Physics of magnetically confined plasmas

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(Published 28 January 2005)

The physics of magnetically confined plasmas has had much of its development as part of the program to develop fusion energy and is an important element in the study of space and astrophysical plasmas. Closely related areas of physics include Hamiltonian dynamics, kinetic theory, and fluid turbulence. A number of topics in physics have been developed primarily through research on magnetically confined plasmas. The physics that underlies the magnetic confinement of plasmas is reviewed here to make it more accessible to those beginning research on plasma confinement and for interested physicists.

CONTENTS

I. INTRODUCTION

A plasma is a gas in which charged particles are of sufficient importance for the gas to be a good electrical conductor. Ordinary matter becomes ionized and forms a plasma at temperatures above about 5000 K, and most of the visible matter in the universe is in the plasma state. The high electrical conductivity implies that currents can flow in a plasma. These currents can interact with magnetic fields to produce the forces that are needed for confinement.

The physics of plasma confinement using magnetic fields has been driven intellectually by the program to develop fusion energy. Fusion has provided a focus for the research, but much of the physics that has been developed has far broader applications—most obviously to space and astrophysical plasmas. In addition, the physical insights and concepts are of intrinsic scientific importance.

A number of topics in physics have had their primary development through research on plasma confinement. These include:

(1) The relation between the field lines of divergence-free fields and Hamiltonian mechanics.

(2) The constraints of magnetic helicity conservation on the rapid evolution of magnetic fields.

(3) Collisionless relaxation phenomena.

(4) Simplified kinetic equations that are based on adiabatic invariants of classical mechanics.

(5) The theory of small-amplitude and short-wavelength turbulence, called microturbulence.

(6) The experimental observation and theoretical explanation of transport barriers where plasma microturbulence is stabilized by a strong gradient in the plasma flow.

(7) The demonstration for shear Alfvén waves that continuum modes are rapidly damped but discrete modes exist that are weakly damped and can be de-

stabilized by particles with a speed near the phase velocity of the waves. Shear Alfvén waves propagate by twisting the magnetic-field lines that are embedded in a plasma.

One of the goals of this review is to make these and other topics in the physics of magnetic confinement accessible to a broader audience.

The physics of magnetic confinement is covered in a number of books. Probably the most complete is *Tokamaks* (Wesson, 2004), which emphasizes the relation between theory and experiment. *Fundamentals of Plasma Physics and Controlled Fusion* (Miyamoto, 1997) gives the details of many of the classic derivations. *The Theory of Toroidally Confined Plasmas* (White, 2001) offers insights on a number of fundamental phenomena, and a standard textbook is *Introduction to Plasma Physics* (Goldston and Rutherford, 1995). The International Thermonuclear Experimental Reactor (ITER) design team summarized the physics of tokamaks in a series of articles in the December 1999 issue of the journal *Nuclear Fusion*. The first of these articles is an overview (ITER Physics Basis Editors *et al.*, 1999).

Despite the number of books and articles written on plasma confinement a review of the fundamental physics that includes derivations of major results has been lacking up till now. This review is designed to fill that gap while being accessible to the general physics community.

Several important types of confined plasmas are not covered in this review. Magnetically confined plasmas occur in many space and astrophysical situations, and much of the physics is shared with laboratory plasmas. However, this review does not provide sufficient context for understanding the breadth of applications to space and astrophysical plasmas. Some texts that do provide such context include *Basic Space Plasma Physics* by Baumjohann and Treumann (1996), *Physics of Space Plasmas* by Parks (1991), and *Plasma Astrophysics* by Tajima and Shibata (1997). Bisnovatyi-Kogan and Lovelace (2001) have written a major review of the plasma issues associated with accretion disks, and Ferrari (1998) has reviewed extragalactic jets. Laboratory plasmas in which all of the magnetic-field lines in the plasma intersect the chamber walls are also not covered. Plasmas either flow out of such open systems at a thermal speed or have a confinement time comparable to a collision time. The fast Z pinch is an open plasma confinement system, which has prominence from its use as a driver for inertial fusion (Ryutov *et al.*, 2000). Open plasma confinement systems are also important for materials processing (Lieberman and Lichtenberg, 1994).

Other major plasma topics that are not covered are waves, which are of particular importance for plasma heating, and the methods of measuring plasma parameters, which is the topic of plasma diagnostics. Standard books on waves in plasmas are those of Stix (1992); Brambilla (1998); and Swanson (2003). Plasma diagnostics are based on a broad range of physics principles. The standard text is *Principles of Plasma Diagnostics* (Hutchinson, 2003), and reviews have been written by Gentle (1995) and the ITER Physics Export Group on Diagnostics *et al.* (1999) in the *Nuclear Fusion* series on ITER.

What is covered in this review is the physics of the magnetic confinement of plasmas that are flowing slowly compared to sonic speeds and at each point in the plasma are close to thermodynamic equilibrium, which implies the confinement is long compared to collision times. Although the fundamental topic is not fusion, an understanding of the basic features of magnetically confined fusion plasmas is useful for placing research on plasma confinement in context. These features are explained in Sec. II.

A review of the physics of magnetically confined plasmas contains too much material to be absorbed at a uniform level. Therefore, paragraphs marked by bullets are given near the beginning of most sections. Readers are encouraged to read just the bulleted sections of the entire review before working through the details of the sections. The main idea in each section can be determined by reading the first sentence of each paragraph. A result is generally given before the derivation, so the derivations can be skipped. However, the derivations are sufficiently complete that a reader should be able to reconstruct them using a table of identities of vector calculus. The review is designed to be read at various levels of detail, and that is the way it should be read.

II. FUSION ENERGY

- The goal of research on fusion energy is a commercially viable source of energy. The primary research effort is on the fusion of two isotopes of hydrogen, deuterium and tritium, to form an isotope of helium, an alpha particle, and a neutron.
- Fundamental considerations imply that in a fusion power plant the particle distribution functions would be close to local Maxwellians with a temperature of about 20 keV and a density of approximately 2×10^{20} nuclei/m^3, and the plasma would have the shape of a torus with the minor radius of the torus a few meters in length.
- Technical limits on the magnitude of the magnetic field that can be used for confining fusion plasmas make the efficiency of the utilization of the magnetic field a central issue.
- Most of the freedom in the design of fusion plasmas is in the plasma shape, which means freedom in the design of the coils that surround the plasma. About 50 shape parameters can be controlled, though only about four of these are consistent with an axisymmetric torus.

Fusion is a potential source of energy, which is essentially unlimited in quantity and produces no greenhouse gases. The fusion reaction that appears technically easiest to harness is between two isotopes of hydrogen, deuterium and tritium, with the product being ordinary helium (an alpha particle) and a neutron. Deuterium occurs naturally in water in sufficient abundance as to be an essentially unlimited resource. Tritium has a half-life of 12 years and must be produced through an interaction of the neutron with isotopes of lithium. Consequently, the fuel for fusion is deuterium and lithium. The waste products of fusion energy production need not be radioactive in principle, but in practice some radioactive products will be produced. The level, lifetime, and nature of the radioactive waste are dependent on the use of appropriate materials and the cleverness of the design. The fusion reaction is easily terminated, so a runaway reaction is not a safety concern. Sheffield (1994) has written a general review of fusion systems and Baker *et al.* (1998) have discussed fusion systems with a focus on the important issue of the materials that would be used in fusion power plants.

The development of fusion power is paced by both physics and engineering considerations. A magnetically confined plasma with an essentially self-sustaining fusion burn has not yet been produced, but this is thought to be essentially an issue of the size of the experiments that have been undertaken. A number of proposals have been made for a burning plasma experiment; the most ambitious currently under consideration is the International Tokamak Experimental Reactor (ITER; Aymar, 2000). Many studies have claimed fusion power could be economically competitive, but the compatibility of the physics and the engineering with energy at an acceptable cost will remain a primary issue until demonstration power plants are built.

Fundamental considerations imply that toroidal fusion power systems would have a power output of order a gigawatt. First, the structure surrounding the plasma must have a minimum thickness, about 1.5 m, for the fusion neutrons to convert lithium into tritium and heat as well as shield the external world from radiation. Second, until material limitations arise at a power density of several megawatts per square meter of chamber wall area, power becomes cheaper the higher the power density. These two considerations plus the need for a toroidal plasma, which is discussed in Sec. III, imply that fusion energy is most economical in units of approximately one gigawatt of electrical power. A larger or smaller unit size would imply a larger or a smaller aspect ratio of the toroidal plasma.

The electrostatic repulsion of the deuterium and the tritium nuclei sets the energy scale at which fusion reactions occur, which is a thermal energy of a few tens of kilovolts, which is a few hundred million degrees kelvin. For the plasma to maintain its burning state, the rate of production of high-energy alpha particles must be sufficient to offset the energy losses of the plasma through electromagnetic radiation and thermal transport through diffusion; see Sec. VI. Since neutrons are electrically neutral they have no significant interaction with the plasma or the magnetic field. The maximum rate of energy production, at fixed plasma pressure, occurs at a temperature of approximately 20 kV, or 2.3×10^8 K. For a steady burn at a temperature of 20 kV, the number density of nuclei times the energy confinement time must equal 2×10^{20} (nuclei/m^3) sec. At that temperature the deuterium and tritium are completely ionized and, therefore, form a plasma. These considerations would be changed if the plasma reaction rate could be substantially increased. Fisch and Herrmann (1994) have proposed a method for increasing the reaction rate by enhancing the transfer of energy from the fusion alphas to the fusing ions.

The required power density on the chamber walls for economic electric power determines the plasma density, about 2×10^{20} nuclei/m^3, and the plasma pressure, about ten atmospheres.

Plasmas of interest for magnetically confined fusion are in a paradoxical collisionality regime. The rate for Coulomb collisions, which relax the particle distribution functions to Maxwellians, times the energy confinement time is about 100 for the ions and 10 000 for the electrons. The rapidity of collisions implies the distribution functions are close to Maxwellian. However, the mean free path of the thermal particles, about 10 km, is enormous compared to the size of the plasma. This paradoxical collisionality regime places requirements on the quality of the collisionless particle trajectories in order to have adequate confinement; see Sec. VI.E.1.

Three considerations set a minimal level for the magnetic-field strength, which is a few tesla.

(1) The magnetic-field pressure must be significantly larger than the plasma pressure to provide a stable force balance; see Sec. V.

(2) A charged particle moves in a circle about magnetic-field lines, and the radius of this circle, the gyroradius ρ, must be sufficiently small that the high-energy alpha particles remain in the plasma and heat it; see Sec. VI.E.1.

(3) The thermal transport coefficients, which are reduced by an increase in the magnetic-field strength, must be sufficiently small to obtain the required energy confinement time; see Sec. VI.F.

Engineering considerations make it desirable to use the lowest magnetic-field strength that is consistent with the physics requirements. The technical limitations that arise for magnetic fields larger than about 10 T make magnetic confinement fusion more difficult than if higher fields could be used. If higher fields were available, power losses from electron cyclotron radiation, which set an upper limit on the electron temperature, would be of increasing importance. A discussion of electron cyclotron losses is given in Sec. VI.A and by Albajar, Bornatici, and Engelmann (2002).

In addition to fusion using magnetic fields to confine the plasma, a large research program is being pursued on inertial confinement fusion. The physics is described in the book *Inertial Confinement Fusion* by Lindl (1998). Inertial confinement means the plasma is confined for a time of order a/C_s, with C_s the speed of sound and a the plasma radius. Confinement is determined by the balance between the inertial and the pressure forces. Magnetically confined plasmas are in essentially a static balance between the magnetic and the pressure forces. The characteristic plasma density during the burn period of an inertially confined plasma is of order 10^{32} particles/m^3, which is about 12 orders of magnitude larger than the density of a magnetically confined fusion plasma. The size of the pellets that are imploded in inertial confinement has a millimeter scale, while magnetic fusion plasmas have a scale of meters. However, the central plasma temperatures are roughly the same in magnetically and inertially confined fusion plasmas in order to obtain an optimal reaction rate.

The focus of much of the research on magnetic fusion during the last decade has been on understanding and extrapolating of transport processes in confined plasmas. Energy transport means the loss of energy by plasma processes rather than by electromagnetic radiation. Confined plasmas are usually, but not always, in a state in which microturbulence dominates the transport processes. Microturbulence (Sec. VI.F) means the fluctuations have a wave number that is greater than, or of order of, the inverse of an ion gyroradius ρ_i and an amplitude of order the ion gyroradius to system size $\rho_i/a < 10^{-2}$. The extrapolation of microturbulent transport from existing to future experiments and to fusion power plants is essential for the scientific planning of fusion research and is an issue in the feasibility of fusion power.

The focus of innovation in fusion research is on ways to obtain more control over fusion plasmas and on improved fusion systems. A plasma equilibrium is determined by the shape of the plasma, the magnetic-field strength, and the profiles of the plasma current and pressure; see Sec. V.A. In a power plant, economics dictates that only about 5% of the fusion power can be used for control. Consequently, the plasma current and pressure profiles are largely determined by internal plasma phenomena. The largest element of control that the designer has over the plasma and its performance is on the plasma shape. Control of the plasma shape is actually control over the design of the coils that surround the plasma. As shown in Sec. V.D.1, coil constraints limit the designer to about 50 shape parameters, of which only four (aspect ratio, ellipticity, triangularity, and squareness) are consistent with an axisymmetric torus. The axisymmetric tokamak (Sec. IV) is the most advanced plasma confinement configuration and is the configuration of choice for all major designs for experiments with a fusion burn. Careful consideration of the four axisymmetric shape parameters is known to be essential for attractive tokamaks. However, tokamaks set more than 90% of the available shape parameters to zero. The use of a larger set of shape parameters allows the designer to sidestep issues in tokamak design, such as current profile control, and is being studied under the topic of stellarator research; see Sec. IV. Other elements of plasma control that may be available include the density profile, through clever fueling techniques, and the interaction between the plasma edge and the surrounding walls.

III. MAGNETIC-FIELD LINES

- Near-Maxwellian plasmas are in force balance between the pressure and the magnetic forces, $\vec{\nabla}p = \vec{j}\times\vec{B}$.
- The magnetic-field lines confining a near-Maxwellian plasma must lie in the surfaces of constant pressure, $\vec{B}\cdot\vec{\nabla}p=0$. These surfaces can be spatially bounded only if they have the topological form of a torus. The pressure $p(\psi_t)$ is a function of ψ_t, which is the flux of the toroidally directed magnetic field that is enclosed by the pressure surfaces.

In plasmas of fusion interest, the ions and the electrons are in near-Maxwellian distributions, which implies the plasma has the pressure of an ideal gas, $p=nT$, where T is the temperature in energy units and $n=n_e+n_i$ is the sum of the number of electrons and ions per cubic meter. Plasma confinement implies a pressure gradient, and $\vec{\nabla}p$ is a force per unit volume. This force is balanced by the electromagnetic force produced by the cross product of the current density in the plasma and the magnetic field,

$$\vec{\nabla}p = \vec{j}\times\vec{B}. \tag{1}$$

Equation (1) gives the force equilibrium of a near-Maxwellian plasma and places fundamental constraints on magnetic confinement systems.

A confined near-Maxwellian plasma is constrained to have the topological form of a torus. Why is this? Equilibrium implies $\vec{B}\cdot\vec{\nabla}p=0$; a magnetic-field line must lie in a constant-pressure surface through its entire length. For example, a constant-pressure surface cannot be a sphere and satisfy Eq. (1). The magnetic-field lines in a constant-pressure surface resemble strands of hair. The hair on a topological sphere, such as a person's head, always has a crown, a place where the hair spirals out from a point. A crown is a point at which Eq. (1) cannot hold. A theorem of topology says a nonsingular vector field $\vec{B}(\vec{x})$ can be everywhere tangent to a spatially bounded function, namely, $p(\vec{x})$, in only one shape, the torus.

Since toroidal surfaces are central to the whole theory of plasma confinement, it is important to have a method for describing them as a basis for terminology. The simplest description uses (R,φ,Z) cylindrical coordinates with spatial positions given by

$$\vec{x}(r,\theta,\varphi) = R(r,\theta,\varphi)\hat{R}(\varphi) + Z(r,\theta,\varphi)\hat{Z}. \tag{2}$$

Expressed in terms of the Cartesian unit vectors, the *radial unit vector* of cylindrical coordinates is $\hat{R}(\varphi)=\hat{x}\cos\varphi+\hat{y}\sin\varphi$ and its derivative $d\hat{R}/d\varphi$ is the *unit vector* $\hat{\varphi}$. Simple circular toroidal surfaces are given by $R=R_o+r\cos(\theta)$ and $Z=-r\sin(\theta)$. The constant R_o is the *major radius*, r is the local *minor radius*, and $\epsilon=r/R_o$, which must be less than one, is the *local inverse* aspect ratio. A function, such as $\vec{x}(r,\theta,\varphi)$, which determines the

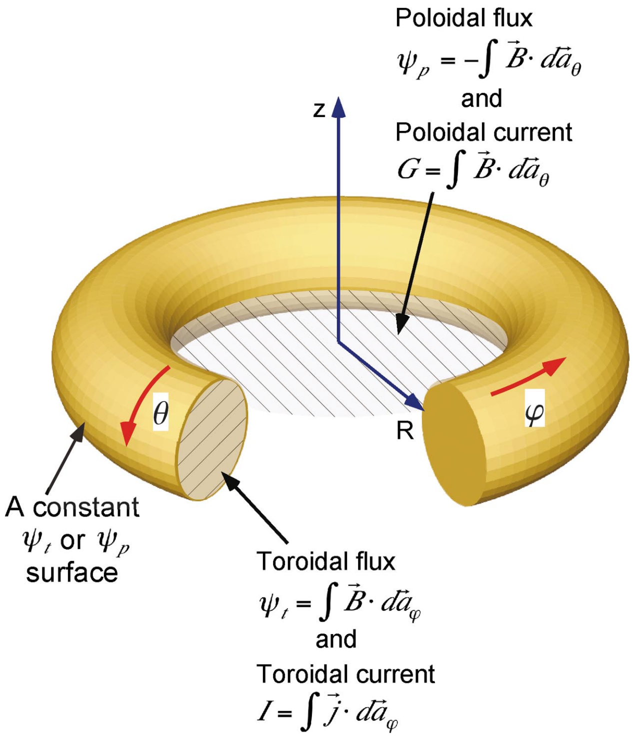

FIG. 1. (Color) Magnetic fluxes and currents defined using the cross-sectional area for the toroidal flux ψ_t and the current I and using the central hole of the torus for the poloidal flux ψ_p and the current G. The poloidal angle is θ and the toroidal angle is φ. (R,φ,Z) are ordinary cylindrical coordinates.

positions in space that are associated with three quantities (r,θ,φ), defines an (r,θ,φ) coordinate system. The Appendix gives the theory of general coordinates that is required for understanding this review. The polar angle of cylindrical coordinates, φ, is also the *toroidal angle*. The angle θ is called the *poloidal angle*, and r is a *radial coordinate* that labels the various toroidal surfaces.

A. Relation to Hamiltonian mechanics

- The field lines of a magnetic, $\vec{B}(\vec{x})$, or any other divergence-free field are the trajectories of a Hamiltonian. (A short tutorial on Hamiltonians is given below just after the bullets.) If $\varphi(\vec{x})$ is a toroidal angle, and $\vec{B}\cdot\vec{\nabla}\varphi \neq 0$, then the magnetic-field lines are given by a one-and-a-half-degree-of-freedom Hamiltonian, which is the poloidal magnetic flux $\psi_p(\psi_t,\theta,\varphi)$. The canonical momentum is the toroidal flux ψ_t, the canonical coordinate is a poloidal angle θ, and the canonical time is the toroidal angle φ; see Fig. 1. The full Hamiltonian system consists of the Hamiltonian, $\psi_p(\psi_t,\theta,\varphi)$, and the transformation function, $\vec{x}(\psi_t,\theta,\varphi)$, which gives the spatial location of each canonical coordinate point (ψ_t,θ,φ).
- When the magnetic-field lines lie in toroidal surfaces, the canonical coordinates can be chosen so that the poloidal flux is a function of the toroidal flux alone, $\psi_p(\psi_t)$. The twist of the field lines, which is called the *rotational transform*, is $\iota \equiv d\psi_p/d\psi_t$. When the rotational transform is a rational number $\iota = n/m$, the field lines close on themselves after n poloidal and m toroidal transits of the torus. The coordinates in which the poloidal flux is a function of the toroidal flux alone are called magnetic coordinates and trivialize the solution of the magnetic differential equation $\vec{B}\cdot\vec{\nabla}f = g$, which arises frequently in plasma physics.
- If a magnetic field $\vec{B}_0(\vec{x})$ that forms perfect surfaces is perturbed by a field $\delta\vec{B}$, then the magnetic surfaces can split to form islands (Fig. 2), where the rotational transform is a rational number, $\iota = n/m$. If $\delta\vec{B}\cdot\vec{\nabla}\psi_t/\vec{B}_0\cdot\vec{\nabla}\varphi = \Sigma b_{mn}\exp[i(n\varphi - m\theta_m)]$, the width of the island that splits the surface $\iota = n/m$ is proportional to $\sqrt{|b_{mn}|}$. If islands from different rational surfaces (ι different rational numbers) are sufficiently wide to overlap, the magnetic-field lines in that region will come arbitrarily close to every point in a volume of space rather than lying on surfaces. Such field-line trajectories are said to be stochastic. Plasma confinement is destroyed in regions of stochastic field lines.
- The opening of an island is a singular process in a toroidal plasma. Let $p_0(\vec{x})$ be any function that satisfies $\vec{B}_0\cdot\vec{\nabla}p_0 = 0$. Without an island $dp_0/d\psi_t$ is generally nonzero near the resonant rational surface $\iota = n/m$, but with an arbitrarily small island only the derivative $dp_0/d\psi_h$ can be nonzero, where the helical flux is defined by $d\psi_h \equiv (\iota - n/m)d\psi_t$.

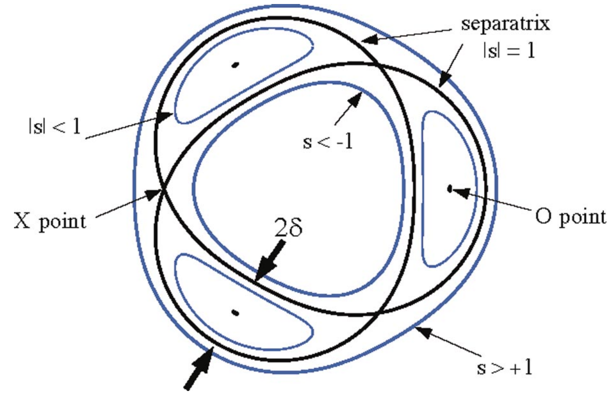

FIG. 2. (Color) The solution to Eq. (15) plotted for an $m=3$ magnetic island in the $\varphi=0$ plane. The surface $|s|=1$ is the island separatrix, and the limit $|s|\rightarrow 0$ gives the island O point. The half width of the island is δ.

The nested toroidal surfaces of magnetic-field lines are reminiscent of the tori formed by particle trajectories of an integrable Hamiltonian $H(p,x,t)$ that is periodic in time with T the period, $H(p,x,t+T) = H(p,x,t)$. Such Hamiltonians are said to have one-and-a-half degrees of freedom. An excellent reference for the Hamiltonian mechanics used in this review is the book *Regular* *and Chaotic Dynamics* (Lichtenberg and Liberman, 1992).

The Hamiltonian formulation of mechanics follows from Newton's equations of motion, $d\vec{p}/dt=-\vec{\nabla}V(\vec{x},t)$ and $d\vec{x}/dt=\vec{p}/m$. Let $H(\vec{p},\vec{x},t)=p^2/2m+V(\vec{x},t)$; then Newton's equations of motion are reproduced by the set of ordinary differential equations $d\vec{p}/dt=-\partial H/\partial\vec{x}$ and $d\vec{x}/dt=\partial H/\partial\vec{p}$. Any set of ordinary differential equations of this form is said to be Hamiltonian with $\vec{x}$ the canonical coordinate, $\vec{p}$ the canonical momentum, and t the canonical time, regardless of the functional form of the Hamiltonian, $H(\vec{p},\vec{x},t)$. If x^i are the components of the canonical coordinate $\vec{x}$, and p_i are the components of the canonical momentum $\vec{p}$, then Hamilton's equations are $dx^i/dt=\partial H/\partial p_i$ and $dp_i/dt=-\partial H/\partial x^i$. The superscripts number the components and are not powers. In Hamiltonian systems with one-and-a-half degrees of freedom $\vec{x}$ and $\vec{p}$ have only one component each.

Magnetic-field lines are the trajectories of a one-and-a-half-degree-of-freedom Hamiltonian (Kerst, 1964; Whiteman, 1977; Boozer, 1983; Cary and Littlejohn, 1983). A magnetic-field line moves through 3-space (x,y,z), so the same number of coordinates are involved as in the (p,x,t) space of Hamiltonian mechanics. The only problem is that Cartesian coordinates (x,y,z) are not the canonical coordinates for magnetic-field lines. To obtain canonical coordinates, one must first write the magnetic field in what is called the symplectic form (Boozer, 1983),

$$2\pi\vec{B}=\vec{\nabla}\psi_t\times\vec{\nabla}\theta+\vec{\nabla}\varphi\times\vec{\nabla}\psi_p, \tag{3}$$

with θ and φ arbitrary coordinates except $\vec{B}\cdot\vec{\nabla}\varphi\neq 0$. In practice, ψ_t is generally the magnetic flux of the toroidally directed field, θ is a poloidal, and φ is a toroidal angle (Fig. 1). When $\vec{B}\cdot\vec{\nabla}\varphi\neq 0$, points in 3-space can be described using (ψ_t,θ,φ) as coordinates, which means points in space are given as the function $\vec{x}(\psi_t,\theta,\varphi)$. Using (R,φ,Z) cylindrical coordinates [Eq. (2)], one defines points in space by giving R and Z as functions of the (ψ_t,θ,φ) coordinates, $\vec{x}(\psi_t,\theta,\varphi)=R(\psi_t,\theta,\varphi)\hat{R}(\varphi)+Z(\psi_t,\theta,\varphi)\hat{Z}$.

The (ψ_t,θ,φ) coordinates are the canonical coordinates of the magnetic-field lines. The mathematics of general coordinate systems is described in the Appendix. When $\vec{B}\cdot\vec{\nabla}\varphi\neq 0$, the Jacobian $\mathcal{J}$ of the canonical coordinates is finite, that is, $1/\mathcal{J}\equiv\vec{\nabla}\psi_t\cdot(\vec{\nabla}\theta\times\vec{\nabla}\varphi)=2\pi\vec{B}\cdot\vec{\nabla}\varphi$. In these coordinates, magnetic-field lines, which are the solutions to $d\vec{x}/d\tau=\vec{B}(\vec{x})$, are easily shown to have the canonical form

$$\frac{d\psi_t}{d\varphi}=-\frac{\partial\psi_p(\psi_t,\theta,\varphi)}{\partial\theta}; \tag{4}$$

$$\frac{d\theta}{d\varphi}=\frac{\partial\psi_p(\psi_t,\theta,\varphi)}{\partial\psi_t} \tag{5}$$

using Eqs. (A4) and (3). The canonical momentum is the toroidal magnetic flux, $\psi_t=\int\vec{B}\cdot d\vec{a}_\varphi$, which is the magnetic field integrated over the cross section of a constant-ψ_t torus (Fig. 1). The canonical coordinate is a poloidal angle θ, and the canonical time is a toroidal angle φ. The Hamiltonian is the poloidal flux that goes down through the hole in a constant-ψ_p torus, $\psi_p=-\int\vec{B}\cdot d\vec{a}_\theta$ (Fig. 1). The field-line Hamiltonian ψ_p can be a function of clock time t in addition to ψ_t, θ, and φ (Sec. III.C), but clock time is a parameter in the Hamiltonian description and not one of the canonical variables.

The derivation of the symplectic representation of the magnetic field, Eq. (3), clarifies the interpretation. Let (r,θ,φ) be an arbitrary set of well-behaved coordinates. For example, let the radial and vertical coordinates of cylindrical coordinates, Eq. (2), have the form $R(r,\theta,\varphi)=R_o+r\cos\theta$ and $Z=-r\sin\theta$, which makes r constant on circular toroidal surfaces. In three dimensions a vector can have only three independent components, so any vector can be written in the form

$$\vec{A}=\vec{\nabla}g+\psi_t\vec{\nabla}\left(\frac{\theta}{2\pi}\right)-\psi_p\vec{\nabla}\left(\frac{\varphi}{2\pi}\right). \tag{6}$$

The functions of position, g, ψ_t, and ψ_p, represent the three components of $\vec{A}$. If $\vec{A}$ is interpreted as the vector potential of the magnetic field, then its curl gives the symplectic representation of the magnetic field [Eq. (3)].

The theory of one-and-a-half-degree-of-freedom Hamiltonian systems (Lichtenberg and Lieberman, 1992) is the same as that of magnetic-field lines. A magnetic-field line has three fundamentally different types of trajectories.

(1) It can close on itself after traversing the torus m times toroidally, in the φ direction, and n times poloidally, in the θ direction.

(2) A field line can come arbitrarily close to every point on a toroidal surface as the number of toroidal traversals goes to infinity.

(3) A field line can come arbitrarily close to every point in a nonzero volume of space as the number of toroidal traversals goes to infinity.

The first of these possibilities is topologically unstable; an arbitrarily small perturbation can destroy the closure of a set of field lines. For example, the closure of a pure toroidal field $\vec{B}=(\mu_0 G/2\pi R)\hat{\varphi}$, with G the current, can be destroyed by an arbitrarily small field in the vertical direction $B_Z\hat{Z}$. The second possibility, each field line coming arbitrarily close to every point on a toroidal surface, is the one desired for magnetic confinement. The Kolmagorov-Arnold-Moser theorem (Kolmogorov, 1954; Arnold, 1963; Moser, 1967) says this possibility is topologically stable over most of a volume filled by field lines if the perturbation is sufficiently small. The statement that a magnetic field lies in toroidal surfaces means the second possibility, except on isolated surfaces of zero measure. The third possibility, magnetic-field lines coming arbitrarily close to every point in a volume, implies the absence of magnetic confinement in that region. Unfortunately, this is the generic situation for a magnetic field. Consequently, magnetic confinement depends on the formation of very special magnetic fields, fields that have lines that lie on nested surfaces through the bulk of the plasma volume.

In regions of magnetic confinement of a near-Maxwellian plasma, the constraint $\vec{B}\cdot\vec{\nabla}p=0$ holds. This constraint greatly simplifies the field-line trajectories. In the language of Hamiltonian mechanics the pressure p is an isolating constant of the motion. Whenever an isolating constant of the motion exists in a one-and-a-half-degree-of-freedom Hamiltonian system, the canonical coordinates can be chosen so the Hamiltonian is a function of only one canonical variable, the canonical momentum, rather than three. These canonical coordinates are called *action-angle variables* and are central to Hamiltonian perturbation theory. For magnetic-field lines, action-angle coordinates are known as magnetic coordinates (Hamada, 1962). Their existence is easily demonstrated. Since $\vec{B}\cdot\vec{\nabla}p=0$, the magnetic field can be written in arbitrary (p,θ,φ) coordinates as $\vec{B}=B_1\vec{\nabla}p\times\vec{\nabla}\theta+B_2\vec{\nabla}\varphi\times\vec{\nabla}p$. The constraint that $\vec{\nabla}\cdot\vec{B}=0$ implies $\partial B_1/\partial\varphi=-\partial B_2/\partial\theta$, which means the expansion coefficients must have the forms $2\pi B_1=\psi_t'(p)(1+\partial\lambda/\partial\theta)$ and $2\pi B_2=\psi_p'(p)-\psi_t'(p)\partial\lambda/\partial\varphi$. The magnetic field, therefore, has the form

$$2\pi\vec{B}=\vec{\nabla}\psi_t\times\vec{\nabla}\theta_m+\vec{\nabla}\varphi\times\vec{\nabla}\psi_p(\psi_t), \tag{7}$$

where the magnetic poloidal angle $\theta_m\equiv\theta+\lambda$. In magnetic coordinates $(\psi_t,\theta_m,\varphi)$, the field-line Hamiltonian, which is the poloidal flux ψ_p, is a function of the toroidal flux ψ_t alone. The rotational transform $\iota(\psi_t)$, the greek letter iota, and its reciprocal, the safety factor $q(\psi_t)$, are defined by

$$\iota\equiv\frac{1}{q}\equiv\frac{d\psi_p}{d\psi_t}. \tag{8}$$

In general, the rotational transform is used in the stellarator and the safety factor in the tokamak literature. We use ι to avoid confusion with the use of q for electrical charge.

When magnetic coordinates exist, $\psi_p(\psi_t)$, the field-line trajectories are said to be integrable and can be given explicitly in terms of the initial conditions: $\psi_t=\psi_0$ and $\theta_m=\theta_0+\iota(\psi_0)\varphi$. The initial poloidal angle of the field lines $\alpha\equiv\theta_0/2\pi$ can be used as a coordinate in place of θ_m. If this is done

$$\vec{B}=\vec{\nabla}\psi_t\times\vec{\nabla}\alpha, \tag{9}$$

which is called the *Clebsch representation*. The theory of this representation predates Clebsch (1859), and in the mathematics literature ψ_t and α are called *Euler potentials*. A history has been given by Stern (1970). The magnetic-field lines are orthogonal to both the toroidal flux ψ_t and to α since $\vec{B}\cdot\vec{\nabla}\psi_t=0$ and $\vec{B}\cdot\vec{\nabla}\alpha=0$, so both are constants of the motion of the Hamiltonian. However, the constancy of α only prevents a field line from coming arbitrarily close to all points on a ψ_t surface if ι is the ratio of two integers, $\iota=n/m$, in other words, a rational number. When iota is a rational number, the field lines close on themselves after m toroidal circuits, which means in the φ direction, and n circuits in the poloidal, which means in the θ direction. If ι is a rational number, α is what is called in Hamiltonian mechanics an *isolating invariant* because it isolates the trajectory so it can approach only a fraction of the spatial points it could otherwise reach. However, when ι is an irrational number, the trajectory of a single field line comes arbitrarily close to every point on a constant-ψ_t surface, and one says α is a *nonisolating invariant*.

The Clebsch representation of the magnetic field, Eq. (9), exists in any region of space in which all field lines pass through a plane. Let (r,α) be any set of coordinates in that plane, and let the trajectory of a field line that passes through the point (r,α) be $\vec{x}(r,\alpha,\ell)$, where ℓ is the distance along a field line from the plane. By construction both r and α are constant along a field line, $\vec{B}\cdot\vec{\nabla}r=0$ and $\vec{B}\cdot\vec{\nabla}\alpha=0$, so $\vec{B}=f(r,\alpha,\ell)\vec{\nabla}r\times\vec{\nabla}\alpha$. Since $\vec{B}$ is divergence free, $\partial f/\partial\ell=0$. If we define ψ_t so $\partial\psi_t/\partial r=f(r,\alpha)$, then $\vec{B}=\vec{\nabla}\psi_t\times\vec{\nabla}\alpha$.

Various choices are commonly made for the third coordinate of Clebsch coordinates. The most common is the distance along the field lines ℓ. The dual relations of general coordinates, Eq. (A7), imply $\vec{B}=\vec{\nabla}\psi_t\times\vec{\nabla}\alpha=(1/\mathcal{J})\partial\vec{x}/\partial\ell$, where $\mathcal{J}$ is the Jacobian of (ψ_t,α,ℓ) coordinates. Since ℓ is the distance along the lines, $(\partial\vec{x}/\partial\ell)^2=(\mathcal{J}B)^2=1$, which implies that the Jacobian of (ψ_t,α,ℓ) coordinates is $1/B$. Any vector in three dimensions can be written in the covariant form, $\vec{B}=\tilde{B}_{\psi_t}\vec{\nabla}\psi_t+\tilde{B}_\alpha\vec{\nabla}\alpha+\tilde{B}_\ell\vec{\nabla}\ell$. The orthogonality relation of general coordinates, Eq. (A3), implies $\tilde{B}_\ell\equiv\vec{B}\cdot\partial\vec{x}/\partial\ell=B$. That is,

$$\vec{B}=B\vec{\nabla}\ell+\tilde{B}_{\psi_t}\vec{\nabla}\psi_t+\tilde{B}_\alpha\vec{\nabla}\alpha. \tag{10}$$

Another choice for the third coordinate is the magnetic scalar potential ϕ, which is defined by the indefinite integral $\phi(\psi_t,\alpha,\ell)\equiv\int B d\ell$ along each field line. Using this coordinate, an arbitrary magnetic field can be written as

$$\vec{B}=\vec{\nabla}\phi+B_{\psi_t}\vec{\nabla}\psi_t+B_\alpha\vec{\nabla}\alpha. \tag{11}$$

The Jacobian of (ψ_t,α,ϕ) coordinates is $1/(\vec{\nabla}\psi_t\times\vec{\nabla}\alpha)\cdot\vec{\nabla}\phi=1/B^2$.

Magnetic coordinates are central to the theory of magnetically confined plasmas because they allow a simple solution to a differential equation that frequently arises, the magnetic differential equation $\vec{B}\cdot\vec{\nabla}f=g$

(Kruskal and Kulsrud, 1958; Newcomb, 1959). In magnetic coordinates, Eqs. (7) and (8), this equation has the form

$$\left(\frac{\partial}{\partial\varphi}+\iota(\psi_t)\frac{\partial}{\partial\theta_m}\right)f=\frac{g}{\vec{B}\cdot\vec{\nabla}\varphi}, \tag{12}$$

which can be solved algebraically using the Fourier expansion of $g/\vec{B}\cdot\vec{\nabla}\varphi=\Sigma\gamma_{mn}e^{i(n\varphi-m\theta_m)}$. The Fourier expansion coefficients of the function f are $f_{mn}=-i\gamma_{mn}/(n-\iota m)$.

A practical example of the use of the magnetic differential equation is the effect of a magnetic perturbation $\delta\vec{B}$ on a field that has perfect surfaces, $\vec{B}_0$. To find the perturbed magnetic surfaces one solves for a function $p=p_0+\delta p$ such that $(\vec{B}_0+\delta\vec{B})\cdot\vec{\nabla}p=0$. The unperturbed system has perfect surfaces so $\vec{B}_0\cdot\vec{\nabla}p_0=0$. Consequently the first-order result is given by the magnetic differential equation

$$\vec{B}_0\cdot\vec{\nabla}\,\delta p=-\,\delta\vec{B}\cdot\vec{\nabla}p_0. \tag{13}$$

For simplicity, assume the Fourier expansion

$$\frac{\delta\vec{B}\cdot\vec{\nabla}\psi_t}{\vec{B}_0\cdot\vec{\nabla}\varphi}=b_{mn}\sin(n\varphi-m\theta_m), \tag{14}$$

where ψ_t is the toroidal flux enclosed by a surface of constant p_0. Then one obtains a finite solution near the resonant rational surface, $\psi_t=\psi_{mn}$, on which $\iota(\psi_{mn})=n/m$ only if the gradient of p_0 vanishes there. More precisely, as $\psi_t\to\psi_{mn}$, one must let $dp_0/d\psi_t=(n-\iota m)c_0$ with c_0 a constant. Then $\delta p=c_0 b_{mn}\cos(n\varphi-m\theta_m)$, and near the rational surface $p_0(\psi_t)=p_0(\psi_{mn})-c_0 m(d\iota/d\psi_t)(\psi_t-\psi_{mn})^2/2$. The equation for the perturbed surfaces is $p_0(\psi_t)+\delta p=$const. Using the identity $\cos(2x)=1-2\sin^2(x)$, the equation for the perturbed surfaces near the resonant rational surface is

$$\psi_t-\psi_{mn}=\frac{s}{|s|}\sqrt{\frac{4b_{mn}}{m\dfrac{d\iota}{d\psi_t}}\left\{s^2-\sin^2\!\left(\frac{n\varphi-m\theta_m}{2}\right)\right\}}. \tag{15}$$

The constant s labels the surfaces of constant p, which are the magnetic surfaces in the perturbed configuration.

Equation (15) for perturbed magnetic surfaces has two topologically distinct regions, which can be studied by holding φ fixed and varying θ_m (Fig. 2). First, the surfaces that have a surface label $|s|\geq 1$ cover the full range of θ_m. The surfaces with $s>1$ and $s<-1$ are different sets of surfaces, which are distorted by the perturbation but not fundamentally changed. However, the surfaces that satisfy $1>s>-1$ cover only a limited θ_m range and are said to form a magnetic island. The two signs of $s=\pm|s|$ give two parts of the same surface for $1>s>-1$.

Each term b_{mn} in a Fourier expansion of a magnetic perturbation, Eq. (14), produces an island if there is a surface $\psi_t=\psi_{mn}$ on which $\iota(\psi_{mn})=n/m$, which is called a *resonant rational surface*. The half-width of the island in toroidal flux is

$$\delta_{mn}\equiv\sqrt{\left|\frac{4b_{mn}}{m\dfrac{d\iota}{d\psi_t}}\right|}. \tag{16}$$

Islands can also be calculated using the Hamiltonian for the field lines. Equation (3) implies the function $b\equiv\delta\vec{B}\cdot\vec{\nabla}\psi_t/\vec{B}_0\cdot\vec{\nabla}\varphi=-\partial\psi_p/\partial\theta$. Therefore, if $\psi_p=\bar{\psi}_p(\psi_t)+(\tilde{\psi}_p(\psi_t))_{mn}\cos(n\varphi-m\theta_M)$, then $b_{mn}=-m(\tilde{\psi}_p)_{mn}$ and Eq. (16) can be rewritten as

$$\delta_{mn}\equiv\sqrt{\left|\frac{4(\tilde{\psi}_p)_{mn}}{\dfrac{d\iota}{d\psi_t}}\right|}. \tag{17}$$

The *Chirikov criterion* (Chirikov, 1979) says that if the half-widths of islands from different resonant rational surfaces become comparable to the separation between these resonant surfaces, the magnetic-field lines become stochastic, which means a single field line comes arbitrarily close to every point in a finite volume of space.

A qualitative understanding of the formation of islands and the breakdown of magnetic surfaces can be gained from a study of what is called the *standard map* (Chirikov, 1979). Given an initial poloidal angle θ_0 and radial position Ψ_0, the standard map assumes that after one toroidal circuit the poloidal angle of a field line becomes $\theta_1=\theta_0+\Psi_0$ and the radial coordinate becomes $\Psi_1=\Psi_0+k\sin(\theta_1)$. When $k=0$, the radial coordinate is 2π times rotational transform, $\Psi=2\pi\iota(\psi_t)$. The parameter k is proportional to the perturbation. A trajectory of the standard map is found by iteration from the original position (θ_0,Ψ_0). That is, the Nth point along the trajectory is at $\theta_N=\theta_{N-1}+\Psi_{N-1}$ and $\Psi_N=\Psi_{N-1}+k\sin(\theta_N)$. The standard map has the essential property for modeling the field lines of a divergence-free field, which is a unit Jacobian $\partial(\Psi_N,\theta_N)/\partial(\Psi_{N-1},\theta_{N-1})=1$. If a large collection of trajectories are followed that were initially in a small area $(\delta\Psi_0)(\delta\theta_0)$, then the area occupied by the trajectories remains the same forever. For $k\neq 0$, islands appear at $\Psi=2\pi n/m$, where n and m are integers, with the width of the islands scaling as $\sqrt{k}$ for small k. For $k<0.9716\ldots$, Greene (1976) has shown that trajectories of the standard map can cover only a limited range of Ψ. However, for larger values of the perturbation k, some trajectories cover an unlimited Ψ range (Fig. 3). Such trajectories are said to be stochastic. The breakdown of the Ψ surfaces with increasing k is analogous to the breakdown in the magnetic surfaces with an increasing perturbation.

In the presence of a perturbation that produces an arbitrarily small island, any function $p_0(\vec{x})$ that satisfies $\vec{B}_0\cdot\vec{\nabla}p_0=0$ must have the form $dp_0/d\psi_t=(n-\iota m)c_0$. This constraint on p_0 implies that the opening of an island is a singular process in a toroidal equilibrium. In the absence of an island, a function such as the pressure can have a nonzero derivative with respect to the toroidal flux, $dp_0/d\psi_t$, but an arbitrarily small island means the derivatives near the island can only be nonzero using the helical flux, $dp_0/d\psi_h \neq 0$, where the helical flux is defined by $d\psi_h \equiv (\iota - n/m)d\psi_t$. The singularities that arise due to the modification of the pressure gradient by a small resonant perturbation are known as the Glasser effect (Glasser *et al.*, 1975). The resolution of these singularities is an area of active research and is usually discussed under the topic of neoclassical tearing modes; see, for example, Rosenberg *et al.* (2002). The singularities associated with the gradient in the electric potential can prevent a small island from opening in a rotating plasma; see Sec. V.B.3.

B. Methods of forming toroidal magnetic surfaces

- A spatial region in which the magnetic-field lines lie in toroidal surfaces requires either a net toroidal current flowing within the region or helical shaping of the bounding toroidal surface.

To form magnetic surfaces, a tokamak (Fig. 4), uses a net toroidal current in the plasma, while a stellarator (Figs. 5–7) uses helical shaping and in some cases a net toroidal current as well. Both of these plasma confinement systems are discussed in Sec. IV. Ignoring the topologically unstable case of all field lines closing on themselves, magnetic-field lines form toroidal surfaces only when there is both a toroidal and a poloidal magnetic field. The toroidal magnetic field can be produced simply by external coils, while the poloidal field is more difficult.

A pure toroidal magnetic field is produced by a wire carrying a current G along the z axis of cylindrical coordinates,

$$\vec{B}_\varphi = \frac{\mu_0 G}{2\pi}\vec{\nabla}\varphi = \frac{\mu_0 G}{2\pi}\frac{\hat{\varphi}}{R}. \tag{18}$$

An infinite straight wire is not a practical coil set, but exactly the same magnetic field is produced by any current distribution that surrounds the toroidal region of interest, runs in a constant-φ plane, and is axisymmetric (no φ dependence). The toroidal field due to external coils is given by a multivalued scalar potential, $\vec{B} = \vec{\nabla}\phi$, that obeys Laplace's equation, $\nabla^2\phi = 0$, which has the solution $\phi = \mu_0 G\varphi/2\pi$. It is impossible to make the toroidal field coils perfectly axisymmetric because that would preclude any direct access to the plasma. The effect of N separate toroidal field coils can be approximated by assuming the coils have N vertical legs located at an outer radius R_c with the current returning along the z axis, that is, $R=0$. The magnetic potential becomes

$$\phi = \frac{\mu_0 G}{2\pi}\left[\varphi + \left(\frac{R}{R_c}\right)^N \frac{\cos N\varphi}{N}\right], \tag{19}$$

which gives a sinusoidal ripple in the strength of the toroidal field. In order to make the ripple sufficiently

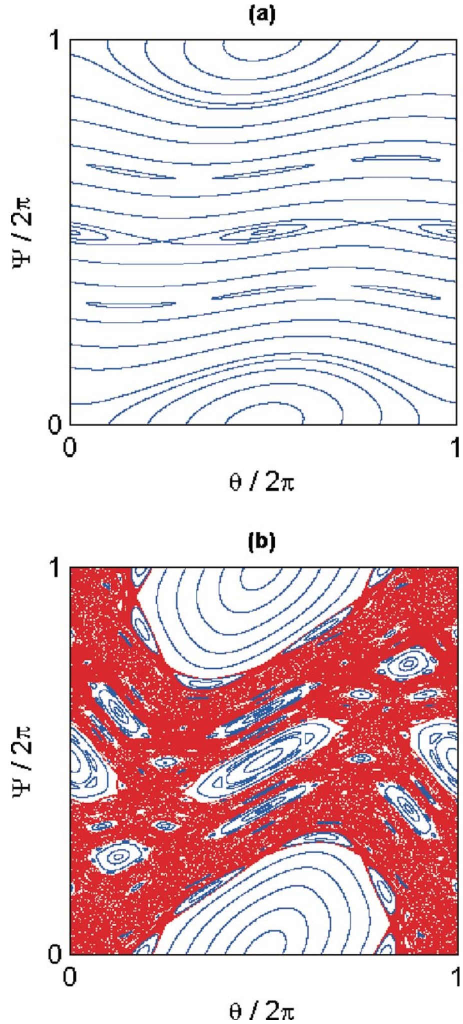

FIG. 3. (Color) Iterates of the standard map plotted for (a) $k = 0.3$ and (b) $k = 1.1$. Islands arise near rational values of $\Psi/2\pi$ for small values of k, but the trajectory followed by iterating the map remains bounded in Ψ space. For $k > 0.9716$, trajectories exist that cover all values of Ψ. The large fuzzy region in (b) is a single stochastic trajectory found by one set of iterations. The periodicity of the standard map was used to plot all iterates in the $0 \leqslant \theta/2\pi < 1$ and $0 \leqslant \Psi/2\pi < 1$ region.

small, one needs the separation between the toroidal field coils to be significantly less than the minor radius of the plasma, which means N must be somewhat larger than $2\pi/\epsilon_c$, with $\epsilon_c \equiv a/R_c$ the inverse aspect ratio of the toroidal surface that forms the plasma edge at $r=a$. The physical reasons that the toroidal ripple must be limited are discussed in Sec. VI.E on particle trajectories and transport.

The production of the poloidal, θ-directed, magnetic field is more difficult. Of course a poloidal field can be produced by a coil that encircles the z axis at a radius R_o and carries a current I. Close to the coil, $r/R_o \ll 1$, the poloidal field is $B_\theta = \mu_0 I/2\pi r$, and the poloidal flux obeys $\partial\psi_p/\partial r = 2\pi R_o B_\theta(r)$. The toroidal magnetic flux is approximately $\psi_t = \pi r^2 B_\varphi$, so the rotational transform, $\iota \equiv d\psi_p/d\psi_t$, is approximated by

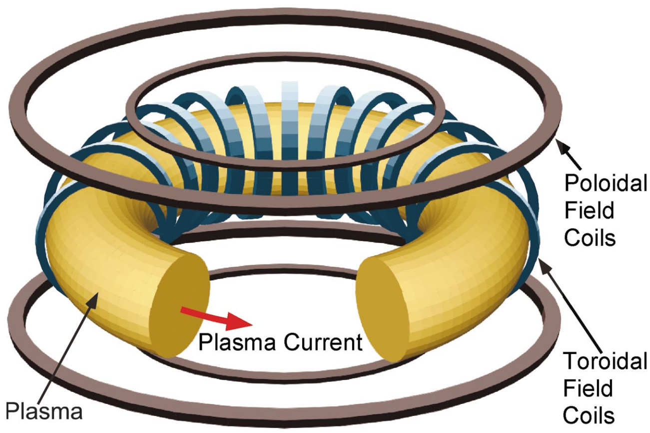

FIG. 4. (Color) A tokamak showing the axisymmetric plasma and the coils necessary to support it. Magnetic surfaces exist in a tokamak only when the toroidal plasma current I is nonzero.

$$\iota \approx \frac{R_o B_\theta}{r B_\varphi}. \tag{20}$$

A circular coil produces magnetic surfaces, but this configuration by itself is not suitable for confining fusion plasmas. Connections are needed between the coil and the outside world so that the heat produced by the slowing of the fusion neutrons within the coil structure can be removed. Such connections would have to pass through the hot fusing plasma and would destroy its confinement.

Although a circular coil cannot be used to produce the poloidal magnetic field, much the same effect can be produced by a current in the plasma that is parallel to the magnetic field, which is called the net plasma current, $\vec{j}_{net}=k(\vec{x})\vec{B}/\mu_0$. This method was suggested by I. Tamm and A. Sakharov in the Soviet Union in the early 1950s and is used to form the poloidal field in tokamaks.

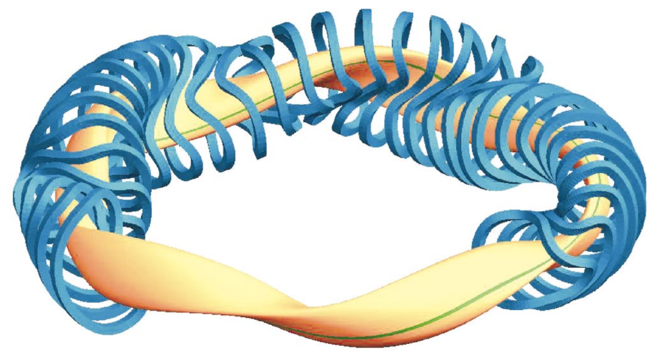

FIG. 6. (Color) The W7-X stellarator with its helically shaped plasma and the coils required to support it. Magnetic surfaces exist in a stellarator even in the absence of a plasma. The average major radius of a plasma is 5.5 m. Figure courtesy of the Max-Planck-Institut für Plasmaphysik, Garching, Germany.

If the plasma pressure gradient is negligibly small, the net current is the total current. Such equilibria are called *force free* and obey the equation

$$\vec{\nabla} \times \vec{B} = k(\vec{x})\vec{B}. \tag{21}$$

Taking the divergence of both sides of this equation, one finds that $k(\vec{x})$ is constrained to be constant along the magnetic field,

$$\vec{B} \cdot \vec{\nabla} k = 0. \tag{22}$$

This implies that a gradient in $k \equiv \mu_0 j_\parallel / B$, with $j_\parallel \equiv \vec{j} \cdot \vec{B}/B$, can only occur in regions of good magnetic surfaces where $k(\psi_t)$ is a function of the enclosed toroidal flux. If the magnetic surfaces are essentially circular, then the rotational transform is given by Eq. (20) with $B_\theta = \mu_0 I / 2\pi r$ and I the current enclosed by a surface of radius r. Usually k can be approximated by a constant

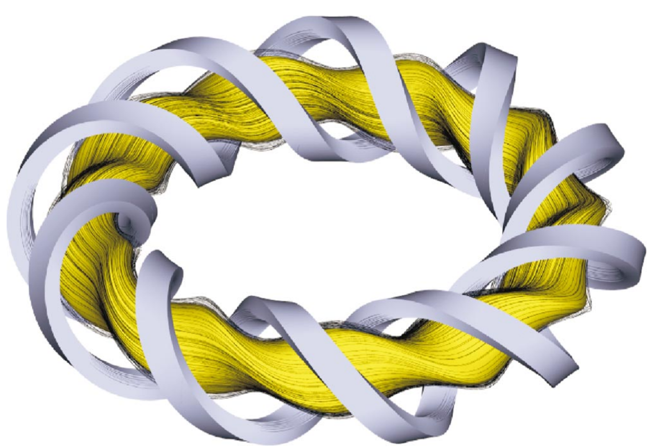

FIG. 5. (Color) The coils and some magnetic-field lines of the Large Helical Device (LHD). The rotational transform in LHD is due to the wobble of the magnetic-field lines produced by the helical coils. A typical plasma has an average major radius of 3.6 m. Figure courtesy of the National Institute for Fusion Science, Japan.

