for $0 \leq x \leq 100\pi$. Sketch the graph you see on your calculator screen. By looking at this graph, estimate the period of $f$.

(d) Have your calculator graph $f$ for $0 \leq x \leq 200\pi$. Sketch the graph you see on your calculator screen. By looking at this graph, estimate the period of $f$.

**3.** We should also see the effect of aliasing on a calculator when we change the period of a function. For each of the following sine functions:

- Determine the period.
- Graph on your calculator with a window where $x$ has a minimum value of $-2\pi$ and a maximum value of $2\pi$.
- Using your graph, approximate the period of the function.
- Note any discrepancies between what the period should be and what it appears to be on the calculator and explain why these discrepancies occurred.

(a) $y = \sin(\pi x)$

(b) $y = \sin(5\pi x)$

(c) $y = \sin(10\pi x)$

(d) $y = \sin(15\pi x)$

(e) $y = \sin(20\pi x)$

(f) $y = \sin(25\pi x)$

(g) $y = \sin(30\pi x)$

This effect has applications in digital recording. A digital recorder samples sound at a certain rate (over 40,000 times each second) and records a picture of the sound wave in much the same way that your calculator pictures a function. The highest sound most people can hear is about 20,000 Hz or a sound wave that has about 20,000 periods per second. By sampling the sound at more than twice that rate the "aliasing" risk is eliminated for sounds that we can hear. If the recorder sampled the sound too slowly it would record a wave much different from the one it was trying to reproduce, much like the graph you made based on Table 1. The result is a wave with a relatively long period and a low frequency, which emits a low rumble when you would expect to hear the high sound made by the wave with the much higher frequency. Aliasing can be such a problem, in fact, that filters are used to eliminate any sounds above 20,000 Hz before they are digitized, thus making sure that every sound the digital recorder tries to sample has a frequency that is less than half of the sampling rate.

## Investigation 2: Days of Our Lives.

The amount of daylight on a given day is an important part of our lives. As the seasons change, so does the amount of daylight. You have probably seen times for the sunrise and sunset in your local newspaper or given in a weather report on television. In this investigation, you will derive a function which will give the amount of daylight in Grand Rapids, Michigan for any day throughout the year. The following table represents the amount of daylight for January 1 and every tenth day after that for Grand Rapids.

| Day | Hrs | Min | Total Minutes |
|---|---|---|---|
| 1 | 9 | 5 | 545 |
| 11 | 9 | 16 | 556 |
| 21 | 9 | 33 | 573 |
| 31 | 9 | 55 | 595 |
| 41 | 10 | 20 | 620 |
| 51 | 10 | 47 | 647 |
| 61 | 11 | 13 | 673 |
| 71 | 11 | 42 | 702 |
| 81 | 12 | 11 | 731 |
| 91 | 12 | 40 | 760 |
| 101 | 13 | 9 | 789 |
| 111 | 13 | 36 | 816 |
| 121 | 14 | 3 | 843 |
| 131 | 14 | 27 | 867 |
| 141 | 14 | 49 | 889 |
| 151 | 15 | 5 | 905 |
| 161 | 15 | 16 | 916 |
| 171 | 15 | 21 | 921 |
| 181 | 15 | 19 | 919 |

| Day | Hrs | Min | Total Minutes |
|---|---|---|---|
| 191 | 15 | 11 | 911 |
| 201 | 14 | 56 | 896 |
| 211 | 14 | 37 | 877 |
| 221 | 14 | 14 | 854 |
| 231 | 13 | 49 | 829 |
| 241 | 13 | 23 | 803 |
| 251 | 12 | 54 | 774 |
| 261 | 12 | 26 | 746 |
| 271 | 11 | 57 | 717 |
| 281 | 11 | 28 | 688 |
| 291 | 10 | 59 | 659 |
| 301 | 10 | 32 | 632 |
| 311 | 10 | 6 | 606 |
| 321 | 9 | 43 | 583 |
| 331 | 9 | 24 | 564 |
| 341 | 9 | 9 | 549 |
| 351 | 9 | 2 | 542 |
| 361 | 9 | 2 | 542 |

Graph the points from the table with the $x$-axis representing the day of the year (numbered 1 through 365) and the $y$-axis representing the number of minutes of daylight.[125] Notice that this data can be modeled by a sine function.

**1.** What is the vertical shift of function that fits the data? What does this number mean in terms of minutes of daylight?

**2.** What is the amplitude of the function that fits the data?

**3.** What is the period of the function that fits the data?

**4.** What is the horizontal shift of a sine function that fits the data? What does this number mean in terms of number of days?

**5.** Find the equation of a sine function that fits the data.

**6.** Graph your function along with the data. If you used a calculator, draw a sketch of your graph.

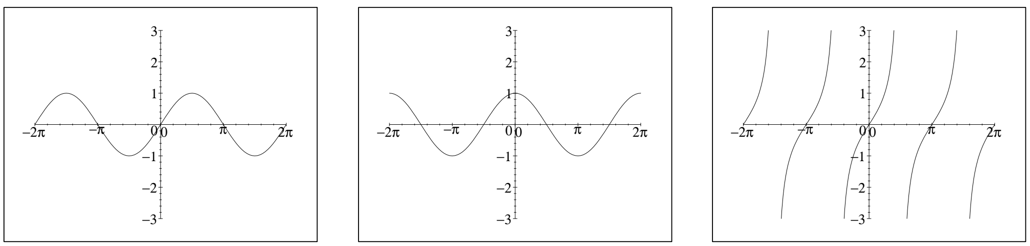
[125] If you prefer, you could use data for your own location. One source of this type of data is the U.S. Naval Observatory. As of June 1, 1998, their web site with this information was located at http://aa.usno.navy.mil/AA/data/docs/RS_OneYear.html.

# 5.4 Trigonometric Identities

There is often more than one way to write a mathematical expression. Sometimes it is obvious that two mathematical expressions are the same, such as $x$ and $\frac{1}{2} \cdot 2x$. Other times, it takes a little more work to see that two expressions are equivalent, such as $(x-4)(x+3)$ and $x^2-x-12$. In this section, we will derive several trigonometric identities and look at how to tell if two expressions are equivalent. There is a summary of trigonometric identities at the end of the section.[126]

## Review of Some Trigonometric Identities

Equivalent trigonometric expressions are called **trigonometric identities**. We have already looked at several identities earlier in this chapter. In Section 5.1, we described three identities that involved the symmetry of the graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$. (See Figure 1.) Because $y = \cos x$ is symmetric about the $y$-axis and $y = \sin x$ and $y = \tan x$ are symmetric about




$y = \sin x$ $\qquad$ $y = \cos x$ $\qquad$ $y = \tan x$

Figure 1: Graphs of three trigonometric functions with a domain of $-2\pi \le x \le 2\pi$.

the origin, we can say that:

- $\cos x = \cos(-x)$.
- $\sin x = -\sin(-x)$.
- $\tan x = -\tan(-x)$.

Another identity mentioned earlier was derived from the Pythagorean Theorem and the unit circle definitions of $y = \sin x$ and $y = \cos x$. Using Figure 2, we can see that:

- $\cos^2 x + \sin^2 x = 1$.

In Section 5.3, we saw that, using a horizontal shift, we could find a relationship between the graph of $y = \sin x$ and the graph of $y = \cos x$. Looking back at Figure 1, you can see that the graph of $y = \cos x$ looks like the graph of $y = \sin x$ shifted to the left $\frac{\pi}{2}$ units. Similarly, the graph of $y = \sin x$ looks like the graph of $y = \cos x$ shifted to the right $\frac{\pi}{2}$ units. This gives us the two identities:[127]

- $\cos x = \sin(x + \frac{\pi}{2})$.
- $\sin x = \cos(x - \frac{\pi}{2})$.

---

[126] In this section, we are assuming the input of the trigonometric function is measured in radians. However, these identities are also true if the input is measured in degrees.

[127] These identities assume $x$ is measured in radians. If $x$ is measured in degrees, substitute 90° for $\frac{\pi}{2}$.

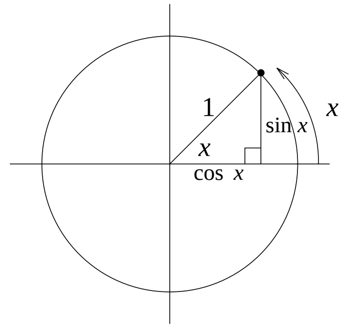



Figure 2: You can use a triangle in the unit circle to derive the identity $\cos^2 x + \sin^2 x = 1$.

