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On the Valuation of Tax-Advantaged Retirement Accounts

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Abstract

Individuals who invest for retirement often hold investment assets in a variety of different types of accounts, some tax-sheltered and some not. The different tax treatments applied to different types of tax-advantaged accounts complicate the task of calculating a single consistent measure of total wealth accumulation. This paper derives models for determining the current after-tax dollar equivalent of assets held in common tax-advantaged retirement accounts so that a prospective retiree can more easily calculate a clear and understandable measure of total wealth accumulation. The results indicate that under certain conditions pre-tax dollars held in a tax-deferred retirement account can be more valuable than an equal number of after-tax dollars, if they are to be used to fund future consumption expenditures. The models developed permit the analysis of many retirement planning decisions using capital investment theory.

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1. Introduction

An important question for individuals approaching retirement is whether their accumulated wealth is sufficient to provide the desired level of retirement income. The usual first step in answering this question is to identify and measure those assets that will be used to fund retirement benefits. Such assets may include stock portfolios, bonds, certificates of

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deposit, mutual funds and other similar assets that have been funded with after-tax dollars, as well as any tax-advantaged retirement accounts.

Typically, the total accumulation of the prospective retiree is measured simply by summing the reported or observed market values of the assets held in the various accounts. For example, Li, Montalto, and Geistfeld (1966); Poterba, Venti, and Wise (1994) and Yuh, Phillips, and Hanna (1998) use this approach in empirical studies of accumulated retirement assets held by elderly Americans. This is a simple and convenient process since accumulation values are often reported monthly or quarterly by fund managers and no other method of valuation is readily apparent. There is, however, a fundamental flaw in this approach. Tax-deferred retirement accounts are typically funded with pre-tax dollars, while ordinary taxable investment accounts are funded with after-tax dollars. Adding together after-tax and pre-tax dollars without any adjustment is a little like adding apples and oranges. The resulting measures do not provide a clear picture of the level of future consumption the various accounts will provide, thus limiting the usefulness and accuracy of retirement planning. Further, incomplete measures of value may bias studies of wealth accumulation, such as those mentioned above.

It may seem strange to talk about valuing retirement accounts that are denominated in dollars. Doesn't the market value of the assets held in a retirement account accurately measure its value? Unfortunately, the answer is no. Not all dollars have the same value or the same ability to fund future consumption. We generally conceive of the dollar value of an asset in terms of after-tax dollars, that is, dollars that can be spent to acquire a dollar's worth of goods and services or may be invested to earn ordinary taxable income. Admittedly, the choice of after-tax dollars as the appropriate measure of wealth is somewhat arbitrary, but these are the dollars that we deal with most of the time and that we understand the best. In this paper, we will refer to them as ordinary dollars or after-tax dollars. Retirement accounts, however, are denominated in a different kind of dollar. For example, 401(k)s, non-deductible IRAs and Roth IRAs are each subject to different tax treatment upon withdrawal, which affects the level of retirement consumption that each will support. As many individuals accumulate funds for retirement in a variety of different types of accounts, some tax-advantaged and some not, clear measurement of accumulated wealth can't be obtained by simply adding together the pre-tax market values of tax-advantaged and ordinary taxable accounts. To accurately measure accumulated wealth, recognition must be given to the tax treatment for income and withdrawals that apply to each investment account.

The purpose of this paper is to derive models for determining the current after-tax dollar equivalent of assets held in tax-advantaged retirement accounts. Three different types of accounts are considered:

1. Tax-advantaged accounts with the characteristic that: (1) contributions are made with pre-tax dollars, (2) taxes on investment income and gains are deferred until withdrawn, and (3) withdrawals are fully taxable at ordinary tax rates at the time of withdrawal. This category includes a wide variety of retirement accounts, including deductible IRAs, 401(k)s, 403(b)s and other similar tax-deferred accounts.
2. Tax-advantaged accounts with the characteristic that: (1) at least some contributions are made with after-tax dollars, (2) taxes on investment income and gains are deferred

until withdrawn, and (3) that portion of any withdrawal representing a return of contributions made with after-tax dollars is free from further taxation, while any withdrawal traceable to pre-tax contributions, as well as any income earned on the account, is fully taxable at ordinary rates. The sole example is the traditional IRA that is funded partially or fully with non-deductible contributions. We will refer to such accounts simply as non-deductible IRAs.

3. Tax-advantaged accounts with the characteristic that: (1) contributions are made with after-tax dollars, (2) income earned on the account is free from current taxation, and (3) qualifying post-retirement withdrawals are also free from taxation. The sole example is the Roth IRA, which was created by the Tax Relief Act of 1997.

The characteristics and tax consequences of each type of account are described in detail in Internal Revenue Service Publication 590 (2001) and summarized in Crain and Austin (1997).