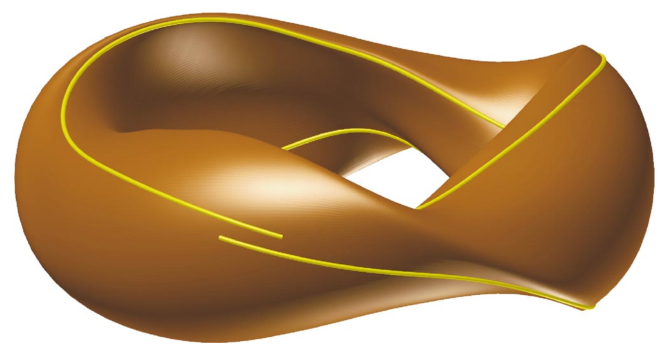

FIG. 7. (Color) The surface of the proposed National Compact Stellarator Experiment (NCSX) quasiaxisymmetric stellarator, shown with three toroidal circuits of a magnetic-field line. The rotational transform ι is just under 2/3, so the line almost closes. Despite the strong helical shaping, the particle drifts are similar to those of an axisymmetric tokamak. Much of the rotational transform in NCSX is due to the torsion of the magnetic axis. The average major radius of a plasma is 1.44 m. Figure courtesy of the Princeton Plasma Physics Laboratory, Princeton, NJ.

for sufficiently small r. The rotational transform in that region is $\iota = R_o k/2$. For radii r larger than the current channel, the rotational transform is $\iota \propto 1/r^2$.

Stellarators (Spitzer, 1958) have magnetic surfaces and a nonzero rotational transform even in the absence of a plasma. Stellarator magnetic surfaces must have helical shaping. The existence of magnetic surfaces without any enclosed currents is demonstrated starting with the axisymmetric magnetic surfaces formed by a coil encircling the z axis at a radius R_o, which is designed so the rotational transform ι is an integer N_p in the center of the region of interest. If this system is modified by a magnetic perturbation in which $b_{m=1,\ n=N_p}$ is nonzero, an island is formed [Eq. (16)]. Since the magnetic surfaces inside an island do not cover the full range of theta (Fig. 2), room exists for connections and supports for the coil. The magnetic surfaces inside the island fill a volume and have no current on them. The integer N_p is the number of periods of this type of stellarator, which is called a Heliac (Boozer *et al.*, 1983).

The poloidal field in a stellarator can be produced in two ways in the absence of a plasma. The first way uses helical wobbles of the magnetic-field lines that are driven by helical currents in coils (Spitzer, 1958). The rotational transform in the Large Helical Device (Yamada *et al.*, 2001) is produced entirely by the field-line wobble produced by helical coils (Fig. 5). The magnetic field that is produced by the helical wobbles acts, when averaged over the wobbles, as if there were a force-free current that filled the space around the coils and that became exponentially small with increasing distance from the coils. To understand this, consider a weak helical perturbation to a spatially constant magnetic field, $\vec{B}/B_0 = \hat{z} + \vec{\nabla}\phi$, with $\phi(x,y,z) = \Delta(x)\cos(k_z z - k_y y)$ in Cartesian coordinates. Since $\nabla^2\phi$ must be zero, the distance $\Delta(x) \propto \exp(kx)$ with $k^2 = k_y^2 + k_z^2$. The magnetic-field lines are given by $dx/dz = B_x/B_z \simeq k\Delta\cos(\cdots)$ or $x \simeq x_0 + (k\Delta/k_z)\sin(\cdots)$, and $dy/dz = B_y/B_z$. That is, $dy/dz = k_y\Delta\sin(\cdots)/\{1 - k_z\Delta\sin(\cdots)\}$, which when expanded to keep the important second-order terms in Δ is $dy/dz \simeq k_y\Delta\sin(\cdots) + \{(k_y k^2/k_z) + (k_y k_z)\}\Delta^2\sin^2(\cdots)$. The magnetic-field lines have an oscillation in the $\hat{y}$ direction, $\delta y = (k_y/k_z)\Delta\sin(\cdots)$, but also a systematic drift in that direction, which is equivalent to a y-directed magnetic field,

$$\frac{\langle B_y\rangle}{B_0} = \frac{1}{2}\frac{k_y}{k_z}\left(1 + \frac{k_z^2}{k^2}\right)(k\Delta)^2. \tag{23}$$

The second-order terms in the dx/dz equation give no systematic drift. Consequently, field lines wobble in and out of a constant-x surface but have a drift in the y direction. The rotational transform per period in the y and x directions is $\iota_p \equiv (k_y/k_z)\langle B_y\rangle/B_z$, or

$$\iota_p = \frac{1}{2}\left(\frac{k_y}{k_z}\right)^2\left(1 + \frac{k_z^2}{k^2}\right)(k\Delta)^2. \tag{24}$$

There is an upper limit on the transform per period since $k\Delta$ must be small compared to 1. The effective force-free current that fills the space around the helical coil is $\mu_0\vec{j}_{eff} = 2k\langle B_y\rangle\hat{z}$.

The second way to produce a poloidal field without a plasma is through torsion in the magnetic axis. The magnetic axis is the field line at $\psi_t = 0$ and is the axis of the poloidal angle. The torsion τ of a curve measures the extent to which the curve fails to lie flat on a plane. The rotational transform due to torsion is

$$\iota = \langle\tau\rangle\frac{L}{2\pi}, \tag{25}$$

with $\langle\tau\rangle$ the torsion of the magnetic axis averaged along its length L. Torsion was used in the early 1950s to produce the poloidal field in Spitzer's Figure-8 stellarators (Spitzer, 1958). The optimization of the particle trajectories in a stellarator (Sec. VI.E.1) requires the use of torsion. Indeed, torsion gives a major part of the rotational transform in both the Wendelstein-7X (W7-X) (Fig. 6; Beidler *et al.*, 1990), and the National Compact Stellarator Experiment (NCSX) (Fig. 7; Zarnstorff *et al.* 2001) stellarator designs. The derivation of the relation between the torsion and the rotational transform uses the Frenet formulas (Mathews and Walker, 1964) to analyze the behavior of field lines near the magnetic axis, with ℓ the distance along the axis. The magnetic axis, $\vec{x}_0(\ell)$, is the closed curve about which the field lines wind. The derivative of a curve is its tangent, $\hat{b}_0 \equiv d\vec{x}_0/d\ell$, which is a unit vector along the magnetic-field line that forms the axis. The derivative of the tangent is $d\hat{b}_0/d\ell = \kappa\hat{\kappa}$, with $\kappa\hat{\kappa}$ the curvature of the axis. The derivative of the curvature unit vector is $d\hat{\kappa}/d\ell = -(\kappa\hat{b}_0 + \tau\hat{\tau})$, with $\tau\hat{\tau}$ the torsion. The derivative of the torsion unit vector is $d\hat{\tau}/d\ell = \tau\hat{\kappa}$. The Frenet unit vectors are mutually orthogonal and satisfy $\hat{b}_0 \times \hat{\kappa} = \hat{\tau}$. Mercier (1964) used the Frenet unit vectors to establish a coordinate system near an arbitrary magnetic axis. We shall simplify his analysis by making the magnetic surfaces circular near the axis, which excludes the rotational transform from helical wobbles. That is, we choose coordinates $\vec{x}(r,\theta,\ell) = \vec{x}_0(\ell) + r\cos\theta\hat{\kappa} + r\sin\theta\hat{\tau}$. Using the methods of general coordinates given in the Appendix, one finds $\vec{\nabla}r = \cos\theta\hat{\kappa} + \sin\theta\hat{\tau}$, $\vec{\nabla}\theta = (-\sin\theta\hat{\kappa} + \cos\theta\hat{\tau})/r + \tau\hat{b}_0/(1 - \kappa r\cos\theta)$, and $\vec{\nabla}\ell = \hat{b}_0/(1 - \kappa r\cos\theta)$. The magnetic field in the absence of plasma currents has the form $\vec{B} = (\mu_0 G/2\pi)\vec{\nabla}\varphi_p$, with G a constant. The potential φ_p, which can be interpreted as a toroidal angle, obeys $\nabla^2\varphi_p \equiv \vec{\nabla}\cdot\vec{\nabla}\varphi_p = 0$. If the magnetic strength is constant along the magnetic axis, then for $r \to 0$ the toroidal angle $\varphi_p = 2\pi(\ell + \kappa\tau r^3\sin\theta + \cdots)/L$, with L the length of the axis. For $r \to 0$, the magnetic-field-line equations are $d\theta/d\varphi_p = \vec{B}\cdot\vec{\nabla}\theta/\vec{B}\cdot\vec{\nabla}\varphi_p = \tau L/2\pi$, and $dr/d\varphi_p = \vec{B}\cdot\vec{\nabla}r/\vec{B}\cdot\vec{\nabla}\varphi_p = 0$. Consequently, the rotational transform near the axis is $\iota = \langle\tau\rangle L/2\pi$, with the axis average of the torsion $\langle\tau\rangle \equiv \oint\tau d\ell/\oint d\ell$.

The magnetic configuration due to the coils alone is called the *vacuum configuration*. It is in principle easy to design vacuum configurations that have magnetic surfaces of any desired shape, but unless the surfaces have helical shaping the rotational transform will be zero. To design a vacuum configuration, first assume there is an axisymmetric toroidal field, $\vec{B}_\varphi = \mu_0 G\hat{\varphi}/2\pi R$, which produces the toroidal magnetic flux. One then defines the desired shape of an outermost magnetic surface using (R,φ,Z) cylindrical coordinates, $\vec{x}_s = R_s(\theta,\varphi)\hat{R}(\varphi) + Z_s(\theta,\varphi)\hat{Z}$, by giving the appropriate Fourier-series representation of R_s and Z_s. The unit normal to this surface is $\hat{n} \propto (\partial\vec{x}_s/\partial\theta)(\partial\vec{x}_s/\partial\varphi)$. Coils just outside the plasma can in principle produce any desired normal magnetic field on the surface $\vec{x}_s$. If the coils are designed so that they produce a normal field equal but opposite in sign to $\hat{n}\cdot\vec{B}_\varphi$, then the surface $\vec{x}_s$ will be a magnetic surface, since the magnetic-field lines do not cross it, $\hat{n}\cdot\vec{B}=0$. When vacuum magnetic fields produce one magnetic surface, the volume enclosed by that surface is generally dominated by regions of good surfaces rather than islands and stochastic regions. Practical limitations on the design of magnetic-field configurations are discussed in Sec. V.D.

C. Evolution of magnetic-field lines

- Clock time is a parameter in the Hamiltonian description of evolving magnetic-field lines. For an evolving magnetic field, the Hamiltonian, $\psi_p(\psi_t,\theta,\varphi,t)$, as well as the coordinate transformation function, $\vec{x}(\psi_t,\theta,\varphi,t)$, depend on time as well as the canonical coordinates. A magnetic field evolves ideally (without a topology change) if the field-line Hamiltonian can be made time independent by an appropriate choice of the coordinate transformation function $\vec{x}(\psi_t,\theta,\varphi,t)$.
- The mathematical condition for an ideal evolution of $\vec{B}(\vec{x},t)$ is that a function $\Phi_B(\vec{x},t)$ exist such that $\vec{B}\cdot\vec{\nabla}\Phi_B = -\vec{E}\cdot\vec{B}$, where the electric field is determined by Faraday's law, $\partial\vec{B}/\partial t = -\vec{\nabla}\times\vec{E}$. Magnetic-field lines evolve ideally under more general conditions than the tying together of the plasma and the field, and a generic magnetic field always evolves ideally in a sufficiently localized spatial region.
- The self-entanglement of magnetic-field lines is measured by the magnetic helicity $K \equiv \int \vec{A}\cdot\vec{B}d^3x$. A spiky current profile causes a rapid dissipation of energy relative to magnetic helicity. If the evolution of a magnetic field is rapid, then it must be at constant helicity.

Ideally a fusion plasma would be in a steady state, but the theory of the evolution of the magnetic field is important for finding the conditions for (1) establishing the field configuration, (2) maintaining the magnetic field, and (3) finding the conditions under which rapid changes in the magnetic field can occur. The evolution of the magnetic field means the evolution of the magnetic-field lines. The equations for the evolution of magnetic-field lines are also of interest for the evolution of other intrinsically divergence-free fields such as the vorticity field of fluid mechanics.

The evolution of a magnetic field $\vec{B}(\vec{x},t)$ is called ideal if it is consistent with the field's being embedded in an ideal, zero-resistivity fluid moving with a velocity $\vec{u}(\vec{x},t)$, which can be interpreted as the velocity of the magnetic-field lines. An ideal evolution does not change the topology of the magnetic-field lines. The conditions for an ideal evolution are obtained by noting that the electric field $\vec{E}(\vec{x},t)$ associated with an evolving magnetic field $\vec{B}(\vec{x},t)$ is given by Faraday's law to within an arbitrary additive gradient of a potential. Mathematics implies that an arbitrary vector $\vec{E}(\vec{x},t)$, and hence the electric field, can be written as

$$\vec{E} + \vec{u}\times\vec{B} = -\vec{\nabla}\Phi_B + V\vec{\nabla}\left(\frac{\varphi}{2\pi}\right) \tag{26}$$

in any region of space in which $\vec{B}\cdot\vec{\nabla}\varphi$ is nonzero. The function V, called the *loop voltage*, is constant along each magnetic-field line and is given by

$$V \equiv \lim_{L\to\infty} \frac{\int_{-L}^{L} \vec{E}\cdot d\vec{\ell}}{\int_{-L}^{L} \vec{\nabla}\left(\frac{\varphi}{2\pi}\right)\cdot d\vec{\ell}}, \tag{27}$$

where $d\vec{\ell}$ is the differential distance along $\vec{B}$. The component of $\vec{E}$ parallel to $\vec{B}$ is balanced by Φ and V, and the components of $\vec{E}$ perpendicular to $\vec{B}$ are balanced by $\vec{u}$. If the loop voltage V is zero, the evolution is ideal.

The relation between the conservation of topological properties and the vanishing of the loop voltage V will be shown in this paragraph, which can be skipped on a first reading. The topological properties of the magnetic-field lines are independent of time if the canonical coordinates (ψ_t,θ,φ) can evolve in such a way that the field-line Hamiltonian does not change, $(\partial\psi_p/\partial t)_c=0$. The subscript c means the partial derivative is performed at a fixed point in canonical coordinates. This conclusion follows from two statements. First, the trajectories of magnetic-field lines in canonical coordinates are determined by the field-line Hamiltonian $\psi_p(\psi_t,\theta,\varphi,t)$ alone [Eqs. (4) and (5)]. Second, topological properties are unchanged by a continuous temporal variation in a coordinate transformation function, $\vec{x}(\psi_t,\theta,\varphi,t)$. Consequently the topological properties of field lines are determined by the field-line Hamiltonian alone, and these properties cannot change unless the field-line Hamiltonian changes. The evolution equation for the magnetic-field-line Hamiltonian is obtained from the evolution of the vector potential in canonical coordinates. Using only the theory of general coordinates (Appendix), one can write (Boozer, 1992) the time derivative of an arbitrary vector

$\vec{A}=\psi_t\vec{\nabla}(\theta/2\pi)-\psi_p\vec{\nabla}(\varphi/2\pi)+\vec{\nabla}g$ in a coordinate system defined by the function $\vec{x}(\psi_t,\theta,\varphi,t)$ [Eq. (A19)] as

$$\left(\frac{\partial\vec{A}}{\partial t}\right)_{\vec{x}}=-\left(\frac{\partial\psi_p}{\partial t}\right)_c\vec{\nabla}\frac{\varphi}{2\pi}+\vec{u}\times\vec{B}+\vec{\nabla}s, \tag{28}$$

where we have let $\vec{B}\equiv\vec{\nabla}\times\vec{A}$ and introduced the velocity of a fixed point in canonical coordinates,

$$\vec{u}\equiv\frac{\partial\vec{x}(\psi_t,\theta,\varphi,t)}{\partial t}. \tag{29}$$

The subscript $\vec{x}$ means a fixed spatial point, and the subscript c means a fixed point in the canonical (ψ_t,θ,φ) coordinates. The function $s=(\partial g/\partial t)_c-\vec{A}\cdot\vec{u}$, Eq. (A20), is the generating function for infinitesimal canonical transformations of Hamiltonian mechanics. Even when the vector potential is independent of time, $(\partial\vec{A}/\partial t)_{\vec{x}}=0$, the canonical coordinates can change, $\vec{u}\neq 0$, using the freedom of infinitesimal canonical transformations s. The relation between Eqs. (26) and (28) can be demonstrated using $\vec{E}=-\partial\vec{A}/\partial t-\vec{\nabla}\Phi$. The two equations have the same content with the identification

$$\left\{\left(\frac{\partial\psi_p}{\partial t}\right)_c-V\right\}\vec{\nabla}\frac{\varphi}{2\pi}=\vec{\nabla}\{s-(\Phi_B-\Phi)\}. \tag{30}$$

If $V=0$, then $\psi_p(\psi_t,\theta,\varphi,t)$ can be made independent of time by the choice of infinitesimal generating function $s=\Phi_B-\Phi$. In a system with magnetic surfaces $\psi_p(\psi_t,t)$, the loop voltage is the change in the poloidal flux outside a given toroidal flux surface,

$$V=\frac{\partial\psi_p(\psi_t,t)}{\partial t}. \tag{31}$$

Remarkably, the evolution of a magnetic field is generically ideal in a sufficiently small spatial region even when the field is embedded in a resistive fluid or is in a vacuum. This follows from the observation that locally $\vec{E}+\vec{u}\times\vec{B}=-\vec{\nabla}\Phi_B$; a nonzero loop voltage V is needed only if the field lines (a) are followed an infinite distance, (b) close on themselves, or (c) intercept surfaces on which the potential Φ_B must obey fixed boundary conditions. The evolution is locally ideal (Boozer, 1992) even near a null field, $\vec{B}=0$, provided the matrix of first derivatives $\mathcal{B}_{ij}\equiv\partial B_i/\partial x^j$ has a nonzero determinant, which is the generic situation at a magnetic-field null. In other words, an arbitrarily small perturbation can make a zero determinant of $\mathcal{B}_{ij}$ nonzero at a null.

The simplest model of a nonideal evolution treats the plasma as a moving conductor. A Lorentz transformation in the nonrelativistic limit implies the electric field in a conductor moving with velocity $\vec{v}(\vec{x},t)$ is $\vec{E}_{mov}=\vec{E}+\vec{v}\times\vec{B}$. In a simple conductor $\vec{E}_{mov}=\overleftrightarrow{\eta}\cdot\vec{j}$, with $\overleftrightarrow{\eta}$ the resistivity tensor, and Ohm's law becomes

$$\vec{E}+\vec{v}\times\vec{B}=\overleftrightarrow{\eta}\cdot\vec{j}. \tag{32}$$

In plasmas the resistivity tensor is diagonal but with distinct values along, $\eta_\parallel$, and across, $\eta_\perp$, the magnetic field. In a quiescent plasma, $\eta_\perp$ is about twice $\eta_\parallel$, but the microturbulence that is present in most confined plasmas enhances $\eta_\perp$ by several orders of magnitude while leaving $\eta_\parallel$ essentially unchanged (see Sec. VI.F). Substituting Ohm's law into the general expression for the electric field, Eq. (26), one finds that

$$V\vec{\nabla}\left(\frac{\varphi}{2\pi}\right)+(\vec{v}-\vec{u})\times\vec{B}=\overleftrightarrow{\eta}\cdot\vec{j}+\vec{\nabla}\Phi_B. \tag{33}$$

The loop voltage, $V=\int\eta_\parallel j_\parallel d\ell/\int\vec{\nabla}(\varphi/2\pi)\cdot d\vec{\ell}$, vanishes if the parallel resistivity is zero. If the resistivity tensor is zero, the magnetic field and the plasma move together, $\vec{u}=\vec{v}$, with the choices $\Phi_B=0$ and $\vec{u}\cdot\vec{B}=\vec{v}\cdot\vec{B}$.

The evolution of a magnetic field that is embedded in a conducting fluid obeys two distinct conservation laws when the resistivity vanishes, $\overleftrightarrow{\eta}=0$: (1) the ideal evolution of $\vec{B}$, and (2) the tying of the magnetic-field lines to the fluid, $\vec{u}=\vec{v}$. A distinction between these two laws is rarely made in the literature, but the first holds under more general conditions. The breaking of the ideal evolution of $\vec{B}$, which changes the magnetic-field-line topology, is determined by the parallel resistivity $\eta_\parallel$ alone. The flow of the plasma relative to the magnetic-field lines, $\vec{v}-\vec{u}$, depends on the perpendicular resistivity $\eta_\perp$ and in general on the parallel resistivity as well.

The distinction between the two conservation laws for magnetic evolution is especially clear in a type-II superconductor with a melted flux lattice (Huebener, 1979). The technically important superconductors are type-II, which means the magnetic field can penetrate into the material, which lowers the magnetic-field energy, but the magnetic-field lines must lie in narrow flux tubes. These flux tubes can form a lattice, which holds the tubes in place, and in the technically important superconductors the flux lattice is rigid. When the lattice is rigid, the superconductor has zero resistivity. However, the flux lattice can melt, which allows the flux tubes to move through the superconductor with a velocity, $\vec{u}-\vec{v}$, proportional to the applied force. This motion is equivalent to a perpendicular resistivity $\eta_\perp$. Therefore a type-II superconductor with a melted flux lattice has a tensor resistivity with $\eta_\parallel=0$ but $\eta_\perp$ nonzero. The first conservation law, the ideal evolution of $\vec{B}$, holds rigorously in a superconductor since $\eta_\parallel=0$. However, if the flux lattice of a superconductor melts, the perpendicular resistivity becomes large, and the second conservation law, the tying of the field lines to the superconductor, is not even approximately valid.

On rational magnetic surfaces the rotational transform is a rational number, $\iota=n/m$, and magnetic-field lines close on themselves. On these surfaces the loop voltage V can have a different value on each magnetic-field line. When this occurs, the field-line Hamiltonian

ψ_p will develop a term that goes as $e^{i(n\varphi - m\theta)}$, and the rational surface will split to form islands. In a highly conducting fluid, this is associated with the important phenomenon of fast reconnection. The currents that arise in a highly conducting fluid, $\eta \to 0$, to preserve the topology are surface currents, which means they are delta functions—nonzero only on the rational surface but infinite there. Such currents lead to rapid dissipation, so perturbations can open islands on a very short time scale compared to the global resistive time, $\tau_\eta \equiv (\mu_0/\eta)a^2$, of a plasma of radius a. The actual time scale in a plasma is a subtle question, a question on which recent progress has been made. See Rogers *et al.* (2001) and reviews by Bhattacharjee *et al.* (2001) and by Priest and Forbes (2000).

In astrophysical and space plasmas, the magnetic-field lines often enter and leave the volume of interest. In such systems nonideal effects in the magnetic field, such as reconnection, are far more subtle than in systems with two periodic directions, where reconnection is focused onto the rational surfaces (Boozer, 2002). A nonzero loop voltage V may be required by boundary conditions. Nonideal effects can also arise if neighboring magnetic-field lines separate from each other exponentially with distance along the lines. In this case, a solution $\Phi_B(\vec{x},t)$ to $\vec{E}\cdot\vec{B} = -\vec{B}\cdot\vec{\nabla}\Phi_B$ may exist but have poor analytic properties due to exponentially large gradients across the field lines. Except when magnetic-field lines lie on surfaces, the exponential separation of neighboring lines is a generic property. Neighboring field lines satisfy $d\vec{x}/d\tau = \vec{B}(\vec{x})$ and $d(\vec{x}+\vec{\delta})/d\tau = \vec{B}(\vec{x}+\vec{\delta})$ with $\vec{\delta}\to 0$. Letting $\mathcal{B}_{ij} \equiv \partial B_i/\partial x^j$ and $\Delta_{ij}(\ell) \equiv \int (\mathcal{B}_{ij}/B)d\ell$, one finds the separation between the lines at distance ℓ down either line is $\vec{\delta}(\ell) = \exp(\overleftrightarrow{\Delta})\cdot\vec{\delta}_0$. Unless the magnitudes of all of the eigenvalues of $\overleftrightarrow{\Delta}(\ell)$ are exponentially small for large ℓ, neighboring field lines separate exponentially.

Similar equations to the magnetic evolution equations can be derived for the vorticity, $\vec{\omega} \equiv \vec{\nabla}\times\vec{v}$, of a fluid. The equivalent of Ohm's law for the vorticity is the Navier-Stokes equation,

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\vec{\nabla}w(p) - \nu\vec{\nabla}\times\vec{\omega}. \tag{34}$$

This form of the equation assumes the density ρ is a function of the pressure alone with $dw(p) \equiv dp/\rho(p)$. The kinematic viscosity is ν. The term $\vec{v}\cdot\vec{\nabla}\vec{v} = -\vec{v}\times\omega + \vec{\nabla}(v^2/2)$, so the Navier-Stokes equation can be written in a form analogous to Ohm's law with $\vec{E}$ replaced by $\partial\vec{A}/\partial t$,

$$\frac{\partial \vec{v}}{\partial t} = \vec{v}\times\vec{\omega} - \vec{\nabla}\left(w + \frac{1}{2}v^2\right) - \nu\vec{\nabla}\times\vec{\omega}. \tag{35}$$

The Hamiltonian of the vorticity field lines can be made time independent if the viscosity vanishes, $\nu = 0$.

The rapidity of the evolution of magnetic fields is limited by the properties of *magnetic helicity* (Woltjer, 1958). Magnetic helicity is defined as

$$K(t) \equiv \int \vec{A}\cdot\vec{B}d^3x. \tag{36}$$

In a system with perfect magnetic surfaces, the helicity is $K(\psi_t,t) = \int(\psi_t\iota - \psi_p)d\psi_t$ with $\iota(\psi_t,t) \equiv \partial\psi_p/\partial\psi_t$ the rotational transform. The helicity is a measure of the topological entanglement of the magnetic-field lines.

The importance of magnetic helicity comes from its rate of dissipation, $2\int\vec{B}\cdot\vec{E}d^3x$. If the current density in a plasma has a large spatial variation, as it does in turbulent situations, the rate of loss of magnetic energy, $\int\vec{j}\cdot\vec{E}d^3x$, is rapid in comparison to the rate of helicity dissipation. It is relatively easy to change the magnetic energy quickly compared to the resistive time scale, $\tau_\eta \equiv (\mu_0/\eta)a^2$, of a plasma of radius a. However, if the magnetic energy changes rapidly compared to τ_η, the change must be at an essentially constant helicity.

The time derivative of the helicity can be put in a convenient form using $\vec{E} = -\partial\vec{A}/\partial t - \vec{\nabla}\Phi$,

$$\frac{dK}{dt} = -2\int \vec{B}\cdot\vec{E}d^3x + S_{es} + S_{in}. \tag{37}$$

The volumetric term, $-(dK/dt)_d \equiv 2\int\vec{B}\cdot\vec{E}d^3x$, is the helicity dissipation. The two surface terms are the external sources of helicity: the electrostatic source

$$S_{es} \equiv -2\oint \Phi\vec{B}\cdot d\vec{a} \tag{38}$$

and the inductive source

$$S_{in} \equiv -\oint \vec{A}\times\frac{\partial\vec{A}}{\partial t}\cdot d\vec{a}. \tag{39}$$

If the bounding surface is a perfect conductor, the external sources of helicity vanish. When the bounding surface is a magnetic surface, the inductive source is $S_{in} = \psi_t d\psi_p/dt - \psi_p d\psi_t/dt$. The poloidal and toroidal loop voltages are defined as $V_p \equiv \oint\vec{E}\cdot(\partial\vec{x}/\partial\theta_m)d\theta_m$ and $V_t \equiv \oint\vec{E}\cdot(\partial\vec{x}/\partial\varphi)d\varphi$, so $S_{in} = V_t\psi_t + V_p\psi_p$, the loop voltages times the fluxes. The toroidal loop voltage is equal to the loop voltage V if the poloidal loop voltage, $V_p = -\partial\psi_t/\partial t$, is zero.

The magnetic energy, $W_B = \int(B^2/2\mu_0)d^3x$, has the time derivative $dW_B/dt = -\int\vec{j}\cdot\vec{E}d^3x$. When the standard Ohm's law, Eq. (32), is used, with $\overleftrightarrow{\eta}$ having distinct parallel and perpendicular components, $-dW_B/dt$ is the sum of two terms: the dissipative loss of energy $(dW_B/dt)_d = \int(\eta_\parallel j_\parallel^2 + \eta_\perp j_\perp^2)d^3x$ and a nondissipative transfer of energy between the magnetic field and the fluid in which it is embedded, $\int\vec{v}\cdot(\vec{j}\times\vec{B})d^3x$. Retaining only the dissipative term, $-(dW_B/dt)_d \geqslant \int\eta_\parallel j_\parallel^2 d^3x$. In contrast, the volumetric dissipation of helicity is $2\int\vec{B}\cdot\vec{E}d^3x = 2\int\eta_\parallel\vec{j}\cdot\vec{B}d^3x$.

The relative rates of dissipation of energy and helicity,

$$-\left(\frac{dW_B}{dt}\right)_d \geqslant \frac{1}{\int \eta_\parallel B^2 d^3x}\left(\frac{1}{2}\frac{dK}{dt}\right)_d^2, \tag{40}$$

can be demonstrated (Berger, 1984) using the Schwarz inequality. This inequality says the average of the square of a function is at least as large as the square of the average of the function. The appropriate average for the relative dissipation rates is the η average, which is defined as $\langle f\rangle_\eta \equiv \int(\eta_\parallel B^2 f d^3x)/\int(\eta_\parallel B^2 d^3x)$. The rate of magnetic energy dissipation satisfies $-(dW_B/dt)_d \geqslant (\int \eta_\parallel B^2 d^3x)\langle (j_\parallel/B)^2\rangle_\eta$, while the rate of helicity dissipation satisfies $(dK/dt)_d = 2(\int \eta_\parallel B^2 d^3x)\langle j_\parallel/B\rangle_\eta$. The Schwarz inequality, $\langle (j_\parallel/B)^2\rangle_\eta \geqslant \langle j_\parallel/B\rangle_\eta^2$, then implies Eq. (40). The greater the spatial variation in $j_\parallel/B$, the stronger the Schwarz inequality becomes. In Sec. V.B.1, we shall find that unstable, or turbulent, plasmas develop very spiky, Dirac delta-function-like, spatial distributions of $j_\parallel/B$. In such situations, the dissipation of the magnetic energy is extremely rapid in comparison to the dissipation of helicity.

Magnetic energy can be dissipated rapidly compared to helicity, so highly unstable plasmas evolve to minimize their energy for fixed helicity. As Woltjer (1958) showed, this evolution relaxes the current density to the form $\vec{j}=(k/\mu_0)\vec{B}$, where k is a spatial constant. This property of the state of minimum energy with fixed helicity is demonstrated by varying $\vec{A}$ in $\int(B^2/\mu_0 - k\vec{A}\cdot\vec{B})d^3x$. The constant k, which is called a Lagrange multiplier, is chosen to make the helicity after minimization equal to its initial value. The terms involving $\delta\vec{B} = \vec{\nabla}\times\delta\vec{A}$ are integrated by parts, and the boundary terms $\oint \vec{B}\times\delta\vec{A}\cdot d\vec{a}$ can be ignored if the boundary is a perfect conductor. One finds $\vec{\nabla}\times\vec{B}=k\vec{B}$.

The physical importance of helicity conservation was demonstrated by Taylor (1974). He showed that turbulent periods in the reversed-field pinch plasma confinement device (Prager, 1999) lead to flattened current profiles with a more quiescent plasma. Since Taylor's important work, the relaxation of the current profile to form a more quiescent plasma has been known as a Taylor relaxation. Taylor's work demonstrated that helicity conservation should be a central element of any theory of rapidly evolving magnetic fields.

A plasma can transition between two states with magnetic surfaces but with different helicity distributions, $K(\psi_t)$, in a time short compared to the resistive time τ_η. However, when this occurs helicity conservation implies that the magnetic surfaces must have broken in the intermediate state.

Spiky current profiles cause a rapid loss of magnetic energy, $\int \vec{j}\cdot\vec{E}d^3x$. On the time scale of energy dissipation, the magnetic helicity can be transported but not dissipated. The absence of helicity dissipation implies $2\vec{E}\cdot\vec{B}$ must be the divergence of a flux, the helicity flux $\vec{\mathcal{F}}_h$ with $2\vec{E}\cdot\vec{B}=\vec{\nabla}\cdot\vec{\mathcal{F}}_h$. Energy dissipation, which in the Schwarz inequality argument takes place through the parallel current $\int(j_\parallel/B)\vec{B}\cdot\vec{E}d^3x$, implies $-\int\vec{\mathcal{F}}_h\cdot\vec{\nabla}(j_\parallel/B)d^3x$ must be positive. This condition is satisfied if $\vec{\mathcal{F}}_h = -\lambda_h\vec{\nabla}(j_\parallel/B)$ (Boozer, 1986). The positive coefficient λ_h is called the hyper-resistivity. The gradient of $j_\parallel/B$ is a source of free energy, which can drive instabilities much as the pressure gradient can; see Sec. V.B.1.

If a system has a large source of free energy other than the magnetic-field energy, the helicity flux $\vec{\mathcal{F}}_h$ can have terms that add energy to the magnetic field. Such terms are needed in a magnetic dynamo, where magnetic fields are generated by taking energy from another source, such as a fluid flow. Taylor relaxation in the reversed-field pinch demonstrates the existence of a dissipative term in the helicity flux, such as $-\lambda_h\vec{\nabla}(j_\parallel/B)$, but not a term that can add energy to the magnetic field, which is needed for a dynamo. The role of helicity and its conservation in limiting the forms of magnetic dynamo theories remains controversial.

The external sources of helicity, the electrostatic S_{es} and the inductive S_{in}, are important for creating the plasma currents needed for magnetic surfaces in axisymmetric devices. The inductive source is usually a toroidal loop voltage V_t supplied by varying the magnetic flux in a solenoid that goes through the central hole of the torus while an essentially constant toroidal flux is supplied by toroidal field coils. If the loop voltage V_t is small, the plasma is relatively quiescent, and this is the mode of operation of the tokamak. In the reversed-field pinch, the loop voltage is made sufficiently large that the plasma has periods of turbulence, which relax the current profile to form transiently quiescent states. Electrostatic helicity injection S_{es} requires magnetic-field lines that penetrate conducting plates, or electrodes. A given magnetic-field line goes from a plate held at one voltage to a plate held at a different voltage with the voltage difference Φ_h driving a current along the magnetic-field line. If Φ_h is small, the magnetic-field structure is affected only slightly by the presence of the voltage. When Φ_h is large, the currents driven by the electric potential difference produce a field that is large compared to the field B_h that penetrates the plates. When Φ_h is large, unstable plasma states can be created that undergo a Taylor relaxation to plasmas that can have closed magnetic surfaces (Jarboe *et al.*, 1983; Raman *et al.*, 2003; Tang and Boozer, 2004).

IV. CONFINEMENT SYSTEMS

- The two most prominent magnetic configurations for confining plasmas are tokamaks and stellarators. An ideal tokamak is axisymmetric and uses a net toroidal current to produce magnetic surfaces. The tokamak has the most extensive database of all magnetic confinement systems and has been the basis for a number of proposals for experiments in which deuterium and tritium are burned under conditions similar to those of a fusion power plant. Stellarators use

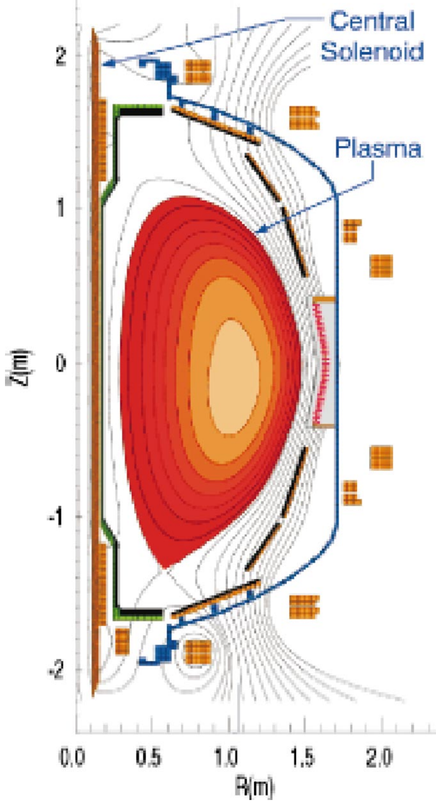

FIG. 8. (Color) The cross section of the NSTX spherical torus (ST; Peng, 2000). All modern tokamaks (Fig. 9) and ST devices have a similar plasma cross section. The field lines that go from the bottom of the plasma to the wall form a divertor; see Sec. VII. Figure courtesy of the Princeton Plasma Physics Laboratory, Princeton, NJ.

helical shaping to produce at least part of the rotational transform.

- The primary control a machine designer has over the performance of a confinement system is the shape of the outermost plasma surface. About 50 properties of the plasma shape can be controlled but only about four of these are consistent with axisymmetry: aspect ratio, ellipticity, triangularity, and squareness.

In the world research effort on magnetically confined plasmas, the tokamak, Fig. 4, and the stellarator, Figs. 5–7, are by far the largest programs. Tokamak plasmas are axisymmetric and carry a net toroidal current in order to form the magnetic surfaces. Stellarator plasmas have the form of a torus with helical shaping, which produces some or all of the rotational transform ι. Of the two, the tokamak has been studied more and more data have been amassed concerning it. It has been the basis for a number of proposals for experiments in which deuterium and tritium are burned under conditions similar to those of a fusion power plant. In addition, the tokamak has a variant, the spherical torus or spherical tokamak (ST; Fig. 8). The ST is like a tokamak at a very tight aspect ratio, $\epsilon_a \equiv a/R_o$, much closer to unity, which has physics advantages (Peng, 2000; Sykes, 2001) as well as a smaller unit size for fusion systems. For tokamaks a tight aspect ratio gives the most desirable physics properties but for stellarators the larger the aspect ratio R_o/a, the easier it is to design desirable physics properties. The helical shaping of the stellarator allows one to design around certain issues of the tokamak and ST. Many of these issues are associated with the maintenance and stability of the net plasma current, issues that can be avoided in the stellarator.

In addition to the tokamak and stellarator, many other magnetic configurations are being actively pursued in the world fusion program. Sheffield (1994) has reviewed many of these configurations. The most prominent are the reversed-field pinch (Ortolani and Schnack, 1993; Prager, 1999), the spheromak (Bellan, 2000), and the magnetic dipole (Garnier, Kesner, and Mauel, 1999; Kesner *et al.*, 2001).

The largest tokamaks have been the Joint European Torus (JET), which is at Culham, England (Keilhacker *et al.*, 1999, 2001); the Tokamak Fusion Test Reactor (TFTR), which was at Princeton, NJ (Hawryluk *et al.*, 1998); and the Japanese Tokamak JT-60U (Kamada *et al.*, 1999). The largest tokamak that is operating in the U.S. is the DIII-D tokamak at General Atomics, Fig. 9, where an investigation is underway to make tokamak plasmas that are better suited for high-pressure, steady-state operation (Chan *et al.*, 2000; Petty *et al.*, 2000). Both the JET and the TFTR tokamaks produced power above the 10-megawatt level by fusing deuterium and tritium (Hawryluk, 1998). However, as expected from the time of their design in the early 1970s, both tokamaks required a high level of externally injected power, comparable to or greater than fusion power, in order to maintain the required plasma temperature, and neither was, therefore, what is meant by a burning plasma experiment.

The largest stellarator (Yamada *et al.*, 2001) is the Japanese Large Helical Device (LHD; Fig. 5). A stellarator of comparable scale, Wendelstein-7X (W7-X), is being constructed in Greifswald, Germany (Beidler *et al.*, 1990). The W7-X stellarator, Fig. 6, has essentially no net current and an unusually small current flowing parallel to the magnetic-field lines. This implies that the shape of the magnetic surfaces is similar with and without plasma. The W7-X design has many innovative features of engineering and physics, which have been studied in the smaller W7-AS stellarator (McCormick *et al.*, 2003) at Garching, Germany. The National Compact Stellarator Experiment (NCSX), Fig. 7, which is under construction at the Princeton Plasma Physics Laboratory (Zarnstorff *et al.*, 2001), is quasiaxisymmetric; see Sec. VI.E.1. Quasiaxisymmetry implies that the particle orbits closely resemble those of a tokamak even though a large fraction of the rotational transform comes from helical shaping. Quasiaxisymmetric stellarators have the feature that they can be designed with an arbitrarily small level of helical shaping, so there is a continuous connection between quasiaxisymmetric stellarator designs and tokamaks. The potential importance is that quasisymmetry allows a minimal modification of the tokamak while giving the freedom to design around phys-

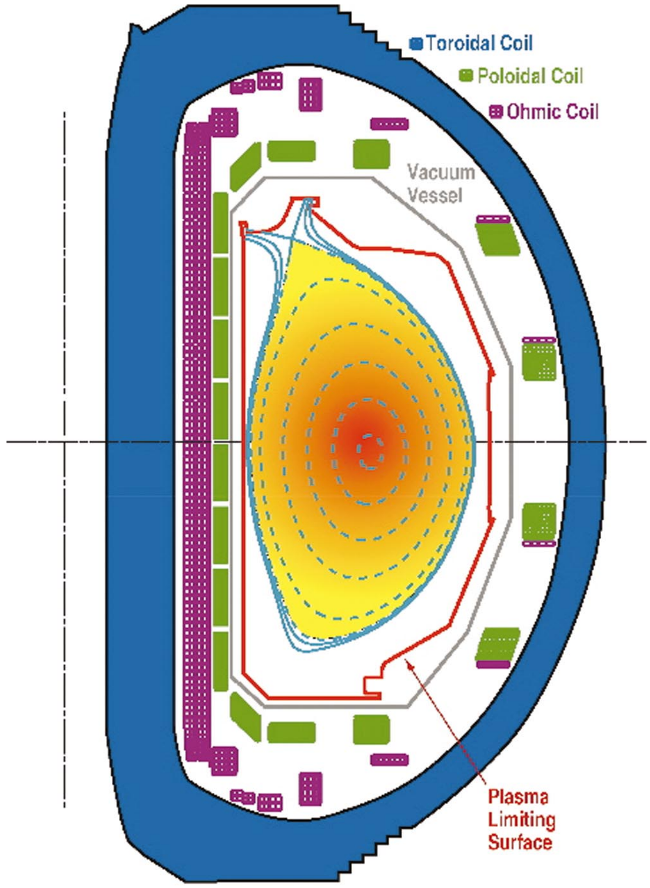

FIG. 9. (Color) Cross section of the DIII-D tokamak. The distance from the machine center line to the plasma center is approximately 1.7 m. The plasma cross section of all modern tokamaks is similar to that of DIII-D. The field lines that go from the top of the plasma to the wall form a divertor; see Sec. VII. Figure courtesy of Edward Lazarus, from Lazarus *et al.*, 1990.

ics issues that could potentially affect the use of the tokamak for practical fusion power.

The primary control that a machine designer has over the plasma properties is the shape of the plasma. As shown in Sec. V.A, a plasma equilibrium is specified (Bauer, Betancourt, and Garabedian, 1984) by giving its toroidal flux content, which is the ψ_t enclosed by the plasma surface; the rotational transform profile $\iota(\psi_t)$; the pressure profile $p(\psi_t)$; and the plasma shape $\vec{x}_s = R_s(\theta,\varphi)\hat{R}(\varphi) + Z_s(\theta,\varphi)\hat{Z}$. The optimal shape for the plasma surface in both tokamaks and stellarators is far from a simple torus with a circular cross section, $R_s = R_o + a\cos\theta$ and $Z_s = -a\sin\theta$. Since a tokamak design is by definition axisymmetric, both R_s and Z_s can depend only on θ. A frequently used expression for the shape of a tokamak plasma is $R_s(\theta) = R_o + a\cos(\theta + \Delta\sin\theta)$ and $Z_s(\theta) = -\kappa a\sin\theta$. The parameter κ is the elongation or ellipticity, and Δ is the triangularity. Stellarators have a much larger design space since R_s and Z_s can depend on both the poloidal angle θ and the toroidal angle φ.

The number of plasma shape parameters that can be controlled by coils at a sufficient distance from the plasma for a fusion power system is limited to about four for a tokamak and about ten times that number for a stellarator (see Sec. V.D.1). The number of free shape parameters for the tokamak is sufficiently small that they all have names: aspect ratio, ellipticity, triangularity, and squareness, though squareness is not defined by an agreed-upon shape function. The performance of tokamaks is markedly improved by a careful choice of these parameters relative to a circular cross section. The number of free parameters that are available to stellarator designers is so large that extensive runs with optimization codes are needed to choose design points. Many of these optimization codes and techniques were developed by Jürgen Nührenberg and co-workers as part of the W7-X design effort (Nührenberg *et al.*, 1995).

V. EFFICIENCY OF MAGNETIC CONFINEMENT

- The cost of fusion power depends on the efficiency with which the magnetic field can be utilized and on the fraction of the power output that must be used to maintain the plasma and the magnetic field, which is called the recirculating power fraction.
- The efficiency of magnetic-field utilization depends on (1) the ratio of the pressure of the plasma to that of the magnetic field, $\beta \equiv 2\mu_0 p/B^2$, and (2) the ratio of the magnetic field at the plasma to that at the coils. The plasma pressure, or β, can be limited by equilibrium, stability, or transport issues.
- The power required to maintain the net current of a tokamak places significant constraints on the design.

The cost of fusion power depends on the efficiency of the confinement. Engineers tend to emphasize the ratio of the power output of a fusion system to its mass. But more physics-oriented efficiency measures are (1) the field strength on the coils needed to confine a given plasma and (2) the ratio of the power output to the power required to sustain the current in the plasma and coils, or more generally, the recirculating power fraction. The power required to sustain the plasma current is discussed in Sec. VI.E.3.

The magnetic field at the coils that is needed to confine a given plasma depends on two numbers. The first is the plasma beta, which is the ratio of the plasma pressure to the magnetic-field pressure,

$$\beta \equiv \frac{2\mu_0 p}{B^2}, \tag{41}$$

and the second is the ratio of the magnetic-field strength on the coils to that in the plasma. The limits on the plasma beta due to equilibrium considerations are discussed in Sec. V.A, and limits from the existence of unstable perturbations of large spatial scale are discussed in Secs. V.B and V.C. The considerations that set the ratio of the magnetic-field strength on the plasma to that on the coils are discussed in Sec. V.D. Section VI will consider limitations on plasma beta, and hence the magnetic-field strength, due to transport.

A. Equilibrium limits

- A plasma equilibrium is determined by the shape of the outermost plasma surface, the toroidal flux enclosed by that surface, and the profiles of the plasma pressure, $p(\psi_t)$, and rotational transform, $\iota(\psi_t) \equiv d\psi_p/d\psi_t$. The rotational transform of a magnetic field is the average number of poloidal transits of the torus a field line makes per toroidal transit. In tokamaks, ι is generally less than unity.
- The plasma $\beta \equiv 2\mu_0 p/B^2$ is limited by the distortions to the plasma shape caused by the current density $j_\parallel$ parallel to the magnetic field that arises to make $\vec{\nabla}\cdot\vec{j}=0$. That is, $\vec{B}\cdot\vec{\nabla}(j_\parallel/B)=-\vec{\nabla}\cdot\vec{j}_\perp$, with $\vec{j}_\perp$ given by the force exerted by the plasma, $\vec{\nabla}p=\vec{j}\times\vec{B}$. These distortions can be important when beta is just a few percent, though much higher betas can be achieved by careful plasma design. In asymmetric plasmas, these distortions of the plasma shape can cause the breakup of the magnetic surfaces that are needed for plasma confinement.
- A magnetic field that lies in surfaces, $\vec{B}\cdot\vec{\nabla}p=0$, has a simple contravariant representation, Eq. (7), which means a representation that uses cross products of pairs of coordinate gradients. A magnetic field associated with a plasma equilibrium simultaneously has a simple covariant representation, Eq. (57) or Eq. (58), which means a representation that uses coordinate gradients. Coordinates in which the magnetic field has both a simple contravariant and a simple covariant representation simplify the analysis of plasmas.

If at each point in a stationary plasma the plasma is close to thermodynamic equilibrium, then the force exerted by a stationary plasma is accurately approximated by its pressure gradient $-\vec{\nabla}p$. The equilibrium between the pressure and electromagnetic forces is $\vec{\nabla}p=\vec{j}\times\vec{B}$, which is known as the equilibrium equation for a plasma.

A toroidal plasma equilibrium is specified by giving (1) the shape of the magnetic surface that bounds the plasma, $\vec{x}_s(\theta,\varphi)$, with θ and φ arbitrary poloidal and toroidal angles, (2) the total toroidal magnetic flux in the plasma, (3) the pressure profile $p(\psi_t)$, and (4) the rotational transform profile $\iota(\psi_t)$ (Bauer, Betancourt, and Garabedian, 1984). This result is proven by showing that equilibria are extrema of the energy in a toroidal region bounded by a fixed surface $\vec{x}_s(\theta,\varphi)$ when the ideal, which means dissipationless, plasma constraints are observed. The energy of the plasma plus the magnetic field is

$$W=\int_{plasma}\left(\frac{B^2}{2\mu_0}+\frac{p}{\gamma-1}\right)d^3x, \tag{42}$$

where γ is the adiabatic index. The variation that is carried out is in the shape of the magnetic surfaces $\vec{x}(\psi_t,\theta,\varphi)$ with the shape of the outermost plasma surface $\vec{x}_s(\theta,\varphi)$ held fixed. That is, $\vec{x}(\psi_t,\theta,\varphi)\to\vec{x}(\psi_t,\theta,\varphi)+\vec{\xi}$, where $\vec{\xi}$ is the displacement of the surfaces with the displacement $\vec{\xi}$ zero on the plasma surface.

Using the expressions that are derived below for the variation in magnetic-field energy and the variation in the pressure in response to a small plasma displacement $\vec{\xi}$, one finds that

$$\delta W=\int_{plasma}(\vec{\nabla}p-\vec{j}\times\vec{B})\cdot\vec{\xi}d^3x, \tag{43}$$

which means W has equilibria, $\vec{\nabla}p=\vec{j}\times\vec{B}$ as its extrema. The minimization of the energy W gives the shape $\vec{x}(\psi_t,\theta,\varphi)$ of all of the magnetic surfaces in the plasma. The best-known code for finding equilibria by minimizing the energy is called VMEC (Hirshman, van Rij, and Merkel, 1986). It should be noted that the adiabatic index γ does not appear in the equilibrium equation, so γ can be chosen arbitrarily. The choice $\gamma=0$ is frequently made because with this choice $dp/dt=0$. When $dp/dt=0$, the pressure profile $p(\psi_t)$ is unchanged by variations $\vec{\xi}$ in the shape of the surfaces.

This paragraph gives the derivations of the variation in magnetic field and the plasma pressure in response to a small plasma displacement $\vec{\xi}$ and can be skipped on a first reading. These derivations have four parts:

(1) First, the relation between the total and the partial time derivative must be explained. In fluid mechanics the total time derivative dg/dt gives the rate of change of any function $g(\vec{x},t)$ in the frame of reference of the moving fluid, while the partial time derivative $(\partial g/\partial t)_{\vec{x}}$ gives the time derivative at a fixed spatial point. The partial derivative $(\partial g/\partial t)_{\vec{x}}$ is usually written as $\partial g/\partial t$. The rate of change of the position vector $\vec{x}$ is the flow velocity of the fluid, $\vec{v}\equiv d\vec{x}/dt$. The chain rule implies $dg/dt=\partial g/\partial t+(d\vec{x}/dt)\cdot\vec{\nabla}g$, so the total time derivative is

$$\frac{dg}{dt}\equiv\frac{\partial g}{\partial t}+\vec{v}\cdot\vec{\nabla}g. \tag{44}$$

(2) Second, the total time derivative of the canonical coordinates must be shown to be zero in an ideal fluid. The total time derivative is zero because the coordinates are carried with the fluid, though the partial time derivatives are nonzero if the fluid is moving. This result is proven using the theory of general coordinates (Appendix), which implies that each canonical coordinate of the transformation function $\vec{x}(\psi_t,\theta,\varphi,t)$ obeys

$$\frac{d\psi_t}{dt}=(\vec{v}-\vec{u})\cdot\vec{\nabla}\psi_t, \tag{45}$$

with $\vec{u}\equiv(\partial\vec{x}/\partial t)_c$ [Eq. (29)]. In an ideal plasma, $\vec{u}=\vec{v}$, so the canonical coordinates satisfy $d\psi_t/dt=0$.

(3) Third, the variation in the magnetic field, $\delta\vec{B}$, in an ideal plasma must be found for a small plasma displacement $\vec{\xi}$. The variation is

$$\delta\vec{B} = \vec{\nabla} \times (\vec{\xi} \times \vec{B}). \tag{46}$$

This equation follows from Ohm's law, Eq. (32), with zero resistivity, $\vec{E}+\vec{v}\times\vec{B}=0$, plus Faraday's law, $\partial\vec{B}/\partial t=-\vec{\nabla}\times\vec{E}$, and $\vec{v}=\partial\vec{\xi}/\partial t$. Actually $\vec{v} \equiv d\vec{x}/dt=\partial\vec{\xi}/\partial t+\vec{v}\cdot\vec{\nabla}\vec{\xi}$, but for a sufficiently small displacement, $\vec{\xi}\to 0$, one has $\vec{v}=\partial\vec{\xi}/\partial t$. The variation in the magnetic energy is determined using a vector identity to write $\vec{B}\cdot\delta\vec{B}=(\vec{\xi}\times\vec{B})\cdot\vec{\nabla}\times\vec{B}+\vec{\nabla}\cdot\{(\vec{\xi}\times\vec{B})\times\vec{B}\}$. The divergence term makes no contribution because the plasma surface has a fixed shape, which means $\vec{\xi}=0$ on the bounding surface. As shown in Sec. III.C, the rotational transform retains its initial profile, $\iota(\psi_t)$, through all variations in the plasma shape when the resistivity is zero.