Let's see how we can use the identities we've described so far to create new identities. We know that $y = \cos x$ is symmetric about the $y$-axis so $\cos x = \cos(-x)$. We also know that $\cos x = \sin(x + \frac{\pi}{2})$. Putting these two together gives us

$$\begin{aligned}\cos x &= \cos(-x)\\ &= \sin\left(-x + \frac{\pi}{2}\right)\\ &= \sin\left(\frac{\pi}{2} - x\right).\end{aligned}$$

Another identity we can derive starts with $\sin x = \cos(x - \frac{\pi}{2})$ and the fact that $\cos x = \cos(-x)$.

$$\begin{aligned}\sin x &= \cos\left(x - \frac{\pi}{2}\right)\\ &= \cos\left(-\left[x - \frac{\pi}{2}\right]\right)\\ &= \cos\left(-x + \frac{\pi}{2}\right)\\ &= \cos\left(\frac{\pi}{2} - x\right).\end{aligned}$$

The two new identities we have derived are:[128]

- $\cos x = \sin\left(\frac{\pi}{2} - x\right)$.
- $\sin x = \cos\left(\frac{\pi}{2} - x\right)$.

We could also have derived these two identities using the right triangle definitions of $y = \sin x$ and $y = \cos x$. Figure 3 shows a right triangle with angles $x$ and $z$. Since the two non-right angles

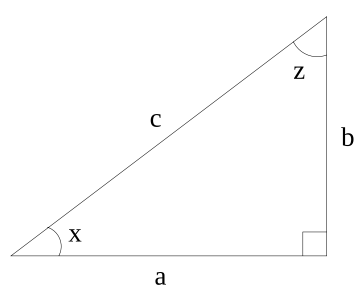



Figure 3: A right triangle can be used to show a relationship between $y = \sin x$ and $y = \cos x$.

[128] These identities assume $x$ is measured in radians. If $x$ is measured in degrees, substitute $90°$ for $\frac{\pi}{2}$.

of a triangle add up to 90 degrees (or $\frac{\pi}{2}$ radians), we know that $z = \frac{\pi}{2} - x$. From the right triangle definitions of $y = \sin x$ and $y = \cos x$, we see that $\sin x = \frac{b}{c}$ and that $\cos z = \frac{b}{c}$. It is clear from Figure 3 that what we proved algebraically, $\sin x = \cos(\frac{\pi}{2} - x)$, can also be derived using right triangles. Similarly, we can use right triangles to show that $\frac{a}{c} = \cos x = \sin z = \sin(\frac{\pi}{2} - x)$.[129] It is common for there to be more than one way to derive a trigonometric identity. Throughout this section, we will alternate between deriving identities algebraically, geometrically, using right triangles, using unit circles, or some combination of the above. There is not one "right way" when it comes to deriving trigonometric identities, but rather an array of approaches which you should become comfortable using.

### Reading Questions

1. Use the definition of $\tan x = \frac{\sin x}{\cos x}$ to show that $\tan x = -\tan(-x)$.
2. Explain, using Figure 2, why $\cos^2 x + \sin^2 x = 1$.
3. What property of the cosine function allows us to write $\cos x = \cos(-x)$?
4. Another identity that was mentioned in Section 5.3 is $\sin x = \sin(x + 2\pi)$. What are the equivalent identities for $y = \cos x$ and $y = \tan x$?

## Cosecant, Secant, and Cotangent

So far, we have concentrated on the three most commonly used trigonometric functions: sine, cosine, and tangent. There are three other trigonometric functions that are not used as frequently. These are the cosecant (abbreviated as $\csc x$), secant ($\sec x$), and cotangent ($\cot x$). These functions can be defined in terms of the other trigonometric functions:

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

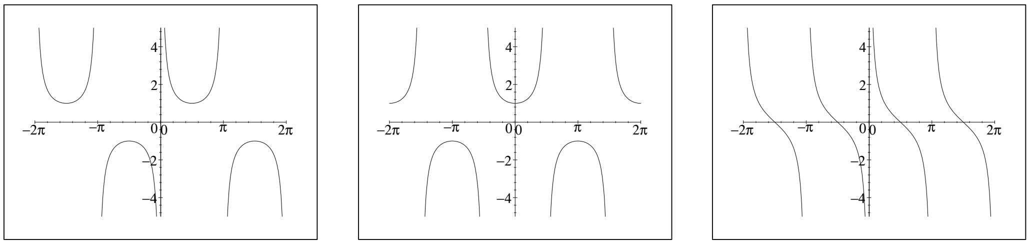
Notice that $y = \csc x$, $y = \sec x$, and $y = \cot x$ are not defined for all real numbers because of division by zero. The domain for $y = \csc x$ is all real numbers except for $x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$. The domain can also be written $x \neq n\pi$ where $n$ is an integer. This is because $y = \sin x$ is zero when $x$ is a multiple of $\pi$. To find the range of $y = \csc x$, remember that the range of $y = \sin x$ is $-1 \leq \sin x \leq 1$. Since you are taking the reciprocal of numbers between $-1$ and 1, the range of $y = \csc x$ is $y \leq -1$ or $y \geq 1$. Similarly, the domain of $y = \sec x$ is all real numbers except for $x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ (or $x \neq \frac{\pi}{2} + n\pi$ where $n$ is an integer). The range is $y \leq -1$ or $y \geq 1$. The domain for $y = \cot x$ is all real numbers except for $x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$ (or $x \neq n\pi$ where $n$ is an integer). To find the range of $y = \cot x$, remember that the range of $y = \tan x$ is all real numbers. Since $y = \cot x$ is the reciprocal of $y = \tan x$, it's range will also be all real numbers. The definitions, domain, and range of these three functions are summarized below. Their graphs are shown in Figure 4.

- $y = \csc x = \dfrac{1}{\sin x}$
  - Domain is all real numbers except for $x = n\pi$ where $n$ is an integer.
  - Range is $y \leq -1$ or $y \geq 1$.
- $y = \sec x = \dfrac{1}{\cos x}$
  - Domain is all real numbers except for $x = \frac{\pi}{2} + n\pi$ where $n$ is an integer.

[129] When using right triangles, there is the assumption that $0 < x < \frac{\pi}{2}$. You can use right triangles to prove identities for other values of $x$ by drawing appropriate triangles in the second, third, or fourth quadrants of the $xy$-plane.

  - Range is $y \leq -1$ or $y \geq 1$.

- $y = \cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$
  - Domain is all real numbers except for $x = n\pi$ where $n$ is an integer.
  - Range is all real numbers.



$y = \csc x$ $y = \sec x$ $y = \cot x$

Figure 4: Graphs of $y = \csc x$, $y = \sec x$ and $y = \cot x$ for $-2\pi \leq x \leq 2\pi$.

Like the other trigonometric functions, $y = \csc x$, $y = \sec x$, and $y = \cot x$ can also be defined in terms of the lengths of the sides of a right triangle. Using Figure 5 as a guide, the right triangle

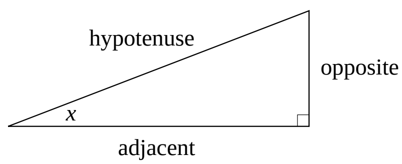



Figure 5: Using a right triangle to find definitions of the secant, cosecant and cotangent functions.

definitions of cosecant, secant, and cotangent are as follows:

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Notice that these are consistent with the definitions we gave earlier. For example,

$$\csc x = \frac{1}{\sin x} = \frac{1}{\frac{\text{opposite}}{\text{hypotenuse}}} = \frac{\text{hypotenuse}}{\text{opposite}}$$

.

Two identities involving these three functions are:

- $1 + \tan^2 x = \sec^2 x$.
- $1 + \cot^2 x = \csc^2 x$.

We can prove these either geometrically or algebraically. We will start with a geometric proof of $1+\tan^2 x = \sec^2 x$. Consider the triangle drawn in a unit circle shown in Figure 6. In this triangle, the side adjacent to angle $x$ has length 1. Since $\tan x = \frac{\text{opposite}}{\text{adjacent}}$ and length of the adjacent side is 1, the opposite side has length $\tan x$. Also, since $\sec x = \frac{\text{hypotenuse}}{\text{adjacent}}$ and the length of the adjacent side is 1, the hypotenuse has length $\sec x$. Using the Pythagorean theorem, we see that

$$1^2 + \tan^2 x = \sec^2 x \quad \text{or} \quad 1 + \tan^2 x = \sec^2 x.$$

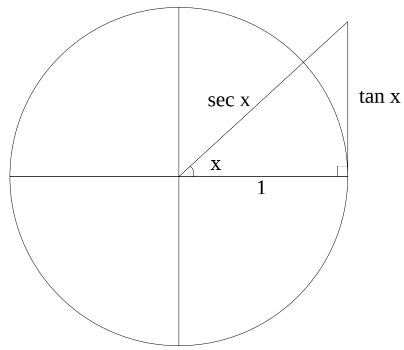



Figure 6: The unit circle and a triangle can be used to show that $1 + \tan^2 x = \sec^2 x$

This proves our identity for $0 < x < \frac{\pi}{2}$. Similar drawings in other quadrants can be used to prove this identity for other values of $x$ where $y = \tan x$ and $y = \sec x$ are defined. We can also prove this identity algebraically, as shown in Example 1.