Valuation models are derived for each kind of account so as to measure the after-tax dollar equivalent of each tax-advantaged dollar. The resulting valuation models can be used to determine the number of ordinary after-tax dollars that would be required to fund the same level of post-retirement consumption as a given tax-advantaged account. This makes it possible to convert the most common types of tax-advantaged accounts into their after-tax dollar equivalent and determine a more accurate and consistent measure of an individual's total wealth accumulation. Such measures should improve the effectiveness of retirement planning, permit more accurate studies of wealth accumulation by prospective retirees, and facilitate a wide range of decisions that require value estimates for tax-advantaged accounts. An interesting finding is that current pre-tax dollars in a tax-deferred account are often more valuable (i.e., capable of funding a higher level of future consumption) than an equal number of after-tax dollars.

2. Previous valuation attempts

The valuation of tax-advantaged retirement accounts is not a topic that has received considerable attention in the literature. The only available model is suggested by Reichenstein (1998, pp. 196–9). To simplify the analysis, Reichenstein assumes a flat tax rate absent deductions and exemptions and concludes that dollars in a tax-deferred retirement account can be converted to after-tax dollars simply by multiplying them by $(1 - \tau)$, where τ is the expected tax rate in retirement. For a retiree with a 28% tax rate, a dollar in a deductible IRA or other similar tax-deferred account would be worth $0.72. This approach is incomplete, however, and provides few useful insights for a real retiree.

Reichenstein's calculates the value of a tax-sheltered account as the discounted present value of its future after-tax cash flows. Although he recognizes the effect of taxes on withdrawals, his model fails to recognize how taxes affect the retiree's opportunity cost or discount rate. The result is to discount *after-tax* cash flows using a *pre-tax* discount rate. This leads to a downward bias in the calculated present value of future withdrawals. Therefore, Reichenstein's approach produces an accurate valuation only if the entire tax-deferred balance is to be immediately withdrawn, in which case no discounting is required. As we will see below, full recognition of taxes may, under some circumstances, make a current tax-deferred dollar more valuable than a current after-tax dollar, which is a totally impossible outcome according to Reichenstein's model.

3. Definitions and assumptions

What is meant by the after-tax value of a tax-advantages retirement account given that law prohibits secondary markets for such accounts? As a tax-advantaged retirement account is used to fund future consumption after retirement, we define its after-tax value as number of ordinary after-tax dollars (invested to earn ordinary taxable income) that would be necessary to provide the same level of future consumption as the tax-advantaged account, thus prompting the owner to be indifferent between the two.

This definition suggests an indirect approach for estimating the after-tax value of a tax-advantaged account. The first step is to determine the future stream of after-tax cash flows a given tax-advantaged account is expected to produce for the retiree. The number of current after-tax dollars (invested to earn ordinary taxable income) that would be required to produce the same stream of future cash flows is then used to measure the after-tax value of the tax-advantaged account. Although this approach does not produce a market value in the conventional sense, since no market exists, it does establish an after-tax value of an account to a prospective retiree, so that accumulated wealth can be measured more accurately and on a consistent basis.

In practice, the fact that two different investment accounts are expected to generate the same future stream of after-tax dollars under a given set of conditions doesn't necessarily mean that a retiree will be totally indifferent between them. It is possible that one account may provide greater flexibility of action or reduced risk of adverse outcome under certain circumstances, thus prompting the retiree to favor one account over the other. Indeed, each type of tax-advantaged and ordinary taxable account has advantages and disadvantages. As these differences are often difficult or impossible to identify and measure, we ignore them for the present and focus only on the ability to generate a stream of future after-tax cash flows under a given set of conditions.

To simplify the analysis and focus on the principal determinants of value, we make the following simplifying assumptions throughout the remainder of the paper:

1. Both tax-deferred and after-tax dollars earn the same geometric average pre-tax annual rate of return, k, until the funds are withdrawn. It is assumed that k is positive.
2. All taxable withdrawals from tax-deferred accounts, as well as investment income and gains from outright taxable investments, are subject to the ordinary tax rate, τ, which is known and constant.
3. We initially assume that tax-advantaged accounts are to be liquidated through a single withdrawal during retirement at a known future date. We then expand the resulting models to accommodate a level annuity of future after-tax withdrawals.
4. We do not consider the consequences of taxes on excessive accumulations and

distributions, estate taxes, and penalties for early withdrawals. Although the vast majority of retirees do not experience such taxes and penalties, they can have a significant impact on the valuation of retirement accounts for some retirees. See Streer, Knight, and Clements (1997) for a discussion of these and related issues.

5. We take the withdrawal plans of the retiree as given and do not attempt to determine optimal withdrawal strategies. Those interested in the determination of optimal strategies should see Ragsdale, Seila, and Little (1994).
6. Because the objective is to estimate the after-tax value of *current* tax-advantaged retirement accounts, no recognition is given to the possibility of future contributions or how they would affect the future value of tax-advantaged accounts.