(4) Fourth, the pressure variation and the displacement must be related. A pressure variation in an ideal plasma conserves the entropy per particle, which implies $(d\ln p/dt)/(d\ln n/dt)=\gamma$. The plasma number density obeys the continuity equation, $\partial n/\partial t+\vec{\nabla}\cdot(n\vec{v})=0$, so $dp/dt=-\gamma p\vec{\nabla}\cdot\vec{v}$. This means the change in the pressure at a fixed point in space, $\delta p \equiv(\partial p/\partial t)_{\vec{x}}\delta t$, is

$$\delta p = -\vec{\xi}\cdot\vec{\nabla}p - \gamma p\vec{\nabla}\cdot\vec{\xi}. \tag{47}$$

The variation in the pressure can be rewritten in a form that makes that part of the energy variation obvious $\delta p=(\gamma-1)\vec{\xi}\cdot\vec{\nabla}p-\vec{\nabla}\cdot(\gamma p\vec{\xi})$. The $(\gamma-1)$ term in δp cancels the similar factor in W and the divergence term in δp makes no contribution to the change in the energy since the change in the shape $\vec{\xi}$ is zero on the plasma boundary.

An important theorem of equilibrium theory is that given an arbitrary set of magnetic surfaces, $\vec{x}(\psi_t,\theta,\varphi)$, and an arbitrary rotational transform profile $\iota(\psi_t)$, the magnetic force $\vec{f}_B\equiv\vec{j}\times\vec{B}$ can always be written as $\vec{f}_B=f_{\psi_t}\vec{\nabla}\psi_t$ with $f_{\psi_t}(\psi_t,\theta,\varphi)$ a known function. The angles θ and φ are arbitrary poloidal and toroidal angles, but ψ_t is the toroidal magnetic flux enclosed by a magnetic surface. If $\vec{x}(\psi_t,\theta,\varphi)$ and $\iota(\psi_t)$ are consistent with an equilibrium, the function f_{ψ_t} depends on ψ_t alone, and the equilibrium pressure profile $p(\psi_t)$ is given by $dp/d\psi_t=f_{\psi_t}$. To prove this theorem, note that if $\vec{\nabla}\psi_t\times\vec{f}_B\equiv(\vec{B}\cdot\vec{\nabla}\psi_t)\vec{j}-(\vec{j}\cdot\vec{\nabla}\psi_t)\vec{B}$ is zero, then $\vec{f}_B=f_{\psi t}\vec{\nabla}\psi_t$. Since $\vec{B}\cdot\vec{\nabla}\psi_t=0$ by the definition of magnetic surfaces, the result is proven if the equilibrium constraint $\vec{j}\cdot\vec{\nabla}\psi_t=0$ can be imposed. The most general divergence-free field, $\vec{\nabla}\cdot\vec{B}=0$, that has the required surfaces, $\vec{B}\cdot\vec{\nabla}\psi_t=0$, and rotational transform profile, $\iota(\psi_t)$, is

$$\vec{B} = \left(1+\frac{\partial\lambda}{\partial\theta}\right)\frac{\vec{\nabla}\psi_t\times\vec{\nabla}\theta}{2\pi} + \left(\iota-\frac{\partial\lambda}{\partial\varphi}\right)\frac{\vec{\nabla}\varphi\times\vec{\nabla}\psi_t}{2\pi}, \tag{48}$$

which is proven using the argument that led to Eq. (7). The constraint $\mu_0\vec{j}\cdot\vec{\nabla}\psi_t=(\vec{\nabla}\times\vec{B})\cdot\vec{\nabla}\psi_t=\vec{\nabla}\cdot(\vec{B}\times\vec{\nabla}\psi_t)=0$ determines λ as the solution to a Poisson-like equation that has derivatives only in the θ and φ coordinates. Periodicity in θ and φ implies that λ has a definite expression, which can be determined magnetic surface by magnetic surface. The function λ is the difference between the arbitrary and the magnetic poloidal angles, $\theta_m=\theta+\lambda$, and determines how the field lines wind through the magnetic surfaces. Once λ is determined, one has a definite expression for $\vec{B}$, which gives the magnetic force $\vec{f}_B$. The calculation of the curl of the magnetic field is complicated but is explained in the Appendix.

The equilibrium with profiles $\iota(\psi_t)$ and $p(\psi_t)$ has been shown to be an extremum of $W=\int\{(B^2/2\mu_0)-p(\psi_t)\}d^3x$. The extremum can also be obtained by allowing arbitrary variations in both the λ function of Eq. (48) and the shape of nested magnetic surfaces $\vec{x}(\psi_t,\theta,\varphi)$ with the outermost plasma surface held fixed. These variations, unlike the variation in shape that led to Eq. (43), are unconstrained and do not use Eqs. (46) or (47). This alternate method of extremizing the energy follows from the magnetic field's obeying the ideal constraints if $\iota(\psi_t)$ is conserved and if the ψ_t surfaces of $\vec{x}(\psi_t,\theta,\varphi)$ remain nested. The same Poisson-like equation for λ is obtained when $\int B^2d^3x$ is minimized with respect to λ.

One can always minimize the energy W over any set of shape functions $\vec{x}(\psi_t,\theta,\varphi)$, so it may at first appear that equilibria always exist. This is not true for two reasons. First, the set of shape functions that are considered may not be rich enough to find an extremum in which f_{ψ_t} is a function of ψ_t alone, as is required for equilibrium. Second, if one considers an arbitrarily rich set of shape functions, the parallel component of the current, $j_\parallel\equiv\vec{j}\cdot\vec{B}/B$, may become singular at the rational surfaces. Rational magnetic surfaces are defined by the rotational transform's being the ratio of two integers, $\iota=n/m$. These are surfaces on which the magnetic-field lines close on themselves after m toroidal and n poloidal circuits of the torus.

The parallel current determines much of the theory of force balance of a plasma embedded in a magnetic field. The parallel current is given by $\vec{\nabla}\cdot\vec{j}=\vec{B}\cdot\vec{\nabla}(j_\parallel/B)+\vec{\nabla}\cdot\vec{j}_\perp=0$, with the perpendicular current determined by the force the magnetic field exerts on the plasma, $\vec{f}_B\equiv\vec{j}\times\vec{B}$. The equation for variation of the parallel current along the field lines is

$$\vec{B}\cdot\vec{\nabla}\frac{j_\parallel}{B} = \vec{\nabla}\cdot\left(\frac{\vec{f}_B\times\vec{B}}{B^2}\right), \tag{49}$$

which is a magnetic differential equation, Eq. (12), for the parallel current. In an equilibrium plasma, $\vec{f}_B=\vec{\nabla}p$.

The subtlety of the parallel current in a plasma equilibrium is clarified by the equation for $j_\parallel/B$ in (ψ_t,α,ϕ) Clebsch coordinates,

$$\frac{\partial}{\partial\phi}\left(\frac{j_\parallel}{B}\right)=-\left(\frac{dp}{d\psi_t}\right)\frac{\partial}{\partial\alpha}\frac{1}{B^2}. \tag{50}$$

On closed magnetic-field lines $u\equiv\oint d\phi/B^2=\oint d\ell/B$ must be independent of α for a single-valued $j_\parallel/B$ to exist. To have a nonzero pressure gradient on a rational surface, u must have the same value on every field line of that surface. In (ψ_t,α,ϕ) coordinates, $\vec{B}=\vec{\nabla}\phi+B_{\psi_t}\vec{\nabla}\psi_t+B_\alpha\vec{\nabla}\alpha$, Eq. (11), which can be simplified in an equilibrium plasma to $\vec{B}=\vec{\nabla}\phi+B_{\psi_t}\vec{\nabla}\psi_t$ since $\vec{\nabla}\times\vec{B}\cdot\vec{\nabla}\psi_t=0$. The divergence of the perpendicular current is $\vec{\nabla}\cdot\{(\vec{B}\times\vec{\nabla}p)/B^2\}=(dp/d\psi_t)\{\vec{\nabla}\phi\cdot(\vec{\nabla}\psi_t\times\vec{\nabla}\alpha)\}\partial(1/B^2)/\partial\alpha$, so $\vec{\nabla}\cdot j_\perp=B^2(dp/d\psi_t)\partial(1/B^2)/\partial\alpha$. The divergence of the parallel current is $\vec{B}\cdot\vec{\nabla}(j_\parallel/B)=B^2\partial(j_\parallel/B)/\partial\phi$. The constraint that $\vec{\nabla}\cdot j=0$ gives Eq. (50).

A solution of Eq. (49) for $j_\parallel/B$ is the sum of (1) a special solution, which varies over the magnetic surface and is given by any solution to the magnetic differential equation, and (2) a homogeneous solution in which $j_\parallel/B$ is constant along field lines, $\vec{B}\cdot\vec{\nabla}(j_\parallel/B)_h=0$. Both the special and the homogeneous solution can be singular at rational surfaces. A singularity of the special solution, $(j_\parallel/B)_s=j_{PS}/B$, where j_{PS} is the *Pfirsch-Schlüter current* (Pfirsch and Schlüter, 1962), can arise if the pressure gradient $dp/d\psi_t$ is nonzero at a rational surface on which $\oint d\ell/B$ varies. This singularity is discussed more extensively below. The homogeneous solution, $(j_\parallel/B)_h=j_{net}/B$, where j_{net} is the net current, has the form

$$\mu_0\frac{j_{net}}{B}=k(\psi_t)+\sum_{m,n}k_{mn}e^{i(n\varphi-m\theta_m)}\delta(\psi_t-\psi_{mn}), \tag{51}$$

with ψ_{mn} defined by $\iota(\psi_{mn})=n/m$ and $\delta(\psi_t-\psi_{mn})$, the Dirac delta function. Each nonzero constant k_{mn} implies a surface current on a rational surface, which produces a normal magnetic field that cancels the resonant part of the normal field due to external perturbations and prevents the formation of an island.

A more complete exploration of the special solution for $j_\parallel/B$ of Eq. (49), the Pfrisch-Schlüter current, requires a covariant representation, Eq. (56), of $\vec{B}$ in general magnetic coordinates. Magnetic coordinates are defined by $\vec{B}$ having the contravariant representation,

$$2\pi\vec{B}=\vec{\nabla}\psi_t\times\vec{\nabla}\theta_m+\iota(\psi_t)\vec{\nabla}\varphi\times\vec{\nabla}\psi_t, \tag{52}$$

which follows from Eq. (7) and $\iota\equiv d\psi_p/d\psi_t$. The general covariant representation can be obtained from the covariant expression in Clebsch coordinates, $\vec{B}=\vec{\nabla}\phi+B_{\psi_t}\vec{\nabla}\psi_t$, that was used in the derivation of Eq. (50) for $\partial(j_\parallel/B)/\partial\phi$. However, the derivation is almost as simple starting from fundamental principles.

The covariant representation of the magnetic field, Eq. (56), follows from the contravariant representation of the current, $\vec{j}=\vec{\nabla}\times\vec{B}/\mu_0$, Eq. (53). The derivation of the contravariant representation of the magnetic field, Eq. (7), depended on two conditions: (1) $\vec{\nabla}\cdot\vec{B}=0$ and (2) that a nonconstant function ψ_t exist with $\vec{B}\cdot\vec{\nabla}\psi_t=0$. These two conditions also hold for the current associated with a plasma equilibrium, since it is divergence-free and in equilibrium satisfies $\vec{j}\cdot\vec{\nabla}\psi_t=0$, so the current associated with an equilibrium must have the contravariant representation

$$\vec{j}=-\frac{\partial G_{tot}}{\partial\psi_t}\frac{\vec{\nabla}\varphi\times\vec{\nabla}\psi_t}{2\pi}+\frac{\partial I_{tot}}{\partial\psi_t}\frac{\vec{\nabla}\psi_t\times\vec{\nabla}\theta_m}{2\pi}, \tag{53}$$

where the radial derivatives of the total poloidal and total toroidal current are

$$\frac{\partial G_{tot}}{\partial\psi_t}=\frac{dG(\psi_t)}{d\psi_t}+\frac{\partial\nu(\psi_t,\theta_m,\varphi)}{\partial\varphi} \tag{54}$$

and

$$\frac{\partial I_{tot}}{\partial\psi_t}=\frac{dI(\psi_t)}{d\psi_t}+\frac{\partial\nu(\psi_t,\theta_m,\varphi)}{\partial\theta_m}. \tag{55}$$

The function $G(\psi_t)=\int\vec{j}\cdot d\vec{a}_{\theta_m}$ is the poloidal current in the region exterior to a constant-ψ_t surface, which is the current through the hole in the torus, Fig. 1, while $I(\psi_t)=\int\vec{j}\cdot d\vec{a}_\varphi$ is the toroidal current in the region interior to a constant-ψ_t surface, which is the current through a cross section of the torus (Fig. 1). The contravariant representation of the current implies the magnetic field has the covariant repesentation

$$\vec{B}=\frac{\mu_0}{2\pi}\{G(\psi_t)\vec{\nabla}\varphi+I(\psi_t)\vec{\nabla}\theta_m-\nu\vec{\nabla}\psi_t+\vec{\nabla}F\}. \tag{56}$$

We shall find that the function $\nu(\vec{x})$ is determined by the equilibrium equation. The function $F(\vec{x})$ is determined by the constraint that $\vec{\nabla}\cdot\vec{B}=0$ and the boundary conditions on the magnetic field.

The contravariant representation of the magnetic field, Eq. (52), is unchanged if one defines new poloidal and toroidal angles by $\theta_m=\theta_n+\iota\omega$ and $\varphi=\varphi_n+\omega$, where ω is any well-behaved function of position. If these forms are substituted into the covariant representation of the magnetic field, Eq. (56), one obtains an equation of the same form but with ν replaced by $\nu_n\equiv\nu+(dG/d\psi_t+\iota dI/d\psi_t)\omega$ and with F replaced by $F_n\equiv F+(G+\iota I)\omega$. There are two obvious choices for ω. The first makes $\nu_n=0$, and the resulting coordinates are called Hamada coordinates ($\psi_t,\theta_H,\varphi_H$; Hamada, 1962), in which

$$\vec{B}=\frac{\mu_0}{2\pi}\{G(\psi_t)\vec{\nabla}\varphi_H+I(\psi_t)\vec{\nabla}\theta_H+\vec{\nabla}F\}. \tag{57}$$

The second makes $F_n=0$, and the resulting coordinates are called Boozer coordinates ($\psi_t,\theta_B,\varphi_B$; Boozer, 1981), in which

$$\vec{B}=\frac{\mu_0}{2\pi}\{G(\psi_t)\vec{\nabla}\varphi_B+I(\psi_t)\vec{\nabla}\theta_B+\beta_*\vec{\nabla}\psi_t\}. \tag{58}$$

The cross product between the contravariant representation of the current, Eq. (53), and of the magnetic field, Eq. (52), must equal the pressure gradient. This equality gives an equation for ν, Eq. (63), and an equation for the average equilibrium on a magnetic surface, Eq. (62). Using arbitrary magnetic coordinates $(\psi_t,\theta_m,\varphi)$,

$$\vec{j}\times\vec{B}=-\frac{\vec{B}\cdot\vec{\nabla}\varphi}{2\pi}\left(\frac{\partial G_{tot}}{\partial\psi_t}+\iota\frac{\partial I_{tot}}{\partial\psi_t}\right)\vec{\nabla}\psi_t. \tag{59}$$

Since $\vec{\nabla}p=(dp/d\psi_t)\vec{\nabla}\psi_t$,

$$\frac{\partial G_{tot}}{\partial\psi_t}+\iota\frac{\partial I_{tot}}{\partial\psi_t}=-\frac{2\pi}{\vec{B}\cdot\vec{\nabla}\varphi}\frac{dp}{d\psi_t}. \tag{60}$$

The contravariant form for the magnetic field, Eq. (52), implies that the Jacobian of magnetic coordinates, $1/\mathcal{J}=(\vec{\nabla}\psi_t\times\vec{\nabla}\theta_m)\cdot\vec{\nabla}\varphi$, is

$$\mathcal{J}=\frac{1}{2\pi\vec{B}\cdot\vec{\nabla}\varphi}, \tag{61}$$

with the volume enclosed by the magnetic surfaces $V(\psi_t)=\int\mathcal{J}d\psi_t d\theta_m d\varphi$. If Eq. (60) is averaged over the poloidal and toroidal angles, one obtains the average equilibrium equation (Kruskal and Kulsrud, 1958),

$$\frac{dG}{d\psi_t}+\iota\frac{dI}{d\psi_t}=-\frac{dV}{d\psi_t}\frac{dp}{d\psi_t}, \tag{62}$$

which is a coordinate-independent equation. The function ν satisfies a magnetic differential equation,

$$\left(\frac{\partial}{\partial\varphi}+\iota\frac{\partial}{\partial\theta_m}\right)\nu=\left(\frac{dV}{d\psi_t}-(2\pi)^2\mathcal{J}\right)\frac{dp}{d\psi_t}. \tag{63}$$

The function ν and hence the current are singular at a rational surface $\iota=n/m$ unless either the resonant Fourier term of $2\pi\mathcal{J}=1/\vec{B}\cdot\vec{\nabla}\varphi$ is zero or the pressure gradient $dp/d\psi_t$ is zero.

Since Eq. (63) for ν holds in any set of magnetic coordinates, it must hold in Hamada coordinates, in which $\nu=0$. This means the Jacobian of Hamada coordinates is independent of the angles, $(2\pi)^2\mathcal{J}_H=dV/d\psi_t$, but also that the coordinate transformation to Hamada coordinates is singular if the function ν is singular when one uses the cylindrical angle φ as the toroidal angle.

The Jacobian of Boozer coordinates can be obtained by dotting the covariant and contravariant representations of $\vec{B}$ together, $\mathcal{J}_B=\mu_0(G+\iota I)/(2\pi B)^2$. The parallel current is particularly simple in these coordinates,

$$\frac{j_\parallel}{B}=\frac{G\dfrac{dI}{d\psi_t}-I\dfrac{dG}{d\psi_t}}{G+\iota I}+\frac{G\dfrac{\partial\nu}{\partial\theta_B}-I\dfrac{\partial\nu}{\partial\varphi_B}}{G+\iota I}. \tag{64}$$

The first term on the right-hand side is $j_{net}/B=k(\psi_t)/\mu_0$ with j_{net} the net current, and the second term is j_{PS}/B with j_{PS} the Pfirsch-Schlüter current. Equation (64) for $j_\parallel/B$ is obtained from the dot product of Eqs. (53) and (58).

In axisymmetric equilibria the expression for the Pfirsch-Schlüter current can be simplified to

$$\frac{j_{PS}}{B}=\frac{1}{\iota}\frac{G}{G+\iota I}\left(\frac{dV}{d\psi_t}-\frac{2\pi}{\vec{B}\cdot\vec{\nabla}\varphi}\right)\frac{dp}{d\psi_t}, \tag{65}$$

with $dV/d\psi_t=\oint d\theta_m/\vec{B}\cdot\vec{\nabla}\varphi$. This equation for j_{PS} follows from Eq. (64) with $\partial\nu/\partial\varphi_B=0$ and Eq. (63), which gives an expression for $\partial\nu/\partial\theta_B$. Equation (65) is written in a form that is valid in any coordinates in which φ is the toroidal angle and ψ_t is the toroidal flux.

Even in simple axisymmetric equilibria, the Pfirsch-Schlüter current j_{PS} provides a significant limitation on the plasma pressure. At large aspect ratio, $R_o/r\gg1$, the vacuum toroidal field is dominant, that is, $G\gg\iota I$, and $1/\vec{B}\cdot\vec{\nabla}\varphi\approx R^2/(R_oB_o)$. The Pfirsch-Schlüter current, Eq. (65), is in the toroidal direction and approximated by

$$j_{PS}\approx-\frac{2(R-R_o)}{\iota B_o}\frac{1}{r}\frac{dp}{dr}, \tag{66}$$

with $R-R_o=r\cos\theta$. If the pressure is assumed to depend on the minor radius r as $p=p_0(1-r^2/a^2)$, one finds the Pfirsch-Schlüter current produces a magnetic field in the vertical or $\hat{Z}$ direction, $\partial\delta B_Z/\partial R=-\mu_0 j_{PS}$, which at the plasma edge, $(R-R_o)^2=a^2$, is $\delta B_Z\approx-2\mu_0p_0/(\iota B_o)$. This vertical field opposes the poloidal field due to the plasma current, $B_\theta\approx\epsilon_a\iota B_o$, on the inboard side with $\epsilon_a\equiv a/R_o$. When the vertical field becomes sufficiently strong to cancel the poloidal field on the inboard side, the equilibrium has reached its pressure limit. In other words, equilibria must have a volume-averaged beta, $\langle\beta\rangle\equiv2\mu_0\langle p\rangle/B_o^2$ with $\langle p\rangle=p_0/2$, which satisfies

$$\langle\beta\rangle\lesssim\frac{1}{2}\epsilon_a\iota^2. \tag{67}$$

Stability limits the average rotational transform in a tokamak to $\iota\approx1/2$ and the inverse aspect ratio is $\epsilon_a\approx1/3$, so $\langle\beta\rangle\lesssim4\%$ when the magnetic surfaces are circular. Actual tokamaks with additional shaping have achieved $\langle\beta\rangle$ greater than 10%, and ST plasmas have achieved $\langle\beta\rangle$ approaching 50%. However, the poor scaling of the beta limit with aspect ratio limits tokamaks to having a tight aspect ratio. The weakening of the poloidal field on the inboard side and the strengthening on the outboard side cause the magnetic surfaces to shift outward, a phenomenon called the *Shafranov shift*, so the same flux passes between the plasma axis and the inboard and outboard edges.

The equilibrium of an axisymmetric plasma can also be found using the Grad-Shafranov equation (Lüst and Schlüter, 1957; Shafranov, 1958; Grad and Rubin, 1959). Using (R,φ,Z) cylindrical coordinates, an arbitrary axisymmetric, divergence-free field can be written in a mixed covariant-contravariant representation as $2\pi\vec{B} = \mu_0 G\vec{\nabla}\varphi + \vec{\nabla}\varphi\times\vec{\nabla}\psi_p$, which follows from $\vec{\nabla}\cdot(\vec{\nabla}\varphi)=0$ in cylindrical coordinates. The curl of this mixed representation gives $2\pi\vec{j}=\vec{\nabla}G\times\vec{\nabla}\varphi+\hat{\varphi}(\Delta^*\psi_p)/(\mu_0 R)$, where

$$\Delta^*\psi_p \equiv R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi_p}{\partial R}\right)+\frac{\partial^2\psi_p}{\partial Z^2} \tag{68}$$

is called the Grad-Shafranov operator and $\hat{\varphi}$ is a unit vector of cylindrical coordinates. From $\vec{j}\times\vec{B} = (dp/d\psi_p)\vec{\nabla}\psi_p$ one finds G is a function of ψ_p alone and

$$\Delta^*\psi_p = -\mu_0^2 G\frac{dG}{d\psi_p} - \mu_0(2\pi R)^2\frac{dp}{d\psi_p}, \tag{69}$$

which is the Grad-Shafranov equation.

Equation (49) for the variation in the parallel current also has important implications for nontoroidal plasmas embedded in magnetic fields. For example, the photosphere of the sun becomes sufficiently tenuous with altitude that the forces exerted by the magnetic field on the plasma must be small, $\vec{f}_B\equiv\vec{j}\times\vec{B}\to 0$. In this situation, which is called a force-free equilibrium, the parallel current satisfies the conservation law that $j_\parallel/B$ be constant along magnetic-field lines. The constancy of $j_\parallel/B$ implies that stars, like the sun, with outer convective zones and a magnetic field, must have a corona, a region in the upper solar atmosphere of very-high-energy electrons compared to the energy that would be expected from thermal diffusion (Boozer, 1999). The reason is that the current $j=env_s$ is the product of the electron charge, number density, and streaming velocity v_s of the electrons relative to the ions. If the streaming velocity exceeds the electron thermal velocity $v_e\equiv\sqrt{T_e/m_e}$, the electrons will break away from a Maxwellian distribution and reach whatever energy is necessary to carry the current. This is the phenomenon of runaway electrons (Dreicer, 1960; Connor and Hastie, 1975), which arises from the collision frequency's dropping as one over the speed of a particle cubed, Eq. (134), when the speed is larger than the thermal velocity. In the absence of a corona in a star like the sun, the density of the atmosphere drops exponentially with height, $n(z) \propto \exp(-z/h)$, due to gravitational force balance, $\vec{\nabla}p=\rho\vec{g}$, with a scale height $h\approx 100$ km. Although the fractional ionization increases with altitude, the total number of electrons that are potentially available to carry the current rapidly drops. The magnetic field above the sun has structures with a scale of 10^4 km, spanning a range of atmospheric densities of e^{100}. Since $j_\parallel/B$ is approximately constant due to the rapid drop in the atmospheric pressure, the streaming parameter v_s/v_e would increase by a factor of roughly $e^{100}\approx 10^{43}$ without a corona. This is of course impossible without the streaming parameter becoming greater than one and the electrons running away. At the bottom of the photosphere, magnetic fields are observed to have variations of 0.1 T over 10^3 km, so local current densities have to be greater than $j \approx 10^2$ A/m^2. The current that could be carried by the full electron density (neutral plus ionized) moving at the electron thermal speed is of order 10^{10} A/m^2, which is only eight orders of magnitude larger. Actually, the local current density on magnetic-field lines emerging from the sun should have a value orders of magnitude larger than the observed current density of 10^2 A/m^2, which by the observation process involves a spatial average. The reason a large current density should be expected is that when field lines are churned around in a conducting fluid, as they are in the outer solar convective zone, they develop strong parallel currents with very short correlation distances (Thiffeault and Boozer, 2003). These currents flow all along the field lines, Eq. (49), relaxing to their equilibrium values via shear Alfvén waves; see Sec. VI.H. Runaway electrons are not generally considered as an explanation for the actual solar corona. Whether the currents along the magnetic-field lines that penetrate the solar surface have sufficient strength and shortness of correlation length to cause the actual solar corona are questions that are not easily answered observationally.

B. Stability limits

- A magnetically confined plasma cannot be in thermodynamic equilibrium, so a potential for instability always exists. Instabilities can be driven by the pressure gradient or the net current in the plasma.
- The pressure gradient is destabilizing when the center of curvature of the magnetic-field lines, $\vec{\kappa} \equiv\hat{b}\cdot\vec{\nabla}\hat{b}$ with $\hat{b}\equiv\vec{B}/B$, is in the direction of higher pressure (bad curvature) and stabilizing when the curvature has the opposite sign (good curvature).
- Stability calculations can be done by preserving the constraints of an ideal, dissipationless, plasma evolution, which prevent islands from opening but give singular currents, or by keeping smooth current distributions, which allow islands to open. Through linear order in the perturbation theory, there is no distinction in these two types of analyses except near rational surfaces $\iota=n/m$. However, the ideal analysis always predicts greater stability.

A magnetically confined plasma always has the thermodynamic free energy that is required for instability. The maximum entropy state of the charged particles that form a plasma is a Gibbs distribution, $\exp(-H/T)$, but the particle energy $H=\frac{1}{2}mv^2+q\Phi$ is independent of the magnetic field $\vec{B}(\vec{x})$. A plasma that is carrying a current cannot be in a Gibbs distribution and therefore always has thermodynamic free energy. The stability of current-carrying plasmas depends on the existence of constraints on the plasma motion that prevent the reduction in the free energy.

The free energy that is available for instabilities allows toroidal plasmas to spontaneously break their symmetry by kinking. This kinking is driven by the gradients in the parallel current distribution, $k \equiv \mu_0 j_\parallel / B$, and in the pressure. Both drives limit the achievable plasma beta (ITER Physics Expert Group on Disruptions, Plasma Control, and MHD *et al.*, 1999). Books by Bateman (1978) and Freidberg (1987) discuss plasma stability under the assumptions of ideal magnetohydrodynamics (MHD), which means in the absence of dissipative effects.

Instabilities driven by the gradient in the parallel current distribution, $d(j_\parallel/B)/d\psi_t$, limit the magnitude of the poloidal magnetic field or more precisely the rotational transform, and both equilibrium and stability limits involve the strength of the poloidal magnetic field.

The pressure limit due to plasma stability for a tokamak plasma is approximated with remarkable success, even for highly shaped plasmas, by the *Troyon limit* (Troyon and Gruber, 1985). The volume-averaged $\beta \equiv 2\mu_0 p/B^2$ can be no greater than

$$\langle \beta \rangle = \frac{\beta_n}{10} \frac{\mu_0}{4\pi} \frac{I}{aB}, \tag{70}$$

where I is the net toroidal current and a is the half-width of the plasma in the $Z=0$ plane. The Troyon coefficient, also called beta normal, is a number which is $\beta_n \approx 3$ in somewhat optimized plasmas but can be a factor of 2 larger. The results of tokamak experiments are often expressed by giving the β_n that was achieved.

The stability of a plasma to a pressure gradient is largely determined by the relative orientation of the pressure gradient $\vec{\nabla} p$ and the curvature of the magnetic field lines, $\vec{\kappa} \equiv \hat{b} \cdot \vec{\nabla} \hat{b}$, where $\hat{b} \equiv \vec{B}/B$ is a unit vector along the magnetic field. For example, the curvature of a circular line $\hat{b} = \hat{\varphi}(\varphi)$ of (R, φ, Z) cylindrical coordinates is $\vec{\kappa} = -\hat{R}/R$. The pressure gradient directly enters the energy change due to a displacement $\vec{\xi}$ of an equilibrium plasma through $-\int (\vec{\xi} \cdot \vec{\nabla} p)(\vec{\xi} \cdot \vec{\kappa}) d^3x$ [Eq. (92)]. Good curvature means $\vec{\kappa} \cdot \vec{\nabla} p < 0$ and bad curvature means $\vec{\kappa} \cdot \vec{\nabla} p > 0$. Good curvature is stabilizing because that geometry compresses plasma motion down the pressure gradient. Unfortunately, topology implies the field-line curvature must be bad somewhere on a toroidal magnetic surface, and pressure-driven instabilities localize in these places. Localization is stabilizing because it implies a bending of the magnetic field. If the bad curvature regions are sufficiently localized, the pressure gradient does not destabilize the equilibrium.

If the plasma is assumed to be ideal, which means zero resistivity, then the evolution equations for the magnetic field imply the rotational transform ι is a time-independent function of ψ_t. In the presence of a perturbation, this assumption generally leads to a singular net plasma current, $\mu_0 \vec{j}_{net} = k\vec{B}$ with $\vec{B} \cdot \vec{\nabla} k = 0$. One can avoid a singular net current by assuming that the distribution of net current $k(\psi_t)$ is independent of time instead of $\iota(\psi_t)$. The assumption of a fixed net current is a resistive stability analysis.

Remarkably, linear perturbation theory makes no distinction between an ideal analysis, $\iota(\psi_t)$ independent of time, and a resistive analysis, $k(\psi_t)$ independent of time, except at resonant rational surfaces, $\iota = n/m$. In an ideal analysis a singular current arises at resonant rational surfaces and in a resistive analysis an island opens. Away from resonant rational surfaces, the change in the rotational transform scales as the amplitude of the magnetic perturbation squared and, therefore, cannot enter a linear stability analysis. To see this consider the perturbed field-line Hamiltonian $\psi_p = \bar{\psi}_p(\psi_t) + \psi_1 \cos(n\varphi - m\theta)$. One can solve for the magnetic-field lines to second order in the perturbation ψ_1 and find that the rotational transform is changed by an amount

$$\Delta\iota(\psi_t) = \frac{m}{4} \left\{ \frac{d}{d\psi_t} \left(\frac{1}{n - \iota m} \frac{d\psi_1^2}{d\psi_t} \right) + \frac{m\iota'' \psi_1^2}{(n - \iota m)^2} \right\}, \tag{71}$$

where $\iota'' \equiv d^2\iota/d\psi_t^2$. The perturbation ψ_1 is linear in the perturbation amplitude for a sufficiently small perturbation, so $\Delta\iota$ is quadratic in the perturbation amplitude where $n - \iota m$ is nonzero.

1. Current-driven instabilities

- The strongest effect of the net plasma current on stability is near the rational surfaces, $\iota = n/m$. Defining a helical flux by $d\psi_h \equiv (\iota - n/m) d\psi_t$, a gradient in the net current is stabilizing if $d(j_\parallel/B)/d\psi_h$ is positive and destabilizing if $d(j_\parallel/B)/d\psi_h$ is negative.
- Only perturbations with low poloidal mode number m can be driven unstable by the net plasma current. These instabilities are called kinks.
- A current density proportional to a Dirac delta function generally arises on the rational surfaces, $\iota = n/m$, in the presence of perturbations that conserve the constraints of an ideal, dissipationless, plasma. The amplitude of these delta-function currents is proportional to the jump in the resonant Fourier term of the radial displacement of the plasma, $[\vec{\xi} \cdot \vec{\nabla} \psi_t]_{mn}$.

The stability properties of a force-free equilibrium, $\vec{\nabla} \times \vec{B} = k(\psi_t)\vec{B}$, illustrates the effect of the distribution of net plasma current $k(\psi_t)$ on plasma stability. The fundamental results are that the net current affects stability primarily through the gradient $dk/d\psi_t$. Near a rational surface $\iota = n/m$ the gradient of the net current distribution enters through the quantity

$$\frac{m}{n - \iota m} \frac{dk}{d\psi_t}, \tag{72}$$

which is destabilizing when positive and stabilizing when negative. In other words, plasma stability is extremely sensitive to the gradient in the current distribution $k \equiv \mu_0 j_\parallel / B$ near rational surfaces, but the effect on stability of the gradient is opposite on the two sides of the rational surface. For example, in a tokamak, a low-order rational surface just outside the plasma is very destabilizing because $dk/d\psi_t$ is negative near the plasma edge while ι is positive but becoming smaller, $d\iota/d\psi_t<0$. The destabilizing effect of the current gradient is obtained on the plasma side of the rational surface, but there is no current to provide a stabilizing effect on the opposite side of the rational surface. This instability, which is called an external kink, is particularly pronounced when $q=1/\iota$ is just above an integer at the plasma edge.

The stability of force-free equilibria is very sensitive to the gradient of the current distribution k due to the properties of the magnetic differential equation $\vec{B}\cdot\vec{\nabla}k=0$, which must hold for both the perturbed and the unperturbed equilibrium. In the perturbed equilibrium, $k=k_0(\psi_t)+\delta k$ and $(\vec{B}_0+\delta\vec{B})\cdot\vec{\nabla}k=0$. The unperturbed equilibrium is $\vec{\nabla}\times\vec{B}_0=k_0(\psi_t)\vec{B}_0$, and the magnetic perturbation is given by $\vec{\nabla}\times\delta\vec{B}=\vec{B}_0\delta k+k_0\delta\vec{B}$, where $\vec{B}\cdot\vec{\nabla}\delta k=-\delta\vec{B}\cdot\vec{\nabla}k_0$. That is, δk is given by a magnetic differential equation and is proportional to $dk_0/d\psi_t$. The perturbed current distribution δk can become very large near a resonant rational surface, $\propto 1/(n-\iota m)$, because of the properties of the magnetic differential equation. The largeness of δk is what allows unstable perturbations to exist.

Instabilities due to the gradient in the net current are driven by the parallel part of the perturbed vector potential, $\delta\vec{A}_\parallel\equiv(\hat{b}\cdot\delta\vec{A})\hat{b}$ with $\hat{b}\equiv\vec{B}/B$. This is proven by noting that the three independent components of any vector, including the perturbed vector potential, can be written at each spatial point as $\delta\vec{A}=(\delta A_B/B)\vec{B}+\delta A_\psi\vec{\nabla}\psi_t+\vec{\nabla}\delta g$. If one chooses a gauge in which $\vec{B}\cdot\vec{\nabla}\delta g=0$, then $\delta A_B=\delta A_\parallel$ and $\delta\vec{B}\cdot\vec{\nabla}k_0(\psi_t)=(\vec{\nabla}\times\delta\vec{A}_\parallel)\cdot\vec{\nabla}k_0$. This representation of $\delta\vec{A}$ must be used with care. Given a change in the poloidal flux, $\delta\psi_p=-\oint\delta\vec{A}\cdot(\partial\vec{x}/\partial\varphi)d\varphi$, the change in the toroidal flux, $\delta\psi_t=\oint\delta\vec{A}\cdot(\partial\vec{x}/\partial\theta)d\theta$, is constrained if δg is a single-valued function of θ and φ. One can satisfy the constraint $\vec{B}\cdot\vec{\nabla}\delta g=0$ and have arbitrary changes in the poloidal and toroidal fluxes if δg has the form $\delta g=(\theta-\iota\varphi)\Delta(\psi_t)$.

The stability of force-free equilibria can be studied with greater simplicity when the aspect ratio is very large, $R_o/r\to\infty$. In a large-aspect-ratio torus with circular magnetic surfaces, one can let $\varphi=z/R_o$ replace the unit vector $\hat{\varphi}$ by $\hat{z}$, and describe the magnetic field using (r,θ,z) cylindrical coordinates, $\vec{B}=B_0[\hat{z}+\iota(r)(r/R_o)\hat{\theta}]$, where the rotational transform ι is assumed to be of order unity. The toroidal flux is $\psi_t=\pi B_0r^2$. The important part of the perturbed vector potential is the parallel component, $\delta\vec{A}=\delta A_\parallel\hat{z}$, so $\delta\vec{B}=\vec{\nabla}\delta A_\parallel\times\hat{z}$. If the parallel component of the vector potential is written as $\delta A_\parallel=\tilde{A}_\parallel(r,t)\sin(n\phi-m\theta)$, then

$$\delta k=\frac{R_o}{rB_0}\frac{m}{n-\iota m}\frac{dk_0}{dr}\delta A_\parallel, \tag{73}$$

and the equation $\vec{\nabla}\times\delta\vec{B}=\delta k\vec{B}_0$ is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\tilde{A}_\parallel}{dr}\right)-\frac{m^2\tilde{A}_\parallel}{r^2}=-\frac{R_o}{r}\frac{m}{n-\iota m}\frac{dk_0}{dr}\tilde{A}_\parallel. \tag{74}$$

The rotational transform and the current distribution are related by

$$\frac{1}{r}\frac{dr^2(\iota-\iota_v)}{dr}=R_ok_0(r), \tag{75}$$

where $\iota_v(r)$ is the rotational transform of the vacuum, or current-free, magnetic field. In a tokamak, ι_v is zero, but it is nonzero in a stellarator.

Externally driven currents are required to perturb the plasma, but it is mathematically simpler to view these as lying just outside the plasma. To understand the need for externally driven currents, note that solutions to Eq. (74) should be well behaved as $r\to 0$, which leaves only one free boundary condition. The solution to Eq. (74) outside of the plasma, $r>a$, is $\tilde{A}_\parallel\propto 1/r^m$ with the assumption that there are no perturbing currents away from the plasma boundary. The matching condition at the plasma boundary at $r=a$ is that $\tilde{A}_\parallel$ be continuous. However, the radial derivative of $\tilde{A}_\parallel$ cannot also be made continuous, for that would be a second boundary condition. The jump in the derivative, which is denoted by $[\partial\tilde{A}_\parallel/\partial r]$, gives the current that must be externally driven at the edge of the plasma to produce the magnetic perturbation.

The stability of a magnetic perturbation to a plasma is determined by the sign of the power that is needed to drive the current on the plasma surface that supports the perturbation. Positive power means stability and negative power instability. The surface current that is required to support the perturbation is

$$\delta\vec{j}=-\delta(r-a)\frac{\hat{z}}{\mu_0}\left[\frac{\partial\delta A_\parallel}{\partial r}\right], \tag{76}$$

where it is assumed that there are no currents outside of the plasma surface, $\delta A_\parallel\propto(a/r)^m$ for $r>a$. This follows from Ampère's law, $\nabla^2\delta A_\parallel=-\mu_0\delta j_z$. The power per unit length in z that is needed to drive the magnetic perturbation is

$$p_d=-\frac{\pi a}{\mu_0}\frac{\partial\tilde{A}_\parallel}{\partial t}\left[\frac{\partial\tilde{A}_\parallel}{\partial r}\right], \tag{77}$$

where the quantities are to be evaluated at $r=a$. To prove this, note that the power per unit volume going into the electromagnetic fields is $-\vec{E}\cdot\delta\vec{j}$. The electric field associated with the magnetic perturbation $\delta\vec{B}=\vec{\nabla}\times(\delta A_\parallel\hat{z})$ is obtained using Faraday's law, $\vec{E}=-\hat{z}\partial\delta A_\parallel/\partial t$, which implies Eq. (77). In the absence of a plasma, $\tilde{A}_\parallel=\mathcal{A}(t)(r/a)^m$ for $r<a$ and $\tilde{A}_\parallel=\mathcal{A}(t)(a/r)^m$ for $r>a$, which requires a power $p_d=(m\pi/\mu_0)d\mathcal{A}^2/dt$ per unit length, which is positive as one would expect. To make the power p_d negative and obtain an instability, $(\partial^2\tilde{A}_\parallel/\partial r^2)/\tilde{A}_\parallel$ must be negative in some part of the plasma. The condition that $(\partial^2\tilde{A}_\parallel/\partial r^2)/\tilde{A}_\parallel$ be negative implies the stability properties of the current-driven modes given in the discussion of Eq. (72) and that only perturbations with low values of m can be unstable.

The restriction of current-driven modes to low mode numbers comes from the stabilizing effect of the term $m^2\tilde{A}_\parallel/r^m$ in Eq. (74), which tends to make $(\partial^2\tilde{A}_\parallel/\partial r^2)/\tilde{A}_\parallel$ positive. Although current-driven instabilities must be low m, pressure-driven perturbations have comparable stability properties at all m. The dominant drive for pressure-driven instabilities is the Pfirsch-Schlüter current that arises in the perturbed plasma state. The destabilization of short-wavelength perturbations by the pressure gradient is discussed in Sec. V.C.2 on ballooning modes.

In a tokamak the vacuum transform ι_v vanishes, and a very simple solution to Eqs. (74) and (75) exists for any $k_0(r)$ when $m=1$. This solution is $\delta A_\parallel=[\iota(r)-n]r\mathcal{A}\sin(n\phi-\theta)$ where $\mathcal{A}$ is a constant. Define r_n by $\iota(r_n)=n$. In an ideal stability analysis, the general $m=1$ solution for $r<r_n$ can be matched to an external solution $\tilde{A}_\parallel=0$ for $r>r_n$. The matching gives a strong surface current, $\mu_0\tilde{j}_z=n\delta(r-r_n)(d\ln\iota/d\ln r)_{r_n}\mathcal{A}$. This $m=1$ perturbation is called a *sawtooth instability*, which in a simple analysis has zero growth rate without plasma dissipation and an infinite growth rate with dissipation. The rotational transform in the center of a tokamak whose current is maintained by a loop voltage tends to rise until it exceeds unity. After the central transform exceeds unity, strong $m=1$, $n=1$ kinklike relaxation oscillations occur that maintain the central transform just above, but close to, unity. These relaxation oscillations are called sawteeth due their appearance on electron temperature diagnostics where they were first observed. Reviews on the $m=1$ mode have been written by Migliuolo (1993) and Hastie (1997).

The location of the resonant rational surface, $\iota(r_{mn})=n/m$, is a special radial position in the differential equation for $\tilde{A}_\parallel$ [Eq. (74)]. In ideal theory an island must be avoided, which means $\delta\vec{B}\cdot\hat{r}=-(m/r)\tilde{A}_\parallel\cos(n\varphi-m\theta)$ must vanish there. The surface current that arises on the resonant surface in an ideal analysis to prevent an island from opening is given by Eq. (76) with a replaced by r_{mn}. The relaxation of this current dissipates energy and allows an island to open.

A jump in the resonant component of a radial displacement, $\vec{\xi}\cdot\vec{\nabla}\psi_t$, is the signature of a singular surface current on a rational surface $\iota=n/m$ in an ideal perturbation analysis. The demonstration is easiest in a large-aspect-ratio torus. The perturbed parallel component of the vector potential $\delta A_\parallel$ can be replaced by the radial component of the displacement,

$$\xi_r=-\frac{R_o}{rB_0}\frac{m}{n-\iota m}\delta A_\parallel. \tag{78}$$

The radial component means $\vec{\xi}\cdot\hat{r}$, where $\hat{r}\equiv\vec{\nabla}r$. The relation between ξ_r and $\delta A_\parallel$ is proven using $\delta B_r=\vec{B}_0\cdot\vec{\nabla}\xi_r$, which follows from $\delta\vec{B}=\vec{\nabla}\times(\vec{\xi}\times\vec{B}_0)$, Eq. (46), and $\delta B_r=\hat{\theta}\cdot\vec{\nabla}\delta A_\parallel$, which follows from $\delta\vec{B}=\vec{\nabla}\delta A_\parallel\times\hat{z}$. The current flowing on a rational surface, $r=r_{mn}$, Eq. (76), can then be written in terms of the jump in the radial displacement,

$$\delta\vec{j}=-\frac{\hat{z}}{\mu_0}\frac{d\iota}{dr}\frac{rB_0}{R_o}\delta(r-r_{mn})[\xi_r]. \tag{79}$$

Much of the theory of instabilities in a cylinder, including the existence of jumps in the radial displacement, was developed by Newcomb (1960). Alan Glasser in unpublished work during the 1990s extended Newcomb's results to general axisymmetric equilibria. Glasser's work can be easily generalized so it applies to plasma equilibria with arbitrary scalar pressure (Nührenberg and Boozer, 2003).

2. Resistive stability

- The stability of a plasma with nonzero resisitivity can be tested by considering the perturbation driven by a delta-function current on a rational surface. If this current increases when power is removed, $\int\vec{j}\cdot\vec{E}d^3x>0$, then the plasma is unstable to the formation of an island with a width that increases linearly in time. Such instabilities are called *tearing modes*.
- The stability of a plasma to perturbations that are sufficiently slow for all current perturbations to relax can be tested by making the distribution of the force-free current a time-independent function of the toroidal magnetic flux ψ_t.

In the presence of a perturbation, a resonant surface current naturally arises on each resonant surface, which means a current density that is proportional to $\delta(\iota-n/m)\cos(n\varphi-m\theta)$. The relaxation of this singular current density due to plasma resistivity leads to an island and potentially to a resistive instability. The paper that sparked research on resisitive instabilities is that of Furth, Killeen, and Rosenbluth (1963).

Magnetic islands are produced by the resonant part of the perturbed parallel component of the vector potential. That is,

$$\frac{\delta\vec{B}\cdot\vec{\nabla}\psi_t}{\vec{B}\cdot\vec{\nabla}\varphi}=\mu_0G(\psi_t)\frac{\partial\delta A_\parallel/B}{\partial\theta}-\mu_0I(\psi_t)\frac{\partial\delta A_\parallel/B}{\partial\varphi}, \tag{80}$$

with (ψ_t,θ,φ) magnetic coordinates in which $\vec{B}$ has a simple covariant representation [Eq. (58)]. A resonant Fourier term in $\delta\vec{B}\cdot\vec{\nabla}\psi_t/\vec{B}\cdot\vec{\nabla}\varphi$, Eq. (14), is related to the half-width of an island by Eq. (16). The derivation of Eq. (80) is straightforward, using the representation for the perturbed vector potential $\delta\vec{A}=(\delta A_B/B)\vec{B}+\delta A_\psi\vec{\nabla}\psi_t$

$+\vec{\nabla}\delta g$ with the gauge δg chosen so $\delta A_\parallel = \delta A_B$ and using Eq. (58) for $\vec{B}$ in $\delta\vec{A}_\parallel = (\delta A_\parallel / B)\vec{B}$. One can easily show that a change in the gauge does not affect the resonant components of Eq. (80).

The rate of growth of an island can be calculated using a Rutherford (1973) analysis. The component of Ohm's law along the resonant magnetic-field line, $\iota = n/m$, is $E_\parallel = -\partial\delta A_\parallel/\partial t = \eta\delta j_\parallel$. As the island grows the force-free current near the rational surface flows on the magnetic surfaces of the island, so the width of the current channel δ_R is proportional to the width of the magnetic island. For a slowly growing island, one can find the correct current distribution by solving the induction equation (Boozer, 1984a). Ampère's law, $\mu_0\delta j_\parallel = -\nabla^2\delta A_\parallel$, implies that the width δ_R of a narrow current channel can be defined by $\delta j_\parallel = -2\Delta'\delta A_\parallel/(\mu_0\delta_R)$, with

$$\Delta' \equiv \frac{1}{\tilde{A}_\parallel}\left[\frac{\partial\tilde{A}_\parallel}{\partial r}\right]. \tag{81}$$

The width of the island and the current channel are proportional to $\sqrt{|\tilde{A}_\parallel|}$, so $\delta_R^2 \propto |\tilde{A}_\parallel|$, and

$$\frac{d\delta_R}{dt} = \frac{\eta}{\mu_0}\Delta'. \tag{82}$$

If $\Delta' > 0$, the width of the island grows until the current singularity disappears, that is, $\Delta' \to 0$.

In the limit of a resistive analysis with no singular currents, both $\tilde{A}_\parallel$ and $\partial\tilde{A}_\parallel/\partial r$ must be continuous. The singularity of the right-hand side of the differential equation for $\tilde{A}_\parallel$, Eq. (74), is sufficiently weak that one can integrate straight through, even numerically, without real difficulty, which makes both $\tilde{A}_\parallel$ and $\partial\tilde{A}_\parallel/\partial r$ continuous. Said more precisely, the solution to $f'' = fx/(x^2+\delta^2)$ is well behaved in the limit as $\delta \to 0$. Because the relaxation of the singular currents that arise in an ideal analysis always takes energy from the field, a perturbation always takes more energy to drive, and is therefore more stable, in an ideal than in a resistive analysis.

3. Robustness of magnetic surfaces

- Plasma rotation prevents an arbitrarily small external perturbation from opening a magnetic island inside a plasma.
- The pressure-driven net parallel current, which is called the bootstrap current, can make an otherwise stable plasma unstable to the opening of an island. This instability is called the neoclassical tearing mode.

An arbitrarily small external magnetic perturbation cannot force an island to open in a rotating plasma (Fitzpatrick and Hender, 1991). To understand why, we must first understand the equations for plasma rotation. The force balance equation of an ion species is $m_i n_i \vec{v}\cdot\vec{\nabla}\vec{v} + \vec{\nabla}p_i = qn_i(\vec{E}+\vec{v}\times\vec{B})$. If the plasma is rotating slowly compared to the thermal speed, $\sqrt{T/m_i}$, but rapidly compared to the ion diamagnetic speed, $|\vec{\nabla}p_i/qn_iB|$, ion force balance is approximated by $\vec{E}+\vec{v}\times\vec{B}=0$ with the electric field given by a potential, $\vec{E} = -\vec{\nabla}\Phi$. The magnitude of the diamagnetic speed is $|\vec{\nabla}p_i/qnB| \approx (\rho_i/a)\sqrt{T/m_i}$. The ion gyroradius is $\rho_i = (m_i/qB)\sqrt{T/m_i}$ with $\rho_i/a \approx 1/500$ under fusion conditions. The equation $\vec{v}\times\vec{B} = \vec{\nabla}\Phi$, plus the requirement that a steady flow be divergence-free, lead to two magnetic differential equations, $\vec{B}\cdot\vec{\nabla}\Phi = 0$ and $\vec{B}\cdot\vec{\nabla}(v_\parallel/B) = \vec{\nabla}\cdot(\vec{\nabla}\Phi\times\vec{B}/B^2)$. These equations act as constraints on the two independent directions of plasma rotation, poloidal and toroidal. Plasma rotation is determined by two functions that depend on only ψ_t. These functions are $\Phi(\psi_t)$ and the solution to the homogeneous equation $\vec{B}\cdot\vec{\nabla}(v_\parallel/B) = 0$. The equation $\vec{B}\cdot\vec{\nabla}\Phi = 0$ holds under very general conditions because of the ease with which electrons flow along the magnetic-field lines. If the magnetic surfaces are good and the rotation is slow compared to the ion thermal speed, the electric potential is accurately given by a function of the toroidal flux alone, $\Phi(\psi_t)$.

Even a small stationary island forces one component of the plasma rotation to be zero; only rotation parallel to the resonant field lines is allowed. On the outermost surface of an island, the separatrix, Fig. 2, the magnetic differential equation $\vec{B}\cdot\vec{\nabla}\Phi = 0$ forces $d\Phi/d\psi_t$ to be zero. This follows from the general result discussed at the end of Sec. III.A that in the vicinity of an arbitrarily small island any function Φ that satisfies $\vec{B}\cdot\vec{\nabla}\Phi = 0$ can only have a nonzero derivative relative to the helical flux, which is defined by $d\psi_h \equiv (\iota - n/m)d\psi_t$. Without the island Φ can be an arbitrary function of ψ_t.

The zeroing of a component of rotation on a surface, which is required if an island is to exist, requires a torque. If an external magnetic perturbation is not strong enough to transport this torque to the current that produces the perturbation, $\delta\vec{B}_x$, then an island cannot open. The z component of that torque, $\mathcal{T}_z$, is associated with toroidal rotation. From the plasma perspective, any external magnetic perturbation in a tokamak can be viewed as being created by a surface current that flows in a thin axisymmetric surface that surrounds the plasma. Torque balance between the plasma and the perturbing external current allows an exact calculation of the total electromagnetic torque, $\int\vec{x}\times(\vec{j}\times\vec{B})d^3x$, that the perturbation exerts within the plasma, by carrying out an integral over the external current-carrying surface. The z component of the torque is

$$\mathcal{T}_z = \frac{1}{\mu_0}\oint_{surf} R[\delta\vec{B}_x]\cdot\hat{\varphi}\,\delta\vec{B}_i\cdot d\vec{a}. \tag{83}$$

Here $\delta\vec{B}_i$ is the field produced by currents internal to the plasma. $\delta\vec{B}_x$ is the external magnetic perturbation, and $[\delta\vec{B}_x]$ is its jump across the current-carrying layer that represents the external current that produces $\delta\vec{B}_x$. Equation (83) is derived by noting that the surface current that flows in the current-carrying layer is $\delta\vec{J}_s=\hat{n}\times[\delta\vec{B}_x]/\mu_0$, where $\hat{n}$ is the normal to the surface, and the force per unit area in the torodial direction is $(\delta\vec{J}_s\times\delta\vec{B}_i)\cdot\hat{\varphi}$. The torque that can be exerted by the current-carrying layer has an upper limit. Since $\delta\vec{B}_i$ for a stable plasma is proportional to the external perturbation $\delta\vec{B}_x$ (Sec. V.D.2), the maximum torque scales as $(\delta B_x)^2$. If the torque that is required to introduce an island is greater than this limit, then the island cannot open. Equation (83) also contains the essence of the phenomenon of the torque on a plasma due to a magnetic-field error becoming much larger as a plasma approaches marginal stability. This is because the ratio $\delta B_i/\delta B_x$ becomes large as marginal stability is approached (Sec. V.D.2).