**Example 1.** Algebraically prove that $1 + \tan^2 x = \sec^2 x$.

**Solution:** One of the identities mentioned earlier is $\cos^2 x + \sin^2 x = 1$. If we divide both sides of this equation by $\cos^2 x$, we get

$$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

which simplifies to

$$1 + \tan^2 x = \sec^2 x.$$

Notice that our proof does not work if $\cos^2 x = 0$. ■

You will be asked to prove the other identity, $1 + \cot^2 x = \csc^2 x$, both geometrically and algebraically in the exercises.

### Reading Questions

**5.** Show that the right triangle definition of $y = \cot x$ is equivalent to the definition $\cot x = \frac{\cos x}{\sin x}$.

**6.** What is the relationship between the range of $y = \sin x$ and the range of $y = \csc x$?

## Double Angle Identities

### A Double Angle Identity for Sine

It is sometimes useful to have trigonometric identities for double angles. We'll start by looking at a double angle identity for $\sin(2x)$. One way to derive this identity geometrically is to start with a triangle drawn in a unit circle. You want to draw a triangle where it is easy to identify an angle of $x$ radians and another angle of $2x$ radians so that you can find a relationship between $\sin(2x)$ and trigonometric functions involving $x$. Two such triangles are shown in Figure 7. Notice that the triangle in Figure 7(b), $\triangle OCD$, is a rotated version of the triangle in Figure 7(a), $\triangle OAB$.

Since we are trying to prove something about $\sin(2x)$, it makes sense to identify a line segment of length $\sin(2x)$ in Figure 7. Using Figure 7(b) and the unit circle definition of $y = \sin x$, we see that $\sin(2x)$ is the $y$-coordinate of point $C$. In other words, $\sin(2x)$ is the length of $CF$. Also notice that $\overline{CF}$ is the height of $\triangle OCD$. Using the fact that the area of a triangle is one-half the base times the height, we see that the area of $\triangle OCD$ is $\frac{1}{2} \cdot CF \cdot OD = \frac{1}{2} \cdot \sin 2x \cdot 1 = \frac{1}{2}\sin(2x)$.[130]

[130] An alternate way of finding the area of $\triangle OCD$ is to use the formula we developed in Section 5.2, Area $= \frac{1}{2}ab\sin\theta$.

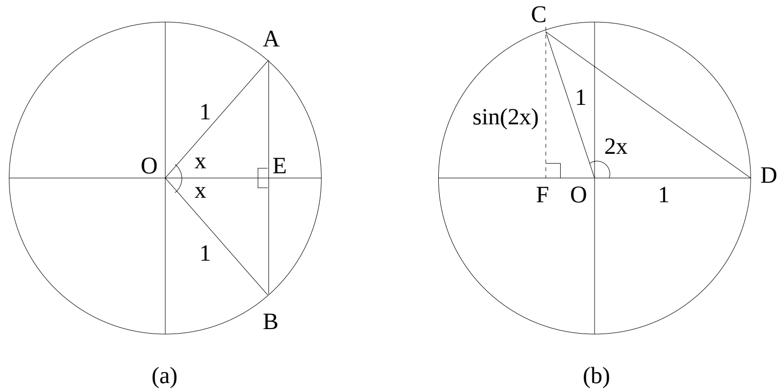



Figure 7: We can use these two triangles to geometrically derive an identity for $\sin(2x)$.

Now let's look at $\triangle OAB$ in Figure 7(a). The area of this triangle is $\frac{1}{2} \cdot OE \cdot AB$. Using the unit circle definition of $y = \cos x$, we see that $\cos x$ is the $x$-coordinate of point $A$. In other words, $\cos x$ is the length of $OE$. To find the length of $AB$, we will use the unit circle definition of $y = \sin x$ to see that $\sin x$ is the $y$-coordinate of point $A$ and $\sin(-x)$ is the $y$-coordinate of point $B$. Since $\sin(-x) = -\sin x$, the $y$-coordinate of $B$ can also be written as $-\sin x$. Therefore, the length of $AB$ is $\sin x - (-\sin x) = 2\sin x$. Putting this all together tells us that the area of $\triangle OAB = \frac{1}{2} \cdot OE \cdot AB = \frac{1}{2} \cdot \cos x \cdot 2\sin x = \cos x \sin x$.

As we mentioned earlier, $\triangle OCD$ is a rotated version of $\triangle OAB$. You can also see this by observing that these are congruent triangles since $\triangle OAB \cong \triangle OCD$ by *side-angle-side*. Congruent triangles have the same area. Therefore,

$$\begin{aligned}
\text{area of } \triangle OAB &= \text{area of } \triangle OCD \\
\cos x \sin x &= \frac{1}{2}\sin(2x) \\
2\cos x \sin x &= \sin(2x).
\end{aligned}$$

This gives us the identity:

- $\sin(2x) = 2\cos x \sin x$

### Double Angle Identities for Cosine

There are three commonly used double angle identities for cosine. They are:

- $\cos 2x = \cos^2 x - \sin^2 x$.
- $\cos 2x = 1 - 2\sin^2 x$.
- $\cos 2x = 2\cos^2 x - 1$.

We will prove the second one geometrically and show how that leads to a simple algebraic proof of the first one. You will be asked to prove the third one in the exercises.

Once again, to start a geometric proof, your goal is to set up a figure where it is easy to identify a line segment of length $\cos(2x)$. We will use the same figure we used to prove the identity for $\sin(2x)$. Figure 8 shows the same triangle as Figure 7(b).[131] We saw earlier that the length of $CF$ is $\sin(2x)$ and the length of $CD$ is $2\sin x$. Using the unit circle definitions of $y = \cos x$, notice that $\cos(2x)$ is the $x$-coordinate of point $C$. Since point $C$ is in the second quadrant, we know

[131] In the exercises, you will be asked to prove this identity using a triangle where $2x < \frac{\pi}{2}$.

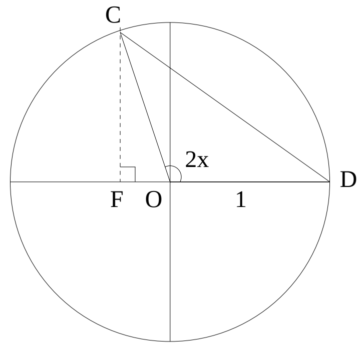



Figure 8: We can use this triangle to find an identity for $\cos(2x)$.

that $\cos(2x)$ is negative. Therefore, the length of $OF$ is $-\cos(2x)$. By the Pythagorean Theorem (using $\triangle FCD$), $\sin^2(2x) + (-\cos(2x) + 1)^2 = (2\sin x)^2$. Simplifying this gives us

$$\begin{aligned} \sin^2(2x) + (-\cos(2x) + 1)^2 &= (2\sin x)^2 \\ \sin^2(2x) + \cos^2(2x) - 2\cos(2x) + 1 &= 4\sin^2 x \end{aligned}$$

Since $\sin^2(2x) + \cos^2(2x) = 1$, we have

$$\begin{aligned} (\sin^2(2x) + \cos^2(2x)) - 2\cos(2x) + 1 &= 4\sin^2 x \\ 2 - 2\cos(2x) &= 4\sin^2 x \\ 1 - \cos(2x) &= 2\sin^2 x \\ 1 - 2\sin^2 x &= \cos(2x). \end{aligned}$$

From this identity, it is easy to derive the other two using $\cos^2 x + \sin^2 x = 1$. We will demonstrate this by algebraically deriving the identity $\cos(2x) = \cos^2 x - \sin^2 x$.

$$\begin{aligned} \cos(2x) &= 1 - 2\sin^2 x \\ \cos(2x) &= (\cos^2 x + \sin^2 x) - 2\sin^2 x \\ \cos(2x) &= \cos^2 x - \sin^2 x. \end{aligned}$$

### A Double Angle Identity for Tangent

There is also a double angle identity for the tangent function. To derive it, we'll start with the definition of $\tan x = \frac{\sin x}{\cos x}$. This means that $\tan(2x) = \frac{\sin(2x)}{\cos(2x)}$. Using the identity we just found for $\sin(2x)$ and one of the identities for $\cos(2x)$, we get

$$\begin{aligned} \tan(2x) &= \frac{\sin(2x)}{\cos(2x)} \\ &= \frac{2\sin x \cos x}{\cos^2 x - \sin^2 x}. \end{aligned}$$

We want to rewrite the right-hand side of this identity in terms of $\tan x$ rather than $\sin x$ and $\cos x$. To do this, we will divide both the numerator and the denominator by $\cos^2 x$ and then simplify.

$$\begin{aligned}
\frac{2\sin x\cos x}{\cos^2 x-\sin^2 x} &= \frac{\frac{2\sin x\cos x}{\cos^2 x}}{\frac{\cos^2 x-\sin^2 x}{\cos^2 x}} \\
&= \frac{2\cdot\frac{\sin x}{\cos x}}{\frac{\cos^2 x}{\cos^2 x}-\frac{\sin^2 x}{\cos^2 x}} \\
&= \frac{2\cdot\frac{\sin x}{\cos x}}{1-\frac{\sin^2 x}{\cos^2 x}} \\
&= \frac{2\tan x}{1-\tan^2 x}
\end{aligned}$$

This gives us the following identity:

- $\tan(2x) = \dfrac{2\tan x}{1-\tan^2 x}$

### Reading Questions

7. In Figure 7, suppose $x < \frac{\pi}{4}$. How would Figure 7(b) look in this case? Explain why this would not change the proof of $\sin(2x) = 2\cos x\sin x$.
8. In Figure 8, explain why the length of $OF$ is $-\cos(2x)$ rather than $\cos(2x)$.
9. Let $a$ be an angle such that $\tan a = \frac{12}{7}$. What is $\tan(2a)$?

