THE INTENSITY OF THE EARTH'S MAGNETIC FORCE REDUCED TO ABSOLUTE MEASUREMENT

by Carl Friedrich Gauss

(Translated from the German by Susan P. Johnson, July 1995)

{The treatise "Intensitas vis magneticae terrestris ad mensuram absolutam revocata" was read by Gauss at the Goettingen Gesellschaft der Wissenschaften (Royal Scientific Society) on December 15, 1832, and printed in Volume 8 of the treatises of this society, pp. 3-44. The translation from the Latin to the German was provided by Herr Oberlehrer Dr. Kiel in Bonn. Edited by E. Dorn, Leipzig, Wilhelm Engelmann Verlag, 1894. Library of Congress, class QC827, book G27.}

[Translator's Note: Wirkung (action, force, effect), Einwirkung (influence, action, effect), Moment (moment, force, impetus), bewegende Kraft (motive force), and beschleunigende Kraft (acceleration) have been left in German.

For the complete determination of the Earth's magnetic force at a given location, three elements are necessary: the deviation (declination) or the angle between the planes, in which it acts, and the meridian plane; the inclination of the direction of the horizontal plane; finally, third, the strength (intensity). The declination, which is to be considered as the most important element in all applications to navigation and geodesy, has engaged astronomers and physicists from the beginning, who for a century, however, have given their constant attention to the inclination as well. In contrast, the third element, the intensity of the terrestrial magnetic force, which is surely just as worthy a subject of science, remained fully neglected until more recent times. Humboldt rendered the service, among so many others, of having been first to direct his attention to this subject, and in his travels assembled a large array of determinations concerning the relative strength of magnetism, which yielded the result of a continuous increase in this strength from the magnetic equator toward the pole. A great many physicists have trod in the footsteps of that great natural scientist, and have already brought together such a large array of determinations, that Hansteen, highly distinguished for his knowledge of terrestrial magnetism, has recently been able to publish a comprehensive isodynamic map.

The method applied in all these investigations consists of observing either the length of time it takes for one and the same magnetized needle to perform the same number of oscillations at different locations, or the number of oscillations of the same needle within the same time period, and the strength is assumed to be proportional to the square of the number of oscillations in a given length of time: in this way all the intensities are compared with one another, when an inclination-needle, suspended at the center of gravity, oscillates on a horizontal axis perpendicular to the magnetic meridian, or the horizontal components, when a horizontal needle swings on a vertical axis. The latter mode of observation leads to greater precision, and the results emerging from it can, after ascertaining the inclination, easily be related to the total intensities.

It is evident that the reliability of this procedure depends on the assumption, that the distribution of free magnetism in the particles of the needle used in this comparison remains unchanged during the individual experiments; that is, if the magnetic force of the needle had undergone any sort of weakening in the course of time, it would subsequently oscillate more slowly, and the observer, who has no knowledge of such an alteration, would attribute too low a value to the strength of the terrestrial magnetism for the subsequent location. If the experiments span only a moderate period of time and a needle manufactured of well-tempered steel and carefully magnetized is used, a significant weakening of the force is not especially to be feared; moreover, the uncertainty will be still further decreased, if several needles are employed for comparison; finally, this assumption will be relied upon more confidently, if it is found, after returning to the first location, that the duration of the needle's oscillation has not changed. Yet whatever precautions may be taken, a slow weakening of the force of the needle can scarcely be prevented, and thus such conformity after a longer absence can seldom be expected. Therefore, in comparing intensities for widely distant locations on the Earth, precision of the degree we must desire cannot be attained.

After all, this disadvantage in the method is less consequential, so long as it is only a matter of comparison of simultaneous intensities or intensities corresponding to time periods not remote from one another. But because experience has taught us, that both the declination and inclination undergo continuous changes at a given location, it cannot be doubted, that the intensity of terrestrial magnetism is subject to analogous secular changes, as it were. It is evident that, as soon as this question arises, the method described above loses all utility. And yet it would be extraordinarily desirable for the progress of science, that this highly important question be fully settled, which certainly cannot occur, if another method does not replace that purely comparative one, and the intensity of the Earth's magnetism is not reduced to fixed units and absolute measurement.

It is not difficult, to specify the fundamental theoretical principles, on which such a long-desired method must be based. The number of oscillations, which a needle carries out in a given length of time, depends on the intensity of the Earth's magnetism, as well as the state of the needle, namely on the static moments of the elements contained in it and on its inertial moment. Because this inertial moment can be ascertained without difficulty, it is evident that observation of the oscillations would provide us with the product of the intensity of the Earth's magnetism into the static moment of the needle's magnetism: but these two magnitudes cannot be separated, if observations of another kind are not made in addition, which yield a different relationship between them. This goal can be attained, if a second needle is added as an auxiliary and is exposed to the influence of the Earth's magnetism and of the first needle, in order to ascertain the relation of these two forces to each other. Each of the two effects will of course depend on the distribution of the free magnetism in the second needle: but the second will further depend on the state of the first needle, the distance between the midpoints, the position of the straight lines connecting their midpoints, and finally on the laws of magnetic attraction and repulsion. Tobias Mayer had been the first to advance the suggestion, that this law accords with the law of gravitation, insofar as those effects also decrease by the square of the distance: the experiments by Coulomb and Hansteen have given this suggestion great plausibility, and the latest experiments elevate it beyond any doubt. But it is well to consider, that this law refers only to the individual elements of free magnetism; the total effect of a magnetic body will be completely different, and, given very great distances, as can be deduced from just that law, will be very nearly proportional to the inverse relationship of the cube of the distance, so that, other conditions being equal, the action of the needle multiplied by the cube of the distance, given ever-increasing distances, approaches a definite limit. This limiting value, as soon as a definite length is taken as the unit, and the distances are expressed numerically, will be of the same type for the effect of the Earth's force and comparable with it.

By means of experiments appropriately constructed and performed, the limiting value of this relationship can be determined. Since the limit contains only the static moment of the magnetism of the first needle, then the quotient of this moment divided by the intensity of the terrestrial magnetism will be obtained; if they are now compared with the already ascertained product of these magnitudes, it will serve to eliminate this static moment, and will yield the value of the intensity of the terrestrial magnetism.

With regard to the possible ways to test the effects of terrestrial magnetism and of the first needle on the second needle, a twofold path is open, since the second needle can be observed either in a state of motion or in a state of equilibrium. The first method consists of observing this second needle's oscillations, while the effect of the terrestrial magnetism is associated with the action of the first needle. This first needle must be set up at a suitable distance such that its axis lies in the magnetic meridian going through the midpoint of the oscillating needle: thereby the oscillations are either accelerated or retarded, according to whether unlike or like poles are turned toward each other, and the comparison of either the oscillation times for each of the two positions of the first needle with each other, or the oscillation time of one of the two positions with the oscillation time which (after the distancing of the first needle) takes place under the exclusive effect of the Earth's magnetism, will show us the relation of this force to the effect of the first needle. In the second method, the first needle is placed so that the direction of its force, which it exerts on the location of the second, freely suspended, needle, forms an angle (for example, a right angle) with the magnetic meridian; by this means the second needle itself will be deflected out of the magnetic meridian, and from the magnitude of the deviation, one can infer the relation between the terrestrial magnetic force and the influence of the first needle.

By the way, the first method essentially coincides with that proposed some years ago by Poisson. But the experiments carried out by certain physicists according to this formulation, at least so far as I know, either totally miscarried, or can at most be considered imperfect approximations.

The actual difficulty lies in the fact that from the observed influences of the needle at moderate distances, a limit must be calculated, which bases itself on, in a sense, an infinitely great distance, and that the eliminations necessary for this purpose are all the more disturbed by the smallest errors in observation, indeed rendered totally useless, the more that unknown [quantities], which depend on the specific condition of the needle, must be eliminated: the calculation can only be reduced to a small number of unknowns, however, when the influences occur at distances, which in relation to the length of the needle become rather large, and therefore they themselves become very small. Yet, in order to measure such small influences the practical expedients employed up to now are insufficient.

I recognized, that I had to turn my efforts above all toward discovering new expedients, whereby the alignments of the needle could be observed and measured with far greater precision than before. The labors undertaken to this purpose, which were pursued for several months, and in which I was assisted by Weber in many ways, have led to the desired goal, to the extent that they not only did not disappoint expectations, but far exceeded them; and that nothing more remains to be desired, in order to make the precision of the experiments equivalent to the acuteness of astronomical observations, but a site fully protected from the influence of nearby iron and air currents. Two pieces of equipment were put at our disposal, which are distinguished no less by their simplicity than by the precision they afford. I must reserve their description for another time, while I submit to the physicists in the following treatise the experiments carried out to date in our observatory toward determining terrestrial magnetic intensity.

1.

To explain magnetic phenomena, we assume two magnetic fluxes: one we call north, the other south. We presuppose, that the elements of the one flux attract those of the other, and that on the other hand, two elements of the same flux mutually repel each other, and that each of the two effects alters in inverse relation to the square of the distance. It will be shown below that the correctness of this law was itself confirmed by our observations.

These fluxes do not occur independently, but only in association with the ponderable particles of such bodies which take on magnetism, and their effects express themselves either when they put the bodies into motion or they prevent or transform the motion, which other forces acting on these bodies, e.g. the force of gravity, would elicit.

Hence the effect of a given amount of magnetic flux on a given amount of either the same or the opposite flux at a given distance is comparable to a given motive force, i.e. with the effect of a given accelerating force on a given mass, and since the magnetic fluxes themselves can be known only through the effects, which they bring forth, the latter must directly serve to measure the former.