After developing valuation models under the foregoing assumptions, we construct tables of valuation factors for a representative range of circumstances and explore their implications for retirement planning.

4. Valuation of fully deductible retirement accounts

Several common tax-deferred retirement accounts have the characteristic that: (1) contributions made to the account (within allowable limits) are fully deductible, (2) income earned on the account is tax-deferred, and (3) all qualifying withdrawals are fully taxable at ordinary tax rates. Such accounts include 401(k)s, 403(b)s, and traditional fully deductible IRAs. We will refer to such accounts simply as fully deductible accounts.

We first develop a valuation model for such accounts under the simplifying assumption that they are to be liquidated through a single future withdrawal. We then expand the model to consider the case where liquidation occurs through an annuity of future withdrawals.

4.1. Liquidation through a single future withdrawal

Let S_0 be the market value of assets an individual has in a fully deductible account at time $t = 0$. For simplicity, assume that the outstanding balance in the account earns a fixed annual rate of return of k per year and that all of the funds are withdrawn at the end of n years. Let W_n^{fd} be the largest after-tax withdrawal possible from a fully deductible account at time $t = n$ given the current size of the account, S_0, and the rate of return, k. The value of W_n^{fd} is calculated as

$$W_n^{fd} = S_0(1 + k)^n(1 - \tau). \tag{1}$$

To determine the current after-tax equivalent of S_0, we must find the current sum of after-tax dollars that will produce the same future after-tax withdrawal as the tax-deferred account. Let A_0 be an amount of ***after-tax*** dollars in an ordinary taxable investment account that earns the same average annual rate of return of k. We define A_0 to be the current after-tax value of S_0 if and only if A_0 will produce the same after-tax withdrawal at time $t = n$ as S_0.

The problem then is to determine the size of A_0 that is necessary to produce an after-tax withdrawal of W_n^{fd}. Assume that A_0 is invested to earn ordinary income and that taxes on the income are paid annually based upon a constant tax rate of τ. At the end of each year (after paying taxes), the balance in the account would consist entirely of after-tax dollars and could be fully withdrawn without any additional taxes. At the end of year 1, after paying taxes, the potential withdrawal from an account funded with after-tax dollars, W_1^A, is calculated as

$$W_1^A = A_0 + kA_0 - kA_0\tau. \tag{2}$$

The first term on the right of the equal sign is simply the after-tax balance at the beginning of the period. The second term is the taxable income earned during the period, and the third term is the tax to be paid on the income. Rearranging the terms of Eq. (2) results in:

$$W_1^A = A_0[1 + k(1 - \tau)]. \tag{3}$$

The terms on the right in brackets in Eq. (3) represents 1 plus the after-tax rate of return. Each year, the after-tax accumulation will grow by this factor. At the end of n years, the after tax withdrawal possible from an account funded with an initial balance of A_0 after-tax dollars can be calculated as:

$$W_n^A = A_0[1 + k(1 - \tau)]^n. \tag{4}$$

The value of A_0 that corresponds to any given S_0 can be found by setting the right hand sides of Eq. (1) and (4) equal to each other and solving for A_0, which produces:

$$A_0 = S_0 \frac{(1 + k)^n(1 - \tau)}{[1 + k(1 - \tau)]^n} \tag{5}$$

If desired, Eq. (5), as well as the other valuation models presented later, can be viewed as a traditional discounted cash flow valuation models. The numerator on the right of Eq. (5) measures the future after-tax cash receipt and the denominator reflects the traditional discount factor, with a discount rate equal to the after-tax rate of return on an outright taxable investment, $k(1\text{-}\tau)$.

Eq. (5) can be modified to produce a relative valuation factor, VF_n, by solving for A_0/S_0. The variable VF_n measures the after-tax dollar equivalent of each tax-deferred dollar. The resulting valuation factor, given by

$$VF_n = \frac{A_0}{S_0} = \frac{(1 + k)^n(1 - \tau)}{[1 + k(1 - \tau)]^n}, \tag{6}$$

is independent of the size of S_0.

From Eq. (6), the after-tax value of a tax-deferred dollar is dependent upon three variables: (1) the investment rate of return (k), (2) the length of time before the tax-deferred dollar is to be withdrawn (n), and (3) and the constant tax rate on ordinary income (τ).

In the case that $n = 0$ (i.e., the tax-deferred account is to be immediately liquidated), the after-tax value of a tax-deferred dollar reduces to \$1(1- τ), which is the well-recognized result found by Reichenstein (1998). It is intuitively clear that after-tax dollars are worth more than tax-deferred dollars if withdrawal is to occur immediately. However, if k and τ are both positive, then the after-tax value of tax-deferred dollars increases monotonically with n.