How does the torque that an island induces prevent the island from opening? For simplicity assume the plasma pressure is zero; then the force balance is $\vec{f}_{vis}=\delta\vec{j}\times\vec{B}$, where $\vec{f}_{vis}$ is the viscous force exerted by the plasma. The parallel current obeys $\vec{B}\cdot\vec{\nabla}(\delta j_\parallel/B)=-\vec{\nabla}\cdot(\vec{B}\times\vec{f}_{vis}/B^2)$. To prevent an external magnetic perturbation, such as that of Eq. (14), from opening an island, the parallel current must have the form $\delta j_\parallel/B=k_r(\psi_t)\cos(n\varphi-m\theta_m)$ so it can cancel the perturbing field on the rational surface. Without the viscous force this current can only flow on the rational surface $\iota=n/m$, which means $k_r(\psi_t)$ must be a delta function. With a viscous force, $(n-\iota m)k_r(\psi_t)$ is given by the resonant Fourier component of $\vec{\nabla}\cdot(\vec{B}\times\vec{f}_{vis}/B^2)$. Since $(n-\iota m)$ vanishes at the rational surface, even a small viscous force can spread out and also maintain the current that is required to prevent an island from opening.

If a plasma is initially confined on perfect magnetic surfaces, then it is a complicated question whether surfaces will split forming islands due to either instabilities or small external perturbations. Rotational shielding of islands, which we discussed earlier, is only one example of several important effects. In the modern literature, these effects are primarily discussed under the topic of *neoclassical tearing modes*; see, for example, Rosenberg *et al.* (2002).

A neoclassical tearing mode causes a magnetic island to open through the formation of a strong gradient in the net plasma current near a rational surface. An island removes the pressure gradient near its resonant rational surface (Sec. III.A), and zeros the bootstrap current, which is a net parallel current due to the pressure gradient [Eq. (220)]. The zeroing of the bootstrap current in the vicinity of the rational surface produces a large gradient $d(j_\parallel/B)/d\psi_h$, which is of the same sign on both sides of the rational surface, where $d\psi_h\equiv(\iota-n/m)d\psi_t$. Depending on the relative signs of the bootstrap current and $d\iota/d\psi_t$, this effect either strongly favors, or mitigates against, the formation of an island (Sec. V.B.1). In a tokamak with ι peaked on axis, the formation of an island is favored.

Magnetic surface quality does seem to have some hysteresis: good surfaces tend to stay good and surfaces split by islands tend to stay split by islands. The preservation of magnetic surfaces is an important design and operational consideration in both tokamaks and stellarators.

In tokamaks instability can lead to a catastrophic loss of surfaces, which is called a *disruption* (ITER Physics Expert Group on Disruptions, …, 1999). Disruption avoidance is critical to the success of magnetic fusion. Stellarators are far more immune to disruptions. There are two reasons for this. First, the free energy in a stellarator equilibrium tends to be smaller because of the contribution of the vacuum magnetic field to the rotational transform. Second, the vacuum field in a stellarator strongly centers the plasma in its surrounding chamber. In a tokamak the plasma is centered in its chamber by a vertical, z-directed, magnetic field. The toroidal current that provides the poloidal field in a tokamak produces a self-force in the $\hat{R}$ direction, which is called the *hoop stress*. This must be balanced by the force of interaction with a vertical field. Changes in the plasma equilibrium cause a change in the plasma position, which may result in the plasma's striking the chamber walls, with a rapid loss of the plasma energy and current. Such an event in a power plant could cause severe damage to the walls.

C. Stability analyses

- Instability of an equilibrium can be demonstrated by finding a plasma displacement $\vec{\xi}(\vec{x})$ that reduces the energy W of the magnetic field and the plasma [Eq. (42)].
- The two components of the plasma displacement that lie in a magnetic surface can be expressed in terms of the radial displacement, $\xi^\psi\equiv\vec{\xi}\cdot\vec{\nabla}\psi_t$, for displacements that minimize the energy W.
- The change in the energy δW produced by a plasma displacement is a quadratic operator on ξ^ψ for perturbations that minimize the energy W.

The theory of long-wavelength plasma stability is closely related to the theory of equilibria. Equilibria can be found by minimizing the energy, Eq. (43), while imposing either torodial symmetry for tokamaks or periodicity for stellarators, $\vec{x}(\psi_t,\theta,\varphi+2\pi/N_p)=\vec{x}(\psi_t,\theta,\varphi)$, where N_p is the number of periods. These equilibria are unstable if the energy can be lowered by a more general shape function $\vec{x}(\psi_t,\theta,\varphi)$. Mathematically, equilibria are extrema and stable equilibria are minima of the energy (Lundquist, 1951; Bernstein *et al.*, 1958). The long-wavelength instabilities found by minimizing the energy are called magnetohydrodynamic (MHD) instabilities as distinct from the microinstabilities that have a wavelength comparable to an ion gyroradius.

1. Expressions for δW

If $\vec{x}_e(\psi_t,\theta,\varphi)$ gives the shape of the magnetic surfaces in an equilibrium plasma, the stability of that equilibrium can be tested by considering the shape function $\vec{x}(\psi_t,\theta,\varphi)=\vec{x}_e(\psi_t,\theta,\varphi)+\vec{\xi}$ with $\vec{\xi}$ a small perturbing displacement. The displacement causes a change in the magnetic field, Eq. (46), and a change in the pressure, Eq. (47). The change in the energy, Eq. (42), due to a small change from equilibrium, Eq. (43), is quadratic in the displacement and given by

$$\delta W=\frac{1}{2}\int_{plasma}(\vec{\nabla}\delta p-\delta\vec{j}\times\vec{B}-\vec{j}\times\delta\vec{B})\cdot\vec{\xi}d^3x, \qquad (84)$$

with $\delta\vec{j}\equiv\vec{\nabla}\times\delta\vec{B}/\mu_0$. The integral is over the volume occupied by the plasma, for outside the plasma δp, $\delta\vec{j}$, and $\vec{j}$ are zero. The factor of 2 arises from the force operator $\vec{F}[\vec{\xi}]\equiv\vec{\nabla}\delta p-\delta\vec{j}\times\vec{B}-\vec{j}\times\delta\vec{B}$ being linearly dependent on the displacement $\vec{\xi}$. Partial integrations demonstrate that the force operator has the important property of being self-adjoint, $\int\vec{\xi}_1\cdot\vec{F}[\xi_2]d^3x=\int\vec{\xi}_2\cdot\vec{F}[\xi_1]d^3x$, where $\vec{\xi}_1$ and $\vec{\xi}_2$ are two arbitrary displacements (Bernstein *et al.*, 1958; Bernstein, 1983). This implies that if the displacement is expanded in a set of vectors, $\vec{\xi}=\Sigma c_j\vec{u}_j$, then $\delta W=\Sigma c_i^* W_{ij}c_j$ where W_{ij} is a Hermitian matrix.

As shown below, Eq. (84) for δW can be written as the sum of a vacuum energy and a plasma energy (Bernstein *et al.*, 1958),

$$\delta W=\delta W_v+\delta W_p. \qquad (85)$$

The vacuum energy is

$$W_v\equiv\int_{ext}\frac{(\delta B)^2}{2\mu_0}d^3x, \qquad (86)$$

with the integral performed over the region exterior to the plasma. The plasma contribution is

$$\delta W_p=\int_{\mathrm{plasma}}\left(w_{\xi_\perp}+\frac{1}{2}\gamma p(\vec{\nabla}\cdot\vec{\xi})^2\right)d^3x, \qquad (87)$$

where

$$w_{\xi_\perp}\equiv\frac{(\delta\vec{B})^2}{2\mu_0}+\frac{1}{2}(\vec{\nabla}\cdot\vec{\xi}_\perp)(\vec{\xi}_\perp\cdot\vec{\nabla}p)-\frac{1}{2}\vec{\xi}_\perp\cdot(\vec{j}\times\delta\vec{B}) \qquad (88)$$

has no dependence on the parallel part of the displacement, $\vec{\xi}_\parallel\equiv(\hat{b}\cdot\vec{\xi})\hat{b}$, where $\hat{b}\equiv\vec{B}/B$. Equation (87) implies that to minimize δW one should choose the parallel component of the displacement so that $\vec{\nabla}\cdot\vec{\xi}=0$.

The derivation of Eqs. (85)–(88) is given in this paragraph, which can be skipped. The derivation follows from noting that the term $(\delta\vec{j}\times\vec{B})\cdot\vec{\xi}$ in δW can be integrated over all of space since $\delta\vec{j}=0$ outside of the plasma. One finds that $\int(\delta\vec{j}\times\vec{B})\cdot\vec{\xi}d^3x=-\int(\delta B)^2d^3x/\mu_0$, where the last volume integral is over all of space. Using Eq. (47),

$$\vec{\xi}\cdot\vec{\nabla}\delta p=-\vec{\xi}\cdot\vec{\nabla}(\vec{\xi}\cdot\vec{\nabla}p)-\vec{\nabla}\cdot(\gamma p\vec{\xi}\vec{\nabla}\cdot\vec{\xi})+\gamma p(\vec{\nabla}\cdot\vec{\xi})^2. \qquad (89)$$

Using Eq. (46) one can easily show $\delta\vec{B}\cdot\vec{\nabla}p=\vec{B}\cdot\vec{\nabla}(\vec{\xi}\cdot\vec{\nabla}p)$, which implies $\vec{\xi}_\parallel\cdot(\vec{j}\times\delta\vec{B})=-\vec{\xi}_\parallel\cdot\vec{\nabla}(\vec{\xi}\cdot\vec{\nabla}p)$. The integral over the plasma volume of a pure divergence vanishes because the plasma pressure and pressure gradient vanish at the plasma edge.

Equation (88) for $w_{\xi_\perp}$ can be rewritten (Furth *et al.*, 1965) as

$$w_{\xi_\perp}=w_++w_\pm, \qquad (90)$$

where

$$w_+\equiv\frac{(\delta B_\perp)^2}{2\mu_0}+\frac{B^2}{2\mu_0}(\vec{\nabla}\cdot\vec{\xi}_\perp+2\vec{\xi}_\perp\cdot\vec{\kappa})^2 \qquad (91)$$

is always positive but the term

$$w_\pm\equiv-\frac{1}{2}\frac{j_\parallel}{B}(\vec{\xi}_\perp\times\vec{B})\cdot\delta\vec{B}_\perp-(\vec{\xi}_\perp\cdot\vec{\kappa})(\vec{\xi}_\perp\cdot\vec{\nabla}p) \qquad (92)$$

has an indefinite sign. The curvature of the magnetic-field lines is

$$\vec{\kappa}\equiv\hat{b}\cdot\vec{\nabla}\hat{b}, \qquad (93)$$

where $\hat{b}\equiv\vec{B}/B$.

The derivation of Eqs. (90)–(92), which is given in this paragraph and can be skipped, follows from an expression for the parallel component of the perturbed magnetic field,

$$\vec{B}\cdot\delta\vec{B}=-(\vec{\nabla}\cdot\vec{\xi}_\perp+2\vec{\xi}_\perp\cdot\vec{\kappa})B^2+\mu_0\vec{\xi}_\perp\cdot\vec{\nabla}p. \qquad (94)$$

The derivation of this identity uses the vector identity $\vec{B}\cdot\vec{\nabla}\times(\vec{\xi}_\perp\times\vec{B})=-\vec{\xi}_\perp\cdot\{(\vec{\nabla}\times\vec{B})\times\vec{B}\}-\vec{\nabla}\cdot(B^2\vec{\xi}_\perp)$ plus Ampère's law, $\vec{\nabla}\times\vec{B}=\mu_0\vec{j}$, and force balance $\vec{\nabla}p=\vec{j}\times\vec{B}$. Force balance plus Ampère's law imply $\vec{\nabla}_\perp(B^2+2\mu_0 p)=2B^2\vec{\kappa}$. Combining results one obtains Eq. (94). The identity of Eq. (94) can be written in a second form, which is

$$\frac{(\delta\vec{B}_\parallel)^2}{2\mu_0}=\Delta^2\frac{B^2}{2\mu_0}+\delta B_\parallel\frac{\vec{\xi}_\perp\cdot\vec{\nabla}p}{2B}-\frac{1}{2}(\vec{\xi}_\perp\cdot\vec{\nabla}p)\Delta, \qquad (95)$$

where

$$\Delta\equiv\vec{\nabla}\cdot\vec{\xi}_\perp+2\vec{\xi}_\perp\cdot\vec{\kappa}. \qquad (96)$$

The last term in Eq. (88) for $w_{\xi_\perp}$ can be rewritten using the fact that the vector triple product in three dimensions is nonzero only when one of the vectors comes from each of the three independent directions. Consequently $\vec{\xi}_\perp\cdot(\vec{j}\times\delta\vec{B})=\vec{\xi}_\perp\cdot(\vec{j}_\parallel\times\delta\vec{B}_\perp)+\vec{\xi}_\perp\cdot(\vec{j}_\perp\times\delta\vec{B}_\parallel)$, which can be rewritten as $\vec{\xi}_\perp\cdot(\vec{j}\times\delta\vec{B})=(j_\parallel/B)\delta\vec{B}_\perp\cdot(\vec{\xi}_\perp\times\vec{B})+(\delta B_\parallel/B)\vec{\xi}_\perp\cdot\vec{\nabla}p$.

A minimization of δW is really a minimization of a quadratic functional of $\xi^{\psi}\equiv\vec{\xi}\cdot\vec{\nabla}\psi_t$ that involves radial derivatives only up to the first order. If the perturbing displacement is written as

$$\vec{\xi}=\xi^{\psi}\frac{\partial\vec{x}}{\partial\psi}+\eta\frac{\partial\vec{x}}{\partial\theta}+\mu\vec{B}, \tag{97}$$

then the only place the coefficient $\mu(\psi,\theta,\varphi)$ enters δW is in the positive definite term involving $(\vec{\nabla}\cdot\vec{\xi})^2$. A minimization of δW makes $\vec{\nabla}\cdot\vec{\xi}=0$, which is a magnetic differential equation for μ in terms of the η and ξ^{ψ} coefficients.

The η coefficient of $\vec{\xi}$ can be expressed in terms of ξ^{ψ} for perturbations that minimize δW. A perturbation that minimizes δW must give a perturbed equilibrium, and in a perturbed equilbrium the current, $\vec{j}+\vec{\nabla}\times\delta\vec{B}/\mu_0$, must be orthogonal to the toroidal flux. At a given spatial point the perturbation causes the toroidal flux to go from ψ_t to $\psi_t-\vec{\xi}_\perp\cdot\vec{\nabla}\psi_t$, which follows from $d\psi_t/dt=\partial\psi_t/\partial t+(\partial\vec{\xi}/\partial t)\cdot\vec{\nabla}\psi_t$ and the constraint that $d\psi_t/dt=0$ in an ideal plasma. See the discussion of Eq. (45). Therefore δW minimizing perturbations obey the constraint $\delta\vec{j}\cdot\vec{\nabla}\psi_t=\vec{j}\cdot\vec{\nabla}(\vec{\xi}_\perp\cdot\vec{\nabla}\psi_t)$, where $\delta\vec{j}\equiv\vec{\nabla}\times\delta\vec{B}/\mu_0$. Now $\vec{j}\cdot\vec{\nabla}(\vec{\xi}_\perp\cdot\vec{\nabla}\psi_t)=\vec{\nabla}\cdot[(\vec{\xi}_\perp\cdot\vec{\nabla}\psi_t)\vec{j}]$. The normal to the surface is $\hat{n}\equiv\vec{\nabla}\psi_t/|\vec{\nabla}\psi_t|$, so one can write $\vec{j}=\hat{n}\times(\vec{j}\times\hat{n})$ and $(\xi_\perp\cdot\vec{\nabla}\psi_t)\vec{j}=\vec{\nabla}\psi_t\times[(\vec{j}\times\hat{n})\vec{\xi}_\perp\cdot\hat{n}]$. This implies $\vec{j}\cdot\vec{\nabla}(\vec{\xi}_\perp\cdot\vec{\nabla}\psi_t)=-\vec{\nabla}\psi_t\cdot\vec{\nabla}\times[(\vec{j}\times\hat{n})\vec{\xi}_\perp\cdot\hat{n}]$. Letting

$$\vec{C}\equiv\delta\vec{B}+(\vec{\xi}_\perp\cdot\hat{n})(\mu_0\vec{j}\times\hat{n}) \tag{98}$$

one finds that

$$(\vec{\nabla}\times\vec{C})\cdot\vec{\nabla}\psi_t=\vec{\nabla}\cdot(\vec{C}\times\vec{\nabla}\psi_t)=0 \tag{99}$$

is the condition for the current in the perturbed equilibrium to lie in the perturbed flux surfaces. This constraint can be imposed ψ_t surface by ψ_t surface, since the only derivatives that arise are in θ and φ. The two constraints $\vec{\nabla}\cdot\vec{\xi}=0$ and Eq. (99) determine both $\mu(\psi_t,\theta,\varphi)$ and $\eta(\psi_t,\theta,\varphi)$ in terms of $\xi^{\psi}(\psi_t,\theta,\varphi)$.

The constraint that $(\vec{\nabla}\times\vec{C})\cdot\vec{\nabla}\psi_t=0$ must vanish for a δW minimizing perturbation can be obtained directly from the energy principle if $w_{\xi_\perp}$ is written as

$$w_{\xi_\perp}=\frac{\vec{C}^2}{2\mu_0}-w_d, \tag{100}$$

where $w_d\equiv(\vec{\xi}\cdot\hat{n})^2(\vec{j}\times\hat{n})\cdot(\vec{B}\cdot\nabla)\hat{n}$ (Bernstein *et al.*, 1958; Bernstein, 1983). For a divergence-free perturbation the only place the coefficient η enters the energy is through $\vec{C}$, and there it enters only through $\delta\vec{B}$. The contravariant representation of the magnetic field in magnetic coordinates, Eq. (52), plus the dual relations imply $(\partial\vec{x}/\partial\theta)\times\vec{B}=\vec{\nabla}\psi_t/2\pi$. Consequently the part of $\vec{C}$ that depends on η is $\vec{C}_\eta=\vec{\nabla}\times(\eta\vec{\nabla}\psi_t)/2\pi$. The minimization of $\int C^2d^3x$ through the variation $\delta\eta$ gives $2\int\{\vec{C}\cdot\vec{\nabla}\times(\delta\eta\vec{\nabla}\psi_t)/2\pi\}d^3x$. An integration by parts then gives the desired result.

2. Pressure-driven ballooning modes

- Perturbations of arbitrarily large wave number can be destabilized by a pressure gradient that is in the same direction as the field-line curvature. These instabilities are called *ballooning modes* since they balloon out, that is, they have their largest amplitude at the least stable place on a constant-pressure surface.
- Plasma instabilities that have a high wave number have a short wavelength in the constant-pressure surfaces across the magnetic-field lines but a long wavelength along the magnetic-field lines. For ballooning modes this anisotropy arises to minimize the stabilizing effect of field-line bending. For microinstabilities (Sec. VI.F), the anisotropy arises to avoid stabilization by Landau damping (Sec. VI.C).

The expression for δW is greatly simplified when the perturbations that are considered have a short wavelength across the magnetic-field lines. Such instabilities are called ballooning modes (Todd *et al.*, 1977; Connor, Hastie, and Taylor, 1978). Despite their short wavelength across the magnetic-field lines, the least stable perturbations of this type have a long wavelength along the magnetic field. These disparate scales can be efficiently represented using a concept from geometric optics, the eikonal $S(\alpha)$, which satisfies $\vec{B}\cdot\vec{\nabla}S=0$. Let

$$\vec{\xi}_\perp=\vec{\Xi}\cos(S(\alpha)), \tag{101}$$

where α is a Clebsch coordinate [Eq. (9)]. For ballooning-mode calculations it is conventional to choose the Clebsch coordinates so $\vec{B}=\vec{\nabla}\alpha\times\vec{\nabla}\psi_p$, where ψ_p is the poloidal magnetic flux and

$$2\pi\alpha=\varphi-q(\psi_p)\{\theta_m-\theta_0(\psi_p)\}, \tag{102}$$

with $q(\psi_p)\equiv1/\iota(\psi_p)\equiv d\psi_t/d\psi_p$ the safety factor. The eikonal S could be a function of both α and ψ_p, but this provides no generality beyond that in the function $\theta_0(\psi_p)$. The eikonal can be written as

$$S=2\pi N\alpha, \tag{103}$$

where N is the toroidal mode number of the perturbation. The short-wavelength limit means $N\to\infty$. The expression that one obtains is

$$\frac{\delta W}{(2\pi N)^2}=\int\left\{\frac{(\vec{\nabla}\alpha)^2}{2\mu_0}\left(\frac{\vec{B}\cdot\vec{\nabla}F}{B}\right)^2-\tilde{\kappa}\frac{dp}{d\psi_p}F^2\right\}d^3x, \tag{104}$$

where the relevant component of the field-line curvature is

$$\tilde{\kappa} \equiv \frac{\vec{\kappa}\cdot\vec{B}\times\vec{\nabla}\alpha}{B^2}. \tag{105}$$

The perpendicular displacement of an unstable perturbation must have the form of Eq. (108), which defines the function F.

The derivation of the short-wavelength form for δW is given in this paragraph and can be skipped for those not interested in the mathematical details. Using Eq. (101) for the perpendicular displacement, derivatives of the displacement enter δW, Eqs. (90)–(92), through

$$\vec{\nabla}\cdot\vec{\xi}_\perp = \cos(S)\vec{\nabla}\cdot\vec{\Xi} - \sin(S)\vec{\Xi}\cdot\vec{\nabla}S \tag{106}$$

and

$$\delta B_\perp = \cos(S)\{\vec{\nabla}\times(\Xi\times\vec{B})\}_\perp. \tag{107}$$

Note that $\delta B = \cos(S)\vec{\nabla}\times(\Xi\times\vec{B}) - \sin(S)\{\vec{\nabla}S\times(\vec{\xi}_\perp\times\vec{B})\}$, but the term $\{\vec{\nabla}S\times(\vec{\xi}_\perp\times\vec{B})\} = -(\vec{\xi}_\perp\cdot\vec{\nabla}S)\vec{B}$ does not contribute to the magnetic perturbation perpendicular to the field lines. When these expressions are inserted into Eqs. (90)–(92), the terms proportional to $\vec{\nabla}S$ and hence N appear only in a positive definite term that is proportional to $(\vec{\nabla}\cdot\vec{\xi}_\perp + 2\vec{\xi}\cdot\kappa)^2$. Consequently instability is possible in the limit as $N\to\infty$ only if one chooses $\vec{\xi}_\perp\cdot\vec{\nabla}S = 0$. By choosing a $1/N$ correction to $\vec{\xi}_\perp$ appropriately, one can eliminate the positive definite term $(\vec{\nabla}\cdot\vec{\xi}_\perp + \vec{\xi}\cdot\kappa)^2$ from δW altogether, since no other term is affected in the lowest nontrivial order by the $1/N$ correction. The perpendicular displacement for unstable short-wavelength modes must have the form

$$\vec{\xi}_\perp = \frac{\vec{B}\times\vec{\nabla}S}{B^2}F \tag{108}$$

since it must be perpendicular to both the magnetic field and $\vec{\nabla}S$. One then finds $\vec{\xi}_\perp\times\vec{B} = F\vec{\nabla}S$ and $\delta\vec{B} = \vec{\nabla}F\times\vec{\nabla}S$. The parallel current term in Eq. (92) vanishes because $(\vec{\xi}_\perp\times\vec{B})\cdot\delta\vec{B}_\perp = F\vec{\nabla}S\cdot(\vec{\nabla}F\times\vec{\nabla}S) = 0$. As discussed earlier, the parallel current cannot be a source of instability for short-wavelength perturbations. The perpendicular part of the magnetic perturbation is $\delta\vec{B}_\perp = (\hat{b}\cdot\vec{\nabla}F)(\hat{b}\times\vec{\nabla}S)$ with $\hat{b}\equiv\vec{B}/B$. Putting the pieces together one obtains Eq. (104).

The form for δW for short-wavelength perturbations implies that unstable modes must have a long wavelength along the magnetic-field lines in order to minimize the $(\vec{B}\cdot\vec{\nabla}F)^2$ term. If one can choose θ_0 in α so $\tilde{\kappa}dp/d\psi_t$ is positive, then δW predicts instability if the variation of F along the field lines is not taken into account. The $(\vec{B}\cdot\vec{\nabla}F)^2$ term can stabilize the mode in places where $\tilde{\kappa}dp/d\psi_t$ is positive, but the mode amplitude is naturally larger there. That is, the plasma displacement balloons out at the locations where $\tilde{\kappa}dp/d\psi_t$ is positive.

If the ballooning expression for δW is extremized, one obtains the differential equation

$$\frac{\partial}{\partial\theta}\left\{\frac{(\vec{\nabla}\alpha)^2}{\mu_0}\frac{\vec{B}\cdot\vec{\nabla}\theta}{B^2}\frac{\partial F}{\partial\theta}\right\} + \frac{2\tilde{\kappa}}{\vec{B}\cdot\vec{\nabla}\theta}\frac{dp}{d\psi_p}F = 0, \tag{109}$$

where $\vec{B}\cdot\vec{\nabla} = (\vec{B}\cdot\vec{\nabla}\theta)\partial/\partial\theta$ and the Jacobian of (ψ_p,θ,α) coordinates is $1/\vec{B}\cdot\vec{\nabla}\theta$ for any poloidal angle θ. Equation (109) predicts instability if a solution F crosses $F=0$ twice. To understand why this prescription works, note that if one of the F's in the F^2 term of Eq. (104) is replaced by the F from Eq. (109), one finds after an integration by parts that $\delta W = 0$. Now add a term $\int\Delta_s F^2 d\psi_p d\theta d\alpha/2$ to both sides of the ballooning expression for $\delta W/(2\pi N)^2$, Eq. (104), with Δ_s a constant. Taking $\delta W_m/(2\pi N)^2 \equiv \delta W/(2\pi N)^2 + \int\Delta_s F^2 d\psi_p d\theta d\alpha/2$ to its extreme, one finds that Eq. (109) is modified by the zero on the right-hand side becoming $\Delta_s F$. One solves this modified equation. If the solution has two places where $F=0$ when $\Delta_s>0$, one replaces an F in the modified equation, $\delta W_m/(2\pi N)^2$, with the F from the differential equation. One then finds $\delta W_m/(2\pi N)^2 = 0$, so $\delta W/(2\pi N)^2 = -\int\Delta_s F^2 d\psi_p d\theta d\alpha/2$, which is negative. That is, one has found a perturbation that reduces the energy, so instability is predicted.

3. Minimization of δW

Given any positive definite normalization of the displacement, $\|\xi^2\|$, a plasma is unstable if one can find a displacement such that

$$\lambda \equiv \frac{\delta W}{\|\xi^2\|} \tag{110}$$

is negative. The sign of λ is independent of the normalization. However, different normalizations provide different types of information. Three normalizations will be discussed, the kinetic-energy norm, the ξ^ψ norm, and the surface norm.

The conventional normalization of the displacement is the kinetic-energy norm, $\|\xi^2\|_k \equiv \int\rho\xi^2 d^3x/2$, which makes $\sqrt{(-\lambda)}$ the growth rate of the instability. This norm is poorly behaved when the plasma is stable because then a minimization of λ can mean a maximization of $\|\xi^2\|_k$. As we have seen, the coefficient μ of the displacement, Eq. (97), is determined by a magnetic differential equation, which means the Fourier component of μ that resonates with a rational surface can be made arbitrarily large in that neighborhood with little change in δW. That is, one can always find a continuum of modes that makes $\lambda\to 0$ for an essentially fixed δW. The existence of this continuum (Grad, 1973; Goedbloed, 1998) makes it difficult to find the points of marginal stability. Even in the unstable region, the displacement that minimizes δW is usually not the perturbation that minimizes λ. Consequently a minimization of λ using displacements that are divergence-free and satisfy $(\vec{\nabla}\times\vec{C})\cdot\vec{\nabla}\psi_t = 0$ does not give the correct growth rate for unstable modes.

The singularity of the displacement $\vec{\xi}$ near rational surfaces is particularly important for the theory of perturbations that are rotating relative to the plasma. A singular plasma displacement rotating through a plasma gives an infinite correction to the energy. This singularity can be resolved (Betti and Freidberg, 1995), but at the cost of an imaginary contribution to the energy, which represents the torque between the perturbation and the plasma. Though not noted by Betti and Freidberg (1995), the resolution of the singularity also changes the real part of the energy (Boozer, 2003), which implies that the critical value of the plasma parameters required for the stability of the perturbation are very sensitive to rotation.

A second choice for the normalization of the displacement is similar to the kinetic energy except only the component ξ^ψ is retained, $\|\xi^2\|_\psi \equiv \int \rho (\xi^\psi)^2 d^3x/2$. The quantity λ has no special interpretation, but the continuum problem in finding points of marginal stability is eliminated.

A third choice of normalization uses ξ^ψ but only its value on the plasma surface, $\|\xi^2\|_s = \oint (\xi^\psi)^2 w\, d\theta d\varphi$, with $w(\theta,\varphi)>0$ an arbitrary weight function. This normalization has pathological features for internal plasma modes, which have a displacement that reduces the energy while keeping the plasma boundary fixed. For these modes $\lambda \to -\infty$. However, if the minimizing λ is bounded, the displacements ξ^ψ of the plasma surface that are associated with a spectrum of λ's give the modified equilibria of the plasma in the presence of external perturbations.

An instability with a negative but bounded λ with a fixed perturbation of the surface shape, the $\|\xi^2\|_s$ norm, is called an *external mode*. External modes would be stabilized if a perfect conductor were close enough to the plasma surface. A magnetic perturbation cannot penetrate a perfect conductor, so the component of the magnetic-field line displacement $\vec{\xi}$ that is normal to the conductor must vanish. Actual plasmas are surrounded by chamber walls, which are conductors. An instability that would be stabilized if these conducting structures were perfectly conducting is called a *wall mode*. Wall modes can grow on the resistive time scale of the conducting structures, which is many orders of magnitude slower than the growth rate determined by plasma inertia. Due to their slow growth, resistive wall modes can be stabilized by plasma rotation and by feedback (Sec. V.D.2). The stabilization of wall modes is considered an important topic for tokamak and ST plasmas that have the profile of net current that is required for steady-state operation.

When one uses a fixed perturbation of the boundary, the $\|\xi^2\|_s$ norm, the Fourier coefficient of ξ^ψ in magnetic coordinates can jump at each resonant rational surface, $\iota=n/m$, and the magnitude of the jump gives the magnitude of the surface current on the rational surface. Once the expansion coefficients η and μ of the displacement, Eq. (97), have been eliminated from δW in favor of ξ^ψ, radial derivatives of the displacement arise only in the form $\partial b/\partial\psi_t$ where

$$b \equiv \frac{\delta\vec{B}\cdot\vec{\nabla}\psi_t}{\vec{B}\cdot\vec{\nabla}\varphi}. \tag{111}$$

Equation (46), $\delta\vec{B}=\vec{\nabla}\times(\vec{\xi}\times\vec{B}_0)$, implies that $b=(\partial/\partial\varphi + \iota\partial/\partial\theta_m)\xi^\psi$. If one allows the resonant Fourier coefficient of ξ^ψ_{mn} to change by a fixed amount over a distance $\delta\psi_t$ about a rational surface $\iota=n/m$, the resonant Fourier coefficient of $\partial b/\partial\psi_t$ has a well-defined limit on each side of the rational surface as $\delta\psi_t\to 0$. One can show that the jump in the resonant coefficient of ξ^ψ or in $\partial b/\partial\psi_t$ is proportional to the surface current on the rational surface (Nührenberg and Boozer, 2003). For a large-aspect-ratio torus, this was shown in the derivation of Eq. (79). The amplitude of the jumps in ξ^ψ at rational surfaces gives a measure of the width of the islands that would open in the absence of a singular current. For a large-aspect-ratio torus with circular surfaces, $\psi_t \propto r^2$ with r the minor radius, one can easily show that the half width of the island that would arise in the absence of the singular current is

$$(\delta r)_{mn} = \sqrt{\left|\frac{2r_{mn}}{m}[(\xi_r)_{mn}]\right|}. \tag{112}$$

One can use an ideal δW method to assess and improve the quality of equilibria and to find the required perturbations to the plasma surface to remove islands (Nührenberg and Boozer, 2003).

D. Interaction of plasmas with coils

- Between the plasma and the surrounding coils, the magnetic field has the form $\vec{B}=\vec{\nabla}\phi$. Since $\vec{\nabla}\cdot\vec{B}=0$, the interaction of a plasma with currents outside the plasma must be through solutions to Laplace's equation, $\nabla^2\phi=0$, which greatly constrains the form of the interaction.
- Practical coils can control only a certain number N_f of features of the plasma shape. For axisymmetric systems $N_f\approx 4$, but for stellarators $N_f\approx 50$.
- Important long-wavelength instabilities of tokamaks would be stabilized if the surrounding chamber walls were perfectly conducting. Such instabilities are called resistive wall modes since the instabilities can grow on a time scale determined by the resistivity of the chamber walls. Their slow growth permits feedback stabilization. The interaction of the plasma with the perturbing currents in walls and in feedback coils can be calculated by coupled circuit equations. The plasma circuit elements have a determinate response to changes in the other circuit elements.

The external coils that are required for the magnetic confinement of a plasma have two distinct functions. First, they provide the required toroidal magnetic flux ψ_t. Second, they ensure $\vec{B}\cdot\hat{n}$ is zero on the plasma surface. These two functions are separated in tokamak coil design: the toroidal field coils provide the flux and the poloidal field coils ensure $\vec{B}\cdot\hat{n}=0$ (Fig. 4). However, in stellarators these functions are often coupled in a single coil set (Fig. 6). Nevertheless, in discussions of the efficiency and practicality of stellarator coils, it is often useful to separate the two functions.

Critical issues in coil design are the ratio of the magnetic field at the coils to that on the plasma, the complexity and the forces associated with the required coil currents, and the production of a field with the required symmetry while providing for ports, for plasma access, and for discrete coils.

A point that is often emphasized in the magnetic fusion literature is that, if all else is equal, the power produced by a fusion plant scales as the square of $\beta \equiv 2\mu_0 p/B^2$. A point less frequently made is that, under the same assumptions as those that give the β^2 scaling, the power production scales as the fourth power of the ratio of the magnetic field on the plasma to that on the coils. If the plasma β is limited, then the optimal plasma temperature is the one that maximizes the fusion power production at a fixed plasma pressure $p=nT$. The power from the fusion of deuterium and tritium scales as $p_{DT} = n^2 f_{DT}(T)$, with n the plasma number density and f_{DT} a function of the plasma temperature. The maximum of p_{DT} at fixed pressure occurs at $d(f_{DT}/T^2)/dT=0$. This condition implies that near the optimal temperature, which is about 20 keV, the fusion power density p_{DT} is proportional to the pressure squared, or equivalently, to β^2.

Between the coils and the plasma, the magnetic field is curl-free,

$$\vec{B} = \vec{\nabla}\phi, \tag{113}$$

as well as divergence-free, so

$$\nabla^2\phi = 0. \tag{114}$$

The interactions of the coils with the plasma must occur through solutions to Laplace's equation, which places strong constraints on the form of this interaction. Much of the theory of coil design, such as the work of Merkel (1987, 1988) on the coil design for the W7-X stellarator, is an application of the theory of Laplace's equation.

1. Freedom of coil design

The choice of plasma shape is the major determinant of the quality of a confined plasma. Unfortunately, the theory of Laplacians says that a general plasma shape cannot be supported by distant coils. What can be done, through the design of coils, is to enforce a certain number of conditions on the plasma shape. This number, which is the number of degrees of freedom in the coil design N_f, is in practice about four for tokamaks and about 50 for stellarators.

The theory of Laplace's equation implies that it is mathematically impossible for a distant coil set to provide a generic normal magnetic field $\vec{B}\cdot\hat{n}$ on the plasma surface, even when $\vec{B}\cdot\hat{n}$ is a smooth analytic function. This statement is proven by giving a generic normal field that cannot be produced by distant coils. Consider the cylindrical problem in which the radial magnetic field that must be produced by coils, $B_r(\theta)=\Sigma B_m \cos(m\theta)$, is given on the surface $r=a$. Between the coils and the plasma the magnetic field obeys Eqs. (113) and (114), so $\phi=\Sigma(aB_m/m)(r/a)^m \cos(m\theta)$. In general, a maximum value of r/a exists for which this Fourier series converges, which means there is a maximum value of r/a for which a solution exists. To find this maximum, consider the convergence properties of the Fourier coefficients $B_m=\int B_r(\theta)\cos(m\theta)d\theta/\pi$. If B_r is an analytic function of θ this integral can be performed using the method of residues of complex analysis. If θ_p is the distance of the nearest pole of $B_r(\theta)$ from the real axis, then as $m\to\infty$ the Fourier coefficients have the form $B_m = B_c \exp(-m\theta_p)$. In other words, the Fourier coefficients of a generic function $B_r(\theta)$ decay exponentially as $m\to\infty$. Now consider the coefficients $B_m(r/a)^m$. As $m\to\infty$ these coefficients are proportional to $\exp\{m[\ln(r/a)-\theta_p]\}$, which means they diverge exponentially with m if $r/a > \exp(\theta_p)$.

Even when distant coils can in principle produce the required magnetic field on the plasma surface, practical coils may not exist. Laplace's equation implies that the intrinsic difficulty of distant coils is exponentially dependent on the wave number k of the distribution of the normal magnetic field that they are producing on the plasma surface. The number of independent distributions of magnetic field that can be produced by practical coils is comparable to the number of Fourier components of the magnetic potential ϕ, Eq. (113), that have a wave number smaller than some critical value k_m. In a large-aspect-ratio circular tokamak, $k=m/a$ and the ratio of magnetic field on the coils at $r=b$ to that on the plasma at $r=a$ is $(b/a)^{(m-1)}$. For a practical separation between the coils and the plasma in a fusion power plant, $b/a\approx 1.7$, the ratio of the field at the coil to that on the plasma is five to one for $m=4$. Practical tokamak coils can control about four properties of the plasma, corresponding crudely to the poloidal Fourier harmonics up to $M_m=4$, so tokamak coil designs have four degrees of freedom, $N_f=4$. For stellarators the wave number of a magnetic-field distribution is $k\simeq\sqrt{(m/a)^2+(n/R_o)^2}$, where a is the minor and R_o is the major radius of the torus. To preserve the N_p-fold periodicity of the stellarator, the toroidal mode number of a field distribution must satisfy $n=n_h N_p$, where n_h is an integer. Summing over the possible combinations of m and n_h that satisfy $k\le M_m/a$, one finds that the number of degrees of freedom of stellarator coils is

$$N_f \approx 1.8\frac{M_m^2}{\epsilon_p}, \tag{115}$$

where the inverse aspect ratio per period of the stellarator $\epsilon_p \equiv N_p a/R_o$. Typically $\epsilon_p\approx 1/2$, so with $M_m=4$, stellarator coils have about 50 degrees of freedom.

Efficient coils for stellarators should exist, since the number of degrees of freedom in stellarator coil design,

$N_f \approx 50$, is larger than the number of shape parameters N_s that are needed to achieve reasonably optimized plasma shapes. The COILOPT code (Strickler, Berry, and Hirshman, 2002) tackles this problem directly by varying parameters that define the coils until a combined optimum is found for the coils and the plasma. However, the large number of degrees of freedom and the nonlinearity of the optimization process means the design of stellarator coils for optimal plasmas is extremely subtle, and it is important to understand the choices that are made during this optimization.

The fundamental choices in the optimization of a stellarator are those of the plasma shape and the coils. The optimized plasma shapes, $\vec{x}_s(\theta,\varphi)$, are found using a set of N_s parameters (Nührenberg *et al.*, 1995; Neilson *et al.*, 2000). For example, the shape parameters could be the Fourier coefficients of $R_s(\theta,\varphi)$ and $Z_s(\theta,\varphi)$ of a representation of the surface in (R,φ,Z) cylindrical coordinates. The optimization process for the plasma shape determines N_s shape parameters, so plasma optimization can only determine N_s constraints on the coils. Practical limits exist on the number of shape parameters that can be determined. These limits arise not only from the increasing difficulty of optimization with additional parameters, but also from the numerical accuracy of the codes that determine the physics properties associated with a given plasma shape. The constraints on the coils should be chosen to ensure the N_s known shape parameters have their desired values but with no constraints imposed for fitting unknown shape parameters. This can be done by studying the relation between changes in the plasma shape and small changes in the fields due to the coils, which is discussed in Sec. V.D.2.

2. Plasma response to coil changes

The relation between perturbations to the plasma shape and small changes in the magnetic field due to the coils is important for stellarator coil design, the elimination of islands due to magnetic-field errors, and the design of feedback systems for resistive wall modes in tokamaks.

A small change in the plasma shape is equivalent to a normal displacement $\vec{\xi}\cdot\hat{n}$ of the plasma surface and has a simple and unique relation to the normal component of the perturbed magnetic field on the original plasma surface $\delta\vec{B}\cdot\hat{n}$. To prove this, we first note that a tangential displacement, which means $\vec{\xi}\cdot\hat{n}=0$ but $\vec{\xi}\neq 0$ with $\vec{x}_s(\theta,\varphi)\rightarrow\vec{x}_s(\theta,\varphi)+\vec{\xi}(\theta,\varphi)$, does not change the plasma shape but does change the (θ,φ) coordinate system that describes that shape. In an ideal plasma $\delta\vec{B}=\vec{\nabla}(\vec{\xi}\times\vec{B})$, Eq. (46), and $\delta\vec{B}\cdot\vec{\nabla}\psi_t=\vec{B}\cdot\vec{\nabla}(\vec{\xi}\cdot\vec{\nabla}\psi_t)$, which relates $\vec{\xi}\cdot\hat{n}$ and $\delta\vec{B}\cdot\hat{n}$ since the normal to the plasma surface is $\hat{n}=\vec{\nabla}\psi_t/|\vec{\nabla}\psi_t|$. Actually the equation $\delta\vec{B}=\vec{\nabla}\times(\vec{\xi}\times\vec{B})$ holds whether or not the magnetic field is embedded in a perfectly conducting plasma, provided the rotational transform on the surface is an irrational number. The reason is $\delta\vec{B}=\vec{\nabla}\times\delta\vec{A}$. But any vector, including the perturbed vector potential, can be written in the form $\delta\vec{A}=\vec{\xi}\times\vec{B}+\vec{\nabla}\delta g$ provided a function $\delta g(\vec{x})$ exists that satisfies the magnetic differential equation $\vec{B}\cdot\vec{\nabla}\delta g=\vec{B}\cdot\delta\vec{A}$. This equation for δg is solvable when the rotational transform ι is an irrational number and the perturbation does not include a loop voltage. A loop voltage was considered by Lüst and Martensen (1960); also see the discussion of Eq. (26). The general validity of $\delta\vec{B}=\vec{\nabla}\times(\vec{\xi}\times\vec{B})$ on irrational surfaces is another way of stating that the rotational transform is not changed in linear order by a perturbation $\vec{\xi}$ except near a rational surface [Eq. (17)].

Functions of θ and φ, such as the magnetic-field perturbation normal to the original plasma surface $\delta\vec{B}\cdot\hat{n}$, are conveniently expressed using orthonormal functions. A set of dimensionless functions $f_i(\theta,\varphi)$ on the plasma surface are orthonormal if

$$\oint f_i f_j^* w\,da = \delta_{ij}, \tag{116}$$

where $w>0$ is an arbitrary weight function and da is the area element of the plasma surface. The normal component of the magnetic perturbation on the original plasma surface is expanded in the f_i's in the form

$$\delta\vec{B}\cdot\hat{n} = w\sum \Phi_j f_j^*. \tag{117}$$

The expansion coefficients have units of flux

$$\Phi_i = \oint f_i \delta\vec{B}\cdot\hat{n}\,da. \tag{118}$$

The derivation of the N_s constraints on stellarator coils that are required to fit N_s parameters in the plasma shape illustrates a response analysis. A small change in each of the N_s shape parameters produces a small change in the plasma shape, which is a normal displacement $\vec{\xi}_s\cdot\hat{n}$ of the plasma surface, and determines a small change in the normal magnetic field $\delta\vec{B}_s\cdot\hat{n}$. The N_s functions $\beta_s(\theta,\varphi)\equiv\delta\vec{B}_s\cdot\hat{n}$ can be transformed into a set of not more than N_s orthonormal functions $f_s(\theta,\varphi)$ by a Gram-Schmidt process such that any of the $\delta\vec{B}_s\cdot\hat{n}$ can be expanded in terms of the f_s's [Eq. (117)]. The constraints on the coils, $\oint f_s\vec{B}\cdot\hat{n}da=0$, then ensure that through linear order in the field error the N_s plasma shape parameters are reproduced but with no constraints on the coils from unknown shape parameters. Good stellarator coils exist if the N_s field distributions that are needed are in the solution space of the N_f field distributions that efficient stellarator coils can produce.

In a number of problems involving coils, it is necessary to know the change in the normal magnetic field on the plasma surface due to the coils, $\delta\vec{B}_x\cdot\hat{n}$, that is required to produce a given normal magnetic field on the plasma surface, $\delta\vec{B}\cdot\hat{n}$. The difference between $\delta\vec{B}\cdot\hat{n}$ and $\delta\vec{B}_x\cdot\hat{n}$ is the normal field produced by the perturbed plasma current $\delta\vec{B}_i\cdot\hat{n}$. The decomposition of $\delta\vec{B}\cdot\hat{n}$ into a part produced by coil currents and a part produced by internal plasma currents can be carried out if there is enough information to solve Laplace's equation for the magnetic potential ϕ between the plasma and the conducting structures. Two boundary conditions are required. In a cylinder, a solution to Laplace's equation has the form $\phi=\{\phi_i(a/r)^m+\phi_x(r/a)^m\}\cos(m\theta)$ and boundary conditions must determine both ϕ_i, which gives the effect of currents inside the plasma, and ϕ_x, which gives the effect of currents outside the plasma.

The response of the plasma to perturbations determines the relation between the perturbed normal field due to external current, $\delta\vec{B}_x\cdot\hat{n}$, and the normal field due to the internal perturbed plasma currents, $\delta\vec{B}_i\cdot\hat{n}$. Since only two boundary conditions are required to determine a unique solution to Laplace's equation, any two independent pieces of information will suffice. One example is the relation between the perturbed normal and tangential magnetic-field components. This information is contained in $\delta\vec{B}$ at the location of the unperturbed plasma surface. Since $\delta\vec{B}$ is continuous at the plasma surface, but $\delta\vec{B}=\vec{\nabla}\phi$ outside the plasma, the vector $\delta\vec{B}$ gives two functions of information on the plasma surface, ϕ and $\hat{n}\cdot\vec{\nabla}\phi=\hat{n}\cdot\delta\vec{B}$, and not three as one might first assume.

Given a plasma model, a linear relation exists at the location of the orginal plasma surface between a small change in the normal component of the magnetic field produced by external currents $\delta\vec{B}_x\cdot\hat{n}$ and the perturbed normal field $\delta\vec{B}\cdot\hat{n}$. This linear relation is conveniently expressed using matrices. The fluxes that are the expansion coefficients of the perturbed normal field on the plasma surface form a matrix vector $\vec{\Phi}\equiv\oint\vec{f}(\delta\vec{B}\cdot\hat{n})da$. The matrix vector $\vec{f}$ has the same functions $f_i(\theta,\varphi)$ as its components. Analogously, the fluxes that are the expansion coefficients of the normal field on the plasma surface due to perturbed coil currents form a matrix vector $\vec{\Phi}_x$. The linear relation between $\delta\vec{B}_x\cdot\hat{n}$ and $\delta\vec{B}\cdot\hat{n}$ determines the permeability matrix,

$$\vec{\Phi}=\overleftrightarrow{P}\cdot\vec{\Phi}_x. \tag{119}$$

The permeability matrix $\overleftrightarrow{P}$ is a property of the plasma and for an ideal plasma can be determined using a δW stability code. Once $\overleftrightarrow{P}$ is known, it is straightforward to uniquely relate any small perturbation in the plasma shape to the change in the normal magnetic field on plasma surface that is produced by the coils.

Tokamak and stellarator plasmas are particularly sensitive to magnetic perturbations that resonate with low-order rational surfaces and cause islands to open. The coils can be modified to eliminate the islands in one plasma state (Hudson *et al.*, 2002), though error correction or trim coils may be needed to eliminate islands over a broad range of plasma states. The issue of islands would not exist in a tokamak with perfect axisymmetry, but it does exist even for an ideal stellarator if the number of periods is small, $N_p\lesssim 6$. As the plasma equilibrium changes, a magnetic field that was nonresonant can develop resonant components. However, there are a relatively small number N_r of low-order rational surfaces that are consistent with the periodicity of a stellarator. This means that any set of trim coils that obeys the periodicity of the stellarator and has N_r sets of leads can control the islands provided the matrix between the fields produced by the trim coils and the resonant fields is nonsingular. In practice, tokamaks are not precisely axisymmetric and stellarators are not precisely periodic, so additional coils may be needed to correct field errors that break these symmetries.

The information required to determine whether a given set of trim coils can control the islands of a plasma equilibrium can be found using an ideal δW stability code. As discussed in Sec. V.C.3, perturbations at the edge of an ideal plasma, $\vec{\xi}\cdot\hat{n}$, can cause a jump in $\vec{\xi}\cdot\vec{\nabla}\psi_t$ at the rational surfaces. These jumps imply islands would open in a resistive plasma with the island width proportional to the square root of the jump. If there are N_r sets of integers n/m that give the rational numbers associated with these jumps, then there are N_r specific forms for $\vec{\xi}\cdot\hat{n}$ on the plasma surface that are proportional to the jumps but otherwise have the smallest possible amplitude, $\oint(\vec{\xi}\cdot\hat{n})^2wda$. All other small displacements of the plasma edge cause no jumps and, therefore, no islands. These N_r specific forms for $\vec{\xi}\cdot\hat{n}$ on the plasma surface give N_r distributions of normal magnetic field, $\delta\vec{B}_r\cdot\hat{n}=\hat{n}\cdot\vec{\nabla}\phi_r$, and have N_r associated perturbed tangential fields on the plasma surface, which give the magnetic scalar potentials ϕ_r there. From the ϕ_r and the $\hat{n}\cdot\vec{\nabla}\phi_r$, one can construct the N_r orthonormal functions $f_r(\theta,\varphi)$ that give the externally produced fluxes $\oint f_r\delta\vec{B}_x\cdot\hat{n}da$ that control the islands. These fluxes are linearly related to the currents in the trim coils. If there are N_t independent sets of trim coils, which means independent currents I_t, then the fluxes required for island control obey $\oint f_r\delta\vec{B}_x\cdot\hat{n}da=\Sigma_t M_{rt}I_t$ with the mutual inductance matrix M_{rt} an $N_r\times N_t$ matrix. If this matrix has N_r nonzero eigenvalues, the trim coils can control the island-causing resonances, though practicality requires that none of the N_r eigenvalues be excessively small.

In tokamaks, important kinklike instabilities, the resistive wall modes, can be stabilized by conducting structures, such as the chamber wall that surrounds the plasma; see Sec. V.C.3. These instabilities grow on the resistive time scale of the wall and can be feedback stabilized using external coils. The permeability matrix $\overleftrightarrow{P}$ contains all of the plasma information that is needed for calculating the stability and the feedback of wall modes. The circuit representation of the plasma that is associated with the permeability matrix has been used to design and interpret feedback systems for a number of major tokamak experiments (Bialek *et al.*, 2001). Prior to the development of the circuit representation of resistive wall modes, Lazarus, Lister, and Neilson (1990) developed a closely related theory for the feedback of the vertical instability of tokamaks. The vertical instability causes a $\hat{Z}$ motion of the plasma and limits the degree to which tokamak plasmas can be shaped to achieve higher beta operation. A number of groups have developed methods for studying feedback of resistive wall modes, starting with the early work of Bishop (1989). Recent discussions of feedback techniques have been given by Fitzpatrick (2001), Bondeson *et al.* (2002), Chance *et al.* (2002), and Boozer (2003).

The energy that is required to drive resistive wall modes comes from the plasma, so a representation of the effects of the perturbed plasma current on the external circuits is critical. Since these effects must be propagated through a solution to Laplace's equation, the effects can be represented by the magnetic field normal to the plasma surface that is produced by the perturbed currents inside the plasma, $\delta\vec{B}_i\cdot\hat{n}$. This means the effects of the perturbed plasma current can be represented by a surface current flowing on the plasma surface that produces the same normal field at the location of the unperturbed plasma surface as the perturbed plasma current.

