THE ORBITAL TOWER

Let us consider a geosynchronous satellite that is in circular orbit around the earth rotating along with it at the same angular velocity. The satellite therefore always stays over the same point on earth, say somewhere over the equator. Let us first calculate the geosynchronous orbit. Since the geosynchronous orbit for earth is several times the radius of the earth, we cannot take the acceleration due to gravity as a constant through this distance. From the familiar inverse square law of gravitation, the acceleration due to gravity g(r) at any distance r from the center of the earth is given by

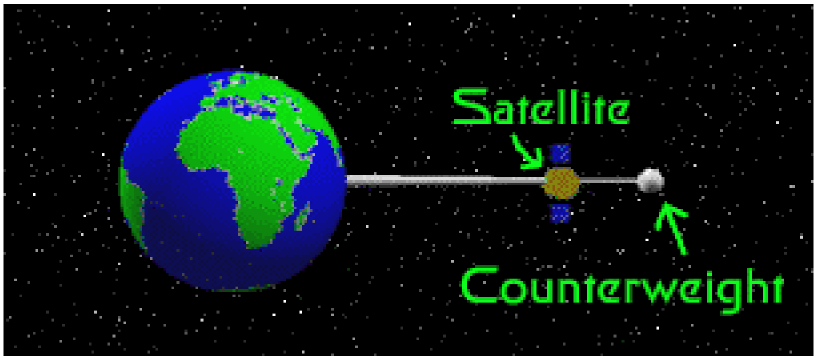
$$g(r) = g\frac{r_e^2}{r^2}$$

where $g=9.81ms^{-2}$ is the acceleration due to gravity on earth's surface at $r = r_e$.
Let the radial distance to the geosynchronous orbit from the center of the earth be $r = \lambda$.
The angular velocity of earth/satellite = ω = 1rev/day = $2\pi/(24*60*60)$ rad/sec=$7.27\text{x}10^{-5}$ rad/sec.

Using Newton's laws in terms of a reference frame that is rotating with the earth (and hence the satellite) we recognize that the forces on the satellite are the weight acting toward the earth and the centrifugal force acting outward, and these balance out:

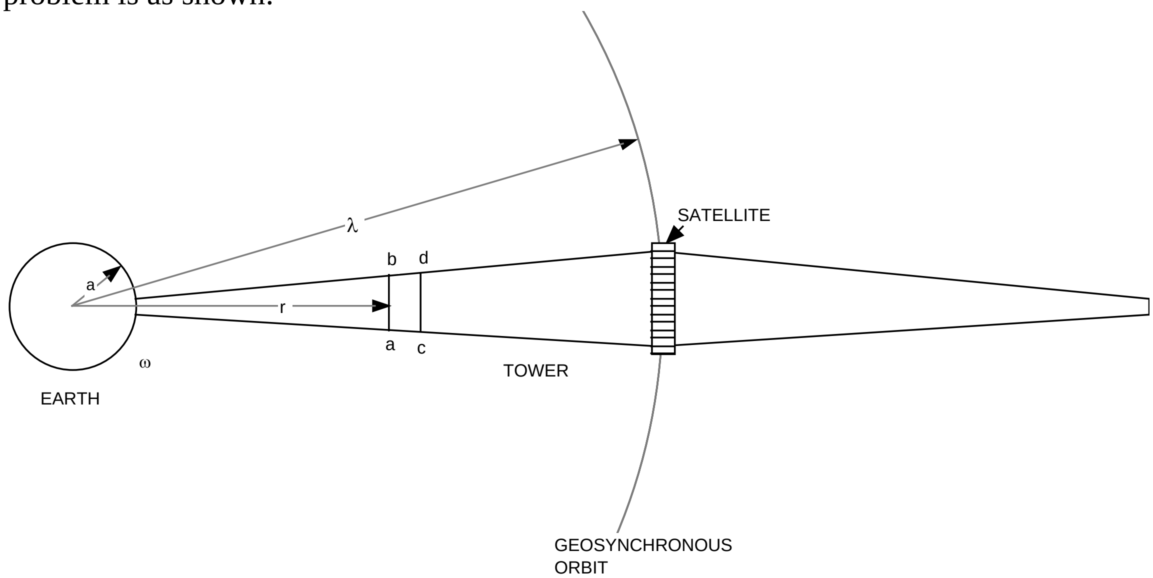
$$mg\frac{r_e^2}{\lambda^2} = m\frac{(\omega\lambda)^2}{\lambda} \Rightarrow \lambda = \left\{\frac{gr_e^2}{\omega^2}\right\}^{1/3} = 42\text{x}10^6\,\text{m}$$