For sufficiently large values of n, current tax-deferred dollars become more valuable than after-tax dollars in terms of their ability to fund future consumption, which is a totally impossible outcome under Reichenstein's approach. The reason for this is clear. Continued investment favors tax-deferred dollars (relative to after-tax dollars) since income earned is not taxed until withdrawal, thus permitting more rapid accumulation. The longer the time until the funds are to be withdrawn, the greater is the advantage to tax-deferred dollars.

The current after-tax value of tax-deferred dollars is also dependent on the average annual rate of return, k, earned until the time of withdrawal, and the tax rate, τ. Because the advantage of tax-deferred accounts is that taxes on income are deferred, this advantage is comparatively small for investors who have low tax rates or who earn low rates of return on their accounts. Indeed, the advantages of tax deferment disappear entirely for investors who earn zero rates of return or who have zero tax rates. Conversely, the advantage of tax deferment is larger for investors who earn higher rates of return or face higher tax rates, since the amount of taxes deferred is correspondingly greater. Thus, tax-deferred dollars are worth more (relative to after-tax dollars) to individuals with higher tax rates than to individuals with lower rates. Likewise, their value is greater for investors who earn higher average rates of return (perhaps from undertaking riskier investments) than for more conservative investors who earn lower average rates of return.

4.2. A level annuity of withdrawals

It is clear that most retirees do not liquidate their retirement accounts with a single withdrawal, but rather, make periodic withdrawals over the life of their retirement. The valuation procedure suggested in the previous section can be modified to accommodate multiple withdrawals, although the process is can be somewhat cumbersome in some cases. In this section, we consider the special case where the prospective retiree anticipates liquidating his retirement account through a future annuity of equal after-tax withdrawals.

Conceptually, the valuation of tax-deferred accounts that are to be liquidated through an annuity of withdrawals involves two steps. First, solve for the largest possible annual after-tax withdrawal, W, which can be funded by the tax-deferred account, S_0, given the desired starting date and length of the annuity. Second, determine the number of current after-tax dollars (invested in ordinary taxable assets) what would be required to fund the same annuity of after-tax withdrawals. Relative valuation factors are then obtained by dividing the after-tax amount found in step two by the tax-deferred amount, S_0. As we shall see below, it is not actually necessary to determine the size of the annual withdrawal, W, because this term drops out in the process of solving for relative valuation factors.

Assume that a prospective retiree has S_0 in a fully deductible tax-deferred account that will be used to fund a future annuity of equal after-tax withdrawals. The annuity is expected to begin at the end of n years and continue for a total of m years. Also, let W be the largest possible annual after-tax withdrawal given S_0, n and m. The relationship between S_0 and the annuity of withdrawals can be identified by first determining the amount of S_0 that is required to find each individual withdrawal. From Eq. (1), the number of current tax-deferred dollars needed to fund the *single* withdrawal expected at the end of year i ($n \leq i \leq n + m - 1$) is given by

$$S_O^i = \frac{W}{(1+k)^i(1-\tau)}. \tag{7}$$

Because we know that $S_O = \sum_{i=n}^{n+m-1} S_O^i$, the number of current tax-deferred dollars needed to fund the entire annuity must be given by

$$S_O = \sum_{i=n}^{n+m-1} S_O^i = \sum_{i=n}^{n+m-1} \frac{W}{(1+k)^i(1-\tau)}. \tag{8}$$

If desired, Eq. (8) can be solved for the annual after-tax withdrawal, W, that corresponds to any combination of S_O, n, and m.

The next step is to determine the number of current after-tax dollars (invested in a non-tax deferred account) that would be required to fund the same annuity of future after-tax withdrawals. Using the same approach describe above, this amount is given by

$$A_O = \sum_{i=n}^{n+m-1} \frac{W}{[1+k(1-\tau)]^i}. \tag{9}$$

Relative valuation factors assuming an m-year annuity of withdrawals that begins at the end of n years are then obtained as

$$VF_n^m = \frac{A_O}{S_O} = \frac{\sum_{i=n}^{n+m-1} 1/[1+k(1-\tau)]^i}{\sum_{i=n}^{n+m-1} 1/(1+k)^i(1-\tau)}. \tag{10}$$

Note that the size of the withdrawal, W, drops out of Eq. (10), reflecting that relative valuation factors are independent of the size of the tax-deferred account, S_O. (This holds true only under our assumption of a flat tax rate.) As should be expected, Eq. (10) reduces to Eq. (6) in the case where the length of the annuity, m, is one (i.e., when the account is to be liquidated through a single withdrawal).

4.3. Some illustrative valuation factors

Table 1 reports illustrative valuation factors for a fully deductible tax-deferred account that is to be liquidated through a single withdrawal. The valuation factors are calculated using Eq. (6) for different combinations of values for the number of years until the withdrawal, the annual pre-tax rate of return earned until withdrawal, and the tax rate.