A surface current can be represented as a matrix vector with elements that are discrete currents J_i. The normal field on the surface due to these currents is linear, so it can be represented using Eq. (118) as a magnetic-flux matrix vector $\vec{\Phi}_J=\overleftrightarrow{L}_p\cdot\vec{J}$. The matrix $\overleftrightarrow{L}_p$ is called the plasma inductance and depends only on the geometric shape and the size of the surface. The surface current that gives the same normal field on the unperturbed plasma surface as the perturbed plasma current is

$$\vec{I}_p=\overleftrightarrow{L}_p^{-1}\cdot(\overleftrightarrow{P}-\overleftrightarrow{1})\cdot\vec{\Phi}_x. \tag{120}$$

The representation of a surface current by a matrix vector $\vec{J}$ is demonstrated by noting that the current flowing on a surface $\psi_t=\psi_s$ has the form

$$\vec{j}_s=\delta(\psi_t-\psi_s)\vec{\nabla}\kappa\times\vec{\nabla}\psi_t. \tag{121}$$

This is the most general vector that is divergence-free, lies in the surface $\vec{j}_s\cdot\vec{\nabla}\psi_t=0$, and is zero except in the surface $\psi_t=\psi_s$. The current potential $\kappa(\theta,\varphi)$ can be expanded in the orthogonal functions, $\kappa=\Sigma J_i^* f_i(\theta,\varphi)$. Each element J_i of the matrix vector $\vec{J}$ has units of amperes and can be viewed as a current in a circuit. Actually, the general expression for the current potential $\kappa(\theta,\varphi)$ is the sum of the single-valued current potential $\Sigma J_i^* f_i$ plus two non-single-valued terms. One of these terms is $-\theta I_s/(2\pi)$, which gives the net toroidal current I_s in the surface, and the other is $\varphi G_s/(2\pi)$, which gives the net poloidal current G_s in the surface. The currents I_s and G_s are not usually retained in analyses of wall modes.

When the plasma is ideal, the permeability matrix $\overleftrightarrow{P}$ has real eigenvalues that are customarily written as $-1/s_i$ (Boozer, 2003). The stability coefficients s_i are proportional to the energy required to drive the perturbation with a positive s_i implying a negative energy. An unstable wall mode arises if the matrix $\overleftrightarrow{P}$ has a negative eigenvalue, which means a positive s_i. The relation between $\overleftrightarrow{P}$ and the energy is demonstrated by considering the power required to drive a perturbing current $\vec{J}$ on a surface infinitesimally outside of the plasma. That power is $\mathcal{P}_d=-\int\vec{j}_s\cdot\delta\vec{E}d^3x$. Using Eq. (121), $\mathcal{P}_d=-\oint\kappa(\vec{\nabla}\times\delta\vec{E})\cdot d\vec{a}$. Since $\partial\delta\vec{B}/\partial t=-\vec{\nabla}\times\delta\vec{E}$, the power is $\mathcal{P}_d=\vec{J}^\dagger\cdot d\vec{\Phi}/dt$. The flux and the current are proportional to each other, $\vec{\Phi}=\overleftrightarrow{P}\cdot\overleftrightarrow{L}_p\cdot\vec{J}$, so the energy δW required to drive the perturbation, $\mathcal{P}_d=d\delta W/dt$, has the explicitly real form $\delta W=\frac{1}{4}(\vec{J}^\dagger\cdot\vec{\Phi}+\vec{\Phi}^\dagger\cdot\vec{J})$. Since the inductance matrix $\overleftrightarrow{L}_p$ is a positive Hermitian matrix, the energy δW is negative only if the permeability $\overleftrightarrow{P}$ has a negative eigenvalue.

The circuit equations for wall modes are particularly simple in the case of primary importance, a single mode passing through marginal stability, which means one stability coefficient s passing through zero. Because of the singularity of the permeability matrix at $s=0$, only the marginal mode is important, so the matrix vectors can be approximated by a single matrix element, the element that represents the marginally stable perturbation. For this problem, $\Phi=-\Phi_x/s$, $\Phi_x=M_{pw}I_w$, $\Phi_w=L_wI_w+M_{wp}I_p$, and $d\Phi_w/dt=-R_wI_w$. Φ_w is the magnetic flux that penetrates the wall, I_w is the current in the wall, L_w is the wall inductance, $M_{pw}=M_{wp}$ is the mutual inductance between the wall and the plasma, and R_w is the wall resistivity. Simple algebra demonstrates that the flux through the wall is

$$\Phi_w=L_w\left\{1-\frac{1+s}{s}\frac{M_{pw}M_{wp}}{L_wL_p}\right\}I_w. \tag{122}$$

When the effective inductance of the wall, Φ_w/I_w, is negative, the wall mode grows at a rate proportional to R_w. By adding sensors and actively driven coils to the circuit equations, one can stabilize the wall mode by feedback.

In a rotating tokamak plasma, the eigenvalues of the permeability matrix (Boozer, 2003) are complex numbers $1/(-s_i+i\alpha_i)$. The quantity α_i gives the toroidal torque between the plasma and the mode i. The toroidal torque exerted by an axisymmetric surface that carries a surface current $\vec{j}_s$ is given by $\tau_\varphi=-\int(\vec{j}_s\times\delta\vec{B})\cdot(\partial\vec{x}/\partial\varphi)d^3x$, which implies $\tau_\varphi=i(N/2)(\vec{J}^\dagger\cdot\vec{\Phi}-\vec{\Phi}^\dagger\cdot\vec{J})$ when the φ dependence of f_i is written as $\exp(-iN\varphi)$. When an external perturbation is applied to a rotating tokamak plasma, the perturbation on the plasma surface can be amplified and can have a toroidal phase shift. The amplification and the phase shift are given by s_i and α_i (Boozer, 2003). Plasma rotation can stabilize the resistive wall mode by dragging the mode toroidally at too rapid a rate for it to penetrate the wall.

VI. RADIATION AND TRANSPORT

- A fusion burn of deuterium and tritium requires that the characteristic time for loss of energy from the plasma satisfy $\tau_E \approx 4\times 10^{21}/n_i T$, where τ_E is in seconds, n_i is the number of deuterium-tritium ions per cubic meter, and T is the plasma temperature in kilovolts. The characteristic energy confinement time of a plasma with few impurities and a temperature of about 20 keV is predominately due to thermal energy transport by plasma processes, not electromagnetic radiation.
- Energy transport determines the minimum size and hence the cost of an experiment to study plasmas that burn deuterium and tritium. For a power plant, the level of transport must be consistent with the required system parameters; see Sec. II.
- The energy confinement time τ_E of proposed experiments is estimated using empirical power-law scalings, which have the fundamental assumption that no critical values are crossed of the parameters that enter the power law.

The scaling of the energy confinement time of toroidally confined plasmas is a primary issue in the feasibility and cost of a burning plasma experiment (ITER Physics Expert Group on Confinement …, 1999). By definition a burning plasma experiment burns deuterium and tritium while requiring little external power. The loss of energy from a fusing plasma at 20 keV with few impurities is dominated by plasma processes, called energy transport, not electromagnetic radiation. The focus on proposals for burning plasma experiments has meant that energy transport has been a major topic for plasma research.

Energy transport is also an issue for the feasibility of fusion power, but differences exist between the transport issues for fusion power and those for a burning plasma experiment. The size of a burning plasma experiment is essentially determined by the magnitude of the energy transport: the smaller the transport, the smaller and cheaper the required experiment. However, the basic size of a power plant is determined by issues that have little to do with plasma transport coefficients. The critical issues in a power plant are whether the energy transport is consistent with what is required to maintain a steady burn—neither too large nor too small—and whether the particle transport is sufficiently large to remove alpha-particle ash. In addition, the feasibility of steady-state tokamak power plants is dependent upon the natural profiles that the temperature, density, and current take in a fusion plasma. The performance of stellarator power plants is far less dependent on the issue of profiles.

Energy transport can provide an upper limit on the plasma beta, $\beta \equiv 2\mu_0 p/B^2$, because transport for given plasma conditions is generally larger, the smaller the magnetic field.

Predictions of the energy confinement times of experiments are usually based on empirical scaling relations (ITER Physics Expert Group on Confinement …, 1999). Empirical scaling relations generally assume the mathematical form of a power law,

$$\tau_E(x_1,x_2,\dots) = a_0 x_1^{a_1} x_2^{a_2}\cdots, \tag{123}$$

so each parameter of the set $(x_1,x_2,\dots)$ enters multiplicatively. The constants $(a_0,a_1,\dots)$ are chosen to obtain the best fit to the data. Mathematics implies that power-law scaling is a precise description if and only if the parameters $(x_1,x_2,\dots)$ have no characteristic values or scale. A function is independent of the scale s if $f(x/s) = f(1/s)f(x)$, which implies $f(x)\propto x^a$. All other functions have a scale, and the properties of the function depend on the size of x relative to s. For example, the scale of $\sin x$ is 2π. The constants of a power law $(a_0a_1,\dots)$ are easily fit to data by a linear regression of the logarithm of τ_E against the logarithms of the parameters.

The most commonly used scaling relations are power laws based on experimental parameters such as input power, plasma current, and plasma size. Less used but more scientifically appealing scaling relations are based on dimensionless parameters (Connor, 1984). The dimensionless energy confinement time is the confinement time times the cyclotron frequency of deuterium, $\Omega \equiv eB/m_d$. Important dimensionless parameters are (a) the gyroradius of deuterium, $\rho \equiv (\sqrt{Tm_d})/(eB)$, divided by the plasma radius, which is called ρ_*; (b) the number of bounces between collisions a deuterium ion makes when trapped in the variation in the magnetic-field strength on a magnetic surface, a ratio called $1/\nu_*$; and (c) $\beta = \langle 2\mu_0 p/B^2\rangle$. In terms of these parameters,

$$\Omega\tau_E = a_0 \rho_*^{a_\rho} \nu_*^{a_\nu} \beta^{a_\beta}. \tag{124}$$

A. Electromagnetic radiation

- Although electromagnetic radiation is usually a subdominant energy-loss mechanism for deuterium-tritium (DT) fusion systems, electromagnetic radiation limits the feasibility of non-DT fusion fuels and the tolerable level of impurities in all fusion systems.
- The most important types of radiative losses from magnetically confined plasmas are bremsstrahlung, cyclotron, and atomic.

Electromagnetic radiation is a subdominant energy-loss mechanism in a deuterium-tritium (DT) fusion plasma operating at 20 keV with a low level of impurities. However, electromagnetic radiation limits the tolerable level of impurities, the range of temperatures, and the types of fuel (Nevins, 1998) for which fusion energy is feasible. Three types of radiative losses are important: bremsstrahlung, cyclotron, and atomic.

Electromagnetic radiation arises when the current density $\vec{j}$ has both a nonzero curl and a time derivative. The time derivative of the current density of a single particle is $\dot{\vec{j}}(\vec{x}) = q\vec{a}\,\delta(\vec{x}-\vec{x}_p)$, where $\vec{x}_p(t)$ is the position of the particle, $\vec{a} = d^2\vec{x}_p/dt^2$ is its acceleration, and $\delta(\cdots)$ is the Dirac delta function. The rate at which a nonrelativistic particle of charge q loses energy H through electromagnetic radiation is given by the Larmor formula,

$$\frac{dH}{dt} = -\frac{q^2 a^2}{6\pi\epsilon_0 c^3}, \tag{125}$$

which is derived in the standard electrodynamics texts.

Bremsstrahlung, which in German means radiation due to deceleration, arises from the electrostatic scattering of one charged particle by another. The power loss due to bremsstrahlung is dominated by the scattering of electrons by ions. The power loss per unit volume due to scattering by ions of charge Ze with a number density n_z is proportional to $Z^2 n_z n_e \sqrt{T_e}$. The typical photon emitted in bremsstrahlung has an energy equal to the electron temperature. In plasmas of fusion interest there is negligible reabsorption of bremsstrahlung radiation because the total radiated power is far below the blackbody level. An accurate derivation of bremsstrahlung is given by Karzas and Latter (1961).

The basic dependences and power loss of bremsstrahlung can be understood starting with the expression for the acceleration of one charged particle by another, $\vec{a}_1 = (q_1 q_2 / 4\pi\epsilon_0 m_1)(\vec{x}_1 - \vec{x}_2)/|\vec{x}_1 - \vec{x}_2|^3$. The maximum acceleration occurs at the point of closest approach of the two particles, $b \equiv \min(|\vec{x}_1 - \vec{x}_2|)$. The characteristic frequency of the bremsstrahlung radiation is $\omega = v/b$, where v is the particle velocity, which implies the characteristic wave number satisfies $kb = v/c$. For nonrelativistic particles, $v/c \ll 1$, the wavelength of the radiation is long compared to the distance between the radiating particles. The time derivative of the current density of two particles is $\dot{\vec{j}}(\vec{x}) = q_1 \vec{a}_1 \delta(\vec{x} - \vec{x}_1) + q_2 \vec{a}_2 \delta(\vec{x} - \vec{x}_2)$. Since the wavelength of the radiation is long compared to the distance between the radiating particles, one can ignore that distance and let $\dot{\vec{j}}(\vec{x}) = (q_1/m_1 - q_2/m_2) m_1 \vec{a}_1 \delta(\vec{x} - \vec{x}_1)$ using Newton's third law to write $m_2 \vec{a}_2 = -m_1 \vec{a}_1$. The expression for $\dot{\vec{j}}$ implies that radiation from electron-electron scattering vanishes in the nonrelativistic limit and that bremsstrahlung is primarily a result of electron-ion scattering.

The change in energy of an electron with speed v that passes by an ion that has a charge Ze at a distance b is $\delta H \approx (dH/dt)(b/v)$, where dH/dt is given by the Larmor formula, Eq. (125), with the acceleration $a \approx (Ze^2/4\pi\epsilon_0 m_e)/b^2$. The bremsstrahlung power p_b emitted per unit volume is given by the number of electrons per unit volume times the number of ions the electrons pass per unit time. The number of ions of charge Z that the electrons pass per unit time at a range between b and $b+db$ is $n_z v 2\pi b db$, so $p_b \approx n_z n_e \int \delta H v 2\pi b db$. This integral is proportional to $1/b_{min}$ with b_{min} the distance of closest possible approach. In classical mechanics the distance of closest possible approach is given by $(Ze^2/4\pi\epsilon_0)/b_{min} = m_e v^2/2$, but for electrons that have energies of importance for fusion systems this distance is much smaller than the limit set by quantum mechanics, which is $b_{min} m_e v \approx \hbar$. Quantum effects determine the closest possible approach when $v/c > 2Z\alpha$, where the fine-structure constant $\alpha \equiv e^2/(4\pi\epsilon_0 c\hbar) \approx 1/137$. The typical energy of an emitted photon is $\hbar\omega \approx \hbar v/b_{min} \approx m_e v^2 \approx T$. The basic dependences of the emitted power, $Z^2 n_z n_e \sqrt{T_e}$, are obtained by combining the results.

Cyclotron radiation results from the acceleration of electrons with velocity $\vec{v}$ by the magnetic field, $\vec{a} = -(e/m_e)\vec{v} \times \vec{B}$. The characteristic frequency of the associated motion is the electron cyclotron frequency $\Omega_e \equiv eB/m_e$, and this is the frequency at which cyclotron radiation is emitted by an electron. A straightforward application of the Larmor formula, Eq. (125), gives the power emitted per unit volume due to the cyclotron motion of Maxwellian electrons, $p_c = \omega_{pe}^2 \Omega_e^2 T/(3\pi c^3)$. Since $\langle m v_\perp^2/2 \rangle$ is the energy in two independent components of velocity, and the energy in three independent components is $\frac{3}{2}T$, one has $\langle m v_\perp^2/2/ \rangle = T$. The plasma frequency is $\omega_{pe} = \sqrt{e^2 n_e/(m_e \epsilon_0)}$, which is the frequency with which the electrons would oscillate if displaced en masse from the ions. Thermodynamics implies that the emission of electromagnetic waves due to the cyclotron motion cannot exceed the energy flux leaving a blackbody in the relevant frequency range. The energy flux leaving a blackbody at frequencies less than ω is $\mathcal{F}_{bb} = \omega^3 T/(12\pi^2 c^2)$ provided the energy per photon is far less than the temperature, $\hbar\omega \ll T$. This is the *Rayleigh-Jeans limit*, which is the relevant limit when $\omega \sim \Omega_e$. To be consistent with thermodynamics, the plasma must reabsorb the radiated power within a distance $L_a \equiv 4\pi \mathcal{F}_{bb}/p_c = (\Omega_e/\omega_{pe})(c/\omega_{pe})$, which is a fraction of a millimeter in a fusion plasma. Due to relativistic effects, electrons also emit radiation at harmonics of the cyclotron frequency with the emission at each higher harmonic reduced by a factor $T/(m_e c^2)$ compared to the previous harmonic. The power emitted by a region is comparable to the blackbody flux $\mathcal{F}_{bb}$ up to a frequency ω that is set by the first cyclotron harmonic for which the power can leave the plasma without strong reabsorption. Accurate calculations of the power losses due to cyclotron emission are complicated. Albajar, Bornatici, and Engelmann (2002) give recent results and historical references.

Atomic radiation arises from electrons switching atomic levels and from the capture of free electrons into bound states. Significant atomic radiation only arises if the electron temperature is low enough for the atom or ion to have bound electrons. The atomic radiation from an element with a number density n_z has the form $p_z = n_e n_z R_z(T)$, with $R_z(T)$ a function that is large at electron temperatures at which ionization occurs. The function $R_z(T)$ for a single element can have large variations with multiple peaks. Under fusion conditions $R_z(T)$ is negligible except for elements with high atomic numbers, such as iron. Functions $R_z(T)$ have been given by Post *et al.* (1977).

At a given impurity level, the power losses due to bremsstrahlung and atomic radiation depend quadratically on the plasma density, as does the nuclear power input, so the nuclear power input minus the radiative losses due to bremsstrahlung and atomic radiation can be viewed as an effective power input, $p_{eff}=n^2 f_{eff}(T)$. The power loss due to cyclotron emission and absorption is more complicated and in some features resembles a diffusive transport process.

Fusion power is feasible only at low impurity levels. In addition to the enhancement of electromagnetic radiation, the ionization of impurities adds many electrons to the plasma. These electrons exert a pressure and transport energy but are unrelated to the production of fusion power.

B. Kinetic theory

- Kinetic theory is needed to calculate transport coefficients such as particle diffusivities, thermal conductivities, and bootstrap currents.
- Collisions in plasmas change the trajectories of particles diffusively. This is in contrast to collisions in ordinary gases, which produce large abrupt changes in the particle trajectories.
- Plasma confinement forces the distribution functions of the particles to deviate from local Maxwellians. The rate of entropy production that is required to hold the distribution functions away from local Maxwellians allows a simple calculation of transport coefficients in the low-collisionality limit.

Transport calculations are carried out using kinetic theory. The fundamental quantity in kinetic theory is the distribution function $f(\vec{x},\vec{p},t)$, which describes the evolution of a large group of identical particles. In a plasma, there is a distribution function for the electrons and for each of the ion species. The distribution function is the density of particles in phase or momentum space $(\vec{x},\vec{p})$. In other words, the number of particles in a region of momentum $\vec{p}$ and ordinary space $\vec{x}$ is the integral of the distribution function over that region, $\int f d^3p d^3x$. A more detailed treatment of kinetic theory than that given here can be found in Helander and Sigmar (2002) and in most plasma textbooks.

The evolution of the distribution function is given by the kinetic equation

$$\frac{df}{dt}=\mathcal{C}(f), \tag{126}$$

where $\mathcal{C}(f)$ is the collision operator. The total time derivative of kinetic theory is an extension of the concept of the total time derivative of fluid mechanics, Eq. (44), which operates on functions $g(\vec{x},t)$ in ordinary space, to functions $f(\vec{x},\vec{p},t)$ in phase space,

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\frac{d\vec{x}}{dt}\cdot\frac{\partial f}{\partial \vec{x}}+\frac{d\vec{p}}{dt}\cdot\frac{\partial f}{\partial \vec{p}}, \tag{127}$$

where $\partial f/\partial\vec{x}\equiv\vec{\nabla}f$. Equation (127) can also be written as

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\frac{\partial}{\partial \vec{x}}\cdot\left(\frac{d\vec{x}}{dt}f\right)+\frac{\partial}{\partial \vec{p}}\cdot\left(\frac{d\vec{p}}{dt}f\right). \tag{128}$$

If $\vec{p}$ is the canonical momentum of Hamiltonian mechanics, Eq. (128) follows from Eq. (127) because of Hamilton's equations $d\vec{x}/dt=\partial H/\partial\vec{p}$ and $d\vec{p}/dt=-\partial H/\partial\vec{x}$. In the presence of a magnetic field, the canonical momentum is $\vec{p}=m\vec{v}+q\vec{A}$ [Eq. (191)]. If $\vec{p}\equiv m\vec{v}$, Eq. (128) follows because $\dot{\vec{x}}=\vec{p}/m$ and $\dot{\vec{p}}=q(\vec{E}+\vec{p}\times\vec{B}/m)$, so $(\partial/\partial\vec{x})\cdot\dot{\vec{x}}=0$ and the momentum space divergence of $\dot{\vec{p}}$ is zero, $(\partial/\partial\vec{p})\cdot\dot{\vec{p}}=0$.

Without collisions, the kinetic equation is called the *Vlasov equation*, $df/dt=0$. The Vlasov equation is easily understood. Given the distribution function at $t=t_0$, the distribution function at $t=t_0+\delta t$ is obtained by advancing each particle in the distribution along its trajectory, $\vec{x}=\vec{x}_0+(d\vec{x}/dt)\delta t$ and $\vec{p}=\vec{p}_0+(d\vec{p}/dt)\delta t$. If $f(\vec{x}_0,\vec{p}_0)$ is the given distribution, then for an infinitesimal interval of time, $\delta t=t-t_0$, the distribution function is $f(\vec{x}-(d\vec{x}/dt)\delta t,\vec{p}-(d\vec{p}/dt)\delta t)$, which implies $\partial f/\partial t=-(d\vec{x}/dt)\cdot\partial f/\partial\vec{x}-(d\vec{p}/dt)\cdot\partial f/\partial\vec{p}$. The Vlasov equation is a hyperbolic, partial differential equation that has one characteristic. That characteristic is the trajectory of a particle.

The collision operator is the momentum-space divergence of a flux $\vec{\mathcal{F}}$ of particles through phase space $(\vec{x},\vec{p})$,

$$\mathcal{C}(f)=-\frac{\partial}{\partial\vec{p}}\cdot\vec{\mathcal{F}}. \tag{129}$$

Collisions are caused by the graininess of the plasma, an effect that is not directly described by the distribution function $f(\vec{x},\vec{p})$.

The graininess of the plasma leads to complicated electric fields that scatter particles and cause the collisional flux $\vec{\mathcal{F}}$. The electric field at position $\vec{x}$ due to a charge q at $\vec{x}_1$ is

$$\vec{E}(\vec{x},\vec{x}_1)=-\frac{q}{\epsilon_0}\int\frac{i\vec{k}}{k^2}e^{i\vec{k}\cdot(\vec{x}-\vec{x}_1)}\frac{d^3k}{(2\pi)^3}, \tag{130}$$

which is demonstrated by checking that $\vec{\nabla}\cdot\vec{E}=(q/\epsilon_0)\delta(\vec{x}-\vec{x}_1)$, $\vec{\nabla}\times\vec{E}=0$, and $\vec{E}(\vec{x}\to\infty,\vec{x}_1)=0$. Note the Dirac delta function has the representation

$$\delta(\vec{x}-\vec{x}_1)=\int e^{i\vec{k}\cdot(\vec{x}-\vec{x}_1)}\frac{d^3k}{(2\pi)^3}. \tag{131}$$

The electric field that is obtained from a uniform density n of charges, $\langle\vec{E}\rangle\equiv\int\vec{E}(\vec{x},\vec{x}_1)nd^3x_1$, is zero. However, the root-mean-square electric field from a uniform density of charge is not zero,

$$\langle E^2\rangle\equiv\int\vec{E}(\vec{x},\vec{x}_1)\cdot\vec{E}(\vec{x},\vec{x}_1)nd^3x_1, \tag{132}$$

and can be written as

$$\langle E^2\rangle = n\left(\frac{q}{\epsilon_0}\right)^2 \int \frac{1}{k^2}\frac{d^3k}{(2\pi)^3}. \tag{133}$$

This electric field causes accelerations $\Delta\vec{a}=(q/m)\vec{E}$ of the particles, which for a fast-moving particle persist for a time $\Delta t=1/kv$ with v the speed of the particle. That is, a particle has essentially random changes in its velocity, $\Delta\vec{v}=(\Delta\vec{a})(\Delta t)$, separated by time intervals of order Δt. These random changes scatter the velocity at an average rate $\nu=\langle(\Delta v)^2/\Delta t\rangle/v^2$. That is, $\nu=\langle(q/m)^2E^2\Delta t\rangle/v^2$, which can be written as

$$\nu \approx \frac{nq^4}{m^2\epsilon_0^2}\frac{1}{v^3}\int \frac{1}{k^3}\frac{d^3k}{(2\pi)^3}. \tag{134}$$

The integral over wave numbers is logarithmically divergent without a cutoff at large or at small wave numbers,

$$\ln\Lambda \equiv \int \frac{d^3k}{4\pi k^2} = \ln\left(\frac{k_{max}}{k_{min}}\right). \tag{135}$$

The cutoff at large wave numbers k_{max} is given by the distance of closest possible approach of two charged particles, $b_{min}=1/k_{max}$. As shown in the discussion of bremsstrahlung, Sec. VI.A, b_{min} is determined by quantum effects at fusion temperatures, $b_{min}=\hbar/(mv)$. The cutoff at small wave numbers is given by the Debye length, $k_{min}=1/\lambda_D$. The Debye length is $\lambda_D \equiv \sqrt{\epsilon_0 T/(nq^2)}$. The Coulomb logarithm is $\ln\Lambda\approx 17$ in laboratory plasmas.

The Debye length, $\lambda_D\equiv\sqrt{\epsilon_0 T/(nq^2)}$, is the shielding distance for the electrostatic potential of a charge. What is meant by this? Let Φ be the potential due to a charge Q that is placed in a plasma. The electric force $-qn\vec{\nabla}\Phi$ on a species with charge q and number density n is balanced by the pressure force of that species $\vec{\nabla}(nT)$. In equilibrium the temperature is constant, so the equilibrium density is $n=n_\infty\exp(-q\Phi/T)$. The Poisson equation for the potential is then

$$\nabla^2\Phi = -\frac{Q}{\epsilon_0}\delta(\vec{x}) - \frac{qn_\infty}{\epsilon_0}(e^{-q\Phi/T}-1). \tag{136}$$

Far from the charge, $q\Phi/T\ll 1$, and Poisson's equation becomes $\nabla^2\Phi=\Phi/\lambda_D^2$. Consequently, for $r\gg\lambda_D$, the electric potential of a charge in a plasma is

$$\Phi = \frac{Q}{\epsilon_0}\frac{\exp\left(-\frac{r}{\lambda_D}\right)}{r}. \tag{137}$$

Conservation laws place three conditions on the collision operator $\mathcal{C}(f)$, and the thermodynamic law of entropy increase places a fourth condition. The discussion of these conditions is simpler using phase-space coordinates in which the momentum is $\vec{p}=m\vec{v}$ rather than the canonical momentum, $\vec{p}=m\vec{v}+q\vec{A}$, of Hamiltonian mechanics, so these are the phase-space coordinates that will be used. The three conservation laws are for the particle,

$$n(\vec{x}) \equiv \int f d^3p; \tag{138}$$

momentum,

$$nm\vec{u}(\vec{x}) \equiv \int \vec{p} f d^3p; \tag{139}$$

and energy conservation,

$$\epsilon(\vec{x}) \equiv \int \frac{p^2}{2m} f d^3p. \tag{140}$$

The entropy per unit volume is

$$s \equiv -\int f\ln(f) d^3p. \tag{141}$$

This definition of the kinetic entropy is shown below to give results that are in agreement with the thermodynamic entropy

The Maxwellian distribution f_M has the maximum entropy per unit volume s, with a fixed number of particles per unit volume n, momentum per unit volume $nm\vec{u}$, and energy per unit volume, $\epsilon=\frac{1}{2}nmu^2+\frac{3}{2}nT$, which defines the temperature T,

$$f_M(\vec{x},\vec{p}) = \frac{n}{(2\pi T/m)^{3/2}}\exp\left(-\frac{(\vec{p}-m\vec{u})^2}{2mT}\right). \tag{142}$$

That is, one takes $s+\lambda_1 n+\vec{\lambda}_2\cdot nm\vec{u}+\lambda_3\epsilon$ to its extreme by considering variations in the distribution function δf while treating the λ's as constants. When the λ's, which are called Lagrange multipliers, are chosen to obtain the correct density, momentum, and energy, the Maxwellian is obtained. To be consistent with thermodynamics, a collision operator must cause the entropy to increase except when the distribution function is a local Maxwellian.

A collision operator, $\mathcal{C}(f)=-(\partial/\partial\vec{p})\cdot\vec{\mathcal{F}}$, that is simple but obeys the four conditions has the phase-space flux

$$\vec{\mathcal{F}} = -\frac{\vec{\vec{\nu}}}{2}\cdot\left\{(\vec{p}-m\vec{u})f + mT\frac{\partial f}{\partial\vec{p}}\right\}. \tag{143}$$

The collision frequency $\vec{\vec{\nu}}$ in this simple collision operator is momentum independent. The correct collision operator is considerably more complicated and in particular the collision frequency $\vec{\vec{\nu}}$ is dependent on the energy of the particles being scattered [Eq. (134)].

The simplified collision operator of Eq. (143) is useful for illustrating the diffusive nature of collisions in plasmas. A distribution function that develops complicated structures in momentum space can be smoothed arbitrarily quickly. More precisely, if the distribution function changes over a range of momenta Δp then that change is smoothed out at a rate of order $\nu(p/\Delta p)^2$.

The change in the entropy density $s(\vec{x},t)$ at each point in a plasma is due to an entropy flux $\vec{\mathcal{F}}_s$ and the creation of entropy by collisions $\dot{s}_c$. A differentiation of the entropy density $s(\vec{x},t)$, Eq. (141), with the use of $df/dt=C(f)$ and Eq. (128), implies

$$\frac{\partial s}{\partial t} + \vec{\nabla}\cdot\vec{\mathcal{F}}_s = \dot{s}_c. \tag{144}$$

The entropy flux is $\vec{\mathcal{F}}_s \equiv -\int \vec{v}(\ln f) f d^3p$ where $\vec{v}=\vec{p}/m$, and the rate of entropy production by collisions is

$$\dot{s}_c \equiv -\int \ln(f)\mathcal{C}(f) d^3p. \tag{145}$$

The definition of entropy per unit volume $s(\vec{x},t)$, Eq. (141), is consistent with the thermodynamic entropy if and only if it satisfies three conditions.

(1) The entropy, $\int s d^3x$, must be additive. That is, entropy in a region that consists of two parts must be the sum of the two entropies, a condition that s satisfies.

(2) The entropy of an isolated system cannot decrease. This condition implies $\dot{s}_c \geqslant 0$ and is a constraint on the collision operator. If the distribution function is written in the form $f=f_M\exp(\hat{f})$ with f_m defined so its $n(\vec{x})$, $\vec{u}(\vec{x})$, and $\epsilon(\vec{x})$ are the same as those of f, then Eq. (143) for simplified collisional flux gives a positive entropy production as long as $\overleftrightarrow{\nu}$ is a positive matrix,

$$\dot{s}_c = \frac{mT}{2}\int \frac{\partial \hat{f}}{\partial \vec{p}}\cdot\overleftrightarrow{\nu}\cdot\frac{\partial \hat{f}}{\partial \vec{p}} f d^3p. \tag{146}$$

(3) The change in the energy due to a transfer of heat is $T\delta S$.

By taking the time derivative of the energy per unit volume $\epsilon(\vec{x},t)\equiv\int(p^2/2m) f d^3p$, one can show, using Eq. (128) and $df/dt=\mathcal{C}(f)$ to eliminate $\partial f/\partial t$, that the energy flux is the sum of two parts: a part proportional to the fluid velocity $\vec{u}$ and a part equal to the heat flux $\vec{\mathcal{F}}_h \equiv \int \frac{1}{2}m(\vec{v}-\vec{u})^3 f d^3p$, where $\vec{v}=\vec{p}/m$. The entropy flux $\vec{\mathcal{F}}_s$ is also the sum of two parts when the plasma is near thermodynamic equilibrium, which means $f=f_M\exp(\hat{f})$ with $|\hat{f}|\ll 1$. One part of the entropy flux is proportional to the fluid velocity $\vec{u}$ and the other part is $\vec{\mathcal{F}}_h/T$. In a continuum system the temperature is well defined only near thermodynamic equilibrium, and in that limit the third condition on the entropy is satisfied.

Thermodynamics gives a simple but rigorous expression for the heating power P required to maintain a stationary, $\vec{u}=0$, steady-state plasma,

$$P(\psi_t) = T(\psi_t)\int \dot{s}_c d^3x, \tag{147}$$

where the volume integral is over a region enclosed by a flux surface ψ_t, and $P(\psi_t)$ is the total heating power in that region. To prove this equation, note that the power per unit volume that must be added to balance the heat flux is $\dot{\epsilon}_h=\vec{\nabla}\cdot\vec{\mathcal{F}}_h$, while the entropy created by collisions must satisfy $\dot{s}_c=\vec{\nabla}\cdot(\vec{\mathcal{F}}_h/T)$.

Equation (147) coupled with Eq. (146) implies the distribution functions must be close to local Maxwellians on each pressure surface to obtain the confinement needed for fusion ignition. The required energy confinement time in a fusion power plant is about 10^2 ion collision times and 10^4 electron collision times. The maintenance of near Maxwellians requires the trajectories of charged particles to remain close to constant-pressure surfaces during the time between collisions.

Thermodynamics relates the collisional entropy production $\dot{s}_c$ and the plasma transport in an even more complete form. Given $\dot{s}_c$, one can read off the transport coefficients. For simplicity, assume that at each point in the plasma the distribution function is well approximated by a stationary Maxwellian, so the flow velocity $\vec{u}=0$. The textbook thermodynamic equation, $dU=TdS-pdV+\mu dN$, relates thermodynamic quantities of an entire system. Plasma studies use the energy $\epsilon\equiv U/V$, the entropy $s\equiv S/V$, and the number of particles $n\equiv N/V$ per unit volume. Substituting these definitions into the thermodynamic equation yields $(d\epsilon - Tds - \mu dn)V = -(\epsilon - Ts + p - \mu n)dV$. The thermodynamic properties of a plasma are independent of the plasma volume V, so both sides of this equation must be zero. The chemical potential μ can be evaluated for a Maxwellian using $\mu=(\epsilon-Ts+p)/n$, and one finds

$$\frac{\mu}{T} = c_0 + \ln\left(\frac{n}{T^{3/2}}\right) \tag{148}$$

with c_0 a constant. Consequently, a stationary Maxwellian has the form

$$f_M = c_M \exp\left(\frac{\mu - mv^2/2}{T}\right), \tag{149}$$

where c_M is a constant. The thermodynamic equation $d\epsilon = Tds + \mu dn$ implies that the time rate of change of the entropy density is $\partial s/\partial t = (1/T)\partial\epsilon/\partial t - (\mu/T)\partial n/\partial t$. Now $\partial n/\partial t = -\vec{\nabla}\cdot\vec{\Gamma}$ with $\vec{\Gamma}$ the diffusive particle flux, and $\partial\epsilon/\partial t = -\vec{\nabla}\cdot\vec{\mathcal{F}}_h$ with $\vec{\mathcal{F}}_h$ the heat flux. Therefore

$$\frac{\partial s}{\partial t} = -\vec{\Gamma}\cdot\vec{\nabla}\frac{\mu}{T} + \vec{\mathcal{F}}_h\cdot\vec{\nabla}\frac{1}{T} - \vec{\nabla}\cdot\left(\frac{1}{T}\vec{\mathcal{F}}_h - \frac{\mu}{T}\vec{\Gamma}\right). \tag{150}$$

The divergence term on the right-hand side of Eq. (150) is the divergence of the entropy flux, but the other terms are due to the irreversible production of entropy, which means production by collisions. Since the chemical potential and the temperature depend only on the ψ_t coordinate, the collisional entropy production is

$$\dot{s}_c = -\Gamma\frac{d\mu/T}{d\psi_t} + \mathcal{F}_h\frac{d1/T}{d\psi_t}, \tag{151}$$

with $\Gamma\equiv\vec{\Gamma}\cdot\vec{\nabla}\psi_t$ and $\mathcal{F}_h\equiv\vec{\mathcal{F}}_h\cdot\vec{\nabla}\psi_t$. The quantities $-d(\mu/T)/d\psi_t$ and $d(1/T)/d\psi_t$ are called thermodynamic forces, and $\mathcal{F}_h$ and Γ are the conjugate fluxes. Near thermodynamic equilibrium, the fluxes are proportional to the forces, $\Gamma=-D_n d(\mu/T)/d\psi_t - D_c d(1/T)/d\psi_t$ and $\mathcal{F}_h = D_c d(\mu/T)/d\psi_t + D_T d(1/T)/d\psi_t$. The cross terms, the terms proportional to D_c, have the same coefficient due to the symmetry of the collision operator, or more generally due to Onsager symmetry (Onsager, 1931). Consequently

$$\dot{s}_c = D_n\left(\frac{d\mu/T}{d\psi_t}\right)^2 + 2D_c\frac{d\mu/T}{d\psi_t}\frac{d1/T}{d\psi_t} + D_T\left(\frac{d1/T}{d\psi_t}\right)^2. \tag{152}$$

By calculating $\dot{s}_c$, one can obtain all three independent transport coefficients.

Much freedom exists in the choice of variables ξ^j that are used to describe the distribution function. The left-hand side of the kinetic equation $df/dt=C(f)$ becomes $df/dt=\partial f/\partial t+\Sigma_j(\partial f/\partial\xi^j)(d\xi^j/dt)$. All that is needed is knowledge of how each of the variables changes along the trajectory of a particle, $d\xi^j/dt$. For example, f is a solution of the collisionless kinetic equation $df/dt=0$, the Vlasov equation, if and only if its variables are constants of the motion of the particles, $d\xi^j/dt=0$. In a system that is independent of time, $f(H)$ is a solution to the Vlasov equation where H is the Hamiltonian or energy of a particle.

The calculation of density perturbations using kinetic theory is subtle when the unperturbed distribution function is given as a function of a constant of the motion such as the Hamiltonian, $f_0(H)$. Consider a collisionless plasma that is perturbed by a change in the electric potential, $\delta\Phi$. The distribution function in the presence of the perturbation is $f=f_0(H)+\delta f$. The density perturbation that is caused by the perturbation $\delta\Phi$ is not $\int\delta f d^3v$, but $\delta n=\int\delta f d^3v+\int q\delta\Phi(df_0/dH)d^3v$. That is, the total change in the distribution function is

$$\Delta f \equiv \delta f + q\delta\Phi\frac{df_0}{dH}. \tag{153}$$

To understand the reason, suppose the plasma is perturbed by an electric potential $\delta\Phi=\tilde{\Phi}\sin(kx-\omega t)$. The kinetic equation of a collisionless plasma, $df/dt=0$, can be solved in two ways, and a subscript of v or H will be placed on δf to indicate which set of variables, (x,v,t) or (x,H,t), is used to find the solution. First, using variables (x,v,t) one can write $d\delta f_v/dt=\partial\delta f_v/\partial t+v\partial\delta f_v/\partial x=-(qE/m)df_0/dv$ since $dv/dt=qE/m$. This equation implies

$$\delta f_v = \frac{k}{kv-\omega}\frac{q\delta\Phi}{m}\frac{df_0}{dv}. \tag{154}$$

Second, using variables (x,H,t) one can write $d\delta f_H/dt=\partial\delta f_H/\partial t+v\partial\delta f_H/\partial x=-(df_0/dH)dH/dt$ where the change in the energy of a particle is $dH/dt=q\partial\delta\Phi/\partial t$. The solution is

$$\delta f_H = \frac{\omega}{kv-\omega}q\delta\Phi\frac{df_0}{dH}. \tag{155}$$

These two solutions are not equal, even after making the substitution $df_0/dv=(df_0/dH)mv$. Indeed, $\delta f_v=\delta f_H+q\delta\Phi df_0/dH$. The resolution of this paradox is that the unperturbed Hamiltonian H_0 goes to $H=H_0+q\delta\Phi$. A first-order Taylor expansion implies $f_0(H)=f_0(H_0)+q\delta\Phi df_0/dH_0$. The term $q\delta\Phi df_0/dH$ is called the adiabatic part of the plasma response.

C. Landau damping and quasilinear diffusion

- A collisionless, but unidirectional, transfer of energy can occur between a wave and the particles that form a plasma. The sign of this transfer, which is called Landau damping, is determined by the sign of the velocity derivative of the distribution function at the place where the particle velocity equals the phase velocity, ω/k, of the wave.
- If the energy density of waves has a continuous dependence on their phase velocity, ω/k, the waves cause a collisionless diffusion of the particles whose velocities resonate with the phase velocities of the waves. This is called quasilinear diffusion.

In fusion, and many other plasmas of interest, the collision frequency is small compared to the time it takes a charged particle to cross a distance comparable to the plasma size. Since collisional effects are weak, it is natural to study the Vlasov equation, $df/dt=0$. This hyperbolic partial differential equation has the particle trajectories as its single characteristic. In other words, the distribution function is carried along by the particle motion, so any function of the constants of motion is a solution to the Vlasov equation.

Since the Vlasov equation, $df/dt=0$, contains only information about the particle trajectories, which are time reversible, it is natural to assume that its solutions are themselves time reversible. This turns out to be false in practice, with the sense of time determined by the nature of the initial conditions. Solutions to the Vlasov equation can evolve from smooth initial conditions into functions with arbitrarily complicated structures. Plasma observations have finite resolution, and the averaging that is implicit in the observation process leads to irreversibility. These time-irreversible collisionless effects are important in plasma physics for the damping of externally driven waves, which gives plasma heating, and for amplifying plasma waves, which gives instabilities. The two collisionless phenomena that will be discussed are Landau damping (Landau, 1946) and quasilinear diffusion (Vedenov, Velikhov, and Sagdeev, 1961; Drummond and Pines, 1962).

Landau damping is a collisionless, but unidirectional, transfer of energy between an electromagnetic wave and a plasma. It is central to the theory of plasma heating by waves and the destabilization of electromagnetic perturbations by non-Maxwellian distributions of the plasma species.

To understand Landau damping, assume that at $t=0$ the distribution function of a plasma species is independent of position, $f_0(v)$, but a weak electric field is introduced that for $t>0$ has the form

$$E = E_k \cos(kx - \omega t). \tag{156}$$

The problem has only one spatial coordinate x and one velocity coordinate, which is v_x, but will be denoted by v for simplicity. The perturbed Vlasov equation, $df/dt=0$, with $f=f_0+\delta f$, is

$$\frac{\partial \delta f}{\partial t} + v\frac{\partial \delta f}{\partial x} + \frac{q}{m}E\frac{\partial f_0}{\partial v} = 0. \tag{157}$$

The solution for the perturbed distribution function δf that obeys the initial condition that $\delta f=0$ is

$$\delta f = \frac{qE_k}{m}\frac{\partial f_0}{\partial v}\frac{\sin(kx-\omega t) - \sin(kx - kvt)}{\omega - kv}, \tag{158}$$

which is nonsingular though growing at $v=\omega/k$,

$$(\delta f)_{v=\omega/k} = -\frac{\partial f_0}{\partial v}\frac{qE}{m}t. \tag{159}$$

Particles moving at the phase velocity of the perturbation, $v=\omega/k$, see a time-independent electric field and are accelerated by it. A spatial average is defined by the limit as L goes to infinity of $\langle g(x)\rangle \equiv (1/2L)\int_{-L}^{+L} g dx$. The spatially averaged power going to the plasma is

$$\langle P\rangle = \int v\langle qE\delta f\rangle dv. \tag{160}$$

Using the trigonometric identity $\sin(kx-kvt)=\sin(kx-\omega t)\cos[(\omega-kv)t]+\cos(kx-\omega t)\sin[(\omega-kv)t]$, one has

$$\langle qE\delta f\rangle = -\frac{q^2E_k^2}{2m}\frac{\partial f_0}{\partial v}\frac{\sin[(\omega-kv)t]}{\omega - kv}. \tag{161}$$

The velocity integral that must be performed to obtain the power, Eqs. (160) and (161), has the form

$$\int F(v)\frac{\sin[(\omega-kv)t]}{\omega-kv}dv = \int F\left[\frac{\omega}{k}\left(1+\frac{\xi}{\omega t}\right)\right]\frac{\sin\xi}{\xi}\frac{d\xi}{|k|}, \tag{162}$$

where $\xi \equiv (kv-\omega)t$. The integral of $\sin(\xi)/\xi$ from minus to plus infinity is π, so when $\omega t\to\infty$

$$\lim_{\omega t\to\infty}\int F(v)\frac{\sin[(\omega-kv)t]}{\omega-kv}dv = \pi\frac{F(\omega/k)}{|k|}. \tag{163}$$

The spatially averaged power is then

$$\langle P\rangle = -\pi\frac{q^2E_k^2}{2m|k|}\left(v\frac{\partial f_0}{\partial v}\right)_{v=\omega/k}. \tag{164}$$

Power is transferred to the plasma (positive power) if there are fewer particles at high energy than low and to the electric perturbation (an incipient instability) if there are more particles at high energy than low.

One can simplify some of the analysis of Landau damping by writing it in the form $\delta f=\tilde f\exp[i(kx-\omega t)]$. The critical step in Landau damping is then the determination of the imaginary part of what appears to be a real integral. The theory of Laplace transforms implies the imaginary part of the integral, the *Landau integral*, is

$$\left(\int\frac{f'(v)}{\omega-kv}dv\right)_{imag} = -i\frac{\pi}{|k|}f'\left(\frac{\omega}{k}\right). \tag{165}$$

Landau damping is closely related to a second phenomenon, *quasilinear diffusion*, which is important for understanding the interaction of waves with plasmas and the effect of short-wavelength perturbations on the plasma. If one identifies the distribution function f_0 of the Landau damping discussion with the spatially averaged distribution function, $f_0\equiv\langle f\rangle$, then a spatial average of the Vlasov equation, $df/dt=0$ [Eq. (128)] with $p=mv$, gives

$$\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial v}\left\langle\frac{q}{m}E\delta f\right\rangle. \tag{166}$$

Equation (161) gives an expression for $\langle qE\delta f\rangle$, which is proportional to $\partial f_0/\partial v$. Therefore

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\mathcal{D}(v)\frac{\partial f_0}{\partial v}, \tag{167}$$

where the velocity diffusion coefficient is

$$\mathcal{D} = \frac{q^2E_k^2}{2m^2}\frac{\sin[(\omega-kv)t]}{\omega-kv}. \tag{168}$$

This expression for the velocity-space diffusion is not useful unless there is a spectrum of perturbations with different k's. The spatially averaged energy density of a single wave is $(\epsilon_0/2)\langle E^2\rangle=(\epsilon_0/4)E_k^2$. If $\int\mathcal{E}(\omega/k)dk$ is the energy per unit volume of a spectrum of waves, one can integrate Eq. (168) over k using the integration formula of Eq. (163) to obtain

$$\mathcal{D} = \frac{2\pi q^2}{\epsilon_0 m^2}\frac{\mathcal{E}(v)}{|v|}. \tag{169}$$

The quantity $\omega(k)/k$ is the phase velocity of the electric perturbation, so a perturbation diffuses particles that have a velocity equal to the phase velocity.

Quasilinear diffusion is closely related to the stochasticity of magnetic-field lines that was discussed in Sec. III.A. In principle, the equations of mechanics are reversible. However, when stochastic, trajectories that have an infinitesimal initial separation increase that separation exponentially with distance along the trajectories. If the trajectories are resolved to finite accuracy, then the information is quickly lost that would be needed to launch the time-reversed trajectories.

The spatial averaging that is involved in deriving the quasilinear diffusion coefficient, destroys information and allows an entropylike quantity $s_0\equiv-\int f_0\ln f_0 d^3p$ to increase. However, s_0 is not the entropy density, nor even the spatial average of the entropy density, $s\equiv-\int f\ln f d^3p$. The spatially averaged entropy density $\langle s\rangle$ cannot change if f is a solution to the Vlasov equation, $df/dt=0$ [Eq. (144)]. Let $f=f_0\exp(\hat f)$, then $\langle\exp(\hat f)\rangle=1$ since $\langle f\rangle=f_0$. Assuming $\hat f$ is small, the relation between $\langle s\rangle$ and s_0 is $\langle s\rangle=s_0-\frac{1}{2}\int\langle\hat f^2\rangle f_0 d^3p$. Consequently s_0 must increase if the magnitude of $\hat f$ does, to avoid a change in

$\langle s \rangle$. Actually, the complicated velocity-space structure that $\hat{f}$ develops, coupled with the diffusive nature of plasma collisions, means that $\hat{f}$ may not reach a large amplitude even when the fractional change in the volume-averaged entropy density is of order unity.

Quasilinear diffusion can represent the heating of a plasma by a fluctuating electric field even in steady state. Of course for a steady state in the presence of heating, there must be a cooling term in the kinetic equation, such as $\partial f_0/\partial t = \partial(\nu_c v f_0)/\partial v$, where $2\nu_c(v)$ is the cooling rate. For this cooling term, the volume-averaged distribution function relaxes to $\partial \ln f_0/\partial(v^2/2) = -\nu_c/\mathcal{D}$, which has the shape of a Maxwellian with an effective temperature $T_{eff}/m \equiv \mathcal{D}/\nu_c$ in velocity regions where $\mathcal{D}/\nu_c$ is independent of velocity.

D. Drift kinetic theory

- The single characteristic of the Vlasov operator, df/dt [Eq. (127)], is the trajectories of the particles. Consequently, one can use approximations to the particle trajectories to simplify kinetic calculations.
- Charged particles make a circular gyration about magnetic-field lines. When the radius of gyration is small compared to the spatial variation of the electric and magnetic fields, the magnetic moment, $\mu = mv_\perp^2/2B$, is a constant of the motion.
- When the gyration radius is small, the particle trajectory can be accurately approximated by tracking the center of the circle, $\vec{x}_g$, about which the particle gyrates, the guiding center. The guiding center drifts at a velocity $\vec{v}_g$ given by Eq. (181). The equation of motion is first order, $d\vec{x}_g/dt = \vec{v}_g$, and not the usual second-order equation, $md^2\vec{x}/dt^2$, equal to a force.
- The drift velocity of the gyrocenters is also given by a Hamiltonian, Eq. (195), which has only four canonical variables $(\theta, \varphi, p_\theta, p_\varphi)$, Eqs. (199) and (200), instead of the six canonical variables of the full trajectories. The form of the Hamiltonian for the drift motion demonstrates that variation in the magnetic-field strength on the magnetic surfaces is the primary determinant of the confinement properties of the particle trajectories.
- Particles with a sufficiently small ratio of their velocity parallel to the magnetic field, $v_\parallel$, to their total velocity v are trapped between maxima of the magnetic-field strength along a magnetic-field line. These trapped particles drift from one field line to another conserving their action, $J = \oint mv_\parallel d\ell$.

The Vlasov operator df/dt in the kinetic equation is determined by the trajectories of the particles, and the determination of these trajectories is a major difficulty in solving the kinetic equation. Charged particles in a magnetic field move in circles with a gyration frequency $\Omega \equiv qB/m$, Fig. 10, which makes a direct calculation inefficient when the circles are small, as they are in a fusion plasma. In addition to being inefficient, a direct calculation obscures fundamental properties of the trajectories. For example, the confinement properties of particle trajectories are essentially determined by the variation of the magnetic-field strength on the constant-pressure surfaces.

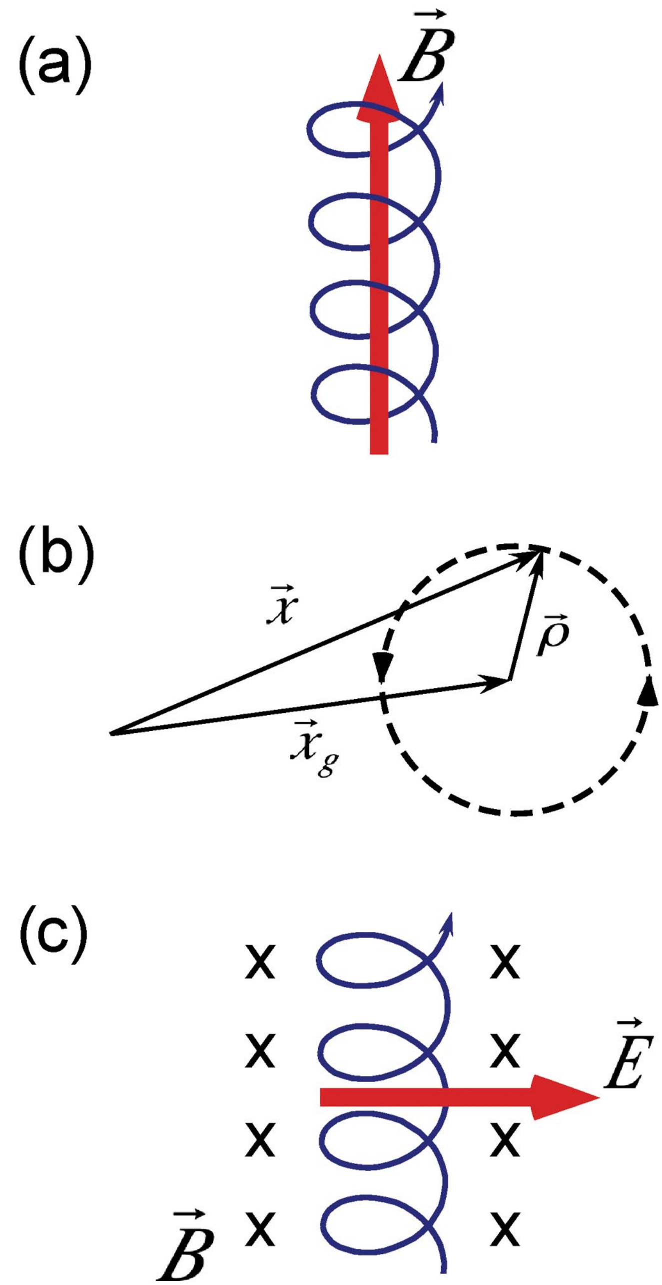

FIG. 10. (Color) Orbit of a charged particle and its guiding center: (a) A charged particle gyrates in a circle about a straight magnetic-field line while moving along the line; (b) the instantaneous center of this circle is the guiding center $\vec{x}_g$, and the difference between the position of the particle $\vec{x}$ and the guiding center is the vector gyroradius $\vec{\rho}$; (c) a uniform electric field perpendicular to the magnetic field causes the particle velocity, and hence its gyroradius, to be slightly larger on one side of its gyro-orbit than the other, causing a drift across the magnetic-field lines.