In order, however, that we may be able to reduce this measurement to definite concepts, units must above all be established for three kinds of magnitudes, namely, the unit of distance, the unit of ponderable mass, and the unit of acceleration. For the third, the force of gravity at the locus of observation can be assumed: if, however, this is not suitable, the unit of time must also enter in, and for us that acceleration will be = 1, which, within the time unit, produces a change of velocity of the body in the direction of its motion, which is equivalent to the unit.

Correspondingly, the unit of the amount of north flux will be that whose repulsive effect on another like it, and whose existing amount of motive force in the unit of distance = 1, i.e. the effect of an accelerating force = 1 on a mass = 1; the same will be true of a unit of the amount of south flux; in this definition, clearly the active flux, as well as that of the effect, must be thought of as, at bottom, united in physical points. Beyond this, however, it must be assumed, that the attraction between given quantities of different kinds of fluxes at a given distance is equal to the repulsion between the same respective quantities of the same kind of flux. Hence the effect of a quantity m of north magnetic flux on a quantity m' of the same flux at distance r (each of the two fluxes being assumed to be united as at one point) will be expressed as $\frac{mm'}{rr}$, or it is equivalent to a motive force = $\frac{mm'}{rr}$, which acts in the direction of the first against the second flux, and evidently this formula holds true in general, when, as from now on we wish to stipulate, a quantity of southern flux will be considered as negative, and a negative value of the force will signify attraction.

Hence if equal quantities of north and south flux are found simultaneously at one physical point, no effect at all will arise; if, however, the amounts are unequal, only the remainder/excess of the one which we wish to term free magnetism (positive or negative) will come under consideration.

2.

To these fundamental presumptions we must add still another, which is confirmed at all times by experience, namely, that every body, in which magnetic flux occurs, always contains an equal amount of each of the two. Experience even shows that this assumption is to be extended to the smallest particle of such a body, which can still be differentiated by our senses. Yet, according to what we have emphasized at the end of the preceding section, an effect can only be present insofar as some separation of the flux takes place, so that we must necessarily assume, that this occurs through such small intersticess, that they are inaccessible to our measurements.

A magnetizable body must thus be conceived of as the union of innumerable particles, of which each contains a certain quantity of north magnetic flux and an equally large amount of south, specifically, so that they are either homogeneously mixed together (the magnetism is latent), or have undergone a lesser or greater separation (the magnetism is developed), a separation, however, which can never involve an overflow of flux from one particle to another. It makes no difference, whether one assumes, that a greater separation originates from a greater amount of freed-up flux or from a greater gap between them: it is evident, though, that, in addition to the size of the separation, its directionality must come into consideration at the same time, for it is according to whether or not this is in conformity in the different particles of the body, that a greater or lesser total effect can arise respecting the points outside the body.

Yet, however the distribution of free magnetism within the body may behave, one can always insert in its place, as a result of a general theorem, according to a specific law, a different distribution on the surface of the body, which exerts fully the same outward force as the first distribution, so that an element of magnetic flux situated anywhere on the outside experiences exactly the same attraction or repulsion from the actual distribution of magnetism inside the body, as from the distribution thought of as on its surface[*]. The same fiction can be extended to two bodies, which, according to the proportion of the free magnetism developed in them, act upon each other, so that for each of them the distribution thought of as on the surface can replace the actual internal distribution. In this way we can finally give its true meaning to the common mode of speech, which, e.g., ascribes exclusively north magnetism to the one end of a magnetized needle, and south magnetism to the other, since it is evident that this turn of phrase is not in harmony with the principle enunciated above, which is unconditionally demanded by other phenomena. But it may suffice to have noted this in passing; we will discuss the principle itself more fully on another occasion, since it is not required for present purposes.

3.

The magnetic state of a body consists in the relation of the distribution of free magnetism in its individual particles. With regard to the variability of this state, we perceive an essential difference between the different magnetizable bodies. In some, e.g. in soft iron, that state changes immediately as a result of the slightest force, and if this force ceases, returns to its previous state: in contrast, in others, especially tempered steel, the force must have attained a certain strength, before it can elicit a perceptible change in the magnetic state, and if the force ceases, either the body remains in the transformed state, or at least it does not

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[*] Cf. Gauss, General Principles..., section 36. Works, vol. V, p. 240 (also Ostwald's Classics, No. 2, p. 49).

fully return to the previous one. Hence, in bodies of the first kind, the magnetic molecules order themselves in such a way that, between the magnetic forces, which originate partly from the bodies themselves, partly from external sources, a complete equilibrium exists, or at least the state differentiates itself barely noticeably from that just described. In bodies of the second kind, in contrast, the magnetic state can be lasting/constant even without complete equilibrium between those forces, provided that stronger external forces remain distant. Even if the source of this phenomenon is unknown, it can nevertheless be represented, as if the ponderable parts of a body of the second kind counterpose to the movement of the magnetic fluxes associated with it an obstacle akin to friction, a resistance, which in soft iron either does not exist at all or is only very slight.

In theoretical investigation, these two cases require a completely different treatment; but in the present treatise, only bodies of the second kind will be spoken of: in the experiments we want to talk about, the unchangeability of the state in the individual bodies will be a fundamental assumption, and hence one must be on guard during the time that the experiments bring other bodies, which could alter this state, into too close proximity.

Yet a certain source of change is at hand, to which the bodies of the second kind are also subjected, namely, heat. Experience indubitably teaches us that the magnetic state of a body changes with its temperature, yet in such a way that, if the body is not immoderately heated, it returns to its former magnetic state when its former temperature returns. This function is to be determined by means of suitable experiments, and if observations at different temperatures are undertaken as part of an experiment, they will be above all reducible to one and the same temperature.

4.

Independently of the magnetic forces, which we see sufficiently proximate individual bodies exert upon each other, another force acts upon the magnetic fluxes, which, since it manifests itself everywhere on Earth, we ascribe to the globe itself, and hence we term it terrestrial (Earth) magnetism. This force expresses itself in a dual way: bodies of the second kind, in which magnetism is induced, are, if supported at the center of gravity, turned toward a definite direction: in bodies of the first kind, in contrast, the magnetic fluxes are automatically separated by that force, a division which can be made very noticeable, if one chooses bodies of appropriate shape and puts them in an appropriate location. Each of the two phenomena will be explained by the interpretation, that at any arbitrary locus, that force drives the north magnetic flux in a definite direction, and drives the south magnetic flux, on the other hand, with equal strength in the opposite direction. The direction of the former is always understood, if we speak of the direction of terrestrial magnetism; hence it will be determined both by the inclination to the horizontal plane as by the deviation from the meridian plane of the vertical plane in which it acts; the former is called the magnetic meridian plane. The intensity of terrestrial magnetism, however, is to be measured by the motive force which it exerts on the unit of the free magnetic flux.

This force is not only different at different spots on the Earth, but also changeable at the same spot, in the course of centuries and years, as well as in the course of decades and hours. This changeability has indeed long been known with regard to direction; but with regard to intensity, until now it has only been possible to observe it in the course of the hours of one day, since we had no experimental device designed for longer periods of time. The reduction of the intensity to absolute measurement will in future provide remedies for this deficiency.

5.

In order to subject the effect of terrestrial magnetism on magnetic bodies of the second kind (from now on, those are always the kind to consider) to calculation, such a body is conceived of as divided into infinitely small parts; let *dm* be the element of free magnetism in a particle, whose coordinates in relation to three fixed planes, perpendicular to each other, in the body, may be designated *x, y, z*: the elements of the south flux we take to be negative. Then it is clear, first of all, that the integral taken for the entire body (even for every measurable part of the body), is $dm = 0$. We wish to further specify $xdm = X$, $ydm = Y$ and $zdm = Z$, which magnitudes may designate the moments of free magnetism in relation to the three orthogonal-planes or in relation to the axis perpendicular to them. Since under the assumption, that *a* refers to an arbitrary constant magnitude, $(x-a)dm$ will be $= X$, it is evident, that the moment in relation to a given axis depends only on its direction, but not on its origin. If we draw a fourth axis through the origin of the coordinates, which forms with it the angle *A, B, C*, the moment of the elements dm in relation to this axis will be $= (x\cos A + y\cos B + z\cos C)dm$, and further the moment of free magnetism in the body as a whole

$$= X\cos A + Y\cos B + Z\cos C = V\ .$$

If we assume $\sqrt{(XX+YY+ZZ)} = M$ and $X = M\cos\alpha$; $Y = M\cos\beta$; $Z = M\cos\gamma$, and draw a fifth axis, which forms the angle α, β, γ with the first three axes, and with the fourth the angle ω, then, since as a consequence of these assumptions $\cos\omega = \cos A\cos\alpha + \cos B\cos\beta + \cos C\cos\gamma$, V will be $= M\cos\omega$. We call this fifth axis simply the <magnetic axis> of the body, and we assume that its <direction> is part of the positive value of the root $\sqrt{(XX+YY+ZZ)}$. If the fourth axis coincides with this magnetic axis, the moment becomes *V* = *M*, which is clearly the greatest of all moments: the moment in relation to an arbitrary other axis is found, by multiplying this greatest moment (which, so that ambiguity need not be feared, can simply be called the moment of magnetism) with the cosine of the angle between it and the magnetic axis. The moment in relation to an arbitrary axis perpendicular to the magnetic axis will = *0*, but will be negative in relation to that axis which forms an obtuse angle with the magnetic axis.