The valuation factors of Table 1 measure the current after-tax dollar equivalent of \$1 in a fully deductible tax-deferred account and can be used to calculate the after-tax dollar equivalent of a tax-deferred account of any given size. For example, consider an individual with \$100,000 in a 401(k) account (a fully deductible account). If the individual expects to liquidate the account with a single withdrawal at the end of thirty years, expects to earn a pre-tax rate of return of 8% on the account assets until withdrawn and has a 30% tax rate, then the tax-deferred account is equivalent to \$100,000(1.3737) = \$137,370 in after-tax

Table 1
After-tax dollar value of $1 in a fully deductible tax deferred account - single withdrawal

	Years until single withdrawal			
	0	10	20	30
4% annual pre-tax rate of return				
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	0.9000	0.9354	0.9721	1.0103
Tax rate = 0.20	0.8000	0.8642	0.9336	1.0086
Tax rate = 0.30	0.7000	0.7861	0.8829	0.9915
8% annual pre-tax rate of return				
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	0.9000	0.9695	1.0443	1.1249
Tax rate = 0.20	0.8000	0.9288	1.0783	1.2519
Tax rate = 0.30	0.7000	0.8764	1.0972	1.3737
12% annual pre-tax rate of return				
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	0.9000	1.0024	1.1164	1.2433
Tax rate = 0.20	0.8000	0.9935	1.2338	1.5322
Tax rate = 0.30	0.7000	0.9705	1.3455	1.8653

dollars. That is, $137,370 of after-tax dollars in an ordinary taxable investment account would be required to provide the same level of consumption at the end of 30 years as the $100,000 in the 401(k) account.

The valuation factors of Table 1 highlight a number of important relationships regarding tax-deferred accounts. For any given tax rate, the after-tax value of a tax-deferred dollar increases with the length of time until withdrawal and with the annual pre-tax rate of return earned on the account. In general, tax-deferred dollars are worth most to those individuals who have considerable time before retirement and who are capable of earning higher average rates of return, perhaps by undertaking more risky investments.

If withdrawal is to occur in the near future, tax-deferred dollars are worth relatively less to those individual in high tax brackets, other things held constant. However, this disadvantage to those in high tax brackets diminishes (and eventually become a valuation advantage) as the time to withdrawal increases. This is especially true for individuals who earn higher average annual rates of return.

Although it is frequently held that tax-deferred dollars are always worth less than after-tax dollars (since the taxes must be paid upon withdrawal), the valuation factors of Table 1 indicate that this is often not the case. Even in a fully deductible account, a tax-deferred dollar may be worth more than an after-tax dollar if the account is to be used to fund retirement expenditures well into the future. This is especially true for individuals with higher tax rates and who earn higher average rates of return.

Table 2, constructed using Eq. (10), reports illustrative valuation factors for a fully deductible tax-deferred account that is to be liquidated through an even annuity of after-tax withdrawals. The relationships and general conclusions found in Table 1 for a single liquidating withdrawal are also observable when multiple withdrawals occur over a span of time.

Table 2
After-tax dollar value of $1 in a fully deductible tax deferred account - annuity of withdrawals

Year of first withdrawal (n)	Year 10			Year 20		
Length of annuity (m)	5 years	10 years	20 years	5 years	10 years	20 years
4% annual pre-tax rate of return (k)						
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	0.9423	0.9506	0.9657	0.9793	0.9879	1.0036
Tax rate = 0.20	0.8772	0.8928	0.9217	0.9476	0.9644	0.9957
Tax rate = 0.30	0.8040	0.8256	0.8666	0.9029	0.9272	0.9733
8% annual pre-tax rate of return (k)						
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	0.9829	0.9980	1.0224	1.0588	1.0750	1.1013
Tax rate = 0.20	0.9549	0.9849	1.0351	1.1087	1.1434	1.2017
Tax rate = 0.30	0.9140	0.9580	1.0344	1.1443	1.1994	1.2951
12% annual pre-tax rate of return (k)						
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	1.0218	1.0423	1.0712	1.1380	1.1608	1.1930
Tax rate = 0.20	1.0329	1.0757	1.1390	1.2827	1.3358	1.4145
Tax rate = 0.30	1.0295	1.0956	1.1987	1.4273	1.5190	1.6619

5. Valuation of non-deductible traditional IRAs

Under current tax law as described in Internal Revenue Service Publication 590 (2001), the extent of deductibility of IRA contributions is dependent on the level of adjusted gross income, the presence and nature of any employer retirement plan, and taxpayer filing status. Depending upon these three variables, it is possible that all, part, or none of an IRA contribution will be deductible in calculating current taxable income. That portion of any contribution that is deductible is effectively made with pre-tax dollars while any non-deductible portion is made with after-tax dollars. For individuals with higher adjusted gross incomes, the entire contribution is typically made with after-tax dollars. Throughout this paper, we refer to traditional IRAs that are funded partially or totally with after-tax dollars as non-deductible IRAs.