## Sum and Difference Identities

Other identities are those involving sums and differences of angles. Some of these are:

- $\sin(x+y) = \sin x\cos y + \cos x\sin y$.
- $\sin(x-y) = \sin x\cos y - \cos x\sin y$.
- $\cos(x+y) = \cos x\cos y - \sin x\sin y$.
- $\cos(x-y) = \cos x\cos y + \sin x\sin y$.
- $\tan(x+y) = \dfrac{\tan x+\tan y}{1-\tan x\tan y}$.
- $\tan(x-y) = \dfrac{\tan x-\tan y}{1+\tan x\tan y}$.

We will give proofs for the identities involving $\cos(x+y)$ and $\sin(x-y)$. You will be asked to prove the other identities in the exercises.

### Proving the Sum Identity for the Cosine Function

We will begin by proving $\cos(x+y) = \cos x\cos y - \sin x\sin y$. Figure 9 is a unit circle with five points of interest. We are assuming $y > x$. The characteristics of these points are as follows:

- Point $A$ is on the $x$-axis,
- Point $B$ is formed by the angle whose measure is $x$,
- Point $C$ is formed by the angle whose measure is $y$,
- Point $D$ is formed by the angle whose measure is $x+y$,
- Point $E$ is formed by the angle whose measure is $-x$.

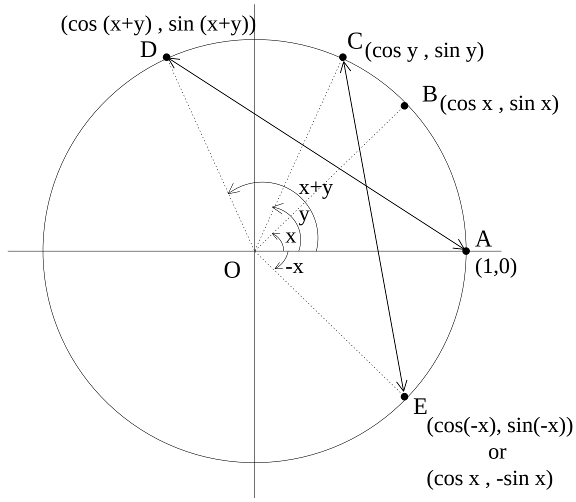



Figure 9: A figure used to prove the identity $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

The coordinates of each point are given. Note that since $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$, the coordinates of point $E$ are also given as $(\cos x, -\sin x)$. The measure of arc $\overset{\frown}{AD}$ is $x + y$. However, the measure of the arc $\overset{\frown}{EC}$ is also $x + y$. Since these two arcs have the same measure, $AD = EC$.

Using the distance formula[132] and setting these two distances equal to each other we get

$$\sqrt{(\cos(x+y)-1)^2 + (\sin(x+y)-0)^2} = \sqrt{(\cos y - \cos x)^2 + (\sin y - (-\sin x))^2}.$$

Squaring both sides and expanding the expressions, we get

$$\begin{aligned}
(\cos(x+y)-1)^2 + (\sin(x+y)-0)^2 &= (\cos y - \cos x)^2 + (\sin y + \sin x))^2 \\
\cos^2(x+y) - 2\cos(x+y) + 1 + \sin^2(x+y) &= \cos^2 y - 2\cos x \cos y + \cos^2 x + \sin^2 y \\
&\quad +2\sin x \sin y + \sin^2 x \\
\left[\cos^2(x+y) + \sin^2(x+y)\right] - 2\cos(x+y) + 1 &= \left[\cos^2 y + \sin^2 y\right] + \left[\cos^2 x + \sin^2 x\right] \\
&\quad -2\cos x \cos y + 2\sin x \sin y
\end{aligned}$$

Using the identity $\sin^2 x + \cos^2 x = 1$, we can replace the expressions in the square brackets with 1. So our equation simplifies to

$$-2\cos(x+y) + 2 = -2\cos x \cos y + 2\sin x \sin y + 2.$$

Subtracting 2 from both sides and dividing both sides by $-2$, we derive our identity,

$$\begin{aligned}
-2\cos(x+y) &= -2\cos x \cos y + 2\sin x \sin y \\
\cos(x+y) &= \cos x \cos y - \sin x \sin y
\end{aligned}$$

### Proving the Difference Identity for the Sine Function

To prove $\sin(x-y) = \sin x \cos y - \cos x \sin y$, we will begin by using the identity $\sin x = \cos(\frac{\pi}{2} - x)$ to convert $\sin(x-y)$ to a cosine function.

$$\begin{aligned}
\sin(x-y) &= \cos\left(\frac{\pi}{2} - (x-y)\right) \\
&= \cos\left(\left(\frac{\pi}{2} - x\right) + y\right).
\end{aligned}$$

[132] The distance between any two points $(x_1, y_1)$ and $(x_2, y_2)$ on the $xy$-plane is $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$.

Using the sum identity for cosine that we just proved, we have

$$\cos\left(\left(\frac{\pi}{2}-x\right)+y\right)=\cos\left(\frac{\pi}{2}-x\right)\cos y-\sin\left(\frac{\pi}{2}-x\right)\sin y.$$

Using the identities $\sin x=\cos(\frac{\pi}{2}-x)$ and $\cos x=\sin(\frac{\pi}{2}-x)$ we derive our result,

$$\begin{aligned}\cos\left(\left(\frac{\pi}{2}-x\right)+y\right) &= \cos\left(\frac{\pi}{2}-x\right)\cos y-\sin\left(\frac{\pi}{2}-x\right)\sin y\\ &= \sin x\cos y-\cos x\sin y.\end{aligned}$$

### Using the Difference Identities for the Tangent Function to Find Angles Formed by Intersecting Lines

An interesting consequence of the difference identity for the tangent function,

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y},$$

is that it allows us to determine the angle formed by the intersection of two lines. You may recall that the unit circle definition of $y=\tan\theta$ is the slope of the line which creates an angle of $\theta$ with the $x$-axis. Using this definition, we can see from Figure 10 that $\tan\theta_1=m_1$ and $\tan\theta_2=m_2$.

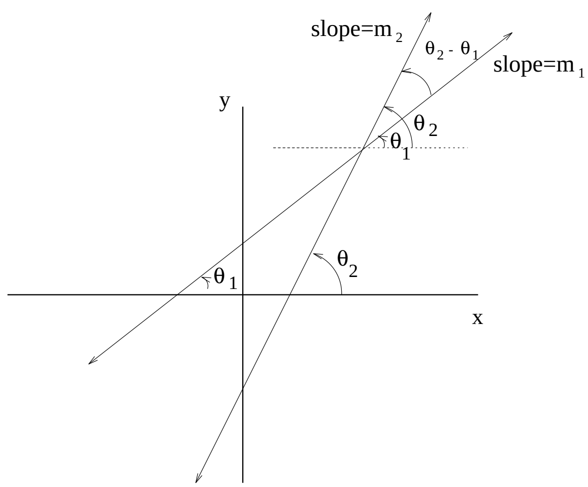



Figure 10: Knowing the slopes of two lines allows us to find the angle between them.

The acute angle formed by the intersection of the two lines in Figure 10 has measure $\theta_2-\theta_1$. Using the identity for the tangent of the difference of two angles and substituting slopes where appropriate, we get

$$\begin{aligned}\tan(\theta_2-\theta_1) &= \frac{\tan\theta_2-\tan\theta_1}{1+\tan\theta_2\tan\theta_1}\\ &= \frac{m_2-m_1}{1+m_2m_1}\end{aligned}$$

Therefore, if two lines intersect to form an angle of $\theta$, then

$$\tan\theta=\frac{m_2-m_1}{1+m_2m_1}$$

where $m_1$ and $m_2$ are the slopes of the two lines. By taking the inverse tangent of both sides, we can find the angle of intersection. Since the range of the inverse tangent is $-\frac{\pi}{2}<y<\frac{\pi}{2}$, this will always give us an acute angle.[133]

[133] The sign of the angle simply indicates direction. In this case, it makes no difference if our angle is measured counter-clockwise or clockwise.