6.

If a force of constant intensity and direction acts on the individual particles of the magnetic flux, the total resultant force on the body can easily be inferred from principles of statics, since in the bodies under consideration these particles have lost their state of flux to some extent, and form a fixed mass with the ponderable body. On an arbitrary magnetic particle *dm* the motive force = *Pdm* may act in a direction *D* (where for the molecule of south flux the negative sign as such signifies the opposite direction); let *A* and *B* be two points on the body lying in the direction of the magnetic axis, and their distance = *r*, positively taken, when the magnetic axis is directed from *A* toward *B*: then one easily sees, that if two new forces are added to these forces, each = $\frac{PM}{r}$, of which the one acts on *A* in direction *D*, the other on *B* in the opposite direction, there will be an equilibrium among all these forces. Therefore the former forces will be equivalent to two forces = $\frac{PM}{r}$, of which one acts on *B* in direction D, the other on A in the opposite direction, and clearly these two forces cannot be united into one.

If, in addition to force P, another similar force P' acts in direction D' on the magnetic fluxes of the body, then they can be replaced by two others, which act either on the same points A and B or more generally on other points A' and B', provided only that $A'B'$ is likewise a magnetic axis, and, if the distance is made $A'B' = r'$, these forces must of course $= \frac{P'M}{r'}$, and must act on B' in direction D', on A' in the opposite direction. The same holds true of several forces.

To the terrestrial magnetic force can safely be ascribed, within a such a small space as that occupied by the body subjected to the experiment, an intensity and direction that are everywhere constant, though variable with time; thus, what we have just said can be applied to it [the terrestrial magnetic force]. However, it can be advantageous, to separate it right at the outset into two forces, a horizontal $= T$ and a vertical $= T'$, which in our situation is directed downward. Since, in case one wants to use for the latter two others acting on points A' and B', it is permissible to arbitrarily posit point A' and also the interval $A'B' = r'$, we want to choose A' for the center of gravity, and, denoting by p the weight of the body, i.e. the motive force of gravity on its mass, we say that $\frac{T'M}{p} = r'$. Hereby the effect of force T' is released into a force $= p$ directed upward on A', and into another of the same magnitude directed downward on B', and since further the former is clearly cancelled by the force of gravity itself, the effect of the vertical component is simply reduced to the transfer of the center of gravity from A' to B'. Moreover, it is clear, that for those situations, where the terrestrial magnetic force forms an acute angle with the vertical force, or, in other words, where its vertical part pushes the magnetic north-fluid upward, a similar shift of the center of gravity occurs from the magnetic axis to the south pole.

In this way of thinking, it is self-evident that, whatever sort of experiments may be conducted with a magnetized needle in a single magnetic state, it is impossible to infer the inclination from this alone, but the locus of the actual center of gravity must already be known by some other means. This locus is ordinarily determined before the needle is magnetized; but this practice is not reliable enough, since as a rule a steel needle already takes on magnetism, though weakly, while it is being manufactured. It is therefore necessary for the determination of inclination to induce another shift of the center of gravity by means of an appropriate alteration in the magnetic state of the needle. In order that this differ as much as possible from the first, it will be necessary to reverse the poles, by means of which a double shift is obtained. However, shifting the center of gravity even in needles which have the most suitable form and are saturated with magnetism, cannot exceed a certain limit, which equals roughly 0.4 mm, and in situations, where the vertical force is at its greatest, an as great mechanical fineness is required in the needle that is to serve to determine the inclination.

7.

When any point C of a magnetic body is assumed to be fixed, the necessary and sufficient condition for equilibrium, is that a plane laid through C, the center of gravity and the magnetic axis coincides with the magnetic meridian, and that, moreover, the moments, with which the terrestrial magnetic force and the center of gravity seek to rotate around point C, cancel each other: the second condition proceeds from the fact that, if T denotes the horizontal part of the terrestrial magnetic force and i the inclination of the magnetic axis toward the horizontal plane, $TM \sin i$ is the product of the weight of the body at the distance of the displaced center of gravity *B'* of the vertical line drawn through *C*: clearly this distance must lie on the north or south side, according to whether *i* is an elevation or depression, and for *i = 0*, *B'* lies in this vertical line itself. If the body is already moved around this vertical line so that the magnetic axis reaches into the vertical plane, whose magnetic azimuth, i.e. its angle with the northerly part of the magnetic meridian (arbitrarily taken as positive toward east or toward west) = *u*, then the terrestrial magnetism will exert a force on the body revolving around the vertical axis, which strives to decrease the angle *u* and whose moment will be $= TM \cos i \sin u$, and the body will perform oscillations around this axis, whose duration can be calculated according to known methods. Namely, if *K* denotes the inertial momentum of the body in relation to the axis of oscillation (i.e., the sum of the ponderable molecules multiplied by the square of the distance from the axis), and, as usual, denotes as π the half-circumference for radius = *1*, then the duration of an infinitely small oscillation will be

$$= \pi \sqrt{\frac{K}{TM \cos i}}$$

namely, in the case that it is based upon the magnitudes *T* and *M* as unit of the accelerating force, which in the time-unit produces the velocity = *1*: the reduction of finite oscillations to infinitely small ones will be calculable in a similar way for the oscillations of the pendulum. Hence, if the duration of <one> infinitely small oscillation is found from observation to = *t*, we will have the equation: $TM = \frac{\pi\pi K}{tt \cos i}$, and when, moreover, as we always assume from now on, the body is suspended in such a way that the magnetic axis is horizontal:

$$TM = \frac{\pi\pi K}{tt}.$$

If one would rather assume gravity as unit of the accelerating forces, one must divide that value by $\pi\pi\, l$, where *l* denotes the length of the simple second-pendulum, so that altogether one would have: $TM = \frac{K}{tt\, l \cos i}$ or for our case $TM = \frac{K}{tt\, l}$.

8.

When these sorts of experiment are performed on magnetized needles, suspended by a vertical thread, then the reaction, which is exerted by the torsion, cannot be neglected in more refined experiments. We want to identify two horizontal diameters in such a thread, the one, *D*, at the lower end, where the needle is attached, parallel to the magnetic axis of the needle, the other, *E*, at the upper end, where the thread is secured, and in fact *E* would be parallel with *D* if the thread were not twisted. We want to assume, that *E* forms the angle *v* with the magnetic meridian, while *D*, or the magnetic axis, forms the angle *u* with the meridian; then, experience shows that the torsion will be proportional, at least approximating the angle *v - u*: hence we will posit the moment, with which this force seeks to make angle *u* equal to angle *v*, as $= (v-u)\Theta$. Since the moment of terrestrial magnetic force, which strives to decrease angle u, is now $= TM \sin u$, the condition for equilibrium is contained in the equation: $(v-u)\Theta = TM \sin u$, which allows for more real solutions, the smaller Θ is in comparison with *TM*: however, so long as only small values of *u* are dealt with, instead of this equation one can safely take the following one: $(v-u)\Theta = TMu$ or $\frac{v}{u} = \frac{TM}{\Theta} + 1$. In our apparatus, the upper end of the thread is fastened to a movable arm, which holds an indicator playing on the periphery of a circle divided into degrees. Hence even if the collimation error (i.e,, the point corresponding to the value $v = 0$) is still not known with sufficient precision, nevertheless this pointer indicates the difference for each second value from v on: just as another part of the apparatus provides the difference between the values of u, corresponding to the equilibrium condition, with the greatest precision, and it is clear, that the value of $\frac{TM}{\Theta}+1$ will be obtained from the division of the difference between two values of v by the difference between the corresponding values of u. If there is a somewhat longer period of time between the experiments conducted for this purpose, it will be necessary, in order to achieve the utmost precision, to take note of the daily change in the magnetic declination, which is easily accomplished with the help of simultaneous observations on a second apparatus, in which the upper end of the thread remains unaffected: it is hardly necessary to recall, that the distance between the two apparatus must be large enough, so that they cannot significantly interfere with each other.

In order to show how great a refinement these sorts of observations permit, we introduce an example from the daily log book. On September 22, 1832, pending the collimation error, the following declinations u and angle v*) were observed:

Experiment	Time	First Needle		Second Needle
		u	v	u
I	9h 33‘ Vm	+0° 4‘ 19.5‘‘	300°	+0° 2‘ 12.1‘‘
II	9h 57‘	-0° 0‘ 19.6‘‘	240°	+0° 1‘ 37.7‘‘
III	10h 16‘	-0° 4‘ 40.5‘‘	180	+0° 1‘ 18.8‘‘

Hence the declinations of the first needle, related to the position of the first observation, are as follows:

I	u = 0° 4‘ 19.5‘‘	v = 300°
II	+0° 0‘ 14.8‘‘	240°
III	-0° 3‘ 47.2‘‘	180°

From this emerges the value of the fraction $\frac{TM}{\Theta}$ from the combination of observations

I and II 881.7
II and III 891.5
I and III 886.6

The daily changes in the magnetic declination are decreased by the torsion in the proportion of unity to $\frac{n}{n+1}$ where $\frac{TM}{\Theta}$ is set = n, a change which, if we use threads of such low torsion as the foregoing example shows, can be considered as insignificant. So far as the duration of the (infinitely small) oscillations, it can easily be concluded from the dynamic principles, that it is decreased by the torsion in proportion of unity to $\sqrt{\frac{n}{n+1}}$. Actually, this relates to the case where $v = 0$. The formulas would hold true, however, in general, if we

*) Both elements expand from left to right.

posited $\frac{TM \cos u\Upsilon}{\Theta} = n$ wherein we denote the value of $u\Upsilon$ by u, which corresponds to the equilibrium: but the difference will certainly be insignificant.