The number of dollars in a non-deductible IRA that were contributed with after-tax dollars represents the tax basis of the account. When withdrawals are made, that portion of the withdrawal that can be traced to an initial contribution of after-tax dollars is considered a return of capital and is free of taxation. The remainder (attributable to contributions of pre-tax dollars or to income earned by IRA assets) is subject to taxation at the ordinary tax rate upon withdrawal. When withdrawals occur, they are taken from the taxable and tax-free portions of the IRA in the proportion that each represents of the total. That is, if α = basis/total, then a withdrawal of W will contain αW tax-free dollars representing a return of capital and $(1\text{-}\alpha)W$ taxable dollars.

5.1. Liquidation through a single future withdrawal

Let W_n^{nd} be the largest single after-tax withdrawal that can be made from a non-deductible account at time $t = n$ given the current size of the account, S_0, and the rate of return, k. Also, let α be the proportion of S_0 that can be traced to contributions of after-tax dollars. Barring future contributions, the pre-tax value of S_0 at the end of n years will equal $S_0(1+k)^n$, and W_n^{nd} can be calculated as:

$$W_n^{nd} = S_0\alpha + [S_o(1 + k)^n - S_0\alpha](1 - \tau). \tag{11}$$

Eq. (11) is a generalization of Eq. (1) found in Horan, Peterson, and McLeod (1997) and Eq. (7) in Crain and Austin (1997).

The first term on the right of Eq. (11), $S_0\alpha$, measures the amount of the withdrawal that is traceable to contributions of after-tax dollars and, therefore, not subject to taxation. The terms in brackets measure the amount of the withdrawal that is attributable to contributions of pre-tax dollars or tax-deferred income earned on the account, all of which is subject to taxation upon withdrawal.

Rearranging the terms on the right of Eq. (11), we have

$$W_n^{nd} = S_0[(1 + k)^n(1 - \tau) + \alpha\tau]. \tag{12}$$

A comparison of Eqs. (1) and (12) highlights the difference between fully deductible and non-deductible IRAs. Because a portion of any withdrawal from a non-deductible IRA is free from taxation at the time of withdrawal, a non-deductible IRA will always produce a larger after-tax withdrawal than a fully deductible account of the same nominal size. The difference is given by $S_0\alpha\tau$ (the reduced tax liability enjoyed by the non-deductible account), regardless of whether the withdrawal occurs immediately or many years in the future.

As in the previous section, the current after-tax value of the tax deferred amount S_0 is the value of A_0 such that $W_n^{nd} = W_n^A$. The value of A_0 that corresponds to any given S_0 can be found by setting the right hand sides of Eqs. (4) and (12) equal to each other and solving for A_0, which produces:

$$A_0 = S_0 \frac{[(1 + k)^n(1 - \tau) + \alpha\tau]}{[1 + k(1 - \tau)]^n}. \tag{13}$$

Relative valuation factors, measuring the after-tax dollar equivalent of one dollar in a traditional non-deductible IRA are given by:

$$VF_n = \frac{A_0}{S_0} = \frac{[(1 + k)^n(1 - \tau) + \alpha\tau]}{[1 + k(1 - \tau)]^n}. \tag{14}$$

Eq. (14) is similar to that of Eq. (6) for the valuation of fully deductible accounts. The only difference is that Eq. (14) recognizes that a portion of any withdrawal represents a return of a contribution that was made with after-tax dollars and is thus free from further taxation at the time of withdrawal. In a sense, the valuation of 401(k)s and traditional IRAs [Eq. (6)], represents a special case of the more general valuation model for non-deductible IRAs. That is, if the proportion of S_0 that can be traced to the contribution of after-tax dollars [a in Eq. (14)] is zero, then Eq. (14) reduces to Eq. (6).

As in the case of fully deductible accounts, tax-deferred dollars in a non-deductible IRA are worth more (relative to after-tax dollars) to individuals with high tax rates than to individuals with lower rates. Likewise, their value is greater for investors who earn higher average rates of return and who anticipate a longer time interval until the funds are to be withdrawn. Assuming a positive tax rate, a dollar in a non-deductible IRA is always worth more than a dollar in fully deductible account since fewer taxes must be paid upon withdrawal.

5.2. Liquidation through an annuity of withdrawals

Following the same derivation approach described in Section 4.2, the relative valuation factor for a non-deductible IRA that is to be liquidated through an m-year annuity of equal after-tax withdrawals beginning in year n is given by

$$VF_n^m = \frac{A_0}{S_0} = \frac{\sum_{i=n}^{n+m-1} 1/[1 + k(1-\tau)]^i}{\sum_{i=n}^{n+m-1} 1/[(1+k)^i(1-\tau) + \alpha\tau]}. \tag{15}$$

As in the other equations in this section, the valuation of non-deductible IRAs is sensitive to the proportion of S_0 that is attributable to contributions of after-tax (nondeductible) dollars, measured here by the variable α. Although this proportion varies over the life of a non-deductible IRA and would almost certainly be different at the time of each withdrawal, it is the value of α at the time the account is being valued that is appropriate for use in Eqs. (14) and (15).