- Let $L_1$ be a line with slope $m_1$ and let $L_2$ be a line with slope $m_2$. The acute angle, $\theta$, formed by the intersection of the two lines is equal to $\tan^{-1}\left(\dfrac{m_2 - m_1}{1 + m_2 m_1}\right)$.

**Example 2.** Determine the angle formed by the intersection of $y = 2x + 3$ and $y = \frac{1}{3}x - 4$.

**Solution:** We will let $m_2 = 2$ and $m_1 = \frac{1}{3}$. Using the formula $\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$, we have

$$\begin{aligned}
\tan\theta &= \frac{m_2 - m_1}{1 + m_2 m_1} \\
&= \frac{2 - \frac{1}{3}}{1 + 2 \cdot \frac{1}{3}} \\
&= \frac{5/3}{5/3} \\
&= 1.
\end{aligned}$$

So $\theta = \tan^{-1} 1 = \frac{\pi}{4} = 45°$. ■

**Reading Questions**

**10.** Show that the sum identity for cosine gives the same result as the double-angle identity for cosine when $x = y$.

**11.** Use the sum identity for sine to prove the difference identity for sine by replacing $y$ with $-y$.

**12.** The formula for finding the angle between two lines is $\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$. What happens if the lines have the same slope (and so are parallel)?

## Proving Trigonometric Identities

It is helpful to be able to prove trigonometric identities. First of all, these type of proofs give you practice in algebraic and geometric skills and reinforce the identities we have already proved in this section. Second, there are often times when two answers involving trigonometric functions look different and yet are actually the same. Being able to use trigonometric identities appropriately allows you to determine whether two expressions are truly different or are actually the same. As you have seen in the proofs we have done so far, there is no general algorithm for proving trigonometric identities. However, there are certain steps which are frequently useful. When doing a geometric proof, try drawing a triangle involving a unit circle. It is a good idea to find a side whose length involves one of the trigonometric functions in your identity. When doing an algebraic proof, try to use the identities we have already shown. It is often a good idea to convert all the functions to sine or cosine functions. Start with one side of the identity (typically the more complicated expression) and try to simplify it until it is obvious that it equals the expression on the other side of the identity. We will do several examples to illustrate ways to prove identities algebraically.

**Example 3.** Prove $\tan x \sin x + \cos x = \sec x$.

**Solution:** Since the left side of the equation is more complicated, we will try simplifying it until it is obvious that it equals $\sec x$. We'll begin by converting the left hand side to an expression involving only sine and cosine functions. This gives us

$$\begin{aligned}
\tan x \sin x + \cos x &= \frac{\sin x}{\cos x} \sin x + \cos x \\
&= \frac{\sin^2 x}{\cos x} + \cos x.
\end{aligned}$$

Finding a common denominator, adding, and using the identity $\cos^2 x + \sin^2 x = 1$, gives us

$$\begin{aligned} \frac{\sin^2 x}{\cos x} + \cos x &= \frac{\sin^2 x}{\cos x} + \frac{\cos^2 x}{\cos x} \\ &= \frac{\sin^2 x + \cos^2 x}{\cos x} \\ &= \frac{1}{\cos x}. \end{aligned}$$

Since we know that

$$\frac{1}{\cos x} = \sec x$$

we have proved that $\tan x \sin x + \cos x = \sec x$. ■

**Example 4.** Prove $\dfrac{1 - \cos^4 x}{1 + \cos^2 x} = \sin^2 x$.

**Solution:** Starting with the left side of the equation, factoring the numerator as a difference of two squares, and simplifying, we get

$$\begin{aligned} \frac{1 - \cos^4 x}{1 + \cos^2 x} &= \frac{(1 - \cos^2 x)(1 + \cos^2 x)}{1 + \cos^2 x} \\ &= 1 - \cos^2 x. \end{aligned}$$

Since $\cos^2 x + \sin^2 x = 1$, $1 - \cos^2 x = \sin^2 x$, which proves $\dfrac{1 - \cos^4 x}{1 + \cos^2 x} = \sin^2 x$. ■

As we mentioned earlier, you may have a solution to a problem that doesn't match someone else's solution and you want to see if they are actually the same. One way to do this is by graphing both solutions. If the graphs look different, then the two solutions are obviously different. If the graphs look the same, then the two solutions *may* be the same. To make sure they are the same, you must provide a proof.

**Example 5.** Graph each of the following equations. If the two graphs appear to be the same, try to prove the two equations are the same.

$$\begin{aligned} f(x) &= (\sin x + \cos x)^2 \\ g(x) &= 1 \end{aligned}$$

**Solution:** The graph of $f(x) = (\sin x + \cos x)^2$ is shown along with the graph of $g(x) = 1$ in Figure 11. Because these graphs are clearly different, $(\sin x + \cos x)^2 \neq 1$. Therefore, this is not an identity. ■

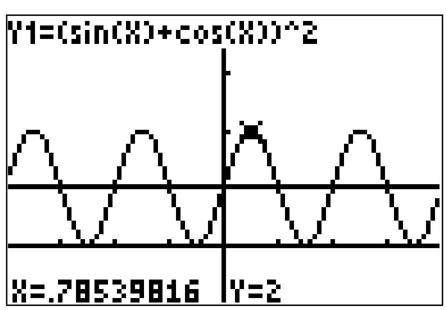
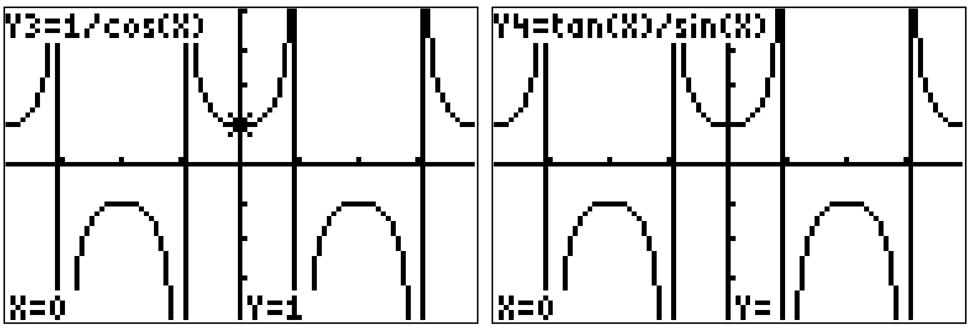



Figure 11: These two graphs are clearly different.

**Example 6.** Graph each of the following equations. If the two graphs appear to be the same, try to prove the two equations are the same.

$$\begin{aligned} h(x) &= \sec x \\ k(x) &= \csc x \tan x \end{aligned}$$

**Solution:** The graph of $h(x) = \sec x$ is shown along with the graph of $k(x) = \csc x \tan x$ in Figure 12. [Note: Since calculators generally don't have a secant button, to graph $y = \sec x$ on your calculator, you will probably have to graph $y = \frac{1}{\cos x}$. Likewise, to graph $y = \csc x \tan x$, you will probably have to graph $y = \frac{1}{\sin x} \tan x$.] The graph of $h(x) = \sec x$ is shown in Figure 12(a) and the graph of $k(x) = \frac{\tan x}{\sin x}$ is shown in Figure 12(b).[134] The two graphs appear to be the same.




(a) (b)

Figure 12: These graphs appear to be the same.

To prove that these two expressions are the same and thus show that this is an identity, we will start by simplifying the right side of the equation to get

$$\begin{aligned} \csc x \tan x &= \frac{1}{\sin x} \cdot \frac{\sin x}{\cos x} \\ &= \frac{1}{\cos x} \\ &= \sec x. \end{aligned}$$

So it appears that $\sec x = \csc x \tan x$. However, there is a subtle difference. The domain of $h(x) = \sec x$ is all real numbers, $x$, such that $x \neq \ldots -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ since you must avoid $x$-values where $\cos x = 0$. However, the domain of $\csc x \tan x$ is all real numbers, $x$, such that $x \neq \ldots -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$ since you must avoid $x$-values where either $\sin x = 0$ or $\cos x = 0$. Therefore, $\sec x = \csc x \tan x$ only *for values where both sides of the equation are defined.* ■

Example 6 illustrates an important, yet subtle, point. It is easy to algebraically simplify an expression to prove an identity yet ignore issues involving the domain. This is similar to saying that $\frac{(x-3)(x+2)}{x+2} = x - 3$ and ignoring that this is only true when $x \neq -2$. Often, when dealing with fractions or square roots, there is a restriction on the values for which the identity is true.

[134] The calculator graphs show vertical lines where there are actually vertical asymptotes.

### Historical Note[135]

While trigonometric identities are most often used in mathematical proofs, they have been used for mathematical computation. Before the invention of logarithms by John Napier in 1614, trigonometric identities were used to aid in the multiplication of large numbers. Similar to using logarithms, appropriate trigonometric identities could convert a multiplication problem to an addition or subtraction problem. This method was know as *prosthaphaeresis*, a Greek word meaning addition and subtraction. This method of multiplying was used by astronomers such as Tycho Brahe (1546-1601) in Denmark. It was learning of Brahe's work with prosthaphaeresis that prompted John Napier to redouble his efforts to develop logarithms, which became an easier method of doing such calculations.

