Problem.

Let x, y, z be real numbers such that $x+y+z=0$. Show that

$$6(x^3+y^3+z^3)^2 \le (x^2+y^2+z^2)^3.$$

Solution.

Let $f(w)=(w-x)(w-y)(w-z)=w^3+pw+q$ be a cubic expression.

Note that

1. $x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-xz)=0$, which implies $x^3+y^3+z^3=3xyz=-3q$.
2. $x^2+y^2+z^2=(x+y+z)^2-2(xy+yz+zx)=-2p$

Now, the desired inequality becomes

$$6(-3q)^2 \le (-2p)^3$$
$$27q^2+4p^3 \le 0$$
$$D_3 \ge 0$$

which is true since $f(w)$ has 3 real roots x, y, z.

Remark.

For (1.), there is an alternative approach to obtain $x^3+y^3+z^3=-3q$. It goes as follows:

$$f(x)+f(y)+f(z)=0 \Rightarrow (x^3+y^3+z^3)+p(x+y+z)+3q=0$$

It follows that $x^3+y^3+z^3=-3q$.