9.

The coefficient Θ depends essentially on the length, the thickness, and the material of the thread, and additionally in metallic threads on the temperature, in satin threads on the humidity: in contrast, it seems to depend not at all, in the former case (perhaps also in the latter, if they are single threads) on the weight they bear. The situation is different when the satin threads are multiplex, as they must be to hold heavy needles: in this case theta increases with the suspended weight, yet it remains far smaller than the value of Θ for a metallic thread of exactly the same length and load capacity. Thus, through a very similar method to that developed in the previous section (although with a different thread and a different needle), the value of *n = 597.4* is found, while the thread holds a needle with the usual equipment alone, where the total weight was *496.2 g*; in contrast = *424.8*, when the weight was increased to *710.8 g*, or in the first case it was $\Theta = 0.0016740TM$, in the second case $\Theta = 0.0023542TM$. The thread, whose length was *800 mm*, is composed of 32 individual threads,[**] which individually hold almost *30 g* securely and are arranged so that they undergo an equal tension. Moreover, it is probable, that the value of Θ consists of a constant element and an element proportional to the weight, and that the constant part will equal the sum of the values of Θ for the individual simple threads. Under this hypothesis (which as of now is not adequately substantiated by experiments), *0.0001012 TM* is found as the constant value for the example above, and thus *0.00000316 TM* as the value of Θ for a simple thread. With recourse to the soon to develop value of *TM*, it will be calculated from this hypothesis, that the resistance of a single thread, wound around a curve equal to the radius *(57° 18')*, is equivalent to the gravity of a milligram pressing on the arm of a lever with the length of approximately *1/17 mm*.

10.

If the oscillating body is a simple needle of regular form and homogeneous mass, the inertial momentum *K* can be calculated according to known methods. If, e.g., the body is a right-angled parallelopiped, whose sides are *a*, *b, c*, whose thickness = *d* and whose mass is *q*, thus = *abcd*, the inertial momentum will be in relation to an axis going through the midpoint and parallel to side $c = \frac{1}{12}(aa + bb)q$: and since in magnetic needles of such a shape, the side which is parallel to the magnetic axis, namely *a*, is usually far longer than *b*, it will, moreover, suffice for crude experiments to make $K = \frac{1}{12}aaq$. But in more refined experiments, even if a simple needle is used, we would hardly be allowed the comfortable assumption of a completely homogeneous mass and a completely regular shape, and for those of our experiments in which, not a simple needle, but a needle involving more complex equipment, it is altogether impossible, to ascertain the state of affairs through such a calculation, and a different procedure was sought for the precise determination of the moment *K*.

To the needle was attached a wooden crossbeam, on which two equal weights hung, by means of which very fine spikes exerted pressure on points *A* and *B* of the beam: these points were on a horizontal straight line, in the same vertical plane as the axis of suspension,

[**] Strictly speaking, these threads are not really simple, but merely the kind that are sold as unspun.

and were equidistant from it on both sides. If the mass of each of the two weights is denoted by p and the distance AB by $2r$, then by adding this apparatus the momentum K will be increased by the magnitude $C + 2prr$, where C is the sum of the momentum of the beam in relation to the axis of suspension, and the momentum of the weights in relation to the vertical axis through the spikes and the center of gravity. Hence, if the oscillations of the un-weighted needle and the needle weighted at two different distances, are observed, namely, for $r = r'$ and $r = r''$, and the duration of oscillations (after they are reduced to infinitely small amplitudes and freed from the effect of torsion) respectively $= t, t', t''$, are found, then from combining the equations:

$$TMtt = \pi\pi K$$
$$TMt't' = \pi\pi(K + C + 2pr'r')$$
$$TMt''t'' = \pi\pi(K + C + 2pr''r'')$$

the three unknowns TM, K and C can be determined. We will attain still greater precision, if, for several values of r, namely $r = r', r'', r'''$ and so forth, we observe the associated duration of oscillations t', t'', t''', and so forth, and, according to the method of least squares, determine the two unknowns x and y such that

$$t' = \sqrt{\frac{r'r' + y}{x}}$$
$$t'' = \sqrt{\frac{r''r'' + y}{x}}$$
$$t''' = \sqrt{\frac{r'''r''' + y}{x}}$$

and so forth.

By this means we obtain:

$$TM = 2\pi\pi px$$
$$K + C = 2py.$$

Regarding this method, the following is to be observed:

I. If a not too smooth needle is used, it suffices, to simply place the wooden beam on it. If, however, the surface is very smooth, so that friction cannot retard the sliding of the beam, it is necessary, so that the whole apparatus may move like a single fixed body, to connect the beam more securely to the rest of the apparatus. In both cases, care should be taken, that points A and B are located with sufficient precision in a horizontal straight line.

II. Since the ensemble of such experiments requires several hours, the variability of the terrestrial magnetism within this time span, at least if the greatest precision is desired, cannot be neglected. Hence, before the elimination is undertaken, the observed times must be reduced to a constant value of T, e.g. to the mean value, corresponding to the first experiment. For this purpose, simultaneous observations of another needle (just as in Section 8) are necessary: if these observations have as the duration of <one> oscillation for the mean time of the single experiment respectively $= u, u', u'', u'''$ and so forth, then instead of using the observed values *t*, *t'*, *t''*, *t'''* and so forth for the calculation, and $\frac{ut'}{u'}, \frac{ut''}{u''}, \frac{ut'''}{u'''}$, so forth respectively are used.

III. A similar comment holds true with regard to the variability of *M*, which comes from the change in temperature, if that occurs during the experiment. But it is clear that the just-described reduction already includes this improvement in and of itself, if each of the two needles is subject to the same temperature, and is influenced in the same way by such a change.

IV. If the task is only to ascertain the value of *TM*, clearly the first experiment is superfluous. Yet it will be useful to conjoin to the experiment conducted with a weighted needle another with an un-weighted needle, in order for the value of *K* to be obtained at the same time, so that it can be taken as a base for these experiments which are performed at another time with the same needle, since it is evident that this value remains constant, even when *T* and *M* undergo a change over time.

11.

To better explain this method, we include here an example from the large quantity of applications. The experiment conducted on September 11, 1832 yielded the following table:

Experiment	Simultaneous oscillations		
	of the first needle		of the second needle
	Load	One oscillation	One oscillation
I	*r = 180mm*	*24.63956‘‘*	*17.32191‘‘*
II	*r = 130mm*	*20.77576‘‘*	*17.32051‘‘*
III	*r = 80mm*	*17.66798‘‘*	*17.31653‘‘*
IV	*r = 30mm*	*15.80310‘‘*	*17.30529‘‘*
V	Without load	*15.22990‘‘*	*17.31107‘‘*

Times were observed on a clock which every day lost *14.24"* mean time, and each of the two weights weighed *103.2572 g*; the distances *r* in milimeters were determined with microscopic precision; the duration of an oscillation, which was ascertained from at least 100 oscillations (in the fifth experiment, even from 677 for the first needle), had already been reduced to infinitely small curves: moreover, these reductions were insignificant, because of the very small amplitude of the oscillations[**)] which it is possible to apply to our apparatus without detriment to the greatest precision. We wanted to reduce these oscillation times, first to the mean value of *TM*, which took place during the fifth experiment, under application of the rules in section II above; then to the values, which would have resulted without torsion, through multiplication by $\sqrt{\frac{n+1}{n}}$ where *n* in the four first experiments = 424.8, in the fifth experiment = 597.4 (cf. section 9); finally to the mean solar time through multiplication by $\frac{864.00}{86385.76}$; our results were:

[**)] Thus the amplitude of the oscillations in the first experiment was 0° 37' 26" at the beginning, 0° 28'34" at the end; in the fifth experiment 1° 10‘ 21‘‘ at the beginning, after 177 oscillations 0° 45‘ 35‘‘, and after 677 oscillations 0° 6‘ 44‘‘.

I. 24.65717‘‘ = t‘ for r‘ = 180mm
II. 20.79228‘‘ = t‘‘ for r‘‘ = 130mm
III. 17.68610‘‘ = t‘‘‘ for r‘‘‘ = 80mm
IV. 15.82958‘‘ = t‘‘‘‘ for r‘‘‘‘ = 30mm
V. 15.24515‘‘ = t for the unloaded needle.

If we take the second, the millimeter and the milligram for the units of time, distance and mass, so that $p = 103257.2$, we infer from combining the first experiment with the fourth:

$TM = 179{,}641{,}070;\ K + C = 4{,}374{,}976{,}000$

and then from the fifth experiment

$K = 4{,}230{,}282{,}000$ and likewise $C = 144{,}694{,}000$.

When, however, it is desired to take into account all the experiments, the method of least squares is the most convenient in the following way. We proceed from the approximate values of the unknowns x and y, which emerge from the combination of the first and fourth experiments, and, designating the still-to-be- added corrections by ξ and η, we say:

$$x = 88.13646 + \xi$$
$$y = 21184.85 + \eta\,.$$

From this are derived as calculated values the times t', t'', t''', t'''' by familiar methods:

$$t' = 24.65717 - 0.13988\xi + 0.00023008\eta$$
$$t'' = 20.78731 - 0.11793\xi + 0.00027291\eta$$
$$t''' = 17.69121 - 0.10036\xi + 0.00032067\eta$$
$$t'''' = 15.82958 - 0.08980\xi + 0.00035838\eta,$$

whose comparison with the observed values, treated according to the method of least squares, yields the results:

$$\xi = -0.03230; \eta = -12.38$$
$$x = 88.10416;\ y = 21172.47.$$

From this finally results

$$TM = 179{,}575{,}250,\ K + C = 4372{,}419{,}000,$$

and then with addition of the first experiment:

$$K = 4{,}228{,}732{,}400,\ C = 143{,}686{,}600.$$

I next present a comparison of the times calculated from the corrected values of the magnitudes x, y, with the observed values.