5.3. Some illustrative valuation factors

Tables 3 and 4, constructed using Eqs. (14) and (15) respectively, provide some illustrative relative valuation factors for non-deductible IRAs with an assumed value of α equal to 0.5. Given that valuation factors are sensitive to the value of α and that α can theoretically vary over a range $0 \leq \alpha \leq 1$, the valuation factors of Tables 3 and 4 must be viewed with caution. Nevertheless, a number of interesting observations are apparent.

As should be expected, the valuation factors in Tables 3 and 4 (for non-deductible IRAs) are higher than their corresponding values in Tables 1 and 2 (for fully deductible accounts). This is due simply to the fact that a portion of any withdrawal from a non-deductible IRA is free from taxation, while 100% of any withdrawal from a fully deductible account is taxable. The tax advantage for a non-deductible IRA dollar is greater for higher tax rates (and disappears entirely in the absence of taxes).

Comparing Tables 1 and 3 or Tables 2 and 4, it is apparent that valuation factors for fully deductible and non-deductible accounts tend to converge as the length of time until withdrawal increases. This tendency is especially apparent if higher rates of return are expected. The explanation for this pattern is found in the favorable tax treatment afforded withdrawals from a non-deductible IRAs. A current dollar in a non-deductible IRA is always worth more than a dollar in a fully deductible account given that it will always generate a larger future withdrawal (assuming a positive tax rate and $\alpha > 0$). The absolute dollar advantage to the

Table 3
After-tax dollar value of $1 in a non-deductible IRA account - single withdrawal

	Years until withdrawal			
	0	10	20	30
4% annual pre-tax rate of return				
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	0.9500	0.9705	0.9968	1.0276
Tax rate = 0.20	0.9000	0.9372	0.9869	1.0474
Tax rate = 0.30	0.8500	0.8999	0.9692	1.0570
8% annual pre-tax rate of return				
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	0.9500	0.9944	1.0567	1.1311
Tax rate = 0.20	0.9000	0.9826	1.1072	1.2674
Tax rate = 0.30	0.8500	0.9634	1.1477	1.4030
12% annual pre-tax rate of return				
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	0.9500	1.0203	1.1228	1.2456
Tax rate = 0.20	0.9000	1.0335	1.2498	1.5386
Tax rate = 0.30	0.8500	1.0374	1.3754	1.8787

current non-deductible IRA dollar is given by $\$1\alpha\tau$ (the difference in tax to be paid upon withdrawal), *irrespective of when the withdrawal occurs.* If withdrawal is to occur immediately or in the near future, this advantage may be significant relative to the size of the entire withdrawal and have a material impact on the valuation factor. If, however, withdrawal of the dollar and its accumulated income is not anticipated until well into the future, the tax advantage will represent an ever-decreasing proportion of the total withdrawal (assuming

Table 4
After tax dollar value of $1 in a non-deductible tax deferred account - annuity of withdrawals

Year of first withdrawal (n)	Year 10			Year 20		
Length of annuity (m)	5 years	10 years	20 years	5 years	10 years	20 years
4% annual pre-tax rate of return (k)						
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	0.9752	0.9811	0.9925	1.0024	1.0093	1.0225
Tax rate = 0.20	0.9460	0.9571	0.9793	0.9978	1.0114	1.0378
Tax rate = 0.30	0.9120	0.9276	0.9593	0.9849	1.0046	1.0437
8% annual pre-tax rate of return (k)						
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	1.0050	1.0175	1.0389	1.0698	1.0848	1.1096
Tax rate = 0.20	1.0032	1.0281	1.0727	1.1346	1.1667	1.2220
Tax rate = 0.30	0.9931	1.0300	1.0986	1.1902	1.2412	1.3325
12% annual pre-tax rate of return (k)						
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	1.0370	1.0553	1.0823	1.1435	1.1655	1.1970
Tax rate = 0.20	1.0673	1.1058	1.1656	1.2965	1.3479	1.4252
Tax rate = 0.30	1.0882	1.1482	1.2465	1.4535	1.5425	1.6834

Note: Above table of valuation factors assumes that 50% of the account is traceable to contributions made with after-tax dollars, and the rest to deductible contributions or income earned on the account.

positive rates of return) and have a correspondingly smaller influence on the valuation factor. If withdrawal does not occur for many years, the taxable portion of the withdrawal will approach 100% of the total, and the valuation factor for a non-deductible IRA will approach that of a fully deductible account.

6. Valuation of Roth IRAs

A Roth IRA is a relatively new type of tax sheltered IRA account. Under normal circumstances, Roth IRAs have the characteristic that contributions are made with after-tax dollars, income earned on the account is free from current taxation, and qualifying post-retirement withdrawals are also free from taxation. While other tax-advantaged retirement accounts allow an individual to defer taxes on investment income, Roth IRAs allow one to avoid taxes on investment income altogether.