We will illustrate the method of prosthaphaeresis. Suppose you were an ancient astronomer who needed to multiply 94,562 by 3253. You could use the identity $\cos x \cos y = \frac{\cos(x+y)+\cos(x-y)}{2}$ and let $\cos x = 0.94562$ and $\cos y = 0.3253$. (You had to convert your numbers to decimals less than one because of the range of the cosine function. To convert back to the original product, you will multiply by $10^9$.) Using the trigonometric tables available at the time, you would find that $x \approx 0.331300886$ and $y \approx 1.239467361$. Using the identity and the trigonometric tables, you calculate

$$\begin{aligned}
94,562 \cdot 3253 &= 10^9 \cdot 0.94562 \cdot 0.3253 \\
&\approx 10^9 \cdot \frac{\cos(0.331300886 + 1.239467361) + \cos(0.331300886 - 1.239467361)}{2} \\
&= 10^9 \cdot \frac{\cos(1.570768247) + \cos(-0.908166475)}{2} \\
&\approx 10^9 \cdot \frac{0.000028080 + 0.615192292}{2} \\
&\approx 10^9 \cdot 0.307610186 \\
&= 307,610,186.
\end{aligned}$$

In this example, we used enough decimal places so that our final answer is exact rather than an approximation. As you can see, this method of multiplying was not a great labor saving device. However, for large numbers, it apparently did save some time. The use of logarithms soon made this method of multiplying obsolete. Now, through the use of calculators, using logarithms (or slide rules) to multiply large numbers also seems like an ancient method which has become obsolete.

## Reading Questions

**13.** Why isn't it sufficient to prove an identity by showing that the graphs of the two expressions are the same?

**14.** (a) Prove $\cot x \sec x = \csc x$.

(b) For what values of $x$ is this identity true?

[135] Boyer, Carl B., *A History of Mathematics*, Princeton University Press, 1968, pp. 339 - 343.

## Summary of Identities

In this section, we have reviewed or introduced the following identities.

**Symmetry Identities**

- $\cos x = \cos(-x)$
- $\sin x = -\sin(-x)$
- $\tan x = -\tan(-x)$

**Horizontal Shift Identities**

- $\cos x = \sin(x + \frac{\pi}{2})$
- $\sin x = \cos(x - \frac{\pi}{2})$

**Complement Identities**

- $\cos x = \sin(\frac{\pi}{2} - x)$
- $\sin x = \cos(\frac{\pi}{2} - x)$

**Definition Identities**

- $\tan x = \frac{\sin x}{\cos x}$
- $\csc x = \frac{1}{\sin x}$
- $\sec x = \frac{1}{\cos x}$
- $\cot x = \frac{1}{\tan x}$

**Pythagorean Identities**

- $\cos^2 x + \sin^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \csc^2 x$

**Double Angle Identities**

- $\sin(2x) = 2\sin x \cos x$
- $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$
- $\cos(2x) = 1 - 2\sin^2 x$
- $\cos(2x) = \cos^2 x - \sin^2 x$
- $\cos(2x) = 2\cos^2 x - 1$

**Sum and Difference Identities**

- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\sin(x - y) = \sin x \cos y - \cos x \sin y$
- $\cos(x - y) = \cos x \cos y + \sin x \sin y$
- $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

## Exercises

**1.** Fill out the following table. All answers are to be exact.

| $x$ (radians) | $x$ (degrees) | $\sec x$ | $\csc x$ | $\cot x$ |
|---|---|---|---|---|
| $\pi/6$ | | | | |
| $\pi/3$ | | | | |
| $\pi/4$ | | | | |

**2.** Each of the following statements are false. Explain why.

(a) $1 + \tan^2 x = \sec^2 x$ for all values of $x$.

(b) The function $y = \cot x$ is undefined for $x = \frac{\pi}{2}$.

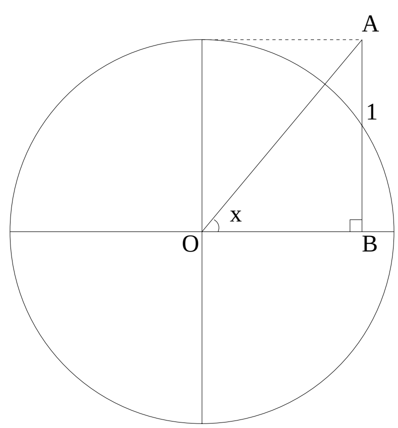
(c) It is possible to find a value of $x$ such that $\csc x = \frac{1}{4}$.

(d) The function $y = \cot x$ is equal to the slope of the line containing the point $(\cos x, \sin x)$ and $(0, 0)$.

(e) Since $y = \cos x$ is symmetric about the origin, I know that $\cos x = \cos(-x)$.

(f) $\sin x = \sin(\frac{\pi}{2} - x)$.

**3.** To prove $1+\cot^2 x = \csc^2 x$ geometrically, consider the triangle drawn in a unit circle shown in the following figure.




In this triangle, the side opposite angle $x$ has length 1.

(a) Explain why the length of $OA = \csc x$.

(b) Explain why the length of $OB = \cot x$.

(c) Prove $1 + \cot^2 x = \csc^2 x$ for $0 < x < \frac{\pi}{2}$.

(d) Draw a similar triangle in the second quadrant to prove $1 + \cot^2 x = \csc^2 x$ for $\frac{\pi}{2} < x < \pi$.

**4.** Prove $1 + \cot^2 x = \csc^2 x$ algebraically.

**5.** Use the identity $\cos(2x) = 1 - 2\sin^2 x$ to prove $\cos(2x) = 2\cos^2 x - 1$.

**6.** Let $a$ be an angle in the first quadrant such that $\sin a = \frac{2}{3}$. Find each of the following.

(a) $\cos a$

(b) $\tan a$

(c) $\sin 2a$

(d) $\cos 2a$

(e) $\tan 2a$

**7.** Let $b$ be an angle in the second quadrant such that $\sin b = \frac{2}{3}$. Find each of the following.

(a) $\cos b$

(b) $\tan b$

(c) $\sin 2b$

(d) $\cos 2b$

(e) $\tan 2b$

**8.** Let $c$ be an angle in the first quadrant such that $\sin c = r$.

(a) What are the restrictions on $r$?

(b) Find each of the following.

i. $\cos c$

ii. $\tan c$

iii. $\sin 2c$

iv. $\cos 2c$

v. $\tan 2c$

**9.** (a) Explain why the identity $\tan(2x) = \dfrac{2\tan x}{1-\tan^2 x}$ does not work for $x = \frac{\pi}{4}$.

(b) For what other values of $x$ does the identity $\tan(2x) = \dfrac{2\tan x}{1-\tan^2 x}$ not work?

**10.** (a) Draw a figure similar to Figure 8 in the section, except assume that $x < \frac{\pi}{4}$. This means that $2x < \frac{\pi}{2}$ so your triangle will be contained in the first quadrant.

(b) Using your figure from part(a), geometrically prove the identity $\cos(2x) = 1 - 2\sin^2 x$ for $0 < x < \frac{\pi}{4}$.

**11.** The value of $\pi$ can be approximated by finding the perimeter of regular polygons inscribed in a circle of radius $\frac{1}{2}$. As the number of sides of the polygon increases, its perimeter becomes closer and closer to the circumference of the circle which is $\pi$. One method of calculating the perimeter of such a polygon with $n$ sides leads to the function

$$q(n) = n\sin\left(\frac{180^\circ}{n}\right).$$

Another method of calculating the same perimeter leads to the function

$$p(n) = \frac{\frac{n}{2}\sin(\frac{360^\circ}{n})}{\sin(90^\circ - \frac{180^\circ}{n})}.$$

Show that these two functions, which look different, are actually the same.

**12.** In this section, we proved the sum identity for cosine and the difference identity for sine. Use them to prove the following identities.

(a) $\cos(x-y) = \cos x\cos y + \sin x\sin y$

(b) $\sin(x+y) = \sin x\cos y + \cos x\sin y$

**13.** Prove the following identities for the tangent function.

(a) $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

(b) $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

**14.** Find the measure of the acute angle formed by the following pairs of lines. Give your answer in degrees and round to the nearest tenth of a degree.

(a) $y = 4x - 7$ and $y = -2x + 9$

(b) $y = \frac{1}{4}x + 3$ and $y = 4x - \frac{1}{2}$

(c) $y = -\frac{3}{5}x - 4$ and $y = \frac{5}{3}x + 19$

**15.** We can find exact values for trigonometric functions when $x = \frac{\pi}{6}, \frac{\pi}{3}$, or $\frac{\pi}{4}$. Use these values and the appropriate sum or difference formulas to complete the following table. All answers are to be exact.

| $x$ (radians) | $x$ (degrees) | $\cos x$ | $\sin x$ | $\tan x$ |
|---|---|---|---|---|
| $5\pi/12$ | | | | |
| $7\pi/12$ | | | | |
| $11\pi/12$ | | | | |
| $\pi/12$ | | | | |

**16.** (a) Graph $f(x) = \cos^2 x - \sin^2 x$.

(b) Using the transformation techniques described in Section 5.3, find the equation of a cosine function whose graph looks like the graph of $f$.