The idea of the orbital tower can now be explained very simply. A geosynchronous satellite always stays over the same point on earth since it rotates along with the earth at the same angular speed. So why not build a bridge across connecting the satellite to the earth, and possibly put in an elevator to ferry people and satellites to space? We might even build condominiums with the ultimate view (at the lower elevations, of course), and throw in a restaurant or two, and make a killing...

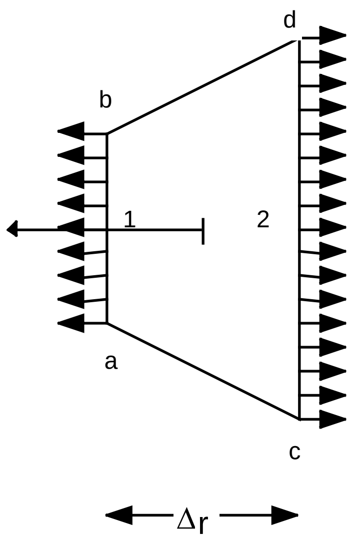


For reasons that will become apparent shortly, it turns out that it is best to build the bridge starting from the geosynchronous satellite on down to the earth. This means that not only do you build the bridge toward earth, you must also have a counterbalancing bridge going out to space in order to keep the satellite in geosynchronous orbit. (If we do not do this, then the center of the mass of the satellite will shift as we build the tower on one side only, and then the satellite will tumble away from the geosynchronous orbit! Besides, the counterbalancing bridge has its uses too.)

Let us analyze the mechanics of the orbital tower in some depth. The geometry of the problem is as shown:



Parameters:
Radius of earth, $r_e = 6.4\text{x}10^6$m
Acceleration due to gravity on earth's surface, $g = 9.81\text{ms}^{-2}$
Angular velocity of earth/tower/satellite = $\omega = 2\pi/(24*60*60)$ rad/sec.
Radius of geosynchronous orbit, $\lambda = 42\text{x}10^6$m.
Density of orbital tower material = ρ
Maximum allowable tensile stress in orbital tower material = σ_{max}.



Let r be the distance from the center of the earth to any point on the tower axis. The cross-sectional area of the tower is A(r), which can vary with position. Thus $A(r_e)$ is the area of the tower on the surface of the earth and $A(\lambda)$ is the area of the tower at the geosynchronous orbit.

Consider the free-body diagram of an element abcd of the tower at a distance r shown:

The forces acting on this element are:
(i) the weight of the element acting inward toward the earth which is equal to:

weight = [mass][acceleration due to gravity at location of the element]

$$\mathbf{W} = \rho \left[\frac{A_1 + A_2}{2}\right] \Delta r \left\{g\frac{r_e^2}{r^2}\right\} \mathbf{e}_n$$

where $\mathbf{e}_n$ is a unit vector pointing towards the earth.

(ii) In addition, we have the centrifugal force acting outward from the earth equal to:

centrifugal force = [mass][(angular velocity)^2.radial distance from center of rotation]

$$\mathbf{F}_C = -\left[\rho\left\{\frac{A_1 + A_2}{2}\right\} \Delta r\right] \left[\omega^2 \; r\right] \mathbf{e}_n$$

Then we have the internal forces acting over the two cut faces of the element. The stresses over the two cross-sections (which are of different areas) could in general be of different magnitude. What we are interested in here is to design the tower using the least amount of material as possible. This can be best done by letting every cross-section be stressed to the maximum extent possible. That is the stress is σ_{max} on all cross-sections. Then the internal forces on each face can be expressed as:

$$\mathbf{F_1} = +\sigma_{\max} \; A_1 \; \mathbf{e}_n$$
$$\mathbf{F_2} = -\sigma_{\max} \; A_2 \; \mathbf{e}_n$$