This section derives the asymptotic expressions for the particle trajectories (Alfvén, 1940) as well as the associated kinetic theory, which holds when the circles made by charged-particle orbits in a magnetic field are smaller than the scale of field variations. The basic results are as follows: (1) the magnetic moment,

$$\mu = \frac{mv_\perp^2}{2B}, \tag{170}$$

is a constant of the motion with $v_\perp$ the magnitude of the velocity components perpendicular to $\vec{B}$. The invariance properties of μ, which is an adiabatic invariant, are discussed in Sec. VI.D.2. (2) The center of the circle, $\vec{x}_g$, about which a particle gyrates (Fig. 10), drifts at a velocity $\vec{v}_g = d\vec{x}_g/dt$. This velocity is called the guiding-center or drift velocity, and $d\vec{x}_g/dt = \vec{v}_g$ gives a first-order ordinary differential equation for the center's trajectories. The guiding-center velocity is given by Eq. (181) as well as by a drift Hamiltonian, Eq. (195). The Hamiltonian of the drift motion, Sec. VI.D.3, has only four canonical variables $(\theta, \varphi, p_\theta, p_\varphi)$—the poloidal and toroidal angles of Boozer coordinates, Eq. (58) and their conjugate momenta [Eqs. (199) and (200); Boozer, 1984b]. This Hamiltonian formulation was utilized in a code by White and Chance (1984). Their code has been the basis of numerous investigations of phenomena that depend on particle drift. In addition to the canonical formulation of the drift equations, which is emphasized here, Littlejohn (1981) has given an important noncanonical, though Hamiltonian, treatment based on Lie theory.

The velocity-space coordinates of the guiding-center motion, which simplify the kinetic equation, $df/dt = C(f)$, are the Hamiltonian H, which is the energy, and the magnetic moment μ, where $H = mv_\parallel^2/2 + \mu B + q\Phi$. An average over the phase angle ϑ of the circular gyromotion of the particles gives the drift equations. The velocity-space volume element in cylindrical velocity-space coordinates is $d^3v = v_\perp dv_\perp d\vartheta dv_\parallel$. Using $(H, \mu, \vartheta, \vec{x})$ as coordinates, $dH = mv_\parallel dv_\parallel$, so

$$d^3v = \frac{B}{m^2}\sum_{\pm}\frac{dH d\mu d\vartheta}{|v_\parallel|}, \tag{171}$$

where the $\Sigma_\pm$ means that the sum is taken over the sign of velocity along the magnetic field, $v_\parallel$. In drift kinetic theory, one can integrate over the gyrophase ϑ, which is equivalent to replacing $d\vartheta$ in Eq. (171) by 2π. The drift kinetic equation is obtained by calculating $df(H, \mu, \vec{x}, t)/dt$ and setting $d\vec{x}/dt = \vec{v}_g$, the drift velocity of the guiding centers,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{v}_g \cdot \vec{\nabla} f + \frac{dH}{dt}\frac{\partial f}{\partial H} = C(f), \tag{172}$$

where the energy exchange between the particles and the electromagnetic fields, dH/dt, is given by Eq. (196).

The use of drift kinetic theory to calculate flows requires care. The current density is not $q\int \vec{v}_g f d^3v$, as one would naively expect, but

$$\vec{j} = q\int \vec{v}_g f d^3v - \vec{\nabla}\times\left(\hat{b}\int \mu f d^3v\right). \tag{173}$$

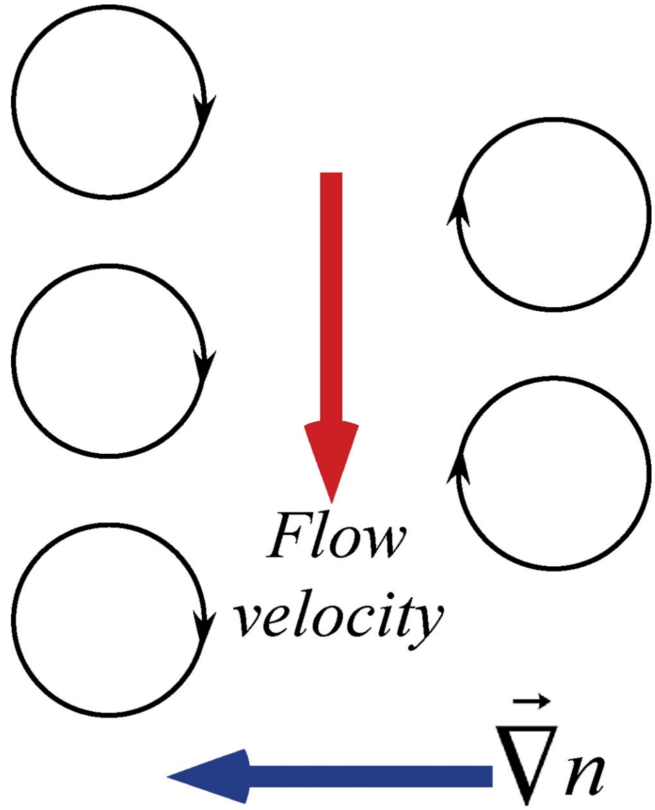

FIG. 11. (Color) Magnetization drift: The magnetic field is out of the figure. In the presence of a density gradient, the number of particles moving in one direction perpendicular to both the magnetic field and density gradient differs from the number of particles moving in the opposite direction. This leads to a divergence-free flow of the particles in the $\vec{B}\times\vec{\nabla}n$ direction even if all the particles are moving in perfect circles. This flow is called the magnetization drift.

The second term is called the *magnetization current* and arises, as does a magnetization current in a solid, from the magnetic moments of the particles that constitute the medium (Jackson, 1999; see Fig. 11). The actual magnetic moment produced by the circular gyromotion is

$$\vec{m} = -\mu\hat{b}. \tag{174}$$

The definition of the magnetic moment of a current distribution is $\vec{m} \equiv \frac{1}{2}\int \vec{x}\times\vec{j}d^3x$, which for a charged particle moving in a gyro-orbit is $\vec{m} = (q/2)\vec{\rho}\times\vec{v}$ with $\vec{\rho}$ the vector gyroradius [Eq. (177)].

1. Alfvén's guiding-center velocity

Alfvén (1940) derived an expression for the guiding-center or drift velocity that follows from the expression for the trajectories of a charged particle in given electric and magnetic fields,

$$m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}). \tag{175}$$

The position of the particle is given by $d\vec{x}/dt = \vec{v}$. In a uniform magnetic field with no electric field, a particle moves in a circle at the frequency

$$\Omega \equiv \frac{qB}{m}, \tag{176}$$

with the circle having a radius (Fig. 10)

$$\vec{\rho} \equiv \frac{\hat{b} \times \vec{v}}{\Omega}, \tag{177}$$

where $\hat{b} \equiv \vec{B}/|\vec{B}|$. The center of the circle $\vec{x}_g$, which is called the guiding center, moves with a velocity $v_\parallel \equiv \hat{b} \cdot \vec{v}$ parallel to the magnetic field. The position of the charged particle is (Fig. 10)

$$\vec{x} = \vec{x}_g + \vec{\rho}. \tag{178}$$

The addition of a constant electric field perpendicular to the magnetic field causes the guiding centers to drift in the $\vec{E} \times \vec{B}$ direction (Fig. 10). Let

$$\vec{v}_{E\times B} \equiv \frac{\vec{E} \times \vec{B}}{B^2}, \tag{179}$$

which is independent of position and time if $\vec{E}$ and $\vec{B}$ are. If $\vec{E} \cdot \vec{B} = 0$, Eq. (175) can be written as $m(d/dt)(v - \vec{v}_{E\times B}) = q(v - \vec{v}_{E\times B}) \times \vec{B}$. The motion of a charged particle in a constant electric field that is perpendicular to the magnetic field lines is identical to the motion without the electric field in a frame of reference moving with the velocity $\vec{v} = \vec{E} \times \vec{B}/B^2$.

The properties of the trajectories become far more complicated when the particles move in electric and magnetic fields that depend on position and time. However, Eq. (178) can be viewed as defining the guiding center of a particle in arbitrary magnetic and electric fields, and the exact motion of the guiding center,

$$\frac{d\vec{x}_g}{dt} = v_\parallel \hat{b} + \frac{\vec{E} \times \hat{b}}{B} + \vec{v} \times \frac{d}{dt}\left(\frac{\hat{b}}{\Omega}\right), \tag{180}$$

is obtained using Eqs. (175), (177), and (178).

When the spatial variation of the electric and magnetic fields is on a scale much longer than the gyroradius $|\vec{\rho}|$, particles can be tracked by following the velocity of the guiding center averaged over the rapid gyromotion, $\vec{v}_g \equiv \langle d\vec{x}_g/dt \rangle$. The expression for the guiding-center velocity is (Alfvén, 1940)

$$\vec{v}_g = v_\parallel \hat{b} + \frac{\hat{b}}{\Omega} \times \left(v_\parallel^2 \vec{\kappa} + v_\parallel \frac{\partial \hat{b}}{\partial t} + \frac{v_\perp^2}{2} \frac{\vec{\nabla} B}{B}\right) + \frac{\vec{E} \times \vec{B}}{B^2}, \tag{181}$$

where the curvature of the magnetic-field lines is

$$\vec{\kappa} \equiv \hat{b} \cdot \vec{\nabla} \hat{b}, \tag{182}$$

with $\hat{b} \equiv \vec{B}/|\vec{B}|$. The curvature of a pure toroidal field $\vec{B} = (\mu_0 G/2\pi R)\hat{\varphi}$ is $\vec{\kappa} = -\hat{R}/R$ since $d\hat{\varphi}/d\varphi = -\hat{R}$. The term involving $\partial \hat{b}/\partial t$ in the guiding-center velocity is usually dropped, since it is small if the time scale T over which $\hat{b}$ varies is long compared to the characteristic time for a particle to cross the system R/v.

In time-independent magnetic and electric fields, Alvén's expression for the guiding-center velocity can be written in an alternative form, which simplifies the theory of particle confinement. One can easily show (Morozov and Solov'ev, 1966) that

$$\vec{v}_g = \frac{v_\parallel}{B} \vec{\nabla} \times (\vec{A} + \rho_\parallel \vec{B}) \tag{183}$$

has the same drift across the magnetic field as Eq. (181). The parallel gyroradius is defined by

$$\rho_\parallel(H, \mu, \vec{x}) \equiv \frac{v_\parallel}{\Omega}, \tag{184}$$

where the energy $H = \frac{1}{2} m v_\parallel^2 + \mu B + q\Phi$ and the magnetic moment μ are treated as constants. That is, the curl of $v_\parallel$ is calculated holding the energy H and the magnetic moment μ constant. The trajectories are obtained by solving first-order equations for the guiding-center position, $d\vec{x}_g/dt = \vec{v}_g$. The only difficulty in the integration is at turning points of the parallel motion, where $v_\parallel$ passes through zero and changes its sign. This difficulty can be removed by the Hamiltonian formulation; see Sec. VI.D.3.

The terms in Alfvén's expression for the drift of the guiding center can be understood by analogy to the $E \times B$ drift [Eq. (179)]. If a force $\vec{F}$ is applied to a charged particle gyrating in a magnetic field, then the same argument that led to the $E \times B$ drift yields a drift velocity $\vec{v}_{F\times B} \equiv \vec{F} \times \vec{B}/(qB^2)$. If the magnetic-field lines have a nonzero curvature $\vec{\kappa}$, the centrifugal force, $\vec{F}_c = -m v_\parallel^2 \vec{\kappa}$, gives the curvature drift. If the magnetic-field strength varies, the quantity μB acts like a potential energy, giving the ∇B force, $\vec{F}_{\nabla B} = -\mu \vec{\nabla} B$, and an associated drift.

The only part of the derivation of the guiding-center velocity, Eq. (181), that is not obvious is the part of $\vec{v}_g$ perpendicular to $\vec{B}$ that follows from the last term in Eq. (180). This part of $\vec{v}_g$ is given by

$$\hat{b} \times \left\{\vec{v} \times \frac{d}{dt}\left(\frac{\hat{b}}{\Omega}\right)\right\} = \vec{v}\hat{b} \cdot \frac{d}{dt}\left(\frac{\hat{b}}{\Omega}\right) - v_\parallel \frac{d}{dt}\left(\frac{\hat{b}}{\Omega}\right). \tag{185}$$

One can let $d/dt = \partial/\partial t + v_\parallel \hat{b} \cdot \vec{\nabla} + \vec{v}_\perp \cdot \vec{\nabla}_\perp$ for $\hat{b} \equiv \vec{B}/|\vec{B}|$ and $\Omega \equiv qB/m$ have no velocity dependence. Since $\hat{b}$ is a unit vector, $\hat{b} \cdot d\hat{b}/dt = 0$ and

$$\hat{b}\times\left\{\vec{v}\times\frac{d}{dt}\left(\frac{\hat{b}}{\Omega}\right)\right\}=\vec{v}_{\perp}\frac{d}{dt}\left(\frac{1}{\Omega}\right)-\frac{v_{\parallel}}{\Omega}\frac{d\hat{b}}{dt}. \tag{186}$$

The direction of the velocity perpendicular to the magnetic field changes over a gyroperiod, so $\langle\vec{v}_{\perp}\rangle=0$ in lowest order, and

$$\left\langle\vec{v}_{\perp}\frac{d}{dt}\frac{1}{\Omega}\right\rangle=\frac{v_{\perp}^{2}}{2}\vec{\nabla}_{\perp}\frac{1}{\Omega}. \tag{187}$$

The gyrophase average is thus

$$\left\langle\frac{v_{\parallel}}{\Omega}\frac{d\hat{b}}{dt}\right\rangle=\frac{v_{\parallel}^{2}}{\Omega}\hat{b}\cdot\vec{\nabla}\hat{b}+\frac{v_{\parallel}}{\Omega}\frac{\partial\hat{b}}{\partial t}. \tag{188}$$

2. Magnetic-moment conservation

The integration of the guiding-center velocity, Eq. (181), to obtain $\vec{x}_g$ is greatly simplified by the existence of an adiabatic invariant, the magnetic moment, which has the approximate form $\mu=mv_{\perp}^{2}/2B$.

An important general principle of Hamiltonian mechanics, which is rarely taught in mechanics courses, is that if the parameters that define a periodic motion of a particle change sufficiently slowly, then the action of the periodic motion is conserved. If $\vec{p}_{\Omega}$ are oscillating components of the canonical momentum of the particle Hamiltonian and $\vec{x}_{\Omega}$ are the conjugate canonical coordinates, then the action is given by an integration over the periodic motion, $\oint\vec{p}_{\Omega}\cdot(d\vec{x}_{\Omega}/dt)dt$. Such invariants are called *adiabatic invariants*, and the magnetic moment μ is an example. The conservation properties of adiabatic invariants is a complicated subject (Kruskal, 1962), closely related to the existence of a magnetic surface (see Sec. III.A). The basic result is that if the parameters of a Hamiltonian are changed on a time scale T, which is slow compared to the frequency Ω of a periodic motion, then the action of that motion is conserved with exponential accuracy. That is, the variation of the action is of order $\exp(-\Omega T)$.

Canonical momenta are defined using the Lagrangian formulation of the equations of motion. The Lagrangian of a particle in electric and magnetic fields is

$$L(\dot{\vec{x}},\vec{x})=\frac{m}{2}\dot{\vec{x}}^{2}+q\dot{\vec{x}}\cdot\vec{A}-q\Phi, \tag{189}$$

where $\dot{\vec{x}}\equiv d\vec{x}/dt$, $\vec{B}=\vec{\nabla}\times\vec{A}$, and $\vec{E}=-\partial\vec{A}/\partial t-\vec{\nabla}\Phi$. The canonical momenta are defined by

$$\vec{p}\equiv\frac{\partial L}{\partial\dot{\vec{x}}}. \tag{190}$$

The time derivatives of the canonical momenta are $\dot{\vec{p}}=\partial L/\partial\vec{x}$.

The Lagrangian gives the standard equations of motion, Eq. (175), in Cartesian coordinates. In these coordinates, the canonical momentum, $\vec{p}=\partial L/\partial\vec{v}$, is

$$\vec{p}=m\vec{v}+q\vec{A}, \tag{191}$$

where $\vec{v}\equiv\dot{\vec{x}}$. The gradient of the Lagrangian is $\partial L/\partial\vec{x}=q\vec{\nabla}(\vec{v}\cdot\vec{A})-q\vec{\nabla}\Phi$. A vector identity implies $\vec{\nabla}(\vec{v}\cdot\vec{A})=\vec{v}\times(\vec{\nabla}\times\vec{A})+\vec{v}\cdot\vec{\nabla}\vec{A}$, where $\vec{v}$ is not differentiated since it is an independent coordinate in a Lagrangian analysis. Now, $d\vec{p}/dt=md\vec{v}/dt+q\partial\vec{A}/\partial t+\vec{v}\cdot\vec{\nabla}\vec{A}$, so $\dot{\vec{p}}=\partial L/\partial\vec{x}$ is equivalent to $md\vec{v}/dt=q(\vec{E}+\vec{v}\times\vec{B})$.

The beauty of the Lagrangian approach is the ease of finding the equations of motion in arbitrary (ψ_t,θ,φ) coordinates, $\vec{x}(\psi_t,\theta,\varphi,t)$. If L is independent of a coordinate φ, as it is in axisymmetric plasmas, then p_{φ} is a constant of the motion, $\dot{p}_{\varphi}=\partial L/\partial\varphi=0$. The φ component of the canonical momentum is

$$p_{\varphi}=m\vec{v}\cdot\frac{\partial\vec{x}}{\partial\varphi}+q\vec{A}\cdot\frac{\partial\vec{x}}{\partial\varphi}. \tag{192}$$

This equation is obtained from $p_{\varphi}\equiv\partial L/\partial\dot{\varphi}$ using $d\vec{x}/dt=\partial\vec{x}/\partial t+(\partial\vec{x}/\partial\psi_t)\dot{\psi}_t+\cdots$, which implies $\partial(d\vec{x}/dt)/\partial\dot{\varphi}=\partial\vec{x}/\partial\varphi$. The poloidal flux is $\psi_p=-2\pi\vec{A}\cdot(\partial\vec{x}/\partial\varphi)$ using Eq. (6) for $\vec{A}$ with the gauge $g=0$. The constancy of p_{φ} ensures the confinement of particles of sufficiently small velocity. Particles cannot cross the field lines by a distance greater than their gyroradius in the poloidal field, $\vec{B}_p\equiv\vec{\nabla}(\varphi/2\pi)\times\vec{\nabla}\psi_p$, alone.

The definition of the adiabatic invariant that is known as the magnetic moment is the integral

$$\mu\equiv\frac{q}{m}\frac{1}{2\pi}\oint\vec{p}_{\perp}\cdot\frac{d\vec{x}_{\perp}}{dt}dt \tag{193}$$

over a gyroperiod where $\vec{p}$ is the canonical momentum, $\vec{p}\equiv m\vec{v}+q\vec{A}$. The quantity $m\mu/q$ is the action of the gyromotion.

The approximate expression for the adiabatic invariant, $\mu=mv_{\perp}^{2}/2B$, which is correct to lowest order in the gyroradius to system size, is obtained from Eq. (193) using $\oint\vec{v}_{\perp}\cdot(d\vec{x}_{\perp}/dt)dt=(2\pi/\Omega)v_{\perp}^{2}$ and $\oint\vec{A}\cdot(d\vec{x}_{\perp}/dt)dt=-\oint\vec{A}\cdot d\vec{x}_{\perp}$. The minus sign arises because a positive charge moves about its circular orbit in a clockwise direction, while the convention for a line integral is counterclockwise. The integral $\oint\vec{A}\cdot d\vec{x}_{\perp}=\int\vec{B}\cdot d\vec{a}=\pi B|\vec{\rho}|^{2}$. The approximate expression for the magnetic moment, $\mu=mv_{\perp}^{2}/2B$, follows.

The magnetic moment is proportional to the magnetic flux enclosed by the circular gyromotion, so magnetic-moment conservation can be viewed as flux conservation. The conservation of the magnetic moment is also equivalent to a fixed Landau level of the quantum theory of particle motion in a magnetic field. That is, the adiabatic invariance of μ follows from the adiabatic approximation of quantum mechanics. The lowest-order conservation of μ can also be demonstrated by differentiating the expression for μ of Eq. (170) with respect to time.

Just as magnetic surfaces can only be broken by resonant perturbations, the invariant μ can only be broken if there is a resonance between the time variation of the gyromotion and the gyromotion itself. The Fourier transform of an analytic function $\int f(t)\exp(-i\Omega t)dt \sim \exp(-\Omega T)$, where T is the distance of the closest pole of $f(t)$ from the real axis; see Sec. V.D.1. T is the characteristic time scale for variations. It is sometimes said that the magnetic moment is conserved to all orders in $\epsilon \equiv 1/\Omega t$. What is meant is that if the function $\exp(-1/\epsilon)$ is Taylor expanded in ϵ about $\epsilon=0$, then every term in the Taylor series is zero. The function $\exp(-1/\epsilon)$ is the most important example in physics of a function that is not zero but has a Taylor series that is identically zero. It serves as a warning that an expansion in a small parameter can be subtle. A complicated variation in the magnetic and electric fields across the field lines is irrelevant to μ conservation if the total time derivatives of these fields, $d/dt=\partial/\partial t+v_\parallel \hat{b}\cdot\vec{\nabla}$, are small compared to the gyrofrequency Ω. The irrelevance of variations across the magnetic field to the conservation of μ is important for the validity of gyrokinetic theory, which is discussed in Sec. VI.G.

The constancy of the magnetic moment implies that the number of independent variables in a guiding-center calculation is four instead of the six required for the full particle motion. The four variables can be taken to be the three components of the guiding-center position and the energy. The energy, or Hamiltonian, of a charged particle is

$$H \equiv \frac{1}{2}mv^2 + q\Phi. \tag{194}$$

Magnetic-moment conservation implies the energy can also be written as

$$H = \frac{1}{2}mv_\parallel^2 + \mu B + q\Phi. \tag{195}$$

As shown below, the gyrophase averaged change in the energy is

$$\left\langle \frac{dH}{dt} \right\rangle = \frac{\partial}{\partial t}(\mu B + q\Phi) - v_\parallel \frac{\partial A_\parallel}{\partial t}, \tag{196}$$

which can be integrated along with $\vec{x}_g$ to obtain the energy H. The parallel velocity is given by the energy and magnetic moment,

$$v_\parallel = \pm\sqrt{2m(H - \mu B - q\Phi)}. \tag{197}$$

Equation (196) for the gyrophase averaged energy change can be derived by writing the Hamiltonian in its canonical variables, $H(\vec{p},\vec{x},t)=(\vec{p}-q\vec{A})^2/(2m)+q\Phi$, with the particle velocity $\vec{v}=(\vec{p}-q\vec{A})/m$. Hamilton's equations imply that $dH/dt=\partial H/\partial t$, so

$$\frac{dH}{dt} = q\left(\frac{\partial\Phi}{\partial t} - \vec{v}\cdot\frac{\partial\vec{A}}{\partial t}\right). \tag{198}$$

The only subtle term in the derivation of Eq. (196) is $\langle q\vec{v}_\perp\cdot\partial\vec{A}/\partial t\rangle$. This term is calculated using the same technique as that for $\oint \vec{A}_\perp\cdot(d\vec{x}_\perp/dt)dt$ in the magnetic moment. That is, $\langle \vec{v}_\perp\cdot\partial\vec{A}/\partial t\rangle = -(\Omega/2)(\partial B/\partial t)|\vec{\rho}|^2$.

3. The drift Hamiltonian

When the scale of the spatial and temporal variation of the magnetic and electric fields is long compared to the gyroradius and the gyrofrequency of a particle, the motion of the particle can be tracked using the drift Hamiltonian, which is the Hamiltonian for the guiding-center motion. This Hamiltonian is the energy, Eq. (195), $H(p_\theta,p_\varphi,\theta,\varphi,t)=\frac{1}{2}mv_\parallel^2+\mu B+q\Phi$. The canonical momenta of the drift Hamiltonian are

$$p_\theta = \frac{\mu_0 I}{2\pi B}mv_\parallel + \frac{q}{2\pi}\psi_t, \tag{199}$$

and

$$p_\varphi = \frac{\mu_0 G}{2\pi B}mv_\parallel - \frac{q}{2\pi}\psi_p. \tag{200}$$

The (ψ_t,θ,φ) coordinate system is the Boozer coordinate system, Eq. (58), with the subscripts omitted to simplify the notation.

To obtain the drift Hamiltonian and its canonical coordinates, we start with a Lagrangian for the guiding-center motion that was given by Taylor (1964),

$$L_T(\vec{x},\dot{\vec{x}}) = \frac{1}{2}m(\dot{\vec{x}}\cdot\hat{b})^2 + q\dot{\vec{x}}\cdot\vec{A} - (\mu B + q\Phi), \tag{201}$$

where $\hat{b}\equiv\vec{B}/B$. To lowest order in the ratio of gyroradius to system size, its trajectories agree with those given by Alfvén's expression for the guiding-center motion [Eq. (181)]. The validity of the Taylor Lagrangian is demonstrated by an explicit calculation. The canonical momentum $\vec{p}\equiv\partial L_T/\partial\dot{\vec{x}}$ is $\vec{p}=mv_\parallel\hat{b}+q\vec{A}$. The energy or Hamiltonian is $H\equiv\vec{p}\cdot\dot{\vec{x}}-L_T$, which gives Eq. (195). The time derivative of the canonical momentum is $d\vec{p}/dt = m\dot{v}_\parallel\hat{b}+mv_\parallel d\hat{b}/dt+qd\vec{A}/dt$, but $d\hat{b}/dt\simeq\partial\hat{b}/\partial t+v_\parallel\vec{\kappa}$ with $\vec{\kappa}\equiv\hat{b}\cdot\vec{\nabla}\hat{b}$, and $d\vec{A}/dt=\partial\vec{A}/\partial t+\vec{v}\cdot\vec{\nabla}\vec{A}$. The equations of motion in Lagrangian dynamics are $d\vec{p}/dt=\partial L_T/\partial\vec{x}$. The gradient of the first term in L_T is zero, that is, $\vec{\nabla}(\vec{v}\cdot\hat{b})\simeq 0$. This follows from $\vec{\nabla}(\vec{v}\cdot\hat{b})=\vec{v}\times(\vec{\nabla}\times\hat{b})+\vec{v}\cdot\vec{\nabla}\hat{b}$, but $\hat{b}\times(\vec{\nabla}\times\hat{b})=-\vec{\kappa}$ and $\vec{v}\cdot\vec{\nabla}\hat{b}\simeq v_\parallel\vec{\kappa}$. Using $\vec{\nabla}(\vec{v}\cdot\vec{A})=\vec{v}\times\vec{B}+\vec{v}\cdot\vec{\nabla}\vec{A}$, one has $\vec{\nabla}L_T\simeq q\vec{v}\times\vec{B}+q\vec{v}\cdot\vec{\nabla}\vec{A}-\vec{\nabla}(\mu B+q\Phi)$. Putting the pieces together, $m\dot{v}_\parallel\hat{b}+mv_\parallel(\partial\hat{b}/\partial t+v_\parallel\vec{\kappa})+q\partial\vec{A}/\partial t\simeq q\vec{v}\times\vec{B}-\vec{\nabla}(\mu B+q\Phi)$. The component of the equations of motion that is along the magnetic field is then $m\dot{v}_\parallel=\mu\hat{b}\cdot\vec{\nabla}B+qE_\parallel$, which is consistent with Eq. (196) for the gyrophase averaged energy change. The components of the velocity that are perpendicular to the magnetic field agree with Eq. (181) for $\vec{v}_g$.

Given a Lagrangian, it is usually trivial to obtain a Hamiltonian description of trajectories. However, a subtlety exists in obtaining a Hamiltonian description of guiding-center motion from the Taylor Lagrangian. The canonical momenta plus the coordinates of the Taylor Lagrangian depend on only four independent variables, which can be taken to be $(\vec{x},v_\parallel)$. Consequently the Hamiltonian can have only four variables, two canonical coordinates and two canonical momenta.

The canonical coordinates of the drift Hamiltonian are closely related to the Boozer magnetic coordinates, Eq. (58), so one needs to transform the Taylor Lagrangian into these coordinates. The expression is

$$L_T = \frac{m}{2}\left(\mu_0 \frac{G\dot{\phi}+I\dot{\theta}}{2\pi B}\right)^2 + q\frac{\psi_t\dot{\theta}-\psi_p\dot{\varphi}}{2\pi} - \mu B - q\Phi_m, \tag{202}$$

where $\Phi_m \equiv \Phi + s$. The function s is determined using Eq. (28) and gives the effect of the motion of the magnetic coordinate system. Usually the distinction between Φ_m and Φ is negligible, and so it will not be retained. The most difficult and subtle point in the transformation of the Taylor Lagrangian into Boozer coordinates is the parallel velocity, $\vec{v}\cdot\vec{B}=\vec{B}\cdot d\vec{x}(\psi_t,\theta,\varphi,t)/dt$, which gives $v_\parallel = (\mu_0/2\pi B)[G(\psi_t)\dot{\varphi}+I(\psi_t)\dot{\theta}]$. To prove this, use the chain rule to write $d\vec{x}/dt=\partial\vec{x}/\partial t+(\partial\vec{x}/\partial\psi_t)\dot{\psi}_t+\cdots$. The term $\vec{B}\cdot\partial\vec{x}/\partial t=\vec{B}\cdot\vec{u}$, Eq. (29), can be taken to be zero, because the flow of the canonical coordinates along the field lines can always be chosen to be zero. The time derivatives $\dot{\theta}\simeq(v_\parallel/B)\vec{B}\cdot\vec{\nabla}\theta$ and $\dot{\varphi}$ are larger by the ratio of the gyroradius to the system size than the derivative $\dot{\psi}_t$, so one can let $\vec{v}\cdot\vec{B}=(\vec{B}\cdot\partial\vec{x}/\partial\theta)\dot{\theta}+(\vec{B}\cdot\partial\vec{x}/\partial\varphi)\dot{\varphi}$. The transformation of $\dot{\vec{x}}\cdot\vec{A}=\vec{A}\cdot\vec{u}+(\psi_t/2\pi)\dot{\theta}-(\psi_p/2\pi)\dot{\varphi}$, while $\vec{A}\cdot\vec{u}=-s$ with the choice of gauge $g=0$ [Eq. (A20)]. Combining the results, one obtains Eq. (202).

Given the Taylor Lagragian in magnetic coordinates, Eq. (202), the determination of the canonical momenta and the drift Hamiltonian are straightforward. The canonical momenta are $p_\theta \equiv \partial L/\partial\dot{\theta}$, which gives Eq. (199) and $p_\varphi \equiv \partial L/\partial\dot{\varphi}$, which gives Eq. (200).

It is also useful to have the Hamiltonian for the guiding-center motion in Clebsch coordinates (ψ,α,ϕ). In these coordinates the magnetic field has the contravariant representation $\vec{B}=\vec{\nabla}\times(\psi\vec{\nabla}\alpha)$, Eq. (9), and the covariant representation $\vec{B}=\vec{\nabla}\phi+B_\alpha\vec{\nabla}\alpha+B_\psi\vec{\nabla}\psi$, Eq. (11). Dotting the two representations together, one finds that the inverse of the coordinate Jacobian is $\vec{B}\cdot\vec{\nabla}\phi=B^2$. The Taylor Lagrangian is transformed into Clebsch coordinates by noting that $\vec{v}\cdot\vec{B}=\vec{B}\cdot(\partial\vec{x}/\partial\phi)\dot{\phi}=\dot{\phi}$ and $\vec{v}\cdot\vec{A}=(d\vec{x}/dt)\cdot\vec{A}=\psi\dot{\alpha}$. The canonical momenta are

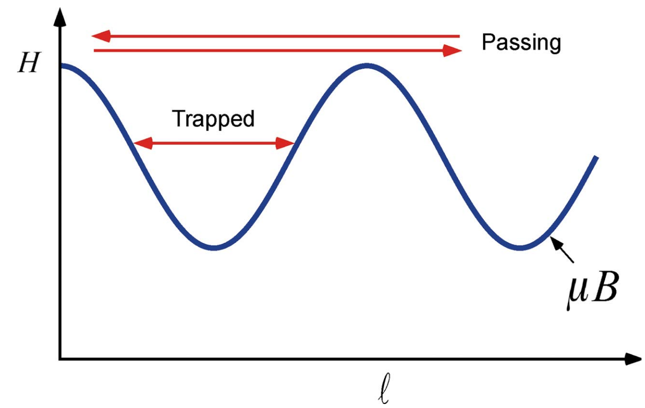

FIG. 12. (Color) Trapped and passing particles: A particle is trapped or passing depending on its energy H relative to the energy μB_{max}. B_{max} is the maximum of the field strength along a magnetic-field line, and μ is the adiabatically conserved magnetic moment. The distance along the line is ℓ. The electric potential Φ is ignored for simplicity.

$$p_\alpha = q\psi \tag{203}$$

and

$$p_\phi = mv_\parallel/B = q\rho_\parallel. \tag{204}$$

The Hamiltonian is the energy [Eq. (195)],

$$H(p_\alpha,p_\phi,\alpha,\phi) = \frac{B^2}{2m}p_\phi^2 + \mu B + q\Phi. \tag{205}$$

A comparison of the trajectories given by the drift Hamiltonian in Clebsch coordinates with those obtained from Eq. (183) is instructive. Equation (183) implies

$$\frac{d\psi}{dt} = \vec{v}_g\cdot\vec{\nabla}\psi = v_\parallel B\left(\frac{\partial\rho_\parallel}{\partial\alpha} - \frac{\partial B_\alpha\rho_\parallel}{\partial\phi}\right). \tag{206}$$

The first of these two terms is reproduced by the Hamiltonian formulation, but the second is not. The change in ψ as the particle moves along its trajectory due to the term $-v_\parallel B\partial(B_\alpha\rho_\parallel)/\partial\phi = v_\parallel\partial(B_\alpha\rho_\parallel)/\partial\ell$ is given by $\delta\psi = -B_\alpha\rho_\parallel$. This follows from $d\phi = Bd\ell$, with ℓ the distance along $\vec{B}$, and $dt = d\ell/v_\parallel$. B_α has dimensions of the magnetic field times a length, which means a length of order a, the scale of the plasma. Consequently, the maximal deviation of ψ along a trajectory is approximately $\delta\psi/\psi \approx \rho_\parallel/a$, which can be viewed as a redefinition of the guiding-center position, not a systematic drift. The location of the guiding center is a question of definition to within a distance of order a gyroradius. That arbitrariness can be used to simplify the equations for the guiding-center motion.

4. The action invariant *J*

The conservation of the magnetic moment μ causes the parallel velocity of a particle to pass through zero and change sign at a point where $H=\mu B+q\Phi$. Particles that are trapped between two such points of high magnetic field or electric potential are called trapped particles (see Fig. 12). They have a periodic oscillation between the points and hence an adiabatic invariant, which is customarily written as

$$J = \oint m v_\parallel d\ell. \tag{207}$$

J is called the *action invariant* (Northrop, 1963). In Clebsch coordinates (ψ,α,ϕ), Eqs. (9) and (11), the action is $J(H,\mu,\psi,\alpha) \equiv \oint p_\phi d\phi = \oint m v_\parallel d\ell$ [Eq. (204)], which means J is an action in the standard sense of Hamiltonian mechanics.

The derivatives of the action, $J(H,\mu,\psi,\alpha)$ give important information about the long-term trajectories of trapped particles. The derivative with respect to the energy gives the bounce time τ_b, the time to go from one turning point to the other. Equation (197) for the parallel velocity implies $\partial v_\parallel/\partial H = 1/m v_\parallel$, so

$$\frac{\partial J}{\partial H} = \oint \frac{d\ell}{v_\parallel} = 2\tau_b. \tag{208}$$

Writing $J = q\oint \rho_\parallel d\phi$ makes the ψ and α derivatives easier to evaluate. Equation (206) implies

$$\frac{\partial \rho_\parallel}{\partial \alpha} = \frac{\vec{v}_g \cdot \vec{\nabla}\psi}{v_\parallel B} + \frac{\partial (B_\alpha \rho_\parallel)}{\partial \phi}. \tag{209}$$

Therefore

$$\frac{\partial J}{\partial \alpha} = q\oint \frac{\vec{v}_g \cdot \vec{\nabla}\psi}{v_\parallel} d\ell = 2q\Delta\psi, \tag{210}$$

where $\Delta\psi$ is the change in ψ going from one turning point to the next. Similarly,

$$\frac{\partial J}{\partial \psi} = -2q\Delta\alpha. \tag{211}$$

The long-term motion of a trapped particle consists of a radial drift,

$$\frac{d\psi}{dt} = \frac{1}{q}\frac{\partial J/\partial \alpha}{\partial J/\partial H}, \tag{212}$$

and a precession,

$$\frac{d\alpha}{dt} = -\frac{1}{q}\frac{\partial J/\partial \psi}{\partial J/\partial H}. \tag{213}$$

E. Particle trajectories and transport

- Particles that have a sufficiently large ratio of the velocity parallel to the magnetic field, $v_\parallel$, to the total velocity v can move all along the field lines (Fig. 12) and are generally well confined when magnetic surfaces exist. Such particles are called passing particles.
- Particles with a small ratio of $v_\parallel/v$ are trapped between maxima of the field strength (Fig. 12) and are well confined only when stringent conditions are met on the variation of the magnetic-field strength in the magnetic surfaces. The trapped particles can be well confined if the field strength depends on only one angle in the magnetic surface, as is the case in axisymmetry, or if the magnetic-field strength is the same at all minima of the field strength in a magnetic surface.
- The electric potential in a confined plasma, with a temperature T, has the characteristic magnitude $|\Phi| \approx T/e$. The reason is that one species is generally more poorly confined, so that a species preferentially leaves the plasma until its pressure gradient is balanced by the electric field $|\vec{\nabla}p| = |en\vec{E}|$. Only a tiny fraction of the particles are lost while setting up this electric field, which is called the ambipolar field, so the plasma is approximately quasineutral.
- A pressure gradient drives a net current along the magnetic-field lines. This current is called the bootstrap current.

In a confined fusion plasma, particles can move a distance more than a thousand times the size of the plasma between collisions. Plasma confinement for times long compared to the collision time requires that the trajectories of all particles that form a Maxwellian distribution stay close to the constant-pressure surfaces. In addition, the alpha particles that are produced by the fusion reaction must remain confined as they slow from their birth energy of 3.5 MeV and heat the plasma (ITER Physics Expert Group on Energetic Particle, …, 1999a). Two questions need to be addressed: (1) Do the trajectories of high-energy particles remain in the plasma? (2) What effect does the straying of near-thermal particles from the pressure surfaces have on the transport coefficients?

Before discussing the motion of particles and the associated transport phenomena, it is useful to have an estimate of the radial electric field in a plasma. The characteristic change in the electric potential across a confined plasma is $|q\Delta\Phi/T|$ of order unity. This potential difference is associated with only a small net charge density qn_Δ with $n_\Delta/n \ll 1$. That is, a confined plasma is generally quasineutral. The reason for the potential difference $\Delta\Phi$ is that one species, ions or electrons, is more poorly confined than the other. The more poorly confined species leaves the plasma until the electric potential becomes sufficiently strong to provide confinement for that species $d\ln p/d\psi_t = -qnd\Phi/d\psi_t$. If the temperature were constant, the density and potential would be related by $n \propto \exp(-q\Phi/T)$. Even a substantial density drop is associated with a modest change in $q\Phi/T$. The net charge density qn_Δ associated with this potential is given by Gauss's law, $\nabla^2\Phi = -qn_\Delta/\epsilon_0$, which implies $qn_\Delta \approx \epsilon_0 \Delta\Phi/a^2$ with a the minor radius of the plasma. The fractional charge imbalance can be written as $n_\Delta/n \approx (\lambda_D/a)^2$, where the Debye length is $\lambda_D \equiv \sqrt{\epsilon_0 T/q^2 n}$. In a fusion plasma, the Debye length is of order a tenth of a millimeter, so the charge imbalance between electrons and ions n_Δ/n is extremely small, and the plasma is said to be quasineutral. The electric force exerted on the overall plasma, $qn_\Delta\vec{E}$, is also a factor of $(\lambda_D/a)^2$ smaller than the pressure force $\vec{\nabla}p$.

A related concept to quasineutrality is ambipolarity, which means the electrons and the ions diffuse at the same rate so there is no radial current. In steady-state situations, plasma transport is generally ambipolar for otherwise the plasma would lose either all of the ions or all of the electrons.

1. Confinement of particle trajectories

The confinement of individual particles is strongly dependent on whether a particle is trapped or passing (Fig. 12). If $B_{max}(\psi_t)$ is the maximum magnetic-field strength on the magnetic surface that contains toroidal flux ψ_t, then a particle is trapped if $\mu B_{max} \geqslant H - q\Phi(\psi_t)$. If a particle is not trapped then it is said to be passing. For passing particles, the parallel velocity retains a given sign between collisions, $v_\parallel > 0$ or $v_\parallel < 0$, and is never zero. For a barely trapped particle, the conservation of μ implies $\mu B_{max} = \mu B_{min} + m v_\parallel^2/2$, where $v_\parallel$ is the parallel velocity at the location of the minimum of the magnetic-field strength. If one defines $2\epsilon \equiv (B_{max} - B_{min})/B_{max}$, then at the magnetic-field minimum $v_\parallel/v = \sqrt{2\epsilon}$ for a barely trapped particle. In a circular-cross-section tokamak, $\epsilon \simeq r/R_o$.

The confinement of passing particles is rarely a problem unless magnetic surfaces are lost. The confinement of passing particles is easily derived using the form for the guiding-center velocity of Eq. (183). Since the parallel velocity never vanishes, the motion of a passing particle is along an effective magnetic field $\vec{B}_*(H,\mu,\vec{x}) \equiv \vec{B} + \vec{\nabla} \times (\rho_\parallel \vec{B})$, where the energy H and the magnetic moment μ are constants of the motion. The field $\vec{B}_*$ is divergence free and differs from the magnetic field $\vec{B}$ only by a small term that is proportional to the gyroradius to the system size. If $\vec{B}$ has magnetic surfaces, then the surface quality of the $\vec{B}_*$ field can be investigated using the methods of Sec. III.A.

The confinement of trapped particles in toroidal plasmas is more difficult than that of passing particles, and constrains the design of confinement systems. In principle, the guiding-center drifts can carry a trapped particle out of the plasma after it has traveled a distance of order $(R/\rho)a$, where R is the radius of curvature of the field lines, ρ is the gyroradius, and a is the minor radius of the plasma. This distance is too short for confining fusion plasmas. The time constant that is associated with unbounded drift motion is $\tau \approx aR/(\rho v)$, which is of order the time for Bohm diffusion [Eq (229)].

Trapped particles are well confined if the magnetic-field strength along the magnetic-field lines satisfies periodicity, $B(\ell) = B(\ell + L)$, where ℓ is the distance along a field line and L is a constant along that line. Magnetic configurations that satisfy this constraint are called quasisymmetric. If the field strength is given in Boozer coordinates, Eq. (58), then periodicity implies the field strength can depend on the poloidal and toroidal angles only though the linear combination $\theta_h \equiv \theta + N_h \varphi$, where N_h is an integer. That is, the field strength has the form $B(\psi_t, \theta_h)$. The helical canonical momentum, $p_h \equiv \vec{p} \cdot (\partial \vec{x}/\partial\varphi)_{\theta_h}$, of the Taylor Lagrangian, Eq. (202), is conserved,

$$p_h = \mu_0 \frac{G - N_h I}{2\pi B} m v_\parallel - q \frac{\psi_p + N_h \psi_t}{2\pi}. \tag{214}$$

The expression for p_h is obtained using $(\partial\vec{x}/\partial\varphi)_{\theta_h} = (\partial\vec{x}/\partial\varphi)_\theta - N_h \partial\vec{x}/\partial\theta$, so $p_h = p_\varphi - N_h p_\theta$. The conservation of p_h means the excursions that trapped particles make from the magnetic surfaces are proportional to the gyroradius.

An axisymmetric tokamak satisfies the condition of quasisymmetry with $N_h = 0$. But the condition of quasisymmetry can also be satisfied to high accuracy in stellarators in which the magnetic surfaces are not symmetric in φ. The Quasi-Helically Symmetric stellarator at the University of Wisconsin, the first operating quasisymmetric stellarator, has four periods, $N_p = 4$ and $N_h = 4$ (Talmadge *et al.*, 2001). The NCSX stellarator, which is being constructed at Princeton (Zarnstorff *et al.*, 2001) is quasi-axisymmetric, $N_h = 0$, just as is a tokamak, but has three periods, $N_p = 3$ (Fig. 7). Garren and Boozer (1991a, 1991b) have shown that quasisymmetry cannot be precisely achieved except in perfect axisymmetry. However, the required breaking of quasisymmetry, which is of order the local inverse aspect ratio cubed, $(r/R)^3$, can be very small.

A different method than quasisymmetry for obtaining trapped-particle confinement can be derived starting with the requirement that a deeply trapped particle remain close to a flux surface. A condition for good confinement of deeply trapped particles is that all the minima of the field strength B_{min} along each field line occur at essentially the same value of B (Mynick *et al.*, 1982). Field minima satisfy $\partial B/\partial\ell = 0$ and $\partial^2 B/\partial\ell^2 > 0$, where ℓ is the distance along a magnetic-field line. Deeply trapped particles have $(v_\parallel/v_\perp)^2 \ll 1$ even when they are near the minimum of the field strength and have esentially zero action, $J = \oint m v_\parallel d\ell = 0$ [Eq. (207)]. Since the action is conserved, the particles must remain deeply trapped throughout their drift motion. That is, they must remain at a minimum of the field strength. The guiding-center drift, Eq. (181), for a deeply trapped particle is $\vec{v}_g = (\vec{B}/qB^2) \times \vec{\nabla}(\mu B + q\Phi)$ so the particles drift on surfaces of constant $\mu B + q\Phi$. Since the potential is a function of the toroidal flux, $\Phi(\psi_t)$ (Sec. V.B.3), the deeply trapped particles stay on a flux surface if $\partial B_{min}/\partial\alpha = 0$. The derivative $\partial B_{min}/\partial\alpha$ is zero at minima if all minima of the magnetic field on a ψ_t surface have the same value.

If deeply trapped particles are confined to the magnetic surfaces, then it is relatively easy to shape the variation in magnetic-field strength along the field lines to confine most of the trapped particles. However, particles near the boundary between trapped and passing tend to have bad orbits unless all field maxima are at the same field strength (Cary and Shasharina, 1997). The condition of all field maxima being at the same field strength is not as important because a radial variation in the average field strength, $B_0(\psi_t) \equiv \sqrt{\int B^2 \mathcal{J} d\theta d\varphi / \int \mathcal{J} d\theta d\varphi}$, and the electric potential $\Phi(\psi_t)$ cause particles to drift in a way that converts particles near the trapped-passing boundary into either trapped or passing particles. In addition, collisions in a plasma are a diffusive phenomenon, and thermal particles near the trapped-passing boundary switch rapidly between the two types of orbits, which means they slowly diffuse rather than rapidly drift out of the confinement region.

The W7-X stellarator is designed (Nührenberg *et al.*, 1995) to make the minima of the field strength on a magnetic surface have the same value. This is accomplished by the plasma's having a pentagonal shape when viewed along the z axis (Fig. 6). Each magnetic surface has five straight sections where the field strength is low and five high-curvature corners where the field strength is high. It is easy to design all field minima to have the same field strength, since they occur in the straight sections. The maxima of the field strength occur in the corners of the pentagonal shape. The fractional variation in the field strength at the maxima is approximately $2\delta R/R$, where δR is the half-width of the plasma along the major radius at the corners, and δR is very narrow in W7-X, $\delta R/R \approx 1/20$. The pressure balance in W7-X approximately satisfies $p(\psi_t)+B_0^2(\psi_t)/2\mu_0$ const, so at the design beta value $\langle 2\mu_0 p/B^2\rangle = 5\%$ the radial variation in B_0 is sufficient to confine most particles near the trapped-passing boundary. The principle used to confine particle orbits in W7-X has been given many names: linked mirrors, quasi-isodynamic, quasi-omnigenious, and most recently quasipoloidal symmetry. Quasipoloidal symmetry is broken by the strong curvature in the corners to first order in the inverse aspect ratio r/R, while quasisymmetry in its more traditional usage is broken in third order, $(r/R)^3$.

A tokamak with too few toroidal field coils offers an example of the bad particle confinement that occurs when the field strength minima on a magnetic surface have differing values. The field minima are created by the space between the individual coils. Even when these minima are shallow, $\partial B_{min}/\partial\alpha$ is large and the particles trapped in these minima drift out of the device on a relatively short time scale $aR/(\rho v)$. The fractional variation in the magnetic field due to toroidal asymmetry is called the *toroidal ripple* δ. The approximate expression for the field strength is $B(\psi_t,\theta,\varphi)=B_0\{1-(r/R_o)\cos\theta + \delta\cos(N\varphi)\}$ [Eq. (19)]. Unless the toroidal ripple satisfies $\delta \ll (r/R_o)(\iota/N)$ it causes a large number of secondary minima at varying values of the field strength.

When the minima of the magnetic field on a pressure surface are not all at the same field strength, the collisionless drift trajectories generally cross a large fraction of the pressure surfaces. From the conservation of action, Sec. VI.D.4, one has

$$\frac{d\psi_t}{d\alpha} = -\frac{\partial J/\partial\alpha}{\partial J/\partial\psi_t}. \tag{215}$$

Since α is the Clebsch angle with a characteristic range of unity, the radial excursion of particles $\Delta\psi_t$ is approximately $(\partial J/\partial\alpha)/(\partial J/\partial\psi_t)$. This can be reduced by enhancing the *precession*, which is the change in the Clebsch angle per bounce, $\Delta\alpha = -(\partial J/\partial\psi_t)/2q$ [Eq. (211)]. The precession of the deeply trapped particles is proportional to $\partial(\mu B + q\Phi)/\partial\psi_t$, which for thermal particles is generally dominated by the radial variation of the potential, $\partial\Phi/\partial\psi_t$ with either sign of the radial electric field $\vec{E} = -(d\Phi/d\psi_t)\vec{\nabla}\psi_t$ enhancing the confinement of drift orbits. However, for superthermal particles a precession zero, or resonance, can occur in which $\partial(\mu B + q\Phi)/\partial\psi_t = 0$. Frequently, $\partial B/\partial\psi_t < 0$ for deeply trapped particles, which means they are in a region of bad field-line curvature. In this case, an electric field that tends to push a charge species out, $q(\partial\Phi/\partial\psi_t) < 0$, cannot have precession resonance and provides better confinement of the individual trajectories than an electric field that pulls that species in. For very-high-energy particles, such as fusion α particles, $\mu = (mv_\perp^2/2)/B$ is sufficiently large that only the term $\mu\partial B/\partial\psi_t$ in the precession is important.

The action J is only an adiabatic invariant and is not conserved if the integrand of Eq. (207) varies on the time scale of the bounce motion τ_b. Even in the absence of collisions, the action invariant can be broken in two ways. First, the drift motion of a particle can take it to a place where the local minimum in which it is trapped no longer confines particles with the action that that particle has. Each region in which a particle can be trapped has a maximum value of the action, J_{max}, that it can confine. As the particle drifts from one field line to another, the maximum action varies, and if the action J exceeds J_{max} the particle escapes from the region where it has been trapped. A particle can also be captured by local minima if it drifts so that J_{max} is increasing.

The second way the action J is broken by the particle drift motion is if a particle drifts a sufficient distance in a full bounce that the magnetic field near a turning point changes significantly from one bounce to the next. This is particularly important for finding the effect of toroidal ripple on trapped alpha particles in tokamaks and can determine the required limitation on ripple. The change in ψ_t per full bounce of a particle is $2\Delta\psi_t(\psi_t,\alpha) = \partial J/\partial\alpha$. If the change in α during a full bounce, which is $-2\partial J/\partial\psi_t$, is sufficiently large, the sign of $2\Delta\psi_t(\psi_t,\alpha)$ is essentially random with each full bounce, and the particle diffuses collisionlessly with a diffusion rate that is approximately $(\Delta\psi_t)^2/\tau_b$, which is proportional to the square of the ripple amplitude δ^2. This effect (Goldston *et al.*, 1981) can occur at arbitrarily small ripple, even ripple that is too small to cause secondary minima $\delta \ll (r/R_o)(\iota/N)$, if the precession rate of the alphas is sufficiently large in the axisymmetric field.

The breaking of the conservation of the action J due to particle drift motion can give trapped-particle trajectories a complexity that can only be studied by numerically integrating the guiding-center equations of motion.

2. Transport at low collisionality

The deviation of trapped-particle trajectories from the pressure surfaces leads to enhanced transport and to a net parallel current that is proportional to the pressure gradient, the *bootstrap current*. The transport phenomena associated with the deviation of the particle drift trajectories from the pressure surfaces are known as *neoclassical transport*.

Usually the transport rates for particles and energy in tokamak plasmas are much larger than their neoclassical values because of microturbulence (see Sec. VI.F), so the neoclassical transport theory is not as important as one might think. However, the bootstrap current is important for steady-state tokamaks and can be a complication in the design of stellarators. The theory of neoclassical transport in tokamaks has been reviewed by Hinton and Hazeltine (1976), and a book on the theory of collisional transport in toroidal plasmas has been written by Helander and Sigmar (2002).

To understand neoclassical transport, suppose the typical deviation of a trapped particle from a pressure surface is a distance $(\Delta r)_t$. In a low-collisionality limit, the distribution function must be consistent with the Vlasov equation, $df/dt=0$. Since the overall confinement is long compared to a collision time, the distribution function must also be close to a local Maxwellian, $f_M \propto n(r)e^{-H/T(r)}$. The deviation from a local Maxwellian is $\hat{f}=(\Delta r)_t\partial \ln(f_M)/\partial r$. If this is inserted into Eq. (146) for the entropy production, ignoring the temperature gradient for simplicity, one finds that

$$\dot{s}_c = \frac{\nu}{\sqrt{2\epsilon}}(\Delta r)_t^2\left(\frac{d\ln(n)}{dr}\right)^2, \tag{216}$$

where $\sqrt{2\epsilon}$ is the fraction of trapped particles. For circular magnetic surfaces, $\epsilon=r/R_o$, the inverse aspect ratio. Actually there are two factors of $\sqrt{2\epsilon}$ in this equation for the entropy production. The first factor is an enhanced effective collision rate $\nu/(2\epsilon)$ because the collision operator is diffusive, and particles need to be scattered through velocity space by a distance of only $\sqrt{2\epsilon}v$ to move all the way across the trapped-particle part of velocity space and become passing particles. The second factor of $\sqrt{2\epsilon}$ comes from the trapped particles' being the only particles that have a large deviation from the pressure surfaces (Sec. VI.E.1). The rate of entropy production is also $[d\ln(n)/dr]^2$ times the diffusion coefficient, Eq. (152), so the diffusion coefficient is $D \approx (\nu/\sqrt{2\epsilon})(\Delta r)_t^2$.