Experiment	Claculated time	Observed time	Difference
I	24.65884‘‘	24.65717‘‘	+0.00167‘‘

II	20.78774	20.79228	-0.00454
III	17.69046	17.68610	+0.00436
IV	15.82805	15.82958	-0.00153

In Goettingen, we set the length of the simple second pendulum = *994,126 mm*, hence the weight, measured by that unit of the accelerating forces, which underlies the preceding calculations, = *9,811.63*; thus if we prefer to take gravity itself as the unit, then *TM* = *18,302.29*: this number expresses the number of milligrams, whose pressure under the influence of gravity on a lever of the length of one millimeter is equivalent to the force with which the terrestrial magnetism seeks to turn that needle around the vertical axis.

12.

After we have completed the determination of the product of the horizontal components *T* of the terrestrial magnetic force into the magnetic moment *M* of the given needle, we proceed to the second part of the experiment, namely, to the determination of the quotient *M/T*. We will accomplish this by comparing the effect of this needle on another needle with the effect of terrestrial magnetism on the very same needle; specifically, this will be observed, as was already discussed in the introduction, either in a state of motion or in a state of equilibrium; we have repeatedly investigated each of the two methods: but for several reasons the latter is preferable by far to the former, and thus we will confine the investigation here to that one, while the first can be dealt with in a similar way without difficulty.

13.

The conditions of equilibrium in a movable body, on which arbitrary forces act, can easily be brought together in one single formula by means of the principle of virtual displacements: namely, the sum of the products of any one force multiplied into the projection of an infinitely small displacement of its working point on the direction of force, must be so constituted that it can attain a positive value for no virtual movement, i.e., a movement complying with the conditions, so that, if the virtual movements are possible collectively in opposite directions, that aggregate, which we wish to designate $d\Omega$, is = *0* for any virtual movement.

The movable body we consider here is the magnetic needle, which is connected at a point *G* to a revolving thread, fastened at the top. This thread merely prevents the distance of point *G* from the fixed end of the thread from becoming greater than the length of the thread, so that here, too, as in the case of a completely free body, the position of the body in space depends on six variables and further the equilibrium itself on six conditions. Since, however, the solution of the problem is only to serve to determine the fraction *M/T*, it suffices to consider that virtual movement, which occurs in the rotation around the vertical axis intersecting *G*; and it is evident that such an axis can be considered as fixed, and only the angle between the vertical planes, in which the magnetic axis is located, and the magnetic meridian plane, as variable. We will take this angle from the north side of the meridian toward the east and designate it *u*.

14.

We will conceive of the volume of the movable magnetic needle as divided into infinitely small elements, letting the coordinates of an arbitrary element be *x*, *y*, *z*, while *e* is the element of free magnetism contained in it. We place the origin of the coordinates in the arbitrary point h inside the needle on the vertical line going through G; the axis of the coordinates x and y is to be horizontal, the former in the magnetic meridian directed toward the north, the latter toward the east; the coordinate z we place as above. Then the effect of the terrestrial magnetism on the element e yields the element $Tedx$ of $d\Omega$.

In a similar way, the volume of the second fixed needle is divided into infinitely small elements, and any element may conform to the coordinates X, Y, Z, and the amount E of free magnetism; finally, r would be $= \sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}$. Under this condition, the effect of element E on element e yields the contribution $\frac{eEdr}{r^n}$ to the sum $d\Omega$, if the power r^n of the distance r is assumed to be inversely proportional. If we denote by N the value of u which corresponds to the un-twisted state of the thread, then the moment of the torsion of the thread can be expressed as $\Theta(N-u)$: this force can be conceived in such a way, as if the tangential force $= \frac{\Theta(N-u)}{D}$ acted on each end of a horizontal diameter of the thread placed through G, where D denotes this diameter, and it is easily seen, that from this as element of the sum is derived: $\Theta(N-u)du$.

The weight of the particles of the needle clearly does not contribute to the sum $d\Omega$, since only u is variable; therefore we have the equation:

$$d\Omega = \quad Tedx + \quad \frac{eEdr}{r^n} + \Theta(N-u)du\ ,$$

in which the summation in the first term relates to all elements e, in the second to all combinations of the individual e with the individual E. Hence it is clear, that the condition of stable equilibrium consists in

$$\Omega = \quad Tex - \quad \frac{eE}{(n-1)r^{n-1}} - \frac{1}{2}\Theta(N-u)^2$$

becoming a maximum.

15.

It suits our purpose, to always construct the experiments so that the magnetic axis of each of the two needles is horizontal, and that both needles are approximately at the same height; hence we wish to confine further calculations to these preconditions.

We will relate the coordinates of the points of the first needle to fixed axes in the needle, which also still intersect point h; that is, the first axis would lie in the direction of the magnetic axis, the second horizontal and to the right of the first, the third vertical and directed upward; the coordinates of the element e in relation to these axis would be a, b, c. Likewise A, B, C would be the coordinates of element E in relation to similarly fixed axes in the second needle, which intersect this needle at a point H: we select this point close to the center of the needle and at the same height as point h.

The position of point H would of course be most conveniently determined by the distance from point h and the direction of the straight line associated with it, if it is a matter of only <one> experiment: since, however, for our purpose several experiments are always necessary, which are related to different positions of point *H*, all of which certainly lie in the same straight line, yet not necessarily in a line through point *h*, it is better to construct the demonstration from the very beginning, in such a way that the system of such experiments depends on one single unknown. Hence we wish to relate point *H* to an arbitrary point *h'* in the same horizontal plane, which lies near *h* and whose coordinates may be $\alpha, \beta, 0$, and wish to make the distance $h'H = R$ and the angle of the lines *h'H* with the magnetic meridian $= \psi$. If we now denote the angle of the magnetic axis of the second needle with the magnetic meridian by *U*, then we will have:

$$
\begin{aligned}
&x = a\cos u - b\sin u\\
&y = a\sin u + b\cos u\\
&y = c\\
&X = \alpha + R\cos\psi + A\cos U - B\sin U\\
&Y = \beta + R\sin\psi + A\sin U + B\cos U\\
&Z = C.
\end{aligned}
$$

Thus everything is in place for developing the sum Ω and the fraction $\frac{d\Omega}{du}$, which has to disappear for the equilibrium state.

16.

First, will $Tex = T\cos u$ $ae - T\sin u$ $be = mT\cos u$, if we denote by m the momentum of the free magnetism of the first needle ae, since be is $= 0$: the element of $\frac{d\Omega}{du}$, which derives from the first term of Ω, will be $= -mT\sin u$.

If, for the sake of brevity, we say:

$$
\begin{aligned}
&k = \alpha\cos\psi + \beta\sin\psi + A\cos(\psi - U) + B\sin(\psi - U) - \alpha\cos(\psi - u) - b\sin(\psi - u)\\
&l = [\alpha\sin\psi - \beta\cos\psi + A\sin(\psi - U) - B\cos(\psi - U) - a\sin(\psi - u) - b\cos(\psi - u)]^2 + (C - c)^2,
\end{aligned}
$$

thus *rr* will be $= (R + k)^2 + l$.

Since the bei verwerthbaren experiments *R* must be far larger than the dimensions of each of the two needles, the magnitude $\frac{1}{r^{n-1}}$ can be evolved into the quickly convergent series

$$R^{-(n-1)} - (n-1)kR^{-n} + \left[\frac{nn-n}{2}kk - \frac{n-1}{2}l\right]R^{-(n+1)} - \left[\frac{1}{6}(n^3 - n)k^3 - \frac{1}{2}(nn-1)kl\right]R^{-(n+2)} + \dots,$$

whose law, if it were worth the effort, could be easily specified. The individual terms of the sum $\frac{eE}{r^{n-1}}$, which derives from inserting the values of the magnitudes *k* and *l*, will contain a factor of the form:

$$eEa^{\lambda}b^{\mu}c^{\nu}A^{\lambda'}B^{\mu'}C^{\nu'};$$

this is equal to the product of the factors $ea^{\lambda}b^{\mu}c^{\nu}$ and $EA^{\lambda'}B^{\mu'}C^{\nu'}$, which respectively depend on the magnetic state of the first and second needles. Taking this into consideration, what we may establish confines itself to the equations:

$$e = 0, \quad ea = m, \quad eb = 0, \quad ec = 0,$$
$$E = 0, \quad EA = M, \quad EB = 0, \quad EC = 0,$$

where we denote by *M* the moment of the free magnetism of the second needle. In the special case, that the shape of the first needle (the moveable one) and the distribution of magnetism in the longitudinal direction are symmetrical, so that two elements always correspond, for which *a* and *e* have equivalent opposite values *b* and *c*, then, as soon as the midpoint coincides with point h, $ea^{\lambda}b^{\mu}c^{\nu}$ will always = *0* for a direct value of the number $\lambda+\mu+\nu$, and similarly for the second needle, if the shape and the distribution of magnetism is symmetrical in relation to point *H*. Hence, in general, in the sum $\frac{eE}{r^{(n-1)}}$ the coefficients of the powers $R^{-(n-1)}$ and R^{-n} disappear; in the special case, where each of the two needles is symmetrically shaped and symmetrically magnetized, while at the same time the midpoint of the first, *h* and *h'*, coincide, and likewise the midpoint of the second and *H* coincide, then the coefficients of the powers $R^{-(n+2)}, R^{-(n+1)}, R^{-(n+6)}$ and so forth will also disappear; every time those conditions occur very approximately, they must at least be very small. The major term, which is derived from the elaboration of the second element of Ω, namely from $-\frac{eE}{(n-1)r^{(n-1)}}$ will be:

$$= -\frac{1}{2}R^{-(n+1)}\left(n \quad eEkk - \quad eEl\right)$$
$$= mMR^{-(n+1)}\left(n\cos(\psi-U)\cos(\psi-u)-\sin(\psi-U)\sin(\psi-u)\right).$$