(c) Using the appropriate sum identity, prove that your cosine function from part(b) is equal to $f$.

**17.** (a) Graph $g(x) = \cos x + \sin x$.

(b) Using the transformation techniques described in Section 5.3, find the equation of a sine function whose graph looks like the graph of $g$.

(c) Using the appropriate sum identity, prove that your sine function from part(b) is equal to $g$.

**18.** Prove the following identities.

(a) $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

(b) $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

(c) $\tan^2 x = \dfrac{1 - \cos 2x}{1 + \cos 2x}$

**19.** Prove the following trigonometric identities. Indicate for which values of $x$ the identity is true.

(a) $\dfrac{\cos x}{\sec x - \tan x} = 1 + \sin x$

(b) $\tan x = \dfrac{1 - \cos 2x}{\sin 2x}$

(c) $\tan x(1 - \sin^2 x) = \frac{1}{2}\sin 2x$

(d) $\sec^2 x \cot x - \cot x = \tan x$

**20.** Simplify each of the following expressions completely. Your answer, in each case, will be a number.

(a) $\cos^4 x + \sin^4 x + \frac{1}{2}\sin^2(2x)$

(b) $\sin^2 x - \sin^4 x - \cos^2 x \sin^2 x$

(c) $1 + \cos(2x) + 2\sin^2 x$

**21.** Prove the following identities.

(a) $\cos(3x) = \cos^3 x - 3\cos x \sin^2 x$

(b) $\sin(3x) = 3\cos^2 x \sin x - \sin^3 x$

**22.** Prove the following identities.

(a) $\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$

(b) $\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$

**23.** (a) Algebraically "prove" $\sin x = \sqrt{1-\cos^2 x}$.

(b) Sketch a graph $f(x) = \sin x$ and $g(x) = \sqrt{1-\cos^2 x}$ for $-2\pi < x < 2\pi$.

(c) The graphs of $f$ and $g$ from part(b) look different yet your "proof" in part(a) says that the graphs should be the same. Explain this discrepancy.

**24.** Graph to determine if each of the following pairs of functions are the same. If they appear to be the same, prove that they are.

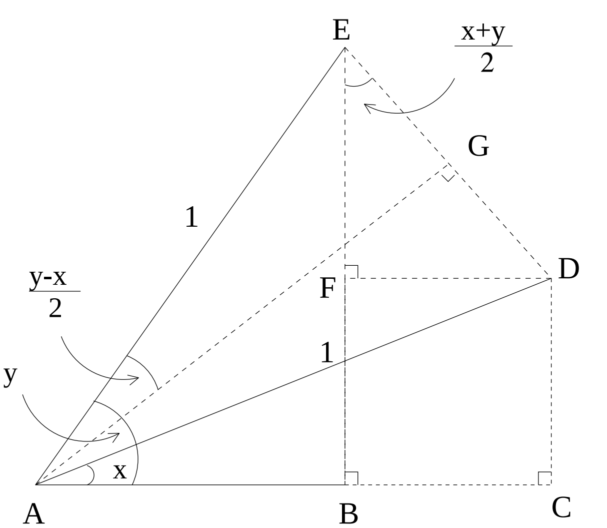
(a) $f(x) = \frac{1}{\cos x}, \qquad g(x) = \frac{\csc(-x)}{\cot(-x)}$

(b) $f(x) = \cos(2x) + \sin(2x), \qquad g(x) = (\cos x - \sin x)^2$

(c) $f(x) = \sqrt{1+\cot^2 x}, \qquad g(x) = \csc x$

(d) $f(x) = 2\sin x + \tan x, \qquad g(x) = \tan x(2\cos x + 1)$

**25.** The following figure consists of two triangles where the measure of $\angle CAD = x$, the measure of $\angle EAB = y$, the measure of $\angle GAE = \frac{y-x}{2}$, and the measure of $\angle GEF = \frac{x+y}{2}$. Right angles are labeled in the figure. Also, the length of $\overline{AE} = 1$ and the measure of $\overline{AD} = 1$. Use this figure to geometrically prove the following identities.[136]




[136] The proofs for these identities was shown by Yukio Kobayashi as a proof without words in *The College Mathematics Journal*, March 1998 on page 133.

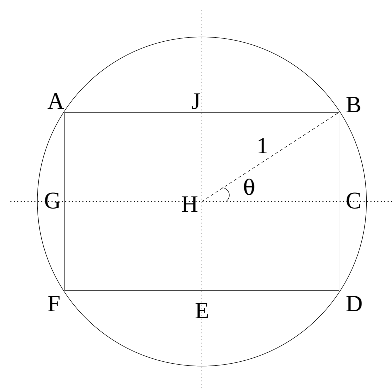
(a) Use $\sin\left(\frac{x+y}{2}\right) = \frac{FD}{ED}$ to prove

$$\cos x - \cos y = 2\sin\left(\frac{y-x}{2}\right)\sin\left(\frac{x+y}{2}\right).$$

(b) Use $\cos\left(\frac{x+y}{2}\right) = \frac{EF}{ED}$ to prove

$$\sin y - \sin x = 2\sin\left(\frac{y-x}{2}\right)\cos\left(\frac{x+y}{2}\right).$$

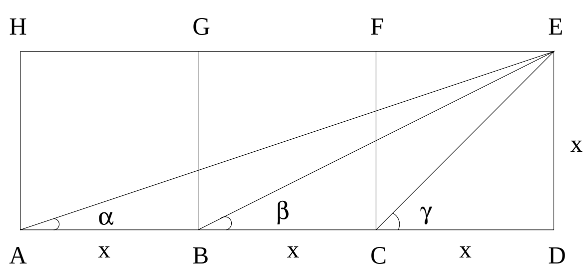
**26.** The following figure shows a rectangle inscribed in a circle of radius 1. Let $\theta$ be the measure of the angle formed by a diagonal of the rectangle and the horizontal line bisecting the rectangle.




(a) Let $A$ be the area of the rectangle. Show $A = 2\sin 2\theta$.

(b) Notice that $0° \leq \theta \leq 90°$. What value of $\theta$ will give you a rectangle with the largest area? What are the dimensions of this rectangle?

**27.** In the following figure, AHGB, BGFC, and CFED are squares with sides of length $x$.




(a) Notice that $\tan\gamma = 1$. Show that $\tan(\alpha+\beta)$ also equals 1. Explain why this shows that $\alpha + \beta = \gamma$.

(b) Geometrically prove that $\alpha + \beta = \gamma$ without using trigonometry.[137] [Hint: Show that $\triangle BCE$ is similar to $\triangle ECA$ by showing that the appropriate sides are proportional.]

[137] Charles Trigg demonstrated 54 proofs of $\alpha + \beta = \gamma$ in *Journal of Recreational Mathematics*, April 1971.

## Investigations

### Investigation 1: Geometric Proofs of Sum Identities.

In the section, we gave a geometric proof for the identity $\cos(x+y) = \cos x \cos y - \sin x \sin y$. In this investigation, you will do an alternate geometric proof for $\cos(x+y)$ and also prove $\sin(x+y)$. These proofs will use the right triangles shown in Figure 1. The measure of $\angle BOD = x$ and the measure of $\angle DOE = y$. Right angles are labeled in the figure. Notice that these triangles are **not** in a unit circle.

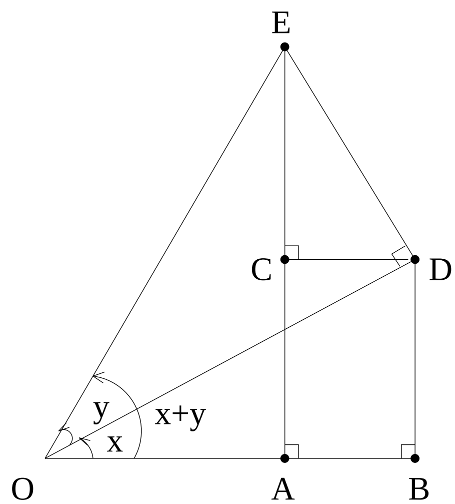



Figure 1

1. Show that the measure of $\angle ODC = x$. Use this to show that the measure of $\angle CED = x$.
2. Using the right triangle definitions of sine and cosine as well as Figure 1, show that
$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$
3. Using the right triangle definitions of sine and cosine as well as Figure 1, show that
$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$
4. Figure 1 assumes what restrictions on $x$ and $y$?