From Newton's law, we have:

$$\mathbf{W} + \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_C = 0$$

$$\left[\rho \left\{\frac{A_1 + A_2}{2}\right\} \Delta r \left\{g\frac{r_e^2}{r^2}\right\} + \sigma_{\max} A_1 - \sigma_{\max} A_2\right] - \left[\rho\left\{\frac{A_1 + A_2}{2}\right\} \Delta r\right] \left[\omega^2 \; r\right] = 0$$

Rearranging terms and dividing by Δr, we get

$$\sigma_{\max}\left[\frac{A_2 - A_1}{\Delta r}\right] = \rho\left\{\frac{A_1 + A_2}{2}\right\}\left[g\frac{r_e^2}{r^2} - \omega^2 r\right]$$

Letting Δr-->0, i.e., shrinking the element, we see that faces 1 and 2 approach each other, and so A_1 and A_2 approach the same value A. Furthermore $\{A_2 - A_1\}/\Delta r \to dA/dr$, ie becomes the derivative of the A with respect to r.

$$\frac{dA}{dr} = \frac{\rho\ A}{\sigma_{\max}}\left[g\frac{r_e^2}{r^2} - \omega^2 r\right]$$

This is a differential equation for the cross-sectional area of the tower, but it is a trivial differential equation that you know how to integrate. Rearranging above equation, and integrating:

$$\int\frac{dA}{A} = \int\frac{\rho}{\sigma_{\max}}\left[g\frac{r_e^2}{r^2} - \omega^2 r\right]dr$$

Integrating:

$$\ln\ A(r) = \frac{\rho}{\sigma_{\max}}\left[-\ g\frac{r_e^2}{r} - \frac{\omega^2\ r^2}{2}\right] + \text{constant}$$

Exponentiating both sides:

$$A(r) = (\text{constant})\ \exp\left[-\frac{\rho}{\sigma_{\max}}\left\{\frac{gr_e^2}{r} + \frac{\omega^2 r^2}{2}\right\}\right]$$

The constant of integration can be determined from choosing the area at some point, say at the geosynchronous satellite A(λ). So,

$$A(r) = A(\lambda)\ \exp\left[-\frac{\rho}{\sigma_{\max}}\left\{gr_e^2\left(\frac{1}{\lambda} - \frac{1}{r}\right) + \frac{\omega^2}{2}\left(\lambda^2 - r^2\right)\right\}\right]$$

Thus the ratio of the area of the tower at the surface of the earth to that at the geosynchronous orbit is:

$$\frac{A(r_e)}{A(\lambda)} = \exp\left[-\frac{\rho}{\sigma_{\max}}\left(4.86\text{x}10^7\right)\right]$$

Note that the tower is much thinner at the surface than at the geosynchronous orbit. This is the reason we want to build it from the satellite down to the earth's surface.

Exercises (these are just for fun)

- Currently the major technical problem is that there are no known materials that are strong enough to withstand the stresses that will develop in a realistic tower with a reasonable cross-sectional area (say $1m^2$ at the surface of the earth). The good news is that these stresses are well within the *theoretical* limit of strength of many materials (which is about two orders of magnitude larger than currently technically achievable).Consider some typical materials such as graphite, diamond, steel, aluminum etc and work out the above ratio. Assume that the theoretical maximum allowable stress in tension is a tenth of the Young's modulus of these materials. Do you think such a tower can be built on earth in the near future? How about on Mars, or on the moon, which are much smaller than the earth?
- What is the volume of material used in constructing the tower, and how much is the cost (at today's prices) of just the material alone for some of the materials typically used in construction today.

References:

- Isaacs, A.C. Vine, H. Bradner and G.E. Bachus, (1966), `Satellite Elongation into a True Sky-Hook,' Science, vol.151, 11th February, 1966.
- Arthur C. Clarke, (1978), `Fountains of Paradise,' (fiction), Ballantine Books, New York.
- Check out the web (http://liftoff.msfc.nasa.gov/academy/tether/spacetowers.html) for variations on the above theme