From this it follows, that the element of $\frac{d\Omega}{du}$, which corresponds to the influence of the second needle, can be expressed by the following series:

$$fR^{-(n+1)} + f'R^{-(n+2)} + f''R^{-(n+3)} + \ldots,$$

in which the coefficients contain rational functions of the cosine and sine of the angles ψ, u, U and the magnitudes are α, β, and beyond that, contain, constant magnitudes, which depend on the magnetic state of the needle; specifically, they will be:

$$f = mM\left(n\cos(\psi-U)\sin(\psi-u)+\sin(\psi-U)\cos(\psi-u)\right).$$

The complete elaboration of the following coefficients *f'*, *f''* and so forth is not necessary for our purpose; it suffices to remark that

1) in the case of complete symmetry, the just-indicated coefficients *f'*, *f''* and so forth disappear;

2) if the remaining magnitudes remain unchanged and is increased by two right angles (or, the same thing, if the distance *R* is taken from the same, backwards-extended straight line on the other side of point *h'*), the coefficients *f*, *f''*, *f''''* and so forth preserve their values, while *f'*, *f'''*, *f'''''* and so forth take on opposite values, or that the series turns into

$$fR^{-(n+1)} - f'R^{-(n+2)} + f''R^{-(n+3)} - \ldots;$$

this is easily inferred from the fact that by means of this change in ψ, *k* becomes -*k*, but *l* is not transformed.

17.

The condition that the movable needle not be revolved around the vertical axis by the combination of forces is thus summed up in the following equation:

$$0 = -mT\sin u + fR^{-(n+1)} + f'R^{-(n+2)} + f''R^{-(n+3)} + \ldots - \Theta(u - N).$$

Since it can easily be effected, that the value of *N*, if not = *0*, is at least very small, and also *u*, for the experiment at hand, remains within narrow limits, then, without having to fear significant error, for the term $\Theta(u - N)$ one can use $\Theta\sin(u - N)$, all the more so, as $\frac{\Theta}{mT}$ is a far smaller fraction. Let $u°$ be the value of *u*, which corresponds to the equilibrium of the first needle in the absence of the second, or let

$$mT\sin u° + \Theta\sin(u° - N) = 0;$$

from this easily follows

$$mT\sin u + \Theta\sin(u - N) = \big(mT\cos u° + \Theta\cos(u° - N)\big)\sin(u - u°),$$

where, in place of the first factor, $mT + \Theta$ can be adopted without reservation. Thus our equation becomes:

$$(mT + \Theta)\sin(u - u°) = fR^{-(n+1)} + f'R^{-(n+2)} + f''R^{-(n+3)} + \ldots.$$

If we keep the term alone, the solution is within reach, namely, we have

$$tg(u - u°) = \frac{mM\big(n\cos(\psi - U)\sin(\psi - u°) + \sin(\psi - U)\cos(\psi - u°)\big)R^{-(n+1)}}{mT + \Theta + mM\big(n\cos(\psi - U)\cos(\psi - u°) - \sin(\psi - U)\sin(\psi - u°)\big)R^{-(n+1)}}$$

where in the denominator of the element, which contains the factor $R^{-(n+1)}$, we can suppress or affirm, with just the same correctness:

$$tg(u-u^\circ) = \frac{mM}{mT+\Theta}\left(n\cos(\psi-U)\sin(\psi-u^\circ)+\sin(\psi-U)\cos(\psi-u^\circ)\right)R^{-(n+1)} = FR^{-(n+1)}.$$

However, when we want to take into account the further terms, then it is clear, that $tg(u-u^\circ)$ can be evolved into a series of the following form:

$$tg(u-u^\circ) = FR^{-(n+1)} + F'R^{-(n+2)} + F''R^{-(n+3)} + \ldots,$$

in which, as a slight reflection shows, the coefficients *F*, *F'*, *F'''* and so forth up to the coefficients of the power $R^{-(2n+1)}$ respectively result inclusively from

$$\frac{f}{mT+\Theta}, \frac{f'}{mT+\Theta}, \frac{f''}{mT+\Theta}, \text{and so forth,}$$

by means of changing *u* into *u°* from the following term on, however, new elements will enter, which for our purpose we need not pursue more precisely. For the rest, it is manifest that *u* – *u°* can be evolved into a series of a similar form, which, up to the power $R^{-(3n+2)}$, coincide with the series for $tg(u-u^\circ)$.

18.[***]

It is now clear that, if the second needle is set up at different points on the same straight line, so that ψ and *U* retain their value, while *R* alone is changed, and the deviation of the moveable needle from the equilibrium position, whereby the second needle is displaced, namely, the angle *u* – *u°* is observed, [and] from this it follows that the values of the coefficients *F*, *F'*, *F''* and so forth, as many as are still significant, can be ascertained through elimination; by this means we will obtain the equation:

$$\frac{M}{T} = \left(1+\frac{\Theta}{Tm}\right)\frac{F}{n\cos(\psi-U)\sin(\psi-u^\circ)+\sin(\psi-U)\cos(\psi-u^\circ)}$$

in which the value of the magnitude $\frac{\Theta}{Tm}$ can be found by the method we demonstrated in section 8. For a more convenient demonstration, however, it will be useful to observe the following:

I. In place of the comparison of *u* with *u°* it is preferable to compare the two opposite deviations with each other, by reversing the position of the second needle, namely, so that *R* and ψ remain unchanged and the angle *U* is increased by two right angles. If the values of *u* corresponding to these positions are denoted by *u'* and *u''*, then in the case of complete symmetry *u''* precisely = *-u'*, if *u°* were = *0*. But it is superfluous, to anxiously keep to these conditions, since it is clear that *u'* and *u''* are determined by similar series, in which the first terms have <precisely> opposite values, and further also $\frac{1}{2}(u'-u'')$, likewise by a similar series, in which the coefficient of the first term is <precisely> = *F*.

[***] S. Anm. 5, Ende.

II. It will be still better, always to combine every four experiments, also after the angle ψ is changed by two right angles or the distance R has been taken on the other side. If the two latter experiments correspond to the values u''' and u'''', then the difference $\frac{1}{2}(u'''-u'''')$ will be expressed by a similar series, whose first term likewise will have a coefficient = F. It should be noted (which follows readily from the foregoing) that, if n were an odd number, the coefficients F, F'', F'''', and so forth would be precisely the same to infinity in any series for $u' - u°$ and $u''' - u°$, and the coefficients F', F''', F''''' and so forth would be precisely opposite to infinity, and the same for $u'' - u°$ and $u'''' - u°$, so that in the series for $u' - u'' + u''' - u''''$ the alternate term would disappear. But in the case of actual reality, where $n = 2$, such a relationship between the series for $u' - u°$ and $u''' - u°$ is, generally speaking, not strictly present, since, for the power R^{-6}, coefficients which are not precisely opposite already arise; however, it can be shown, that for this term as well a complete cancellation occurs in the combination $u' - u'' + u''' - u''''$, so that $tg\frac{1}{4}(u'-u''+u'''-u'''')$ has the form:

$$LR^{-3} + L'R^{-5} + L''R^{-7} + \ldots$$

or, more generally, if we leave the value of n undetermined for the time being, the following form:

$$LR^{-(n+1)} + L'R^{-(n+3)} + L''R^{-(n+5)} + \ldots,$$

where $L = F$.

III. It will be useful to choose the angles Ψ and U in such a way that small errors occurring in the process of measurement itself, do not significantly change the value of F. For this purpose, the value of U for a given value of Ψ must be posited in such a way, that F will be a maximum; namely, it must be

$$ctg(\psi - U) = ntg(\psi - u°).$$

Then

$$F = \pm\frac{mM}{mT+\Theta}\sqrt{(nn\sin(\psi - u°)^2 + \cos(\psi - u°)^2)}.$$

The angle Ψ is to be chosen, however, such that this value of F becomes either a maximum or a minimum: the former occurs for $\psi - u° = 90°$ or 270°, in which case $F = \pm\frac{nmM}{mT+\Theta}$, the latter occurs for $\psi - u° = 0$ or 180°, where $F = \pm\frac{mM}{mT+\Theta}$.

19.

Hence two methods are available which are best suited to carrying out our task. Their elements are shown in the following schema.

First Method.

The midpoint and the axis of the second needle lie in the straight line perpendicular to the magnetic meridian.***)

Deflection	Position of the needle		Midpoint toward	North pole toward
$u = u'$	$\psi = 90°$	$U = 90°$	East	East
$u = u''$	$\psi = 90$	$U = 270$	East	West
$u = u'''$	$\psi = 270$	$U = 90$	West	East
$u = u''''$	$\psi = 270$	$U = 270$	West	West

Second Method.