### Investigation 2: Identities for the Sum of Sines or Cosines Functions.[138]

In addition to having identities for sine and cosine of the sum of two angles, there are also identities for the sum of two sine functions and the sum of two cosine functions. These are

$$\sin x + \sin y = 2\cos\left(\frac{y-x}{2}\right)\sin\left(\frac{x+y}{2}\right).$$

and

$$\cos y + \cos x = 2\cos\left(\frac{y-x}{2}\right)\cos\left(\frac{x+y}{2}\right).$$

In this investigation, we will geometrically prove these identities using Figure 2. The measure of $\angle BAF = x$, the measure of $\angle CBF = y$, the measure of $\angle EAB = \frac{x+y}{2}$. Right angles are labeled in the figure. The length of $\overline{AF} = 1$ and the length of $\overline{FH} = 1$.

[138] The proofs for these identities was shown by Yukio Kobayashi as a proof without words in *The College Mathematics Journal*, March 1998 on page 157.

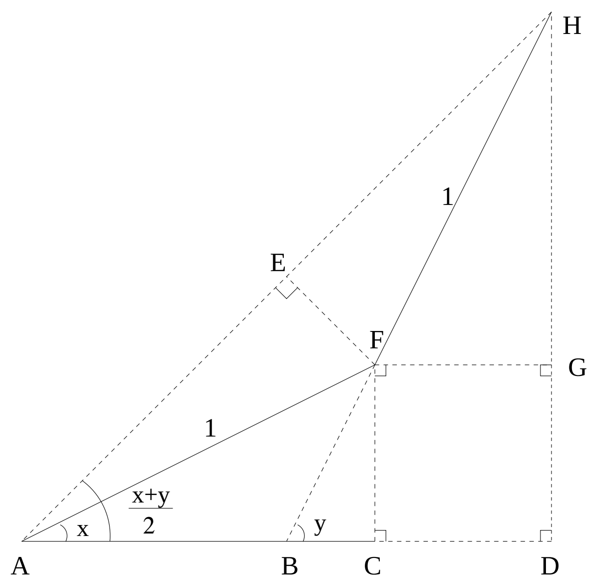



Figure 2

1. Show that the measure of $\angle GFH = y$.

2. Show that the measure of $\angle EAF = \frac{y-x}{2}$.

3. Use $\cos\left(\frac{x+y}{2}\right) = \frac{AD}{AH}$ to prove

$$\cos x + \cos y = 2\cos\left(\frac{y-x}{2}\right)\cos\left(\frac{x+y}{2}\right).$$

4. Use $\sin\left(\frac{x+y}{2}\right) = \frac{DH}{AH}$ to prove

$$\sin x + \sin y = 2\cos\left(\frac{y-x}{2}\right)\sin\left(\frac{x+y}{2}\right).$$

## Investigation 3: Projectile Motion.

Suppose a projectile was fired into the air at an angle of $\theta$. The vertical distance of the projectile (in feet) is given by $v(t) = -16t^2 + v_0 t \sin\theta$ where $t$ is time (in seconds) and $v_0$ is the intial velocity (in feet per second). The horizontal distance of the projectile (in feet) is given by $h(t) = v_0 t \cos\theta$. We are interested in determining the total horizontal distance (known as *range*) traveled by the projectile.

1. Let $r$ be the range of the projectile. Show that $r = \frac{1}{32}v_0^2 \sin 2\theta$. [Hint: First use the vertical distance to find an expression for the time when the projectile has landed.]

2. Calculate the range of a projectile shot at an angle of 60° that has an initial velocity of 120 feet per second.

3. Calculate the range of a projectile shot at an angle of 30° that has an initial velocity of 120 feet per second.

4. Suppose the first projectile is shot at an angle of $\theta$ and a second projectile is shot at an angle of $90° - \theta$. Assume the initial velocity is the same for both projectiles. Which one travels further? Justify your answer.

5. Suppose, given a fixed initial velocity, you want the projectile to travel as far as possible. At what angle should you fire the projectile? Justify your answer.

**6.** Suppose you have two projectiles, the first of which is shot at an angle of $\alpha$ and a second which is shot an an angle of $\beta$. Assume the initial velocity is the same for both projectiles. If you want the first projectile to travel the farthest, what do you know about the relationship between $\alpha$ and $\beta$?

# 5.5 Project: Looking Out to Sea

The Chicago to Mackinac sailboat race is held every summer on Lake Michigan. The residents along the shore of the lake have an opportunity to view the boats as they make their way from Chicago, on the southern end of the lake, to Mackinac Island, on the northern end. One year the winds died down and stranded many of the boats. If you were standing on the shore you could see about twenty boats stalled out in the lake. However, if you climbed up one of the tall bluffs near the shore, your vision was greatly enhanced as almost one hundred boats came into view. Going just a little bit higher dramatically increases how far you can see. Exactly how much farther? That is the question answered in this project.

**1.** Study Figure 1. The circle represents the circumference of the earth. The line segment labeled $a$ is the distance one is above the surface of the earth. The line segments labeled $b$ are the radii of the earth, and $c$ is the line of sight distance. Since $c$ is tangent to the circle and $b$ is a radius, the angle where they intersect is a right angle.

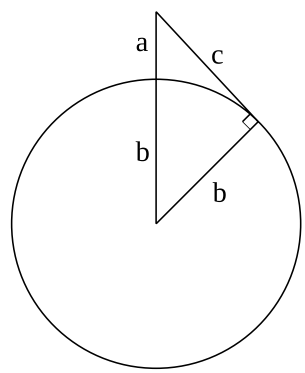



Figure 1

(a) The radius of the earth is about 3950 miles. Using this, find a function whose input is $a$ and whose output is $c$.

(b) In the function you found in part (a), $a$ and $c$ are in terms of miles. This is a bit cumbersome since we usually don't talk about being so many miles off the surface of the earth. Rewrite your function so that $c$, your output, is in terms of miles and $a$, your input, is in terms of feet.[139]

**2.** Let's explore the properties of the function from question 1(b).

(a) Complete Table 1. The first column is how high you are above the earth. The second column is your line of sight distance. The third column is the difference between your current and previous output divided by the difference between your current and previous input, i.e. an approximate "slope" of the function at that point.

| height (feet) | distance (miles) | diff in output/diff in input (miles/feet) |
|---|---|---|
| 5 | 2.7352 | — |
| 10 | | |
| 15 | | |
| 20 | | |

Table 1

[139] 5280 feet = 1 mile and $5280^2 \text{ft}^2 = 1 \text{ mile}^2$.

(b) Using the information in the chart above, could the graph of this function be a line? Why or why not?

(c) Graph the function you found in question 1(b). What is the relationship between the general shape of the graph and the information given by the numbers in the third column of the chart in question 2(a)?

(d) How high above the earth would you have to be in order to see 20 miles out?

**3.** The distances we have found so far have been the distance from the observer to the farthest point in a straight line, i.e. the line of sight distance. However, often what people refer to when talking about how far they can see is the distance along the horizon of the earth, i.e. the ground distance. This distance is the arc length, $s$, in Figure 2 below. Find a formula to express this arc length in terms of $a$ where the input is in miles. Also, find another version of this formula where the input is in feet.

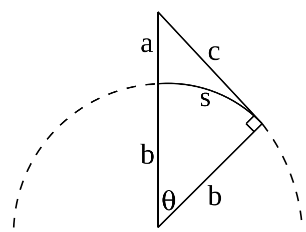



Figure 2

**4.** We want to compare the line of sight distance to the ground distance along the horizon.

(a) Compare the ground distance to the line of sight distance when $a$ is 10 feet, 1000 feet, 10,000 feet, and 200 miles by completing Table 2. What do you notice about the difference between the arc length and the line of sight distance as $a$ gets larger? [Note: The formula commonly used for arc length assumes the angle is measured in *radians*, not degrees. Be sure to set your calculator appropriately.]

| height | straight line dist. (miles) | arc length (miles) |
|---|---|---|
| 10 ft. | | |
| 1000 ft. | | |
| 10,000 ft. | | |
| 200 miles | | |

Table 2

(b) Using the functions where the input is in miles, graph both the line of sight distance and the ground distance on the same set of axes using a domain of 0 to 500 miles. Then graph using a domain of 0 to 50,000 miles.

(c) We are interested in exploring what happens to both the line of sight distance and the ground distance as $a$ gets larger. Some functions increase without bound, i.e. keep getting bigger and bigger, and other functions have a limit or upper bound, i.e. a limit as to how large they will get.

i. Think of the physical situation. (Refer back to Figure 2.) The line of sight distance will increase without bound and the ground distance will have a limit. Explain why. What is the limit for the ground distance?

ii. Look at the second graph for question 4(b). Explain how this graph is compatible with your answer to question 4(c)i.

iii. Look at the behavior of your symbolic formulas as $a$ gets large. Explain how this behavior is compatible with your answer to question 4(c)i.

**5.** The space shuttle orbits approximately 200 miles above the surface of the earth. Is it possible for the someone in the space shuttle to view the entire continental United States? Explain your answer. Use the included map of the United States as a guide as shown in Figure 3.


Figure 3