For quasisymmetric confinement systems, the deviation of the trapped-particle trajectories from a constant-pressure surface can be calculated using the conservation of the helical canonical momentum p_h [Eq. (214)]. The deviation of a particle from a magnetic surface depends on the variation in its parallel velocity $v_\parallel$, which is maximized by the barely trapped particles. For a barely trapped particle, $v_\parallel/v=\pm\sqrt{2\epsilon}$. Consequently a barely trapped particle deviates by an amount

$$\Delta\psi_{bt} = \sqrt{2\epsilon}\mu_0\frac{G-N_hI}{\iota+N_h}\rho \tag{217}$$

on either side of the flux surface on which its turning points are located, where $\rho=mv/qB$ is the gyroradius. At large aspect ratio, $\psi_t=\pi Br^2$, $I\ll G=2\pi R_oB/\mu_0$, and the deviation of a barely trapped particle is

$$(\Delta r)_{bt} = \sqrt{\frac{2}{\epsilon}}\frac{\rho}{\iota+N_h}. \tag{218}$$

This deviation of trapped particles is called a *banana orbit* because of its shape when projected on a constant-φ plane. The diffusion, $D\approx(\nu/\sqrt{2\epsilon})(\Delta r)_t^2$, is then approximately

$$D \approx \frac{\nu}{\epsilon^{3/2}}\left(\frac{\rho}{\iota+N_h}\right)^2, \tag{219}$$

which is called the *neoclassical diffusion coefficient* (Galeev and Sagdeev, 1968). In tokamaks and quasiaxisymmetric stellarators, $N_h=0$, but N_h is nonzero in quasihelically symmetric stellarators (Talmadge *et al.*, 2001).

As shown by Kovrizhnikh (1969), the statement that Eq. (219) gives the neoclassical diffusion in a quasisymmetric system is misleading. This equation is approximately correct for the diffusion of heat, but the diffusion of particles in a quasisymmetric system can only arise from unlike particle collisions. The reason is the momentum-conserving properties of the collision operator. Three expressions are important: (1) the collisional entropy production, Eq. (145), (2) the relation between the collisional entropy production and the transport coefficients, Eq. (152), and (3) $\hat{f}=(\Delta\psi_t)\partial\ln(f_M)/\partial\psi_t$. The deviation of a particle from a magnetic surface, $\Delta\psi_t$, is calculated using p_h conservation, Eq. (214), which for a density gradient implies $\hat{f}\propto(dn/d\psi_t)mv_\parallel$. The momentum conservation properties of the collision operator then make the entropy production zero, $\dot{s}_c=0$, so no particle transport occurs. For a temperature gradient, $\hat{f}$ has additional factors of the velocity, so the entropy production $\dot{s}_c$ is nonzero, and the diffusion of heat is approximated by Eq. (219). Particle transport does not vanish in quasisymmetric systems because ions and electrons can exchange momentum. For ions this rate of momentum exchange is a factor of approximately $\sqrt{m_e/m_i}$ smaller than the rate of ion-ion collisions, though for electrons the rates of momentum exchange with like and unlike particles are comparable. Momentum conservation means that the transport of ions and electrons must be at the same rate, to lowest nontrivial order in gyroradius-to-system-size ratio, independent of the radial electric field. This phenomenon is called *intrinsic ambipolarity*.

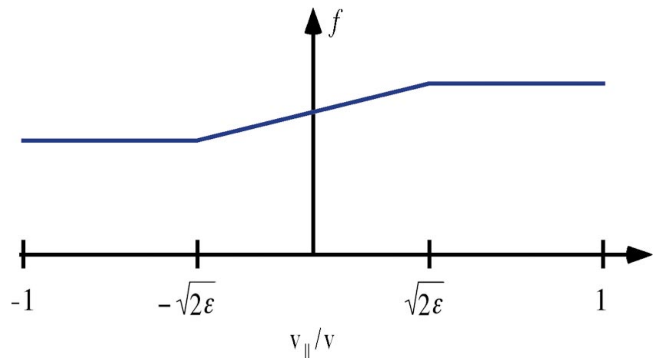

FIG. 13. (Color) Trapping of particles: In the presence of a density gradient, the number of trapped particles with a parallel velocity greater than zero differs from the number with a parallel velocity less than zero. This variation in density is transmitted to the passing particles by the diffusive property of collisions in a plasma. A particle is trapped if $-\sqrt{2\epsilon}<v_{\parallel}/v<\sqrt{2\epsilon}$ at the minimum of the magnetic-field strength along a magnetic-field line with ϵ the variation in the field strength. Otherwise the particle is passing.

In systems that are not quasisymmetric, the constraint of ambipolarity sets the radial electric field. In quasisymmetric systems, the radial electric field, and hence the rate of toroidal rotation, are only weakly affected by neoclassical transport.

One neoclassical effect that is very important for steady-state tokamaks is the bootstrap current (Bickerton *et al.*, 1971). The conservation of p_h, Eq. (214), implies that the deviation of a trapped-particle trajectory from the pressure surface has one sign, say positive, $(\Delta r)_t>0$, when the parallel velocity is positive and another, say negative, when the parallel velocity is negative. In the presence of a density gradient, $dn/dr<0$, this implies that at a fixed radius there are more barely trapped particles with a positive than with a negative parallel velocity. The passing particles that have a positive parallel velocity interact through a diffusive collision operator with passing particles that have a negative parallel velocity only through the trapped particles. This and the fact that there are more trapped particles moving in the positive direction along the magnetic-field lines than in the negative direction leads to an excess of passing particles moving in the positive direction (see Fig. 13). The excess of passing particles moving in the positive direction produces a net parallel current $j_b \approx q v_T(\Delta r)_t dn/dr$, where $v_T \equiv \sqrt{T/m}$ is the thermal speed. That is,

$$j_b \approx \frac{1}{\sqrt{\epsilon}B}\frac{T}{\iota+N_h}\frac{dn}{dr}. \quad (220)$$

The calculation of transport coefficients in nonquasisymmetric configurations is complicated. Transport coefficients as well as the effect of collisions on the particle trajectories can be determined using the Monte Carlo equivalent to the collision operator (Boozer and Kuo-Petravic, 1981). The Monte Carlo collision operator represents the effects of collisions during a time step by a change of the velocity coordinates that has a random component for each particle. The simplest Monte Carlo collision operator represents the Lorentz collision operator,

$$\mathcal{C}(f)=\frac{\nu}{2}\frac{\partial}{\partial\lambda}(1-\lambda^2)\frac{\partial f}{\partial\lambda}, \quad (221)$$

where $\lambda \equiv v_{\parallel}/v$ is the pitch of the particle relative to the magnetic field. The Monte Carlo equivalent is

$$\lambda_n=(1-\nu\tau)\lambda_o \pm \sqrt{(1-\lambda_o^2)\nu\tau}, \quad (222)$$

where λ_n and λ_o are the new and the old values of the pitch with the change caused by collisions during a time interval τ. The symbol $\pm$ means the sign is chosen at random. Since the particle trajectories are the characteristics of the operator df/dt, the inclusion of effects of collisions by Monte Carlo methods means one can find the solution f to the equation $df/dt-\mathcal{C}(f)=g$ by the *method of characteristics*. This is the basis of δf Monte Carlo studies of transport (Lin *et al.*, 1995; Sasinowski and Boozer, 1995), in which one calculates transport coefficients by using Monte Carlo methods to determine the deviations from a local Maxwellian of the particle distribution functions. An alternative to the Monte Carlo codes for numerical evaluation of transport coefficients is the DKES code (van Rij and Hirshman, 1989) which solves an approximate form of the drift kinetic equation by a variational principle.

3. Power required for maintaining fields

- The net current in a tokamak can be maintained in steady state using externally produced radio-frequency waves. However, the required power is unacceptably large for practical fusion power if more than about a third of the total current is wave driven.
- Tokamaks can be designed so that more than two-thirds of the net plasma current is maintained by the bootstrap current, which is the net current driven by the pressure gradient. The requirement of a large bootstrap current makes the feasibility of steady-state tokamaks more sensitive to the pressure profile given by plasma transport processes than is a steady-state stellarator.

An important issue for steady-state tokamaks is the power that is required for maintaining the plasma current (Fisch, 1987; ITER Physics Expert Group on Energetic Particles, …, 1999b). The traditional way to maintain the current in a tokamak is to have a solenoid within the central hole of the torus. A change in the magnetic flux in this solenoid produces a loop voltage, $V=\partial\psi_p/\partial t$, Eq. (31), which drives the current. A loop voltage can only be maintained transiently, so a different method must be adopted for long-pulse or steady-state tokamaks.

The most developed method of steady-state current drive uses waves to maintain a distribution of electrons in which more electrons are moving in the direction required for the current than in the opposite direction.

Unfortunately, too much power is required to maintain the full tokamak current. However, the total current need not be externally driven because a pressure gradient drives a net toroidal current, the bootstrap current [Eq. (220)]. From the point of view of nonequilibrium thermodynamics, the bootstrap current arises from a cross term in the transport matrix. The bootstrap current can provide most of the current in a tokamak, but the magnitude and profile of this current is dependent on the plasma pressure and density profiles. Since plasma profiles are difficult to control in the absence of external input power, the requirement of a large bootstrap current places an additional uncertainty on the feasibility of steady-state tokamaks, an uncertainty that does not exist for steady-state stellarators.

Waves in various frequency ranges can provide the power to maintain a current. In the frequency range of the so-called lower hybrid waves, this power was derived by Fisch (1978) and in the electron cyclotron frequency range by Ohkawa (1970) and Fisch and Boozer (1980). The required power to maintain a current using waves can be approximated by a simple argument (Boozer, 1988). The most efficient steady-state current drive has the current carried by high-energy, mildly relativistic electrons that form a tail on the background Maxwellian. The current density is $j=en_tc$, where n_t is the number of the tail electrons per unit volume and c is the speed of light. The power per unit volume that is required to maintain this current cannot be smaller than the power required to maintain the tail, $p_w=(\gamma-1)n_tm_ec^2\nu(\gamma)$, where $\gamma\equiv1/\sqrt{1-v^2/c^2}$ and $\nu(\gamma)$ is the slowing-down rate of electrons with kinetic energy $(\gamma-1)m_ec^2$. The power per unit volume p_w is proportional to a driven current of density j. The quantity $\mathcal{E}_w\equiv p_w/j$ has units of volts per meter. For electrons at $\gamma\approx2$, which is the most efficient energy for current drive, $\mathcal{E}_w\approx\mathcal{E}_r$ where

$$\mathcal{E}_r\equiv\frac{e\ln(\Lambda)}{4\pi\epsilon_0(c/\omega_{pe})^2}\simeq\left(0.087\frac{\text{V}}{\text{m}}\right)\left(\frac{n}{10^{20}\dfrac{1}{m^3}}\right), \tag{223}$$

the background electron density is n, the electron plasma frequency is $\omega_{pe}\equiv\sqrt{ne^2/\epsilon_0m_e}$, and $\ln(\Lambda)\approx17$ is the Coulomb logarithm [Eq. (135)]. Equation (223) is obtained within a numerical factor if one uses the collision frequency ν of Eq. (134) in the calculation of $\mathcal{E}_w\equiv p_d/j$ with $v=c$.

The power required to maintain a current using subrelativistic electrons can be calculated by similar arguments, with the current density $j=en_tv$ and the power per unit volume $p_w=(n_tm_ev^2/2)\nu(v)$. The rate of slowing of high-energy particles $\nu(v)$ is proportional to $1/v^3$, Eq. (134), so

$$\mathcal{E}_w\approx\mathcal{E}_r\frac{c^2}{v^2}. \tag{224}$$

The total power to maintain the current is obtained by multiplying $\mathcal{E}_w$ by $2\pi\bar{R}$, where $\bar{R}$ is the average major radius on a magnetic surface, to obtain, $\mathcal{V}_w\equiv2\pi\bar{R}\mathcal{E}_w$, which has units of voltage. The power is $P_w=\int\mathcal{V}_w(\psi_t)(dI/d\psi_t)d\psi_t$, where $I(\psi_t)$ is the net toroidal current inside a magnetic surface that contains toroidal flux ψ_t. The quantity $\mathcal{V}_w$ can be tens of volts, which makes the power requirements for driving the total current—more than ten megamperes—unacceptable.

The power required to maintain a current using waves scales differently than $p_\eta=E_\|j_\|$, the power required using a loop voltage, which means an electric field $E_\|=\eta j_\|$. The power required for maintaining a current with waves, p_w, is proportional to the driven current, while p_η scales as the current squared. For the total current, the ratio $p_\eta/p_w\approx(m_ec^2/T)\{(c/\omega_{pe})/a\}/\sqrt{\beta_\theta}$, where $\omega_{pe}^2\equiv nq^2/\epsilon_0m_e$ is the square of the electron plasma frequency and $\beta_\theta\equiv2\mu_0p/B_\theta^2$ is the plasma beta in the poloidal field alone. For small tokamaks p_η/p_w can be larger than one, and currents are efficiently carried by high-energy electrons, such as runaway electrons. In a power-plant-scale device p_η is much less than p_w. The more important ratio is the ratio of p_w to the fusion power, which is $\nu_T\tau_E(T/m_ec^2)(c/\omega_{pe})/(a\sqrt{\beta_\theta})$, where ν_T is the collision frequency of thermal electrons and τ_E is the energy confinement time. In a power plant, this ratio is comparable to unity.

F. Microstability

- Confined plasmas are generally unstable to perturbations that have a wavelength across the magnetic field comparable to, or smaller than, the ion gyroradius ρ_i, but with a wavelength along the field lines that is comparable to the overall system size. For near-Maxwellian plasmas, the characteristic growth rates and frequencies of microinstabilities are C_s/a, where $C_s\equiv\sqrt{(T_e+T_i)/m_i}$ is the sound speed and a is the plasma radius.
- Microinstabilities lead to microturbulence with an associated particle diffusion of order the gyro-Bohm rate, $D_g=\rho_i^2C_s/a$. The relative amplitude of the fluctuations in microturbulence is small, roughly equal to the ratio of the ion gyroradius to system size, $\delta n/n\approx\rho_i/a$.
- The most prominent microinstabilities in the modern literature are the ion and the electron temperature gradient modes. They are called the ITG or η_i mode and the ETG or η_e mode.
- The logarithmic temperature gradients, $d\ln T/dr$, can have critical values at which the transport greatly increases. Plasma temperature gradients may remain close to these critical values, which makes the temperature throughout the plasma proportional to the temperature near the plasma edge.

Even when a plasma is stable to perturbations that have a wavelength comparable to the plasma size, the plasma may be unstable to perturbations with wavelengths comparable to or smaller than the gyroradius of the ions, ρ_i. Such instabilities are called *microinstabili-*

ties. Microinstabilities do not cause a sudden loss of the plasma equilibrium but can greatly enhance the plasma transport across the magnetic-field lines. Long-wavelength instabilities, such as the instabilities discussed in Sec. V, can be so catastrophic in their effect that the only issue of interest is whether they are stable or not. With microinstabilities the primary issue is the nonlinear, or saturated, state, which is generally turbulent. In other words, the primary issue is the transport caused by the microturbulence. Unlike the large fluctuations associated with turbulence in ordinary fluids, the fluctuations associated with plasma microturbulence are small, $\delta n/n \approx \rho_i/a \sim 1/500$. Horton (1999) has reviewed the theory of microinstabilities associated with drift waves, which is the type of microinstability of most relevance to toroidal plasmas. Yoshizawa *et al.* (2001) have reviewed the theory of turbulence in fluids and plasmas with an emphasis on plasma microturbulence.

Although collisions play an important role in some microinstabilities, generally the issue is whether the Vlasov equation, $df/dt=0$, is consistent with the growth of electromagnetic perturbations. Gardner's theorem (Gardner, 1963) gives a condition under which the Vlasov-Maxwell equations can have no unstable solutions. Unfortunately, this condition is violated in current-carrying plasmas, such as magnetically confined plasmas. Gardner (1963) noted that the distribution function in the Vlasov equation is the density of a conserved fluid in phase space $(\vec{x},\vec{p})$. If $f(\vec{x},\vec{p})$ has a form such that the plasma energy is increased by the interchange of any packets of this fluid, then no energy can be removed from the plasma to support electromagnetic fluctuations, and the system must be stable. If the distribution function of a single plasma species in the direction $\hat{r}$ is

$$F(u,\vec{v}_0) \equiv \int \delta[u - \hat{r}\cdot(\vec{v}-\vec{v}_0)]f(\vec{v})d^3v, \qquad (225)$$

Gardner's theorem says the Vlasov-Maxwell equations have no unstable solutions if a $\vec{v}_0$ exists such that $u\partial F/\partial u \leqslant 0$ for all species, for all values of u, and for all directions $\hat{r}$. Distribution functions can be far from local Maxwellians and satisfy Gardner's condition for stability. Unfortunately, Gardner's condition is not satisfied for a current-carrying plasma, so microinstabilities are an issue in magnetic confinement.

Rosenbluth and Rutherford (1981) argued that only low-frequency microinstabilities are energetically possible when the source of their free energy is the pressure gradient. By low frequency they mean no higher than approximately C_s/a, where $C_s \equiv \sqrt{T_s/m_i}$, T_s is the sum of the electron and ion temperatures, a is the plasma radius, and m_i is the ion mass. The energy per unit volume associated with the oscillations of an instability is roughly $nm_i(\omega\Delta)^2$ with ω the frequency and Δ the spatial scale of the microinstability. The maximum energy that the oscillation can tap from the pressure gradient is approximately $nT_s(\Delta/a)^2$, where a is the plasma radius. The first-order term $nT_s\Delta/a$ vanishes because the average motion of the plasma $\langle\Delta\rangle$ is zero in an oscillation. The instability is not energetically favored unless $nm_i(\omega\Delta)^2 \leqslant nT_s(\Delta/a)^2$, or $|\omega| \leqslant C_s/a$.

A heuristic model clarifies some of the properties of the plasma transport caused by the turbulence associated with fully developed microinstabilities. The perturbed electric potential $\delta\Phi$ of a fully developed microinstability has a scale across the magnetic-field lines in the magnetic surface of $1/k_\perp$ and Δ in the radial direction. The variation of the potential $\delta\Phi$ along the magnetic-field lines is very weak, $k_\parallel/k_\perp \ll 1$, in order to avoid Landau damping (Sec. VI.C). The variation in the electric potential causes a drift velocity $\vec{E}\times\vec{B}/B^2$. The magnitude of the radial component of this velocity is $\delta v_r \approx k_\perp \delta\Phi/B$. Let τ_c be the correlation time of the microturbulence, which means the time scale over which significant changes in the pattern of the perturbed potential $\delta\Phi$ occur. If $\delta_r \lesssim \Delta$, then during a correlation time particles in the plasma take a radial step, $\delta_r \approx \delta v_r \tau_c$, with equal probability of the step's being inwards or outwards. The random steps cause diffusion with the diffusion coefficient $D \approx \delta_r^2/\tau_c$, which can be rewritten as $D \approx k_\perp^2(\delta\Phi/B)^2\tau_c$.

The correlation time of microturbulence has different values in two limits. If the microturbulence is extremely weak the correlation time is determined by the growth rate of the underlying microinstability, and the theory is essentially that of quasilinear diffusion (Sec. VI.C). However, when the microturbulence is fully developed the correlation time is determined by the turbulence itself changing the potential $\delta\Phi$. In this strong-turbulence limit, the correlation time is determined by how long it takes particles to diffuse across the potential contours, $\tau_c \approx 1/(k_\perp^2 D_s)$. The diffusion coefficient D_s is the coefficient for diffusion in the magnetic surfaces across the field lines. In this direction, the $E\times B$ velocity is $\delta v_s \approx (\delta\Phi/\Delta)/B$, so the diffusion in the magnetic surfaces is related to diffusion across the surfaces by $D_s \approx D/(k_\perp\Delta)^2$. The correlation time is $\tau_c \approx \Delta^2/D$, and the radial step is $\delta_r \approx \Delta$. Inserting this expression for τ_c into $D \approx k_\perp^2(\delta\Phi/B)^2\tau_c$, one finds

$$D \approx k_\perp \Delta \left|\frac{\delta\Phi}{B}\right|. \qquad (226)$$

In strong microturbulence the particle diffusion is linear in the perturbation amplitude rather than quadratic as it is in quasilinear diffusion.

The perturbation amplitude that is reached in a microturbulent plasma is bounded by $|\vec{\nabla}\delta n| \approx |\vec{\nabla} n|$. That is, the microturbulence cannot create steeper gradients than the ambient gradient while transferring energy from the gradients. If one of the species is responding adiabatically, which means $e\delta\Phi/T = \delta n/n$, the bound on the fluctuation amplitude is

$$\frac{e\delta\Phi}{T} = \frac{\delta n}{n} \approx \frac{1}{k_\perp a}, \tag{227}$$

where $1/a \equiv d\ln(n)/dr$ is the scale of the ambient density gradient. The radial diffusion coefficient is then

$$D \approx \frac{\Delta}{a}\frac{T}{eB}, \tag{228}$$

where $T/(eB) = \rho_i C_s$, which is the ion gyroradius times the speed of sound.

The transport caused by microturbulence is dependent on the radial extent Δ of the constant $\delta\Phi$ contours. There are two extreme assumptions about Δ. The more pessimistic is that Δ is proportional to the size of the plasma, $\Delta \propto a$. This assumption makes the diffusion proportional to the Bohm diffusion coefficient, which in the modern literature is usually defined as

$$D_B \equiv \frac{T}{eB}, \tag{229}$$

although David Bohm's unpublished work, which first gave this coefficient, included an unjustified factor of $\frac{1}{16}$. The confinement time given by Bohm diffusion, $\tau_B \approx a^2/D_B$, is similar to the time it takes particles to drift out of a torus due to unconfined particle trajectories, $\tau_d = a/v_g$, where a is the plasma radius and $v_g \approx (\rho_i/a)C_s$ is the guiding-center drift velocity. The more optimistic assumption is that Δ is of order the ion gyroradius ρ_i. The assumption that $\Delta \approx \rho_i$ gives the gyro-Bohm rate,

$$D_g \equiv \frac{\rho_i}{a}\frac{T}{eB} = \frac{\rho_i^2 C_s}{a}. \tag{230}$$

The Bohm time is far too short for a fusion power plant. The gyro-Bohm confinement time, $\tau_g = a^2/D_g$, is marginal,

$$\tau_g \approx 7\ \text{ms}\frac{a^3B^2}{(T/10\ \text{keV})^{3/2}}, \tag{231}$$

where the plasma radius is in meters and the magnetic field is in tesla.

What sets the radial scale Δ of the perturbed potential $\delta\Phi$? First, consider a microinstability, such as the ion temperature gradient (ITG) instability discussed below, in which the electrons have an adiabatic response, which means $\Delta n/n = e\Phi/T$, but the ions behave nonadiabatically. In linear theory, Δ can be much larger than the ion gyroradius. However, the long radial contours of the potential are broken up in the nonlinear microturbulent state by what are known as *zonal flows* (Diamond and Kim, 1991). These flows come from the $E\times B$ drift in the part of potential perturbation that has a nonzero average over the magnetic surface, $\delta\bar{\Phi}(\psi_t,t) \equiv \langle\delta\Phi\rangle$. The surface-averaged fluctuation in the potential, $\delta\bar{\Phi}(\psi_t,t)$, cannot be damped by electrons flowing along the field lines, unlike the rest of the variation in the potential, $\delta\Phi - \delta\bar{\Phi}$. Indeed, the adiabatic response of the electrons, if written correctly, is $\delta n/n = e(\delta\Phi - \delta\bar{\Phi})/T$. The drive for the zonal flows is the surface-averaged inertial force of the fluctuating $E\times B$ velocity. The left-hand side, or inertial part, of the Navier-Stokes equation is $\rho(\partial\vec{v}/\partial t + \vec{v}\cdot\vec{\nabla}\vec{v}) = \partial(\rho\vec{v})/\partial t + \vec{\nabla}\cdot(\rho\vec{v}\vec{v})$, where the continuity equation $\partial\rho/\partial t + \vec{\nabla}\cdot(\rho\vec{v}) = 0$ was used to place the inertial terms in the second form. The force that drives the zonal flows is the average over the magnetic surfaces of $\langle\vec{\nabla}\cdot(\rho\vec{v}\vec{v})\rangle$, where $\vec{v} = (\vec{B}\times\vec{\nabla}\delta\Phi)/B^2$ is the $E\times B$ velocity of the plasma in the turbulence. The tensor $\rho\vec{v}\vec{v}$ is known as the Reynolds stress tensor, so zonal flows are driven by the Reynolds stresses. When the microturbulence is strong, the zonal flows are sufficiently robust to make $\Delta \approx 1/k_\perp$. When the ions are the nonadiabatic species, the perpendicular wave number $k_\perp$ tends to be comparable to the ion gyroradius ρ_i. The reason is that when $k_\perp \ll 1/\rho_i$ the instability grows faster the larger $k_\perp$. However, when $k_\perp \gg 1/\rho_i$ the ions respond adiabatically to changes in the potential by moving across the field lines (Sec. VI.G), which removes the drive for the instability. The typical fluctuation amplitude, Eq. (227), is of order the ion gyroradius to system size, ρ_i/a.

Microinstabilities, such as the electron temperature gradient (ETG) mode that is discussed below, also exist in which the ions respond adiabatically but electrons nonadiabatically. The ions behave adiabatically when the wave number of the perturbations satisfies $k_\perp\rho_i \gg 1$, so the ions are free to cross the field lines in response to changes in the potential. For this case zonal flows are not important because the adiabatic ion response, $\delta n/n \approx -e\delta\Phi/T$, is to the full variation in the electric potential and not just the part that varies on the magnetic surfaces, as is the case with adiabatic electrons. The radial extent Δ of the constant-potential contours is still limited by the ion gyroradius ρ_i because otherwise the ions could not respond adiabatically, although the perpendicular wave number for the ETG mode is comparable to the electron gyroradius, $k_\perp\rho_e \approx 1$. Because of the great anisotropy, $k_\perp\Delta \approx \rho_i/\rho_e$, ETG microturbulence can produce transport comparable to ITG microturbulence with zonal flows (Dorland *et al.*, 2000; Jenko *et al.*, 2000). For both, the diffusion is comparable to the gyro-Bohm rate. However, it should be noted that a number of experiments see a reduction in the diffusion with ion mass rather than the increase that would be expected from gyro-Bohm diffusion. This effect, which is discussed by Bessenrodt-Weberpals *et al.* (1993), is not understood theoretically.

An important feature of recent experiments has been the observation of transport barriers, which are narrow regions in which transport is greatly reduced. This topic has been recently reviewed (Terry, 2000; Wolf, 2003). Despite the narrowness of the transport barriers, the reduction in transport is sufficiently great to significantly enhance the overall plasma confinement. The theoretical explanation (Biglari *et al.*, 1990) is a stabilization of the microturbulence by a strong radial gradient, or shear, in the $E\times B$ flow in the magnetic surfaces. The shearing rate of the flow is $\gamma_s \equiv |d[(\vec{E}\times\vec{B})/B^2]/dr|$. If the shearing rate γ_s is greater than the growth rate of a microinstability, the instability is stabilized by being torn apart by the sheared flow and is stabilized. Since the jump in the potential across a shear layer cannot be greater than roughly $\Delta\Phi \approx T_s/e$, and the typical growth rate is C_s/a, one finds the width of a shear layer is roughly $\delta_s \approx \sqrt{a\rho_i}$. Neoclassical diffusion in a tokamak, Eq. (219), divided by the gyro-Bohm diffusion, Eq. (230), is

$$\frac{D_g}{D_{nc}} \approx \frac{\iota^2 \epsilon^{3/2}}{\nu} \frac{C_s}{a}, \tag{232}$$

which is roughly 10^2 in a fusion plasma. A simple estimate of the temperature drop that one can obtain across a transport barrier $\Delta T/T$ is $(\delta_s/a)(D_g/D_{nc})$. Since $(\delta_s/a)(D_g/D_{nc})$ is generally greater than unity under fusion conditions, one can obtain a large temperature drop across a transport barrier. The difference between zonal flows and transport barriers is that zonal flows have a radial scale and a time variation determined by the microturbulence. Transport barriers are quasistatic features with the strong variation in the electric potential related to the strong variation in the pressure, through the tendency in plasmas for $|\vec{\nabla} p| \approx |en\vec{E}|$ for one of the two species.

The diffusion coefficients that have been discussed are more properly considered transport coefficients for heat than coefficients for particles. The reason is that if either species responds adiabatically there can be no particle transport. Radial particle transport averaged over a magnetic surface is $\Gamma_r = \langle (n+\delta n)\delta v_r \rangle$. For an adiabatic response, $\delta n/n = \pm e\delta\Phi/T$ while $v_r = \hat{r}\cdot(\vec{B}\times\vec{\nabla}\delta\Phi)/B^2$, so δn and v_r are out of phase, which means their average over the surface is zero. Although individual particles of the nonadiabatic species diffuse, so that heat can be transported, the electric field arranges itself so the net particle flux is zero in order to preserve quasineutrality, $qn_i \simeq en_e$.

Neither species responds fully adiabatically in fully developed microturbulence, so microturbulence generally leads to particle transport across the magnetic-field lines, which is effectively an enhancement of the perpendicular component of the resistivity tensor $\eta_\perp$ [Eq. (32)]. The component of the resistivity along the magnetic field, $\eta_\parallel$, is, however, rarely enhanced by microturbulence, for two reasons. First, the parallel component of the fluctuating electric field $\delta E_\parallel = -ik_\parallel \delta\Phi$ is small compared to the electric field across the magnetic-field lines because $|k_\parallel/k_\perp| \ll 1$, so little scattering of the parallel motion of the particles occurs. Second, if any group of ions and electrons can diffuse rapidly across the magnetic-field lines, then $\eta_\perp$ is enhanced, but if any group of electrons can flow freely along the magnetic-field lines, then $\eta_\parallel$ remains close to its quiescent plasma value.

Our discussion of microturbulence has assumed that only the electric potential is perturbed. Microturbulence couples to shear Alfvén modes (Sec. VI.H), when the Alfvén frequency, $\omega_A \equiv v_A \iota / R_o$, is comparable to that of microinstabilities, C_s/a. Since the Alfvén velocity is $v_A \equiv \sqrt{B^2/\mu_0 m_i n}$, the coupling occurs when the plasma pressure satisfies $\beta \equiv 2\mu_0 p/B^2 > (a\iota/R_o)^2$. Alfvén coupling means that the component of the vector potential parallel to the magnetic field, $\delta A_\parallel$, is perturbed. If $\delta A_\parallel/B$ is Fourier decomposed in magnetic coordinates that have a simple covariant form, Eq. (58), then the magnetic surfaces are broken to form an island unless each Fourier component $(\delta A_\parallel/B)_{mn}$ vanishes at its rational surface, $\iota = n/m$ [Eq. (80)]. It is unclear whether the resonant components of $(\delta A_\parallel/B)_{mn}$ are nonzero under standard plasma conditions, but if they were nonzero they would produce a qualitative change in the microturbulence due to electron transport along the stochastic magnetic-field lines. On the other hand, if the Fourier components $(\delta A_\parallel/B)_{mn}$ exactly vanish at their resonant rational surfaces, then the effect of the coupling of the microturbulence to the shear Alfvén modes would be to cause the magnetic surfaces to wobble, which makes calculations more difficult but causes no qualitatively new physical effects.

The microinstability that has received the most attention in recent years is the ion temperature gradient (ITG) instability, also called the η_i mode,

$$\eta_i \equiv \frac{d\ln(T)}{d\ln(n)}, \tag{233}$$

where $T(\psi_t)$ is the ion temperature and $n(\psi_t)$ is the ion density. The ITG mode, which has been reviewed by Horton (1999), appears to be responsible for the enhanced transport of heat by ions in tokamak plasmas and can be viewed as a kinetic version of sound waves that are destabilized by the ion temperature gradient.

The simplest version of the ITG instability (Kadomtsev and Pogutse, 1970) occurs in a uniform magnetic field $\vec{B} = B\hat{z}$ with the plasma having temperature and density gradients in the x direction. The fluctuations depend on y, z, and time as $\exp[i(k_y y + k_z z - \omega t)]$ and only weakly on x. In a fluid model, the ions obey the continuity equation, $\partial n/\partial t + \vec{\nabla}\cdot(n\vec{v}) = 0$, with the perpendicular components of the velocity given by $\vec{v}_\perp = \vec{E}\times\vec{B}/B^2$. Using a tilde to denote perturbed quantities, the continuity equation implies

$$\omega\frac{\tilde{n}}{n} + \frac{k_y}{B}\frac{d\ln n}{dx}\tilde{\Phi} = k_z \tilde{v}_\parallel. \tag{234}$$

The parallel component of the velocity is given by force balance along the field lines, $nm_i dv_\parallel/dt = -\partial p/\partial z - en\partial\Phi/\partial z$, or

$$\omega m_i n \tilde{v}_\parallel = k_z(\tilde{p} + en\tilde{\Phi}). \tag{235}$$

The ions are assumed to respond adiabatically, which means $p/n^{5/3}$ is carried with the flow, $\partial(p/n^{5/3})/\partial t + \vec{v}\cdot\vec{\nabla}(p/n^{5/3}) = 0$, and

$$\frac{\tilde{p}}{p} = \frac{5}{3}\frac{\tilde{n}}{n} - \frac{k_y}{\omega B}(\eta_i - 2/3)\frac{d \ln n}{dx}\tilde{\Phi}. \tag{236}$$

The electrons are of sufficiently low mass that they can move rapidly along the magnetic-field lines and remain in thermodynamic equilibrium,

$$\frac{\tilde{n}}{n} = \frac{e\tilde{\Phi}}{T_e}, \tag{237}$$

which is called an adiabatic response. In giving the electron response we have assumed that the perturbation has a long spatial scale compared to the Debye length, which implies the electron and ion densities are essentially equal in both the unperturbed and the perturbed states. In other words, the perturbed plasma is quasineutral. Combining results, one finds

$$1 - \frac{\omega_{*e}}{\omega} = \left(\frac{k_z}{\omega}\right)^2 \frac{T}{m_i}\left\{\frac{T_e}{T} + \frac{5}{3} + \frac{\omega_{*e}}{\omega}\left(\eta_i - \frac{2}{3}\right)\right\}, \tag{238}$$

where the electron drift frequency is

$$\omega_{*e} \equiv -\frac{k_y T_e}{eB}\frac{d \ln n}{dx}. \tag{239}$$

If the frequency, ω, of the perturbation is high compared to the drift frequency ω_{*e}, then Eq. (238) gives sound waves

$$\omega^2 = \frac{T_e + \frac{5}{3}T}{m_i}k_z^2. \tag{240}$$

However, when the frequency of the perturbation is low compared to the drift frequency, $\omega \ll \omega_{*e}$, the frequency of the perturbation is

$$\omega^2 = \frac{T}{m_i}\left(\frac{2}{3} - \eta_i\right)k_z^2, \tag{241}$$

which has an exponentially growing ITG mode for $\eta_i > 2/3$. In a plasma such as a fusion plasma in which collisions are weak, neither the sound wave nor the ITG mode are treated realistically in this analysis unless $\omega/k_\parallel \gg \sqrt{T/m_i}$, because otherwise strong-ion Landau damping (Sec. VI.C) implies that only decaying solutions exist. In other words, the analysis is only realistic for sound waves if $T/T_e \ll 1$ and for the ITG mode if $\eta_i \gg 1$. To the extent the electrons respond adiabatically, the η_i mode causes ion heat transport but no electron transport and, because of quasineutrality, no particle transport.

Here we have ignored not only kinetic effects, but also two other important determinants of ITG stability, shear and the $\vec{B}\times\vec{\nabla}B$ drift. Magnetic shear, the change in direction of magnetic-field lines from one pressure surface to another, is stabilizing. Magnetic shear effects will be considered in the discussion of Alfvén instabilities in Sec. VI.H.2. The $\vec{B}\times\vec{\nabla}B$ drift destabilizes the ITG drift in regions of bad magnetic-field-line curvature, which means the center of curvature is on the higher-pressure side of the field lines. This destabilization of the ITG mode is closely related to the destabilization of ballooning modes by bad curvature (Sec. V.C.2). The version of the ITG mode in which curvature effects dominate is called the *toroidal branch*, and the version in which curvature effects are subdominant is called the *slab branch*.

An important feature of ITG turbulence is that it becomes strong only when a critical gradient is reached. In the simplest version of the theory, the critical gradient is a critical value of η_i. When curvature effects are retained, turbulence can arise when the radius of curvature R times $1/L_T \equiv |d \ln T/dr|$ exceeds a critical value. The R/L_T critical gradient is counterintuitive, since the weaker the curvature the easier it is for curvature to destabilize the mode. However, the growth rate of the mode and the maximum rate of transport become small when R is large. In many experiments, in particular those with a high ion temperature, the plasma is thought to operate just above the critical gradient. In this situation, the logarithmic ion temperature gradient, $d \ln T/dr$, is given by the critical gradient with the magnitude of the heat flux determined by conditions at the plasma edge. That is, the ion temperature throughout the plasma is proportional to the ion temperature near the plasma edge. A similar phenomenon occurs in the earth's atmosphere where the temperature profile is determined by the critical gradient for convection, $d \ln(p)/d \ln(n) \geq \gamma$, with the magnitude of the heat flux determined by the boundaries of the convection zone.

A second instability, the electron temperature gradient (ETG) mode, has very similar physics, only the role of the electrons and ions is reversed. The ETG mode has wave numbers comparable to the electron gyroradius ρ_e. On this scale the ions have only weak magnetic effects, $\rho_i \approx 60\rho_e$, and the ions respond adiabatically, $\delta n/n = -e\Phi/T$. The ETG mode may be responsible for the enhancement of the electron heat transport above its neoclassical value (Dorland *et al.*, 2000; Jenko *et al.*, 2000), and the combination of the ITG and the ETG modes may give the enhanced particle transport that is observed in experiments.

G. Gyrokinetic theory

- The simplification of the particle trajectories and kinetic theory that occurs when the gyroradius ρ is small compared to the system size can be extended to include perturbations of the magnetic and electric fields that have arbitrarily large wave numbers perpendicular to the magnetic field, $k_\perp\rho$ arbitrary. This approximate kinetic theory is called gyrokinetic theory.

Computational studies of microinstabilities, which are on the scale of the gyroradius of one of the species, are carried out using the gyrokinetic equations. This and other computational studies of plasmas have been discussed by Tang (2002). The gyrokinetic equations are the kinetic equations but with the particle velocity represented by the gyrokinetic drift velocity. The derivation of these equations was developed by Rutherford and Frieman (1968), Taylor and Hastie (1968), and Antonsen and Lane (1980).

The gyrokinetic drift velocity consists of two parts. The first part is the guiding-center drift of the particles in the large-scale electric and magnetic fields. This is just the ordinary guiding-center drift, $\vec{v}_g$, which was discussed in Sec. VI.D. The second part, $\delta\vec{v}_g$, is the modification of the drift of the guiding center by the small-scale perturbations. The perturbations that arise in microturbulence are small, of order the gyroradius-to-system-size ratio ρ/a, but the drifts that the perturbations cause can be of order of other guiding-center drifts due to the strong spatial gradients, of order $1/\rho$.

Despite having a wavelength across the magnetic-field lines comparable to the gyroradius of one of the species, the perturbations that arise in a microturbulent plasma have a wavelength along the magnetic field comparable to the plasma size. The long parallel wavelength arises to minimize Landau damping (Sec. VI.C). This complicated spatial structure can be accommodated mathematically by writing perturbed quantities, such as the electric potential, in eikonal form,

$$\delta\Phi = \tilde{\Phi}(\vec{x},t)e^{iS(\psi_t,\alpha)}, \tag{242}$$

just as one does for ballooning modes. The eikonal $S(\psi_t,\alpha)$ depends on the two Clebsch coordinates, where $\vec{B}=\vec{\nabla}\psi_t\times\vec{\nabla}\alpha$, Eq. (9), and represents the rapid variation of $\delta\Phi$ across the magnetic-field lines. $\tilde{\Phi}$ varies on a much longer spatial scale, which involves the plasma size, and with a slow time scale, of order the thermal sound speed divided by the plasma radius C_s/a. Although

$$\vec{k}_\perp \equiv \vec{\nabla}S \tag{243}$$

is very large, $k_\perp\rho\approx 1$, the spatial variation of $\vec{k}_\perp$ itself is much longer, involving the overall scale of the plasma.

Three electromagnetic quantities affect the particle drifts due to their rapid spatial variation: the perturbed electric potential, $\delta\Phi=\tilde{\Phi}\exp(iS)$, the perturbed parallel component of the vector potential, $\delta A_\parallel=\tilde{A}_\parallel\exp(iS)$, and the perturbed parallel component of the magnetic field, $\delta B_\parallel=\tilde{B}_\parallel\exp(iS)$. As shown below, their effect on the drifts is given by

$$\delta\vec{v}_g \equiv \left\langle e^{-iS_g}\frac{d\vec{x}_g}{dt}\right\rangle = -\frac{i\vec{k}_\perp\times\hat{b}}{B}\chi, \tag{244}$$

with the generalized potential

$$\chi = (\tilde{\Phi} - v_\parallel\tilde{A}_\parallel)J_0(k_\perp\rho) + \frac{\mu\tilde{B}_\parallel}{q}\frac{2J_1(k_\perp\rho)}{k_\perp\rho} \tag{245}$$

defined using the time derivative of the particle energy or Hamiltonian,

$$q\frac{\partial\chi}{\partial t} \equiv \left\langle e^{-iS_g}\frac{dH}{dt}\right\rangle. \tag{246}$$

Here S_g means the eikonal is evaluated at the guiding-center position, and $\langle\cdots\rangle$ means an average over the gyrophase ϑ. J_0 and J_1 are the zeroth and the first cylindrical Bessel functions, $\mu\equiv mv_\perp^2/2B$ is the magnetic moment, which is an adiabatic invariant, and $\hat{b}\equiv\vec{B}/B$. The time derivative of the guiding-center position $\vec{x}_g$, Eq. (180), depends on the electric and magnetic fields at the true position of the particle, $\vec{x}$, rather than at the location of the guiding center, $\vec{x}_g$. The difference between the true position $\vec{x}$ and the guiding center $\vec{x}_g$ is the vector gyroradius $\vec{\rho}=\vec{x}-\vec{x}_g$, Eq. (177). The difference between the eikonal evaluated at the guiding-center position and the actual position of a particle is $S_g-S=-\vec{k}_\perp\cdot\vec{\rho}$.

The Bessel functions that appear in the gyrokinetic equations measure the importance of the interaction of a charged particle with the magnetic field. As $k_\perp\rho\to 0$ the interaction is strong, and the Bessel functions have the limits $J_0(k_\perp\rho)\to 1$ and $2J_1(k_\perp\rho)/(k_\perp\rho)\to 1$. For $k_\perp\rho\to\infty$, the interaction becomes weak, but at a slow rate, $1/\sqrt{k_\perp\rho}$. As $z\equiv k_\perp\rho\to\infty$, the Bessel functions have the asymptotic forms $J_n(z)=\sqrt{(2/\pi z)}\cos[z-(n+1/2)(\pi/2)]$.

The distribution function of gyrokinetic theory is evaluated at the guiding-center position of the particles $\vec{x}_g$ rather than their actual position $\vec{x}$ and has the form $f(H,\mu,\vec{x}_g,t)=f_0(H,\vec{x}_g)+\delta f$, where $\delta f=\tilde{f}(H,\mu,\vec{x}_g,t)\times\exp(iS_g)$ and f_0 is the equilibrium distribution function.

The gyrokinetic equation for the amplitiude of the perturbed distribution function, $\delta f=\tilde{f}\exp(iS_g)$, is

$$\left(\frac{\partial}{\partial t} + \vec{v}_g\cdot\vec{\nabla} + i\vec{k}_\perp\cdot\vec{v}_g\right)\tilde{f} + \mathcal{Q} + F_I = C_L(\tilde{f}), \tag{247}$$

where the inhomogeneous term is

$$F_I \equiv \delta\vec{v}_g\cdot\vec{\nabla}f_0 + q\frac{\partial\chi}{\partial t}\frac{\partial f_0}{\partial H}, \tag{248}$$

with f_0 the distribution function for the unperturbed equilibrium. $C_L(\tilde{f})$ is a linearized collision operator, and the quantity $\mathcal{Q}$ is a quadratic nonlinearity. The dominant nonlinearity in gyrokinetic theory is generally taken to be the $(d\vec{x}_g/dt)\cdot\vec{\nabla}\delta f$ nonlinearity, for which $\mathcal{Q}$ has the form

$$\mathcal{Q} = \frac{\hat{b}\cdot\langle\langle e^{-iS_g}\vec{\nabla}(\chi e^{iS_g})\times\vec{\nabla}(\tilde{f}e^{iS_g})\rangle\rangle}{B}. \tag{249}$$

The notation $\langle\langle\cdots\rangle\rangle$ means an average over the gyrophase and a projection of the expression using the orthogonality of the solutions to the linear gyrokinetic equation. The function $\chi\exp(iS_g)$ comes from converting the electric and magnetic fields that appear in $d\vec{x}_g/dt$ from the particle position to the guiding-center position plus an average over the gyrophase. The gyrophase average is discussed below in the derivations of $\delta\vec{v}_g$ and χ.

The nonlinear part $\mathcal{Q}$ of the gyrokinetic equation requires a definition of orthogonality among the solutions to the linear gyrokinetic equation, which means solutions that have different eikonals S. For example, in axisymmetric systems different solutions in eikonal form can have different toroidal mode numbers, as discussed in Sec. V.C.2 on ballooning modes. The gradient of the eikonal is the effective perpendicular wave number. As is well known from Fourier analysis, perturbations with different wave numbers are orthogonal. The definition of orthogonality and the evaluation of the required integrals to obtain $\mathcal{Q}$ is the major subtlety of the study of microturbulence with the gyrokinetic equation. An approximate method, which is relatively easy to follow, has been given by Beer, Cowley, and Hammett (1995).

The linear part of the gyrokinetic equation follows from Eqs. (244) and (246) and the kinetic equation $df/dt=C(f)$, where $df/dt=df_0/dt+d\delta f/dt$. The time derivative of the perturbed distribution function is $d\delta f/dt=(d\tilde{f}/dt)e^{iS_g}+i\tilde{f}e^{iS_g}dS_g/dt$, and the time derivative of the equilibrium distribution function is $df_0/dt=(df_0/dH)dH/dt+(d\vec{x}_g/dt)\cdot\vec{\nabla}f_0$. The linear gyrokinetic equation is obtained by multiplying both sides of the kinetic equation by e^{-iS_g} and then averaging over the gyrophase ϑ. The rapidly varying part of δf is in the eikonal, so $\tilde{f}$ is slowly varying, and $d\tilde{f}/dt=\partial\tilde{f}/\partial t+\vec{v}_g\cdot\vec{\nabla}\tilde{f}$. The gyrophase average is $\langle dS_g/dt\rangle=\langle\vec{v}_g\cdot\vec{\nabla}S_g\rangle=\vec{k}_\perp\cdot\vec{v}_g$.

To have a complete set of equations, relations are needed between the perturbed distribution function $\tilde{f}$ and the perturbed electric potential $\tilde{\Phi}$, the parallel component of the vector potential $\tilde{A}_\parallel$, and the parallel component of the magnetic field $\tilde{B}_\parallel$. For purposes of taking the averages, the perpendicular velocity of the particle will be written as

$$\vec{v}_\perp = v_\perp\{\cos\vartheta\hat{k}_\perp\times\hat{b}+\sin\vartheta\hat{k}_\perp)\}. \tag{250}$$

The perturbed electric potential is given by Poisson's equation, which can be approximated as $k_\perp^2\delta\Phi=(q/\epsilon_0)\delta n$, or

$$k_\perp^2\tilde{\Phi}=\frac{q}{\epsilon_0}\tilde{n}. \tag{251}$$

The perturbed density of a species is $\delta n=\int\Delta f d^3v$, where $\Delta f\equiv q\delta\Phi(\partial f_0/\partial H)+\delta f$ [Eq. (153)]. Since $\delta f=\tilde{f}\exp(iS_g)$ but $\delta n=\tilde{n}\exp(iS)$ with $S_g-S=-k_\perp\rho\cos\vartheta$, the perturbed density is

$$\tilde{n}=q\tilde{\Phi}\int\frac{\partial f_0}{\partial H}d^3v+\int J_0(k_\perp\rho)\tilde{f}d^3v, \tag{252}$$

where we used the integral expression for the zeroth-order Bessel function,

$$J_0(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-iz\cos\vartheta}d\vartheta, \tag{253}$$

to carry out the integration over the gyrophase ϑ, which is one of the integration variables in d^3v [Eq. (171)].

The perturbed parallel component of the vector potential can be calculated using Ampère's law, $\vec{\nabla}\times\vec{\nabla}\times\delta\vec{A}=\mu_0\delta\vec{j}$, or

$$k_\perp^2\tilde{A}_\parallel=\mu_0\tilde{j}_\parallel. \tag{254}$$

The perturbed current density is

$$\tilde{j}_\parallel=q\int v_\parallel J_0(k_\perp\rho)\tilde{f}d^3x, \tag{255}$$

where we have assumed df_0/dH is symmetric in $v_\parallel$.

The equation for $\tilde{B}_\parallel$ is given by the force balance across the field lines, which for short-wavelength perturbations has the form $\hat{k}\cdot\vec{\nabla}(p_\perp+B^2/2\mu_0)=0$. This follows from $\mu_0\vec{j}\times\vec{B}=\vec{B}\cdot\vec{\nabla}\vec{B}-\vec{\nabla}B^2/2$ since $\vec{B}\cdot\vec{\nabla}\vec{B}=B^2\hat{b}\cdot\vec{\nabla}\hat{b}+\hat{b}\vec{B}\cdot\vec{\nabla}B$ is negligible when $k_\perp\gg k_\parallel$. The perturbed perpendicular pressure is $\tilde{p}_\perp=\langle\int m(\vec{v}\cdot\hat{k}_\perp)^2\Delta f d^3v\rangle$, which means

$$\tilde{p}_\perp=\left(q\tilde{\Phi}\int\mu\frac{\partial f_0}{\partial H}d^3v+\int\mu\tilde{f}\frac{2J_1(k_\perp\rho)}{k_\perp\rho}d^3v\right)B, \tag{256}$$

where $\mu\equiv mv_\perp^2/2B$ is the magnetic moment and the first-order Bessel function obeys

$$\frac{J_1(z)}{z}=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-iz\cos\theta}\sin^2\theta d\theta. \tag{257}$$

The adiabatic term in $\tilde{p}$, which is the term proportional to $\tilde{\Phi}$, is equal and opposite for the electrons and the ions provided the Debye length satisfies $k_\perp\lambda_D\ll 1$, so $e\delta n_e=q\delta n_i$. The force balance is $\tilde{p}_\perp+B\tilde{B}_\parallel/\mu_0=\text{const}$, so

$$\tilde{B}_\parallel=-\mu_0\int\mu\tilde{f}\frac{2J_1(k_\perp\rho)}{k_\perp\rho}d^3v. \tag{258}$$

The remainder of the section gives the derivations of Eq. (245) for χ, which is defined by $q\partial\chi/\partial t\equiv\langle e^{-iS_g}dH/dt\rangle$ and $\delta\vec{v}_g\equiv\langle e^{-iS_g}d\vec{x}_g/dt\rangle$ [Eq. (244)].

First, we need the approximate relation between the vector potential and the magnetic field that holds when $k_\perp a\gg 1$ with a the plasma radius. The magnetic field is $\delta\vec{B}=\vec{\nabla}\times\delta\vec{A}$ where $\delta\vec{A}=\tilde{\vec{A}}\exp(iS)$. Dotting this expression with the unperturbed magnetic field $\vec{B}$, one finds that $\vec{B}\cdot\delta\vec{B}=\vec{\nabla}\cdot(\delta\vec{A}\times\vec{B})+\delta\vec{A}\cdot\vec{\nabla}\times\vec{B}$. The only term on the right-hand side that is large is the first term, through its dependence on $\vec{k}_\perp\equiv\vec{\nabla}S$,

$$\delta B_\parallel=-(i\vec{k}_\perp\times\hat{b})\cdot\delta\vec{A}_\perp. \tag{259}$$

The components of the perturbed magnetic field that are perpendicular to the unperturbed field are given by $\vec{B}$

$\times \delta\vec{B} = \vec{\nabla}(\vec{B}\cdot\delta\vec{A}) - \delta\vec{A}\times\vec{\nabla}\times\vec{B} - \delta\vec{A}\cdot\vec{\nabla}\vec{B} - \vec{B}\cdot\vec{\nabla}\delta\vec{A}$. The large term is the first term, which gives

$$\hat{b}\times\delta\vec{B} = i\vec{k}_\perp \delta A_\parallel. \quad (260)$$

Second, we derive the expression for χ, Eq. (245), from its definition, Eq. (246). The exact equation for the change in the energy of the particles is $dH/dt = q(\partial\Phi/\partial t - \vec{v}\cdot\partial\vec{A}/\partial t)$ [Eq. (198)]. The required gyrophase average $\langle e^{-iS_g} dH/dt\rangle$ is the sum of two terms. The first term is

$$q\left(\frac{\partial\tilde{\Phi}}{\partial t} - v_\parallel \frac{\partial \tilde{A}_\parallel}{\partial t}\right)\langle e^{i(S-S_g)}\rangle. \quad (261)$$

The gyrophase average $\langle \exp[i(S-S_g)]\rangle$ can be written as $\langle \exp(ik_\perp \rho\cos\vartheta)\rangle = J_0(k_\perp\rho)$. The second term is

$$-\left\langle \vec{v}_\perp \cdot \frac{\partial}{\partial t}(\delta\vec{A}_\perp e^{-iS_g})\right\rangle = \frac{\mu}{q}\frac{\partial\tilde{B}_\parallel}{\partial t}\frac{2J_1(k_\perp\rho)}{k_\perp\rho}, \quad (262)$$

which is derived using Eqs. (250) and (259), $\vec{v}_\perp\cdot\delta\vec{A}_\perp = v_\perp \hat{k}_\perp\cdot\delta\vec{A}_\perp \sin\vartheta + i(v_\perp/k_\perp)\delta B_\parallel\cos\vartheta$. The gyrophase average $\langle \sin\vartheta \exp[i(S-S_g)]\rangle$ is zero because

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\cos\vartheta}\sin\vartheta d\vartheta = 0, \quad (263)$$

and the gyrophase average $\langle \cos\vartheta \exp[i(S-S_g)]\rangle$ is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\cos\vartheta}\cos\vartheta d\vartheta = iJ_1(z), \quad (264)$$

where $z = k_\perp\rho$ and $\rho = mv_\perp/qB$. Assembling the parts, one obtains $\langle e^{-iS_g}dH/dt\rangle = q\partial\chi/\partial t$ with χ given by Eq. (245).