The midpoint of the second needle lies in the magnetic meridian.

Deflection	Position of the needle		Midpoint toward	North pole toward
$u = u'$	$\psi = 0°$	$U = 270°$	North	West
$u = u''$	$\psi = 0$	$U = 270$	North	East
$u = u'''$	$\psi = 180$	$U = 90$	South	West
$u = u''''$	$\psi = 180$	$U = 90$	South	East

If we set $\frac{1}{4}(u'-u''+u'''-u'''') = v$ and $tgv = LR^{-(n+1)} + L'R^{-(n+3)} + L''R^{-(n+5)} +$ and so forth, then for the first method

$$L = \frac{nmM}{mT + \Theta},$$

for the second

$$L = \frac{mM}{mT + \Theta}.$$

20.

From the theory of elimination it will easily be concluded, that the calculation is more imprecise because of the unavoidable errors in observation, the more coefficients must be determined by elimination. Therefore the method set forth in section 18, II is greatly to be prized, because it suppresses the coefficients $R^{-(n+2)}$, $R^{-(n+4)}$. In the case of complete symmetry, these coefficients would of course fall out by themselves, but it would be too uncertain to rely upon that case occurring. Moreover, a small deviation from the symmetry would have a far lesser influence in the first method than in the second, and if at least care is taken that point h', from which the distances will be measured, lies with sufficient precision in the magnetic meridian going through point h, there will scarcely be a significant difference between $u' - u''$ and $u''' - u'''$. However, things stand otherwise in the second method,

***) More precisely, to the vertical plane, which corresponds to the value $u = u°$, i.e. in which the magnetic axis finds itself in equilibrium, when the second needle is not there. For the rest, in practice, the difference can be safely ignored, both because of the smallness and directly because of the relationships we discussed in section III above.

especially if the apparatus requires an eccentric suspension. This method, whenever space does not permit observations from both sides, will always attain far less precision. Moreover, the first method is also especially preferable, because it yields a value of L twice as large as the second, in the case of reality, $n = 2$. If, by the way, one wants to discard, as much as possible in the second method, the term dependent on $R^{-(n+2)}$ in the case of the eccentric suspension, point h' should be chosen, so that the midpoint of the needle (for $u - u°$) lies in the center between h and h': the computation which showed this, however, I must omit for the sake of brevity.

21.

In the foregoing computations, we have left the exponent n undetermined: during the days from June 24 to June 28, 1832 we carried out two series of experiments, under extension to the greatest distances as the room permitted, through which it is shown most intelligibly, which values Nature demands. In the first series, the second needle (according to the method in Section 19) was placed in a straight line perpendicular to the magnetic meridian, in the second series, the midpoint of the needle was itself placed in the meridian. Here is an overview of these experiments, in which the distances R are expressed in fractions of meters and the values of the angle $\frac{1}{4}(u'-u''+u'''-u'''')$ are denoted for the first series by v, for the second by v'.

R	v	v'
1.1m		1° 57‘ 24.8‘‘
1.2		1 29 40.5
1.3	2° 13‘ 51.2‘‘	1 10 19.3
1.4	1 47 28.6	0 55 58.9
1.5	1 27 19.1	0 45 14.3
1.6	1 12 7.6	0 37 12.2
1.7	1 0 9.9	0 30 57.9
1.8	0 50 52.5	0 25 59.5
1.9	0 43 21.8	0 22 9.2
2.0	0 37 16.2	0 19 1.6
2.1	0 32 4.6	0 16 24.7
2.5	0 18 51.9	0 9 36.1
3.0	0 11 0.7	0 5 33.7
3.5	0 6 56.9	0 3 28.9
4.0	0 4 35.9	0 2 22.2

Even a superficial glance shows that for larger values, the numbers in the second column nearly twice as big as the numbers of the third, and also the numbers in each row approximate the inverse of the cube of the distances, so that no doubt can remain as to the accuracy of the value $n = 2$. In order to confirm this law still further by means of specific experiments, we have dealt with all the numbers according to the method of least squares, from which the following values of the coefficients emerged:

$$tgv = 0.086870R^{-3} - 0.002185R^{-5}$$
$$tgv' = 0.043435R^{-3} + 0.002449R^{-5}.$$

The following overview shows the comparison of the values computed by this formula with the observed values.

Computed values.				
R	v	Difference	v‘	Difference
1.1m			1° 57‘ 22.0‘‘	+2.8‘‘
1.2			1 29 46.5	-6.0
1.3	2° 13‘ 50.4‘‘	+0.8‘‘	1 10 13.3	+6.0
1.4	1 47 24.1	+4.5	0 55 58.7	+0.2
1.5	1 27 28.7	-9.6	0 45 20.9	-6.6
1.6	1 12 10.9	-3.3	0 37 15.4	-3.2
1.7	1 0 14.9	-5.0	0 30 59.1	-1.2
1.8	0 50 48.3	+4.2	0 26 2.9	-3.4
1.9	0 43 14.0	+7.8	0 22 6.6	+2.6
2.0	0 37 5.6	+10.6	0 18 55.7	+5.9
2.1	0 32 3.7	+0.9	0 16 19.8	+4.9
2.5	0 19 2.1	-10.2	0 9 38.6	-2.5
3.0	0 11 1.8	-1.1	0 5 33.9	-0.2
3.5	0 6 57.1	-0.2	0 3 29.8	-1.0
4.0	0 4 39.6	-3.7	0 2 20.5	+1.7

22.

The foregoing experiments were mainly undertaken with the intention of securing the law of magnetic action against any doubt, and further, of examining how many terms of the series are to be taken into account, and what degree of precision the experiments permit. They have shown that, if we do not make the distances smaller than four times the length of the needle, two terms suffice[****]. Furthermore, the differences which the computation has yielded, may no way be simply held to be observational errors: several precautionary measures, from whose application a greater conformity may be expected, were at that time not yet in readiness. These include corrections for the hourly variability of the intensity of the terrestrial magnetism, of which heed must be taken in using another needle for comparison, according to the method we have spoken of in section 10. However, so that one gets to know the value of the terrestrial magnetism, insofar as it can be inferred from these experiments, we append a synopsis of the remaining experiments in this group.

The value of the fraction $\frac{\Theta}{Tm}$ for the first needle and the thread on which it hangs, is ascertained by the method described in section 8 to be $= \frac{1}{251.96}$.

From this comes the result:

$$\frac{M}{T} = 0.0436074.$$

[****] The length of the needles used in these experiments was approximately *0.3 m*; if we had tried to include the term R^{-7} in the computation, the precision would have been decreased rather than increased.

This number is based on the meter as the unit of length. If we prefer to take the millimeter, this number should be multiplied by the cube of 1,000, so that

$$\frac{M}{T} = 43607400.$$

For the second needle, experiments were performed on June 28 resembling those we have described for another needle in section 11; millimeters, milligrams, and the seconds of the mean sun time are taken as units, resulting in:

$$TM = 135{,}457{,}900$$

and from this, by eliminating the magnitude *M*,

$$T = 1.7625.$$

23.

If experiments are performed to determine the absolute value *T* of the terrestrial magnetism, it is of great significance to take care that the entirety of these experiments be completed within not too long a time span, so that no significant change in the magnetic state of the needles used in the experiments is to be feared. It is recommended, that in observing the deviations of the movable needle, the first procedure in section 20 alone be applied, after only two different distances are appropriately chosen, assuming that two terms of the series suffice. We choose as an example one of several applications of this method, namely, the one to which the most exacting care was devoted, by measuring the distances with microscopic precision.

The experiments were performed on September 18, 1832 with two apparatus, which we wish to denote by *A* and *B*, and specifically with three needles, denoted 1, 2, 3. Needles 1 and 2 are the same ones referred to as first and second in section 11. The experiments are divided into two parts.

First the simultaneous oscillations of needle 1 in apparatus *A* and needle 2 in apparatus *B* were observed. As the duration of an oscillation, reduced to infinitely small amplitudes, resulted

for needle 1....... 15.22450"
for needle 2....... 17.29995",

the former for 304 oscillations, the latter for 264 oscillations.

Next, needle 3 was suspended in apparatus *A*, while needle 1 was placed in the straight line perpendicular to the magnetic meridian, eastward and also west and on both sides in a double manner, and the deviation of needle 3 for the single position of needle 1 was observed. This experiment, which was repeated for two different distances *R*, gave the following values of the angle *v*, whose significance is the same as in section 19 and section 21:

R = 1.2m	*v = 3° 42‘ 19.4‘‘*
R‘ = 1.6m	*v‘ = 1° 34‘ 19.3‘‘.*

During this experiment, too, the oscillations of needle 2 in apparatus *B* were observed. The mean time corresponded to the value, computed from 414 oscillations, of the oscillation duration for infinitely small curves = 17.29484.

The time periods were observed on a clock, whose daily retardation was 14.24". If *M* and *m* denote the momentum of the free magnetism for needle 1 and 3, and Θ the torsion constant of the threat in apparatus *A*, while it held needle 1 or 3 (whose weight is almost identical), then we have:

$$\frac{\Theta}{TM} = \frac{1}{597.4},$$

as in section 11,

$$\frac{\Theta}{TM} = \frac{1}{721.6};$$

because needle 3 was more strongly magnetized than needle 1.