Third, we calculate the part of $\delta\vec{v}_g$, Eq. (244), that is across the magnetic field. The time derivative of the guiding-center position is given by Eq. (180). Because of the slowness of the time variation, the perturbed electric field is given by the gradient of the potential,

$$\delta\vec{E} = -i\vec{k}_\perp\delta\Phi. \quad (265)$$

The contribution of the $E\times B$ drift to $\delta\vec{v}_g$ is

$$\left\langle e^{-iS_g}\frac{\delta\vec{E}\times\vec{B}}{B^2}\right\rangle = -\frac{i\vec{k}_\perp\times\hat{b}}{B}J_0(k_\perp\rho)\tilde{\Phi}, \quad (266)$$

where we used $\langle \exp[i(S-S_g)]\rangle = J_0(k_\perp\rho)$. The effect of the perturbation on the term is

$$\delta\left\langle e^{-iS_g}\vec{v}_\perp\times\hat{b}\frac{d}{dt}\frac{1}{\Omega}\right\rangle = -\frac{i\tilde{B}_\parallel}{\Omega B}\left\langle e^{i(S-S_g)}\vec{v}_\perp\times\hat{b}\frac{dS}{dt}\right\rangle. \quad (267)$$

The large term is $dS/dt = \vec{v}_\perp\cdot\vec{\nabla}S = \vec{k}_\perp\cdot\vec{v}_\perp$, which means $dS/dt = k_\perp v_\perp \sin\vartheta$. Equation (250) implies $\vec{v}_\perp\times\hat{b} = v_\perp(\hat{k}_\perp\times\hat{b})\sin\vartheta - v_\perp\hat{k}_\perp\cos\vartheta$. The gyrophase integral that involves the second term in $\vec{v}_\perp\times\hat{b}$ is zero, and the gyrophase integral that involves the first term is calculated using Eq. (257),

$$\left\langle e^{-iS_g}\vec{v}_\perp\times\hat{b}\frac{d}{dt}e^{iS}\right\rangle = -i(\vec{k}_\perp\times\hat{b})v_\perp^2\frac{J_1(k_\perp\rho)}{k_\perp\rho}. \quad (268)$$

Fourth, we calculate the part of $\delta\vec{v}_g$, Eq. (244), that is along the perturbed magnetic field. Motion along the magnetic field is given by $v_\parallel\hat{b} + v_\parallel\delta\hat{b}$ with the change in the direction of the magnetic field, $\delta\hat{b} = \delta\vec{B}_\perp/B$. Equation (260) implies

$$\delta\langle e^{-iS_g}v_\parallel\delta\hat{b}\rangle = \frac{i\vec{k}_\perp\times\hat{b}}{B}v_\parallel\tilde{A}_\parallel J_0(k_\perp\rho). \quad (269)$$

Putting the pieces together, one finds the effect of the perturbations on the drift equation (244) with the generalized potential χ given by Eq. (245).

H. Alfvén instabilities

- Shear Alfvén waves are a twisting of the magnetic-field lines. These waves have a continuous spectrum of possible frequencies but are generally heavily damped. The damping occurs if there is sufficient field-line shear, which means variation in the direction of the magnetic-field lines or sufficient variation in the Alfvén velocity, $B/\sqrt{\mu_0\rho}$ with ρ the plasma mass density, transverse to the magnetic field.
- If the Alfvén velocity varies along the magnetic-field lines and the magnetic-field lines have shear, then the continuous spectrum of Alfvén waves has gaps, and weakly damped discrete (sharp-frequency) shear Alfvén modes can exist in these gaps.
- Energy is transferred between shear Alfvén modes and particles with a resonant velocity, $\omega/k_\parallel$, along the field lines. A destabilizing transfer of energy from the particles to the waves occurs if the diamagnetic drift frequency of the interacting particles, ω_* of Eq. (290), is larger than the frequency ω of the Alfvén mode; otherwise the transfer is stabilizing. Only particles, such as fusion alphas, that have a gyroradius in the poloidal magnetic field alone comparable to their density gradient have a destabilizing interaction.

If a magnetic field is twisted and released, the twist travels down the magnetic-field lines at the Alfvén velocity,

$$v_A \equiv \sqrt{\frac{B^2}{\mu_0\rho}}, \quad (270)$$

and is called a shear Alfvén wave. Most shear Alfvén perturbations are strongly damped, but certain perturbations called gap modes have weak natural damping. The weakly damped gap modes are susceptible to being driven to high amplitude by interactions with particles that are moving along the field lines with a velocity comparable to the Alfvén velocity. This instability of the gap modes goes under the name of the toroidal Alfvén eigenmode (TAE) instability. The theory of short-wavelength TAE instability was developed by Cheng *et al.* (1985), and the theory of the more important low-mode-number TAE instability by Cheng and Chance (1986). The concern is that alpha particles produced by the fusion reaction will drive gap modes to a high amplitude causing a loss of the alphas from the plasma before they can transfer their energy to the bulk plasma (ITER Physics Export Group on Energetic Particles, …, 1999a). Only at a low plasma beta, $\beta\equiv 2\mu_0 p/B^2$, is the Alfvén velocity above the speed v_α at which fusion alphas are produced, $\beta\simeq 1.8\,\%\,(T/20\text{ keV})(v_\alpha/v_A)^2$, with T the plasma temperature. Wong (1999) has reviewed the relation between the experiments and theory for TAE instabilities. The properties of shear Alfvén modes that are central to this theory will be discussed in this section.

1. Continuum Alfvén wave damping

Shear Alfvén waves have a continuous spectrum that is heavily damped (Tataronis and Grossmann, 1973). The physics is illustrated by a model in which the equilibrium magnetic field is uniform and in the $\hat{z}$ direction, $\vec{B}=B\hat{z}$, but the plasma mass density ρ depends on x. The force balance is $\rho(x)\partial\vec{v}/\partial t=\delta\vec{j}\times\vec{B}$ with $\mu_0\delta\vec{j}=\vec{\nabla}\times\delta\vec{B}$. The perturbation to the magnetic field is given by the ideal Ohm's law, $\partial\delta\vec{B}/\partial t=\vec{\nabla}\times(\vec{v}\times\vec{B})$. The two components of velocity that enter the calculation are written as $v_x=\tilde{v}_x(x)\exp[i(k_y y+k_\parallel z-\omega t)]$ and $v_y=\tilde{v}_y(x)\exp[i(k_y y+k_\parallel z-\omega t)]$. The wave number k_y is in the surface of constant Alfvén velocity, and the wave number $k_\parallel$ is along the magnetic field.

If the model equations for Alfvén waves are analyzed, keeping all three components of $\delta\vec{B}$, one finds

$$[\omega^2-k_\parallel^2 v_A^2(x)]\tilde{v}_x+v_A^2\frac{d^2\tilde{v}_x}{dx^2}+ik_y v_A^2\frac{d\tilde{v}_y}{dx}=0 \qquad (271)$$

and

$$[\omega^2-(k_\parallel^2+k_y^2)v_A^2(x)]\tilde{v}_y+ik_y v_A^2\frac{d\tilde{v}_x}{dx}=0. \qquad (272)$$

These equations, which are more general than the derivation given here, can be combined into a single equation for $\tilde{v}_x$,

$$\frac{d}{dx}\left(\frac{\omega^2-k_\parallel^2 v_A^2}{\omega^2-(k_\parallel^2+k_y^2)v_A^2}\frac{d\tilde{v}_x}{dx}\right)+\frac{\omega^2-k_\parallel^2 v_A^2}{v_A^2}\tilde{v}_x=0. \qquad (273)$$

When the Alfvén velocity is independent of position, this equation has two types of solutions. One type is the compressional Alfvén mode with $v_A^2 d^2 v_x/dx^2+\{\omega^2-(k_\parallel^2+k_y^2)v_A^2\}v_x=0$, which is a propagating wave if $k_x^2\equiv(\omega^2/v_A^2)-(k_y^2+k_\parallel^2)\geqslant 0$ and evanescent (exponential dependence on x) otherwise. The other solution is the shear Alfvén wave with $\omega^2=k_\parallel^2 v_A^2$, which has an arbitrary dependence on x. When the Alfvén velocity depends on position, these two types of solutions are coupled, and the point where $v_A^2(x)=\omega^2/k_\parallel^2$ is a singular point, a point at which energy is absorbed.

Equation (273) has a singular point, which will be denoted by $x=0$, where $v_A^2(x)=\omega^2/k_\parallel^2$. At the singular point, the velocity $\tilde{v}_y$ has a $1/x$ singularity, which means there is an infinite amount of energy in the vicinity of the singular point, an unphysical result. This singularity in the energy means an arbitrarily large amount of energy can accumulate, a buildup of energy that can be represented by letting the frequency be complex, $\omega=\omega_0+i\gamma$. Assume γ is small, let $x=0$ be the point where $k_\parallel^2 v_A^2=\omega_0^2$, and let

$$\delta\equiv\frac{2\omega_0\gamma}{k_\parallel^2 dv_A^2/dx} \qquad (274)$$

evaluated at $x=0$. Then near its singular point Eq. (273) can be approximated as

$$\frac{d}{dx}(x-i\delta)\frac{d\tilde{v}_x}{dx}=(x-i\delta)k_y^2\tilde{v}_x, \qquad (275)$$

while Eq. (272) implies $k_y\tilde{v}_y=id\tilde{v}_x/dx$.

To calculate the power per unit area flowing into the singular point of Eq. (273), we study the two independent solutions of Eq. (275) in the vicinity of the singular point. These solutions are the cylindrical zeroth-order modified Bessel functions of the first, $I_0(\xi)$, and the second, $K_0(\xi)$, kind, where $\xi\equiv k_y(x-i\delta)$. As $\xi\to 0$, $I_0\to 1$ and $K_0\to-\ln\xi$. The regular, or I_0, part of the solution is undamped, but the singular, or K_0, part is damped. Focusing on the singular part of the solution, let

$$\tilde{v}_x=V_s K_0(\xi), \qquad (276)$$

where V_s is a constant. Then near $\xi=0$, $\tilde{v}_y=-iV_s/\xi$. The average power per unit $y-z$ area that goes into the velocity singularity is $P=\mathcal{R}_e\{\int\rho\tilde{v}_y^*(\partial\tilde{v}_y/\partial t)dx/2\}$. Here we used the easily proven result that if a function $f(y)$ is the real part of $f_c\exp(ik_y y)$ and another function $g(y)$ has the same form, then the y average of $f(y)g(y)$ is $\mathcal{R}_e\{f_c^* g_c\}/2$. Since $\partial\tilde{v}_y/\partial t=-i\omega\tilde{v}_y$, $P=(\gamma/2)\int\rho\tilde{v}_y^*\tilde{v}_y dx$. The integral that must be performed has the form

$$\int_{-\infty}^{\infty}\frac{dx}{x^2+\delta^2}=\frac{\pi}{|\delta|}, \qquad (277)$$

so $P=(\pi/2)(\gamma/|\delta|)\rho_0 V_s^2/k_y^2$ where $\rho_0=\rho(x=0)$. In other words, the power per unit area going into the sigularity is

$$P=\frac{\pi}{2}\omega_0\frac{\rho_0 V_s^2}{k_y^2}\left|\frac{d\ln v_A(x)}{dx}\right|. \qquad (278)$$

2. The weakly damped gap mode

The existence of weakly damped gap modes requires both a shear in the magnetic-field lines across the lines and a variation in the magnetic-field strength along the magnetic-field lines. An equation for the shear Alfvén mode, which contains both shear and variation in the Alfvén velocity, is relatively simple when the perturbation is localized to a magnetic-field line. This occurs when the perpendicular wave number $k_\perp$ is large in comparison to the wave number along the magnetic field lines $k_\parallel$. The assumption $k_\perp \gg k_\parallel$ leads to the eikonal approximation as in the discussion of ballooning modes, Sec. V.C.2, and gyrokinetics, Sec. VI.G.

All the perturbed quantities in a shear Alfvén wave can be expressed in terms of the perturbation to the electric potential $\delta\Phi=\tilde{\Phi}\exp(iS)$, where $S(\alpha)=2\pi m\alpha$ is the eikonal and m is a large integer. The perpendicular wavelength is $k_\perp \equiv \vec{\nabla}S$. The Clebsch angle $\alpha=(\theta_m-\iota\varphi)/2\pi$ is defined so $\vec{B}=\vec{\nabla}\psi_t\times\vec{\nabla}\alpha$ [Eq. (9)]. The equation for the perturbed electric potential, which is derived below, is

$$\frac{\partial}{\partial\zeta}\left(\frac{k_\perp^2}{B}\frac{\partial\delta\Phi}{\partial\zeta}\right)=-\frac{k_\perp^2}{B}\Omega(\zeta)^2\delta\Phi, \tag{279}$$

where ζ is a dimensionless coordinate along the magnetic-field line. The distance along a line is $\ell=(L/2\pi)\zeta$ with L a characteristic distance. Ω is a dimensionless, or normalized, frequency,

$$\Omega(\zeta)^2\equiv\omega^2\left(\frac{L}{2\pi v_A(\zeta)}\right)^2, \tag{280}$$

with ω the frequency of the perturbation $\tilde{\Phi}\propto\exp(-i\omega t)$. The characteristic form is $\Omega^2(\zeta)=\Omega_0^2(1+2\epsilon\cos\zeta)$. In a large-aspect-ratio tokamak, $\epsilon=r/R_o$ gives the variation in the magnetic-field strength, and $L=2\pi R_o/\iota$ is the periodicity length of the field strength. The period of the magnetic-field strength in a tokamak is $\theta_m=2\pi$, but along a field line $\theta_m=\iota\varphi$ with $d\ell=R_o d\varphi$.The frequency $\omega_A\equiv(2\pi/L)v_A$ is known as the Alfvén frequency, so $\Omega=\omega/\omega_A$. For a large-aspect-ratio tokamak, $\omega_A=v_A\iota/R_o$.

A magnetic field has global shear if $d\iota/d\psi_t$ is nonzero. When the field is sheared, the characteristic dependence of $k_\perp^2/B$ on ζ is

$$\frac{k_\perp^2}{B}\propto(1+s^2\zeta^2), \tag{281}$$

with $s\propto d\iota/d\psi_t$. This follows from $2\pi\vec{\nabla}\alpha=\vec{\nabla}\theta_m-\iota\vec{\nabla}\varphi-\varphi(d\iota/d\psi_t)\vec{\nabla}\psi_t$ with $\varphi\propto\zeta$. For a large-aspect-ratio tokamak, $\zeta=\iota\varphi$ and $s=d\ln\iota/d\ln r$.

This paragraph contains the derivation of Eq. (279) for $\delta\Phi$. The derivation starts with the ideal Ohm's law, $\delta\vec{E}+\vec{v}\times\vec{B}$. The magnetic perturbation of a shear Alfvén wave is perpendicular to the main field, and when $k_\perp\to\infty$ Eq. (260) shows that $\delta\vec{B}_\perp=i\vec{k}_\perp\times\hat{b}\delta A_\parallel$ with $\hat{b}\equiv\vec{B}/B$. Therefore $\delta\vec{E}=-\hat{b}\partial\delta A_\parallel\partial t-\vec{\nabla}\delta\Phi$, which implies $\vec{v}=(\vec{B}\times i\vec{k}_\perp)\delta\Phi/B^2$ and $\partial\delta A_\parallel/\partial t=-(\vec{B}\cdot\vec{\nabla}\delta\Phi)/B$. The force balance $\rho\partial\vec{v}/\partial t=\delta\vec{j}\times\vec{B}$ implies $\delta\vec{j}_\perp=(\vec{B}\times\rho\partial\vec{v}/\partial t)/B^2$ and that $\vec{\nabla}\cdot\delta\vec{j}_\perp=k_\perp^2\rho(\partial\delta\Phi/\partial t)/B^2$. Since the current is divergence-free, $\vec{B}\cdot\vec{\nabla}(j_\parallel/B)=-\vec{\nabla}\cdot\delta\vec{j}_\perp$. Therefore $\vec{B}\cdot\vec{\nabla}(j_\parallel/B)=-k_\perp^2(\rho/B^2)\partial\delta\Phi/\partial t$. Ampere's law gives $k_\perp^2\delta A_\parallel=\mu_0 j_\parallel$, and $\partial^2\delta\Phi/\partial t^2=-\omega^2\delta\Phi$. Let $\vec{B}\cdot\vec{\nabla}=(2\pi B/L)\partial/\partial\zeta$ with L a characteristic length along the field line. Ω^2 is defined by Eq. (280). Combining results, one obtains Eq. (279).

The energy density of a shear Alfvén wave,

$$w=\frac{\rho k_\perp^2}{4B^2}\left(\delta\Phi^*\delta\Phi+\frac{1}{\Omega^2}\frac{\partial\delta\Phi^*}{\partial\zeta}\frac{\partial\delta\Phi}{\partial\zeta}\right), \tag{282}$$

is the sum of the kinetic energy $\rho\vec{v}^*\cdot\vec{v}/4$ and the magnetic energy $\delta\vec{B}^*\cdot\delta\vec{B}/4\mu_0$. The factors of four, where two is expected, come from the use of complex arithmetic, as discussed just above Eq. (277).

A shear in the magnetic-field lines, $s\neq0$, causes a rapid loss of energy from Alfvén waves if the field strength is constant, $\epsilon=0$. Consider the Fourier transform of Eq. (279) with $k_\perp^2/B$ replaced using Eq. (281). That is, let $\delta\Phi=\int\tilde{\phi}\exp(i\kappa_\parallel\zeta)d\kappa_\parallel$, which implies $d\delta\Phi/d\zeta\to i\kappa_\parallel\tilde{\phi}$ and $\zeta\delta\Phi\to-id\tilde{\phi}/d\kappa_\parallel$. Then when the variation of the Alfvén speed along the field lines is ignored, $\Omega^2=$const, Eq. (279) implies

$$s^2\frac{d}{d\kappa_\parallel}(\Omega^2-\kappa_\parallel^2)\frac{d\tilde{\phi}}{d\kappa_\parallel}=(\Omega^2-\kappa_\parallel^2)\tilde{\phi}. \tag{283}$$

This equation is singular at the place where $\kappa_\parallel=\Omega$, which is the equation for a shear Alfvén wave in the absence of shear, $s=0$. The energy density in $\kappa_\parallel$ space, w_κ, is defined so $\int w_\kappa d\kappa_\parallel=\int w d\zeta$, and is given by

$$w_\kappa=\frac{\pi}{2}\rho\left(\frac{k_\perp}{B}\right)_0^2\left(1+\frac{\kappa_\parallel^2}{\Omega^2}\right)\left(\tilde{\phi}^*\tilde{\phi}+s^2\frac{d\tilde{\phi}^*}{d\kappa_\parallel}\frac{d\tilde{\phi}}{d\kappa_\parallel}\right) \tag{284}$$

with the subscript naught on $(k_\perp/B)_0^2$ implying evaluation at $\zeta=0$. Near the singular point $\kappa_\parallel=\Omega$, the dominant component in the energy density w_κ scales as the square of $sd\tilde{\phi}/d\kappa_\parallel$, and an infinite amount of energy is located near this point in $\kappa_\parallel$ space. These equations and the resolution of their singularity exactly follow the derivation that led to Eq. (278) for the dissipation of Alfvén wave energy. The singular point, $\kappa_\parallel=\Omega$, of Eq. (283) represents the transfer of energy to the region $\zeta\to\infty$, which in steady state has infinite energy content, while the singular point $k_\parallel v_A=\omega$ of Eq. (273) represents the transfer of energy to an arbitrarily small region in x, which means to $k_x\to\infty$.

The existence of gaps in the spectrum of Alfvén waves and the existence of weakly damped modes in those gaps is easier to demonstrate if the perturbed quantities are expressed in terms of $u\equiv(ik_\perp/\sqrt{B})\delta\Phi$, which places the energy density in a nonsingular form,

$$w=\frac{\rho}{4B}\left(|u|^2+\frac{1}{\Omega^2}\left|\frac{du}{d\zeta}-\frac{s\zeta}{1+s^2\zeta^2}su\right|^2\right). \tag{285}$$

The equation for u has the characteristic form

$$\frac{\partial^2u}{\partial\zeta^2}-\frac{s^2}{(1+s^2\zeta^2)^2}u=-(1+2\epsilon\cos\zeta)\Omega_0^2u. \tag{286}$$

The solution to this equation for values of Ω_0 is given in Fig. 14. The equation obtained from transforming Eq. (279) is $\partial^2u/\partial\zeta^2-\sigma(\zeta)u+\Omega^2(\zeta)u=0$, where $\sigma(\zeta)\equiv(\sqrt{B}/k_\perp)\partial^2(k_\perp/\sqrt{B})/\partial\zeta^2$. Using $k_\perp/\sqrt{B}\propto\sqrt{1+s^2\zeta^2}$ and $\Omega^2(\zeta)=\Omega_0^2(1+2\epsilon\cos\zeta)$, one obtains the characteristic

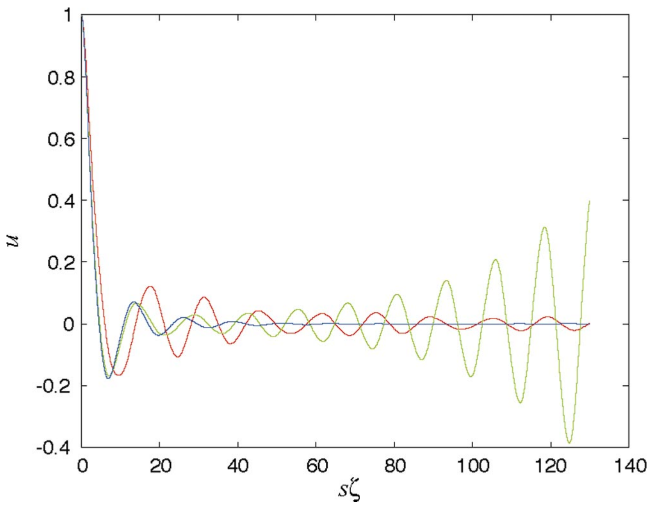

FIG. 14. (Color) The solution to Eq. (286), plotted for three values of the eigenvalue: red, $\Omega_0=0.42$ gives a solution in the continuum, green, $\Omega_0=0.4775$ gives a singular solution, and blue, $\Omega_0=0.487$ gives a discrete mode. The parameters $s=0.5$ and $\epsilon=0.2$ are the same for all three solutions.

equation for u. If $\Omega^2=$const, then the solution is $u \to \exp(i\Omega\zeta)$ for either $s\to 0$ or $s\zeta\to\infty$. This means as $\zeta \to\infty$ the Fourier transform of the shear Alfvén wave satisfies $\kappa_\parallel \to \Omega$. Since the amplitude of u is constant as $\zeta \to\infty$, the solution $u\to\exp(i\Omega\zeta)$ has infinite energy.

Now consider the effect of a variation in the Alfvén speed along the magnetic-field lines. The characteristic equation for u, Eq. (286), reduces to the Mathieu equation (Bender and Orszag, 1978),

$$\frac{d^2u}{dx^2}+(1+2\epsilon\cos x)\Omega_0^2 u(x)=0, \tag{287}$$

for either zero shear, $s=0$, or for $|s\zeta|\gg 1$. The Mathieu equation has two types of solutions, depending on the value of the eigenvalue Ω_0^2. For certain ranges of Ω_0^2, the regular regions, the equation yields oscillatory solutions, and for others, the singular, or gap, regions, there are exponentially growing solutions (Fig. 14). The smallest $\Omega_0^2>0$ region that is singular occurs in the vicinity of $\Omega_0^2=1/4$ for $|\epsilon|\ll 1$, which means $\omega=\omega_A/2$ with $\omega_A \equiv v_A\iota/R_o$ the Alfvén frequency. More precisely, the singular, or gap, region is $\Omega_0^2=\frac{1}{4}(1\pm\epsilon)$ for $|\epsilon|\ll 1$. The regular ranges of Ω_0^2 give continuum shear Alfvén waves. These solutions extend over the full range of ζ and, therefore, require an infinite energy to drive. The singular regions, or gaps, are values of Ω_0^2 for which physical shear Alfvén waves do not exist. The energy in a singular solution is in the region $|\zeta|\to\infty$, since the amplitude of u increases exponentially with $|\zeta|$.

The characteristic equation for u, Eq. (286), also has spatially bounded solutions, which means the solution is concentrated in the region where $|s\zeta|$ is small (Fig. 14). These solutions are the gap modes because they occur in the singular gap regions of the Mathieu equation and have little intrinsic damping. They are the modes of interest for Alfvén instabilities. Since they have finite energy, they can be driven unstable by their interactions with high-energy particles. The existence of discrete modes in the gaps of a frequency spectrum is a well-known phenomenon in condensed-matter physics. For a simple discussion see Allen *et al.* (2003).

3. The particle-Alfvén wave interaction

The interaction of the particles that form the plasma with the shear Alfvén wave is given by the kinetic equation. This interaction is weaker than one might at first expect for modes, such as the discrete or gap mode, that have no parallel electric field. When $E_\parallel=0$, particles in resonance with a shear Alfvén wave, $v_\parallel=\omega/k_\parallel$, do not exchange energy with the wave unless there is a cross-field drift v_d in the equilibrium magnetic field.

To simplify the analysis of the interaction of particles with a shear Alfvén wave, the perturbed electric potential will be assumed to have the form

$$\delta\Phi=\tilde{\Phi}e^{i(k_\parallel z+k_y y-\omega t)}. \tag{288}$$

The distribution function f of the species that has a resonant interaction with the Alfvén wave can be written as the sum of an equilibrium distribution and a perturbation, $f=f_r(H,x)+\delta f$ and obeys the Vlasov equation $df/dt=0$. The time-averaged power to the Alfvén wave per unit volume will be shown to be

$$p_w=\frac{\pi}{2}\frac{k_y^2v_d^2}{|k_\parallel|}\left(\frac{\omega_*}{\omega}-1\right)\frac{q^2}{T_r}\tilde{\Phi}^2\bar{f}_r\left(\frac{\omega}{k_\parallel}\right), \tag{289}$$

where v_d is the guiding-center drift velocity of particles in the $\hat{y}$ direction in the unperturbed magnetic field,

$$\omega_*\equiv\frac{k_yT_r}{qB}\frac{\partial\ln f_r}{\partial x} \tag{290}$$

is the diamagnetic drift frequency, $\bar{f}_r(v_z)\equiv\int f_r dv_x dv_y$, and $T_r\equiv-\partial\ln f_r/\partial H$ is the effective temperature of the interacting particles, which have a Hamiltonian H.

The power to the shear Alfvén wave, Eq. (289), has a destabilizing sign only if ω_* is greater than ω, which is equivalent to the requirement that the gyroradius ρ_θ of the interacting particles in the poloidal field alone be sufficiently large to be comparable to their density gradient. The condition on the poloidal gyroradius is derived by first noting that the gap mode has a typical frequency $\omega=\omega_A/2$ with the Alfvén frequency $\omega_A \equiv v_A\iota/R_o$. For the interacting species $T_r\sim m_r v_A^2$, and $k_y \simeq m/r$ with m the poloidal mode number. Defining the poloidal gryroradius $\rho_\theta\equiv(R_o/\iota r)\rho$, one finds that the condition $\omega^*>\omega$ implies $\rho_\theta d\ln n_r/dr \gtrsim 1/m$. Only particles, such as fusion alphas, that have a poloidal gyroradius comparable to their density gradient can destabilize Alfvén modes. Extremely high m-number Alfvén modes are not of such great concern because of the small radial extent of the perturbations that they produce.

To derive the power going to an Alfvén wave, the time derivative of both the unperturbed and the perturbed parts of the distribution function are needed. The time derivative of the unperturbed part of the distribution function f_r is given by

$$\frac{df_r}{dt} = \frac{dx_g}{dt}\frac{\partial f_r}{\partial x} + \frac{dH}{dt}\frac{\partial f_r}{\partial H}, \tag{291}$$

which depends on $dH/dt = \partial H/\partial t$ and the $\hat{x}$ component of the guiding-center velocity, dx_g/dt. The time rate of change of the Hamiltonian is

$$\frac{\partial H}{\partial t} = q\left(\frac{\partial \delta\Phi}{\partial t} - v_\parallel \frac{\partial \delta A_\parallel}{\partial t}\right) = q\left(1 - \frac{k_\parallel v_\parallel}{\omega}\right)\frac{\partial \delta\Phi}{\partial t}, \tag{292}$$

where we used $\delta E_\parallel = -\partial\delta\Phi/\partial z - \partial\delta A_\parallel/\partial t = 0$, which means $\omega\delta A_\parallel = k_\parallel \delta\Phi$. The relation between $\delta A_\parallel$ and $\delta\Phi$ also gives the guiding-center velocity in the x direction,

$$\frac{dx_g}{dt} = \frac{\delta E_y}{B} + v_\parallel \frac{\delta B_x}{B} = -\left(1 - \frac{k_\parallel v_\parallel}{\omega}\right)\frac{1}{B}\frac{\partial \delta\Phi}{\partial y}, \tag{293}$$

and the time derivative

$$\frac{df_r}{dt} = i(\omega - k_\parallel v_\parallel)\left(1 - \frac{\omega_*}{\omega}\right)\frac{q\,\delta\Phi}{T_r} f_r, \tag{294}$$

where we have written $\partial \ln f_r/\partial H = -1/T_r$ and used the diamagnetic drift frequency [Eq. (290)].

The time derivative of the perturbed part of the distribution function δf is given by $d\delta f/dt = \partial\delta f/\partial t + \vec{v}_g \cdot \vec{\nabla}\delta f$, which can be written as $d\delta f/dt = -i(\omega - k_\parallel v_\parallel - k_y v_d)\delta f$, where v_d is the drift of particles in the y direction in the unperturbed magnetic field. Since f obeys the Vlasov equation, $df/dt = 0$, the perturbed distribution function is given by $d\delta f/dt = -df_r/dt$, or

$$\delta f = \left(1 - \frac{\omega_*}{\omega}\right)\left(1 + \frac{k_y v_d}{\omega - k_\parallel v_\parallel - k_y v_d}\right)\frac{q\,\delta\Phi}{T_r} f_r. \tag{295}$$

The time-averaged power per unit volume going to the Alfvén wave is $p_w = \langle -\delta\vec{j}\cdot\delta\vec{E}\rangle$. The perturbed electric field is $\delta\vec{E} = -ik_y \delta\Phi\hat{y}$, and the part of $\delta j_y = \int v_d \delta f d^3v$ that is in phase is the imaginary part, which is given by the Landau integral, Eq. (165),

$$i\delta j_y^i = -i\frac{\pi}{|k_\parallel|} q k_y v_d^2 \left(1 - \frac{\omega_*}{\omega}\right)\frac{q\,\delta\Phi}{T_r}\bar{f}_r\left(\frac{\omega}{k_\parallel}\right), \tag{296}$$

where $\bar{f}_r(v_z) \equiv \int f_r dv_x dv_y$. The time-averaged power per unit volume to the Alfvén wave is $p_w = \langle -\delta\vec{j}\cdot\delta\vec{E}\rangle = \langle \delta\vec{j}_\perp \cdot \vec{\nabla}\delta\Phi\rangle$, which implies $p_w = k_y \tilde{j}_y^i \tilde{\Phi}/2$. Consequently the power to the wave per unit volume is given by Eq. (289).

VII. PLASMA EDGE

- Control of the plasma edge is important to (a) maintain plasma purity, (b) limit the maximum heat flux to the chamber walls, and (c) achieve a high edge temperature if the plasma transport is associated with a critical temperature gradient.
- The plasma edge can be defined by a solid object, called a limiter, or by a separatrix between the magnetic-field lines that lie on toroidal surfaces and on open magnetic-field lines that intercept the chamber walls. Most modern plasma confinement devices define the edge with a separatrix. This method of defining the plasma edge is called a divertor.

The edge is the interface between the hot plasma and the walls. This interface must remove impurities, such as the helium ash, and the waste heat. Ideally the waste heat is removed by electromagnetic radiation because that avoids hot spots on the walls. In addition, the edge is more important to plasma confinement than one might expect because transport often appears to obey a critical-gradient theory, which means the temperature throughout the plasma is proportional to the edge temperature.

In modern plasma confinement devices the plasma edge is generally defined by the separatrix between magnetic-field lines that lie on toroidal magnetic-surfaces and open magnetic field lines that go into specially designed regions on the chamber walls (Fig. 9). This method of defining the plasma edge is called a divertor (ITER Physics Expert Group on Diverter Modelling …, 1999; Loarte, 2001).

Simple features of divertors can be understood from a fluid model, $\rho\vec{v}\cdot\vec{\nabla}\vec{v} + \vec{\nabla}p = \vec{j}\times\vec{B}$, with $\vec{\nabla}\cdot(\rho\vec{v}) = 0$ and $\rho = mn$ the mass density of the plasma. The plasma flow is rapid along the magnetic field. Let $\rho\vec{v} = \Gamma\hat{b} + \rho\vec{v}_\perp$, then

$$\vec{B}\cdot\vec{\nabla}\left(\frac{\Gamma}{B}\right) = S, \tag{297}$$

where the source $S \equiv -\vec{\nabla}\cdot(\rho\vec{v}_\perp)$. Equation (297) gives the flux of plasma Γ along a magnetic-field line. Since the flow is essentially parallel to the magnetic field, the parallel component of the force balance is $\Gamma\hat{b}\cdot\vec{\nabla}(\Gamma/\rho) + \hat{b}\cdot\vec{\nabla}p = 0$, which can be written as

$$\Gamma(\ell)\frac{d(\Gamma/\rho)}{d\ell} + \frac{dp}{d\ell} = 0, \tag{298}$$

where $d/d\ell \equiv \hat{b}\cdot\vec{\nabla}$. If the density $\rho(\ell)$ is used as the independent variable,

$$\frac{d\Gamma^2}{d\rho} = -2\rho\left(C_s^2 - \frac{\Gamma^2}{\rho^2}\right), \tag{299}$$

where $C_s^2 \equiv dp/d\rho$. That is, the effective sound speed is $C_s^2 \equiv (dp/d\ell)/(d\rho/d\ell)$. Equation (299) is well known from the theory of nozzles in fluid mechanics and says the density drops as the flux Γ increases when the flow starts with zero speed, $\Gamma/\rho = 0$. The maximum flux that can be obtained is $\Gamma = \rho C_s$, which implies a flow at the speed of sound C_s. If the divertor chamber exerts a sufficiently small back pressure, then the plasma flow will reach the sonic rate as it flows down the field lines. In the other limit in which the back pressure p_b keeps the flow slow compared to the sound speed, the flux reaches

$\Gamma^2 = 2\rho_0(p_0 - p_b)$ with p_0 the pressure where $\Gamma = 0$. Divertors have been operated in both the high and the low back-pressure limit. The advantage of a high back pressure is that more of the energy moving down the divertor channel can be radiated away primarily through atomic radiation (Sec. VI.A), which reduces the peak heat loads. The advantage of a sonic flow is that the plasma is efficiently swept into the divertor chamber.

An important issue is the ability of materials to withstand the high energy fluxes in the energy of the impinging particles. The energy of the impinging particles is greatly modified by the sheath potential between a plasma and a material surface. The flow of a plasma into a wall is roughly at the speed of sound. However, the electrons would naturally flow into the wall at the electron thermal speed, which is a factor $\sqrt{m_i/m_e}$ faster. To preserve the quasineutrality of the plasma, a jump in the electrostatic potential occurs on a Debye length scale between a plasma and a wall with the potential holding back the electrons. This means $\sqrt{T_e/m_e}\exp(e\Delta\Phi) \approx C_s$, or $\Delta\Phi/eT_e \approx \ln(\sqrt{m_i/m_e}) \approx 4$. The ions impinge on the wall not only with their thermal energy but also with the kinetic energy obtained from the sheath potential $q\Delta\Phi$.

The required divertor flux is $\Gamma = \bar{n}\pi a^2/(\tau_p\delta)$, where $\bar{n}$ is the average density in the plasma, τ_p is the confinement time for particles, and δ is the width of the divertor outflow channel.

Stellarators use an island chain about a rational surface, ι equal to a rational number, at the plasma surface to divert the plasma into the divertor chamber (Renner *et al.*, 2002). This idea is analogous to that used in tokamaks where the separatrix that lies between field lines that encircle the plasma and those that go to the divertor chamber is at the $\iota = 0$ rational surface, which is the only rational surface on which an island can form in axisymmetry.

The variation in the electrostatic potential across the divertor region has important implications for the overall confinement of the plasma. The ions can sense the presence of open magnetic-field lines once they are within a banana orbit width of the plasma edge. Because of their higher thermal speed, electrons tend to leave as soon as they cross onto open field lines, while the ions penetrate deeper into the open field-line region. Consequently, it is natural for a strong shear in the $E\times B$ flow to occur near the plasma edge, which can stabilize the microtubulence. A region of greatly reduced transport at the plasma edge is called a high, or H, mode of confinement and was the first transport barrier observed (Wagner *et al.*, 1982).

ACKNOWLEDGMENTS

The author would like to thank many colleagues and friends who have answered questions, read drafts, and made helpful suggestions. Three physicists, who work with me at Columbia, deserve special thanks: Dmitry Maslovsky, who drew or placed in appropriate form all of the figures, David Lazanja, and Ronald Schmitt. Figure 5 was provided by the National Institute for Fusion Science in Japan, Fig. 6 by the Max-Planck-Institut für Plasmaphysik in Germany, Figs. 7 and 8 by the Princeton Plasma Physics Laboratory in the United States, and Fig. 9 by Edward Lazarus. This work was supported by Grant Nos. DE-FG02-95ER54333 and DE-FG02-03ER5496 from the U.S. Department of Energy.

APPENDIX: GENERAL COORDINATE SYSTEMS

Clever choices of coordinates are an important part of classical physics. Any three quantities, which are conventionally denoted by (x^1,x^2,x^3), can be used as coordinates if they are well-behaved functions of the Cartesian coordinates and if positions in Cartesian coordinates are well-behaved functions of (x^1,x^2,x^3). The superscripts on the x's number the coordinates and are not powers. The position in space associated with each coordinate point is defined by the transformation function $\vec{x}(x^1,x^2,x^3)$. The transformation function can be given in Cartesian coordinates as

$$\vec{x} = x(x^1,x^2,x^3)\hat{x} + y(x^1,x^2,x^3)\hat{y} + z(x^1,x^2,x^3)\hat{z}. \qquad \text{(A1)}$$

For example, cylindrical coordinates (R,φ,Z) are defined by $\vec{x}(R,\varphi,Z) = R\cos\varphi\hat{x} + R\sin\varphi\hat{y} + Z\hat{z}$.

Most people find it surprising that a coordinate transformation is defined by giving $\vec{x}(x^1,x^2,x^3)$ rather that $x^i(\vec{x})$ with the index i going from 1 to 3. The reason $\vec{x}(x^1,x^2,x^3)$ is given is that we want to convert functions of position, like the temperature $T(\vec{x})$, into functions of (x^1,x^2,x^3). Given $\vec{x}(x^1,x^2,x^3)$ one has $T(x^1,x^2,x^3) = T(\vec{x}(x^1,x^2,x^3))$.

The quantities (x^1,x^2,x^3) are valid coordinates only if the Cartesian coordinates (x,y,z) are well-behaved functions of (x^1,x^2,x^3). This implies the Jacobian

$$\mathcal{J} \equiv \frac{\partial\vec{x}}{\partial x^1}\cdot\left(\frac{\partial\vec{x}}{\partial x^2}\times\frac{\partial\vec{x}}{\partial x^3}\right) \qquad \text{(A2)}$$

cannot be infinite. The condition that the quantities (x^1,x^2,x^3) be well-behaved functions of the Cartesian coordinates implies that the Jacobian cannot be zero.

The gradients of the three coordinates $\vec{\nabla}x^i$ with the index $i = 1,2,3$, and the three tangent vectors $\partial\vec{x}/\partial x^i$ are related by the orthogonality relation,

$$\vec{\nabla}x^i\cdot\frac{\partial\vec{x}}{\partial x^j} = \delta^i_j. \qquad \text{(A3)}$$

This relation, which is fundamental to the whole theory of general coordinates, can be proven using the chain rule. First consider each coordinate to be a function of the Cartesian coordinates $x^i(x,y,z)$, so $\vec{\nabla}x^i = \partial x^i/\partial\vec{x} = (\partial x^i/\partial x)\hat{x} + \cdots$. The derivatives of the transformation equations, called tangent vectors, are $\partial\vec{x}/\partial x^j = (\partial x/\partial x^j)\hat{x} + (\partial y/\partial x^j)\hat{y} + (\partial z/\partial x^j)\hat{z}$. When one takes a dot product between the gradient of a coordinate and a tangent vector, one finds using the chain rule that $\vec{\nabla}x^i\cdot\partial\vec{x}/\partial x^j = \partial x^i/\partial x^j$. If $i=1$ and $j=2$, one has $\vec{\nabla}x^1\cdot\partial\vec{x}/\partial x^2 = (\partial x^1/\partial x^2)_{x^1,x^3}$, which is zero. In words, the derivative of x^1 with respect to x^2 while holding x^1 constant is zero. If one lets $i=1$ and $j=1$, one has $\vec{\nabla}x^1\cdot\partial\vec{x}/\partial x^1=(\partial x^1/\partial x^1)_{x^2,x^3}$, which is one.

The use of general coordinates is illustrated by the derivation of the equations for field lines. Field lines are defined by $d\vec{x}/d\tau=\vec{B}(\vec{x})$ where τ is a parameter that defines positions along the line. We want the trajectory of the field lines in general coordinates, which means we want the functions $x^i(\tau)$. Now $d\vec{x}/d\tau=\Sigma(\partial\vec{x}/\partial x^i)dx^i/d\tau$. Dotting both sides of the defining equation for field lines by the various coordinate gradients and using the orthogonality relation, one finds that

$$\frac{dx^i}{d\tau}=\vec{B}\cdot\vec{\nabla}x^i. \tag{A4}$$

If $\vec{B}\cdot\vec{\nabla}x^3$ is nonzero, one can use x^3 as the parameter that defines positions along the line. The chain rule implies $dx^1/dx^3=\vec{B}\cdot\vec{\nabla}x^1/\vec{B}\cdot\vec{\nabla}x^3$.

The orthogonality relation allows one to write the gradients of the coordinates in terms of the tangent vectors, for example,

$$\vec{\nabla}x^1=\frac{1}{\mathcal{J}}\frac{\partial\vec{x}}{\partial x^2}\times\frac{\partial\vec{x}}{\partial x^3}. \tag{A5}$$

This relation is called a dual relation and is important in practical calculations, for otherwise the calculation of a coordinate gradient would require a function inversion of $\vec{x}(x^1,x^2,x^3)$ to obtain x^1 as a function of the Cartesian coordinates (x,y,z). A vector in three dimensions can be expanded using any three independent vectors, so

$$\vec{\nabla}x^1=a_1\frac{\partial\vec{x}}{\partial x^2}\times\frac{\partial\vec{x}}{\partial x^3}+a_2\frac{\partial\vec{x}}{\partial x^3}\times\frac{\partial\vec{x}}{\partial x^1}+a_3\frac{\partial\vec{x}}{\partial x^1}\times\frac{\partial\vec{x}}{\partial x^2}. \tag{A6}$$

Dotting this equation with $\partial\vec{x}/\partial x^1$, one finds that $a_1=1/\mathcal{J}$. Dotting the equation with $\partial\vec{x}/\partial x^2$, one finds $a_2=0$. Similarly, one finds $a_3=0$.

Dual relations also exist that give the tangent vectors in terms of the gradients, for example,

$$\frac{\partial\vec{x}}{\partial x^1}=\mathcal{J}\vec{\nabla}x^2\times\vec{\nabla}x^3. \tag{A7}$$

These dual relations are derived in an analogous manner.

Given a transformation function $\vec{x}(x^1,x^2,x^3)$, one can expand any vector using the tangent vectors $\partial\vec{x}/\partial x^i$ as basis vectors. This expansion, $\vec{B}=\Sigma B^i(\partial\vec{x}/\partial x^i)$, is called the *contravariant representation* of the vector $\vec{B}$. The orthogonality relation implies the expansion coefficients are given by $B^i=\vec{B}\cdot\vec{\nabla}x^i$.

An arbitrary vector can also be expanded using the gradients of the coordinates as the expansion vectors, $\vec{B}=\Sigma B_i\vec{\nabla}x^i$, which is called the *covariant representation*. The expansion coefficients are given by $B_i=\vec{B}\cdot\partial\vec{x}/\partial x^i$. The dot product of two vectors is given by $\vec{A}\cdot\vec{B}=\Sigma A_iB^i=\Sigma A^iB_i$.

The covariant and contravariant representation of vectors have distinct roles in both differential and integral vector calculus. First consider the gradient of a scalar,

$$\vec{\nabla}f=\sum\frac{\partial f}{\partial x^i}\vec{\nabla}x^i, \tag{A8}$$

which is a covariant vector.

Next consider the curl of a vector. This is easy for a vector written in the covariant representation, which is the expansion using the coordinate gradients,

$$\vec{\nabla}\times\vec{B}=\sum\vec{\nabla}B_i\times\vec{\nabla}x^i. \tag{A9}$$

One can expand $\vec{\nabla}B_i$ in terms of the coordinate gradients and use the dual relations to show that

$$\vec{\nabla}\times\vec{B}=\frac{1}{\mathcal{J}}\sum\epsilon^{ijk}\partial_iB_j\frac{\partial\vec{x}}{\partial x^k}, \tag{A10}$$

where

$$\partial_i\equiv\frac{\partial}{\partial x^i} \tag{A11}$$

and ϵ^{ijk} is the contravariant fully antisymmetric tensor. This tensor is defined by $\epsilon^{1,2,3}=1$ with the sign changing if the order of two adjacent indices are changed. For example, $\epsilon^{2,1,3}=-1$. If two indices are identical, the fully antisymmetric tensor is zero, $\epsilon^{1,1,3}=0$. The curl of a covariant vector is a contravariant vector.

The divergence can be calculated using the contravariant vector. One uses the dual relations to write

$$\vec{B}=\frac{\mathcal{J}}{2}\sum\epsilon_{ijk}B^i\vec{\nabla}x^j\times\vec{\nabla}x^k, \tag{A12}$$

where the components of the covariant antisymmetric tensor ϵ_{ijk} have identical values to the components of the contravariant tensor ϵ^{ijk}. The divergence is simple because the divergence of cross gradients is zero, and $\vec{\nabla}\cdot(f\vec{v})=\vec{v}\cdot\vec{\nabla}f+f\vec{\nabla}\cdot\vec{v}$. Writing the Jacobian as $1/\mathcal{J}=\vec{\nabla}x^1\cdot(\vec{\nabla}x^2\times\vec{\nabla}x^3)$, which can be proven using the dual relations, one finds

$$\vec{\nabla}\cdot\vec{B}=\frac{1}{\mathcal{J}}\sum\frac{\partial\mathcal{J}B^i}{\partial x^i}. \tag{A13}$$

The curl can be taken only if a vector is in covariant form and the divergence can be taken only if a vector is in contravariant form. How can one convert a vector from one form to another? The covariant components are given by $B_i=\vec{B}\cdot(\partial\vec{x}/\partial x^i)$, which if $\vec{B}$ is known in contravariant form yields $B_i=\Sigma(\partial\vec{x}/\partial x^i)\cdot(\partial\vec{x}/\partial x^j)B^j$. The metric tensor is defined as

$$g_{ij}\equiv\frac{\partial\vec{x}}{\partial x^i}\cdot\frac{\partial\vec{x}}{\partial x^j}, \tag{A14}$$

so $B_i=\Sigma g_{ij}B^j$.

The name metric tensor comes from its role in determining distances between two coordinate points. The vector between two adjacent coordinate points is $\delta\vec{x} = \Sigma(\partial\vec{x}/\partial x^i)\delta x^i$. The square of the distance is $(\delta\vec{x})^2 = \Sigma g_{ij}\delta x^i \delta x^j$.

A vector in covariant form can be rewritten in contravariant form using $B^i = \Sigma g^{ij} B_j$ where $g^{ij} \equiv \vec{\nabla} x^i \cdot \vec{\nabla} x^j$. Using the dual relations one can show that g^{ij} is the matrix inverse of g_{ij}.

There are three types of integrals: line, area, and volume. A line integral is performed along a curve, which means one of the three coordinates is varied while two are held constant. Let us assume the first coordinate is varied, then $d\vec{x} = (\partial\vec{x}/\partial x^1)dx^1$ and $\int \vec{B}\cdot d\vec{x} = \int \vec{B}\cdot(\partial\vec{x}/\partial x^1)dx^1$, which can also be written as $\int \vec{B}\cdot d\vec{x} = \int B_1 dx^1$. In other words, a line integral is an ordinary integral of a covariant coefficient.

The only difficulty in area, or surface, integrals is becoming comfortable with the definition of the area element. A surface is defined by holding one coordinate constant, say x^1, and varying the other two. The area element is then

$$d\vec{a}^1 \equiv \frac{\partial\vec{x}}{\partial x^2} \times \frac{\partial\vec{x}}{\partial x^3} dx^2 dx^3 = \vec{\nabla} x^1 \mathcal{J} dx^2 dx^3, \tag{A15}$$

where a dual relation was used to obtain the second form. An area integral is then the double integral $\int B\cdot d\vec{a} = \int B^1 \mathcal{J} dx^2 dx^3$, where $B^1 = \vec{B}\cdot\vec{\nabla} x^1$ is the contravariant coefficient.

The volume element is obtained by dotting the area element with the distance across the surface, $d^3x \equiv (\partial\vec{x}/\partial x^1)dx^1 \cdot d\vec{a}^1$, which can be written as $d^3x = \mathcal{J} dx^1 dx^2 dx^3$. In other words, the integral of a function f over a volume is the triple integral $\int f \mathcal{J} dx^1 dx^2 dx^3$.

The expression for the time derivative of a general covariant vector, namely, the vector potential, is important for the theory of the evolution of magnetic fields. Let $\vec{A} = \psi_t \vec{\nabla}(\theta/2\pi) - \psi_p \vec{\nabla}(\varphi/2\pi) + \vec{\nabla} g$. The time derivative at a fixed spatial point $\vec{x}$ is

$$\left(\frac{\partial\vec{A}}{\partial t}\right)_{\vec{x}} = \left(\frac{\partial\psi_t}{\partial t}\right)_{\vec{x}} \vec{\nabla}\frac{\theta}{2\pi} - \left(\frac{\partial\theta/2\pi}{\partial t}\right)_{\vec{x}} \vec{\nabla}\psi_t + \cdots + \vec{\nabla} s, \tag{A16}$$

where the $\cdots$ stands for terms involving ψ_p and φ, which are of the same form as those for ψ_t and θ. The quantity

$$s \equiv \left(\frac{\partial g}{\partial t}\right)_{\vec{x}} + \psi_t \left(\frac{\partial\theta/2\pi}{\partial t}\right)_{\vec{x}} - \psi_p \left(\frac{\partial\varphi/2\pi}{\partial t}\right)_{\vec{x}}. \tag{A17}$$

Time derivatives holding the coordinates $(\psi_t, \theta, \varphi)$ fixed, which are denoted by a subscript c, are related to time derivatives holding the spatial position $\vec{x}$ fixed by the chain rule,

$$\left(\frac{\partial f}{\partial t}\right)_c = \left(\frac{\partial f}{\partial t}\right)_{\vec{x}} + \left(\frac{\partial\vec{x}}{\partial t}\right)_c \cdot \vec{\nabla} f. \tag{A18}$$

The velocity of the $(\psi_t, \theta, \varphi)$ coordinates through space is $\vec{u} \equiv (\partial\vec{x}/\partial t)_c$, so the time derivative of the coordinates has the form $(\partial\psi_t/\partial t)_{\vec{x}} = -\vec{u}\cdot\vec{\nabla}\psi_t$, while the derivative of ψ_p has the form $(\partial\psi_p/\partial t)_{\vec{x}} = (\partial\psi_p/\partial t)_c - \vec{u}\cdot\vec{\nabla}\psi_p$. Let $\vec{B} \equiv \vec{\nabla}\times\vec{A}$, then $\vec{u}\times\vec{B} = [\vec{u}\cdot\vec{\nabla}(\theta/2\pi)]\vec{\nabla}\psi_t - (\vec{u}\cdot\vec{\nabla}\psi_t)\vec{\nabla}(\theta/2\pi) + \cdots$, and the time derivative of $\vec{A}$ can be written

$$\left(\frac{\partial\vec{A}}{\partial t}\right)_{\vec{x}} = -\left(\frac{\partial\psi_p}{\partial t}\right)_c \vec{\nabla}\frac{\varphi}{2\pi} + \vec{u}\times\vec{B} + \vec{\nabla} s. \tag{A19}$$

The expressions $\vec{u}\cdot\vec{\nabla}\theta = -(\partial\theta/\partial t)_{\vec{x}}$ and $\vec{u}\cdot\vec{\nabla}\varphi = -(\partial\varphi/\partial t)_{\vec{x}}$ plus Eq. (A17) imply

$$\vec{A}\cdot\vec{u} = -s + (\partial g/\partial t)_c. \tag{A20}$$

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