The inertial moment of needle 1 was already known from earlier experiments (see section 11), which had yielded: *K = 4228732400*, in which millimeter and milligrams were taken as the units.

The change in the thermometer in both rooms where the apparatuses were installed, was so slight during the entire period of the experiments, that it is superfluous to consider it.

We now wish to proceed to the computation of these experiments, in order to infer from it the intensity *T* of the terrestrial magnetism. The variation in the oscillations of needle 2 indicate a slight change in this intensity: hence, in order for us to speak of a definite value, we will reduce the observed duration of the oscillations of the first needle to the mean state of the terrestrial magnetism during the second part of the observations. This duration requires still another reduction on account of the retardation of the clock, and a third on account of the torsion of the thread. In this way the reduced duration of one oscillation produced:

$$= 15.22450 \leftrightarrow \frac{17.29484}{17.29995} ? \frac{86400}{86385.76} ? \sqrt{\frac{598.4}{597.4}} = 15.23500'' = t.$$

From this follows the value of the product

$$TM = \frac{\pi\pi K}{tt} = 179770600.$$

The slight difference between this value and the one in section 11 found on September 11, is to be ascribed to the change in the terrestrial magnetism and also to the change in the magnetic state of the needle.

From the observed deviations we derive:

$$F = \frac{R'^5\, tgv' - R^5 tgv}{R'R' - RR} = 113056200,$$

if we take the millimeter as the unit, and thence

$$\frac{M}{T} = \frac{1}{2}F\left(1+\frac{\Theta}{Tm}\right) = 56606437.$$

Finally, the comparison of this number with the value of *TM* yields

$$T = 1.782088$$

as value of the intensity of the horizontal terrestrial magnetic force on September 18 at 5 o'clock.

24.

The foregoing experiments were made at the observatory, where the location for the apparatus was sought out in such a way that iron was kept away from the vicinity as much as possible. Nevertheless, it cannot be doubted, that the iron masses, which were abundantly distributed in the walls, windows, and doors of the building, indeed, even the iron components of the large astronomical instruments, in which magnetism is induced by the terrestrial magnetic force, exert an in no way insignificant effect on the suspended needles. The forces arising from this alter not only the direction, but also the intensity of the terrestrial magnetism by a small amount, and our experiments do not provide the pure value of the intensity of the terrestrial magnetism, but the value modified for the locus of apparatus *A*. As long as the iron masses remain in place and the elements of the terrestrial magnetism itself (namely, the intensity and direction) do not change very considerably, this modification must remain significantly constant, but what magnitude is reached, is indeed unknown up to now; nevertheless, I am scarcely inclined to believe, that this exceeds one or two hundredths of the total value. However, it should not be difficult to determine the magnitudes, at least approximately, by means of experiments, namely, through observation of the simultaneous oscillations of two needles, of which the one would be at the usual locus of observation, the other meanwhile at a rather great distance from the building and from other disturbing masses of iron, and which then would have to exchange places. But up to now it was not possible to carry out this experiment. The safest remedy, however, must be to construct a special building devoted to magnetic observations, which as a result of royal graciousness will soon be erected, and from whose construction iron is to be altogether excluded.

25.

In addition to the experiments described, we have carried out many other similar experiments, even if we took less care with the earlier ones. It will, nevertheless, be of interest here to assemble the results in a table, in which, however, those results are uebergangen, which before the installation of the more refined apparatus, were obtained by means of other, cruder adjuncts to needles of the most disparate dimensions, although all of them have provided at least an approximation of reality. Through repeated experiments, the following successive values of *T* resulted:

Number	Time, 1832	*T*
I	May 21	1.7820
II	May 24	1.7694
III	June 4	1.7713

IV	June 24-28	1.7625
V	July 23, 24	1.7826
VI	July 25, 26	1.7845
VII	September 9	1.7764
VIII	September 18	1.7821
IX	September 27	1.7965
X	October 15	1.7860

Experiments V-IX were performed together in the same place, but I-IV in different places; experiment X is actually a composite, since the deviations were observed at the usual place, but the oscillations at another place. In experiments VII and VIII, an almost equal precision was applied; in experiments IV, V, VI and X, however, a somewhat lesser precision, and in experiments I-III a far lesser. In experiments I-VIII, in fact, different needles were used, although they had the same weight and the same length (the weight was between 400 *g* and 440 *g*); experiment X, in contrast, used a needle, whose weight was 1,062 *g* and whose length was 485 *mm*. Experiment IX was only undertaken in order to see what degree of precision can be attained by means of a very small needle. The weight of the needle employed was only 58 *g*, otherwise, however, the precision was no less than in experiments VII and VIII. There exists no doubt, that the refinement of the observations will be significantly increased, when still heavier needles are used, e.g. needles weighing up to 2,000 or 3,000 *g*.

26.

If the intensity *T* of the terrestrial magnetic force is expressed by a number *k*, then this is based on a certain unit *V*, namely, a force that is the same as the other force, whose connection with the other directly given units is certainly contained in the above, but in a somewhat complicated way: therefore, it will be worth the trouble, to demonstrate this connection here de novo, in order to show with elementary clarity, what change the number *k* undergoes, if we start with other units instead of the original units.

To establish unit *V*, one must proceed from the unit of free magnetism *M*[****)] and the unit of distance *R*, and we make *V* equal to the force exerted by *M* at distance *R*.

As unit *M*, we have assumed that quantity of magnetic flux, which elicits, in an equally large quantity *M* at distance *R*, a motive force (or, if one prefers, a pressure), which is equivalent to that *W*, which serves as the unit, i.e., equivalent to the force, which the acceleration *A* taken as the unit exerts on the mass *P* taken as the unit.

To establish unit *A*, two paths are available: namely, *A* can either be derived from a similar immediately given force, e.g. from the gravity at the locus of observation, or from the effect of *A*, which manifests itself in the movement of bodies. The second method, which we followed in our calculations, requires two new units, namely, the unit of time *S* and the unit of velocity *C*, so that that accelerating force is taken as the unit which, if it acts in time *S*, elicits the velocity *C*; finally, for the latter, that velocity is assumed, which corresponds to uniform movement through space *R* in time *S*.

Thus it is clear, that the unit *V* depends on three units, either *R, P, A* or *R, P, S*.

[****)] It will scarcely be necessary to recall, that the earlier denotation given to the letters stops here.

If we now assume that in place of units V, R, M, W, A, P, C, S, others are taken: V', R', M', W', A', P', C', S', which are connected with each other like the previous ones, and that the terrestrial magnetism will be expressed by the use of the metric V', then it is to be investigated, how this [k'] relates to k.

If we say:

$$V = vV'$$
$$R = rR'$$
$$M = mM'$$
$$W = wW'$$
$$A = aA'$$
$$P = pP'$$
$$C = cC'$$
$$S = sS',$$

then v, r, m, w, a, p, c, s will be absolute numbers, and

$$kV = k'V' \text{ or } kv = k',$$

$$v = \frac{m}{rr},$$

$$\frac{mm}{rr} = w = pa,$$

$$a = \frac{c}{s},$$

$$c = \frac{r}{s}.$$

From the combination of these equations we find:

I.	$k' = k\sqrt{\frac{p}{rss}}$
II.	$k' = k\sqrt{\frac{pa}{rr}}.$

As long as we keep to the path we have followed in our observations, we are obliged to use the first formula: if, for example, we assume the meter and the gram as units instead of millimeters and milligrams, then r will be $= \frac{1}{1000}$, $p = \frac{1}{1000}$, thus $k = k'$; if the Paris line and the Berlin pound [are the units], then we will have: $r = \frac{1}{2.255829}$, $p = \frac{1}{467711.4}$, consequently

$$k' = 0.002196161\, k,$$

and thus, e.g., experiment VIII gives the value $T = 0.0039131$.

If one prefers to follow another path, and assumes gravity as the unit of the accelerating force, then, for the Goettingen observatory, *a* will $= \frac{1}{9811.63}$; then, if we keep the millimeter and milligram, the numbers *k* are to be multiplied by 0.01009554 and the alteration of the former unit is to be handled according to formula II.

27.

The intensity of the horizontal terrestrial magnetic force *T* is to be multiplied by the secant of the inclination, in order to yield the total intensity. The fact that the inclination is variable in Goettingen and has undergone a diminution in the recent period, has been shown by the observations of Humboldt, who in the month of December 1805 found the value of 69° 29', but in the month of September 1826 found 68° 22' 26". Likewise I found on June 23, 1832, with the help of the same inclinatorium which der s. Mayer once used, 68° 22' 52", which seems to signify a retardation in the decline; yet I am inclined to put less trust in this observation, not only because of the imperfection of the instrument, but also because of the circumstance, that the observation performed in the observatory was not adequately secured from disturbance by iron masses. However, these factors will also gain greater precision in future.

28.

In this treatise, we have followed the generally accepted manner of explaining magnetic phenomena, not only because it is completely adequate, but also because it progresses in far simpler computations than the other view, which ascribes to magnetism, galvano-electric orbits around the particles of the magnetic body; it was our intention, neither to confirm nor to refute this view, which, to be sure, recommends itself in several respects; this would have been inopportune, since the law of mutual effect among the elements of such orbits does not yet seem to be sufficiently investigated. Whichever view is accepted, in the future as well, for magnetic and electromagnetic phenomena: the first theory [of orbits] must always lead to the same results as the usual theory, and what is developed on the basis of the usual theory in the foregoing treatise, will be in a position to be changed merely in form, but not in essentials.

END OF DRAFT