6.1. Liquidation through a single future withdrawal

The valuation of Roth IRAs is relatively simple because of the fact that post-retirement withdrawals are free from taxation. Let S_0 be the number of dollars an individual has in a Roth IRA account at time $t = 0$. Again, assume that the outstanding balance in the account earns a fixed annual rate of return of k per year and that all of the funds are withdrawn at the end of n years. Let W_n^R be the largest withdrawal possible at from a Roth IRA at time $t = n$ given the nominal size, S_0, and the rate of return, k. For a Roth IRA, the value of W_n^R is calculated simply as

$$W_n^R = S_0(1 + k)^n. \tag{16}$$

Note that Roth IRA dollars enjoy the very substantial benefit that all qualified distributions are excluded from taxable income. Thus, Roth IRA dollars are always more valuable that ordinary after-tax dollars if they are to be used to fund future consumption and the holder of the account is subject to a positive tax rate.

Again, the current after-tax value of the tax sheltered amount S_0 is the value of A_0 such that $W_n^R = W_n^A$. The value of A_0 that corresponds to any given S_0 can be found by setting the right hand sides of Eqs. (4) and (16) equal to each other and solving for A_0, which produces:

$$A_0 = S_0 \frac{(1 + k)^n}{[1 + k(1 - \tau)]^n} \tag{17}$$

Relative valuation factors, measuring the after-tax dollar equivalent of one dollar in a Roth IRA are given by:

$$VF_n = \frac{A_0}{S_0} = \frac{(1 + k)^n}{[1 + k(1 - \tau)]^n}. \tag{18}$$

Table 5
After-tax dollar value of $1 in a roth IRA - single withdrawal

	Years until withdrawal			
	0	10	20	30
4% annual pre-tax rate of return				
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	1.0000	1.0393	1.0801	1.1226
Tax rate = 0.20	1.0000	1.0803	1.1670	1.2607
Tax rate = 0.30	1.0000	1.1231	1.2613	1.4165
8% annual pre-tax rate of return				
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	1.0000	1.0772	1.1603	1.2499
Tax rate = 0.20	1.0000	1.1610	1.3479	1.5648
Tax rate = 0.30	1.0000	1.2520	1.5675	1.9624
12% annual pre-tax rate of return				
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	1.0000	1.1137	1.2404	1.3815
Tax rate = 0.20	1.0000	1.2419	1.5422	1.9152
Tax rate = 0.30	1.0000	1.3864	1.9221	2.6648

If $n > 0$ and $\tau > 0$, then $VF_n > 1$. That is, an individual will value Roth dollars more highly than ordinary after-tax dollars if he/she plans to use them for future retirement income and anticipates a positive tax rate in at least one year before or including the year of withdrawal.

The after-tax value of Roth dollars depends on three variables: (1) the number of years until withdrawal, (2) the rate of return earned, and (3) the tax rate. In the special case where $n = 0$ (the funds are to be withdrawn immediately during retirement), $VF_n = 1$. In this special case, Roth dollars are equivalent to ordinary after-tax dollars. Likewise, Roth dollars are equivalent to ordinary after-tax dollars if the constant tax rate (τ) is zero, even if the funds are to be withdrawn many years in the future.

6.2. Liquidation through an annuity of withdrawals

Again, the same approach described in section 4.2 can be used to derive relative valuation factors for a Roth IRA that is to be liquidated through an m-year annuity of equal after-tax withdrawals beginning in year n. The resulting valuation factors are given by

$$VF_n^m = \frac{A_0}{S_0} = \frac{\sum_{i=n}^{n+m-1} 1/[1 + k(1 - \tau)]^i}{\sum_{i=n}^{n+m-1} 1/(1 + k)^i}. \tag{19}$$

6.3. Some illustrative valuation factors

Tables 5 and 6, prepared using Eqs. (18) and (19), contain illustrative valuation factors for Roth IRAs for a range of circumstances. The results parallel a number of relationships found

Table 6
After-tax dollar value of $1 in a roth IRA - annuity of withdrawals

Year of first withdrawal (n)	Year 10			Year 20		
Length of annuity (m)	5 years	10 years	20 years	5 years	10 years	20 years
4% annual pre-tax rate of return (k)						
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	1.0470	1.0562	1.0729	1.0882	1.0977	1.1151
Tax rate = 0.20	1.0965	1.1160	1.1521	1.1845	1.2056	1.2446
Tax rate = 0.30	1.1485	1.1795	1.2380	1.2899	1.3246	1.3904
8% annual pre-tax rate of return (k)						
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	1.0921	1.1089	1.1360	1.1764	1.1945	1.2237
Tax rate = 0.20	1.1937	1.2311	1.2939	1.3858	1.4293	1.5021
Tax rate = 0.30	1.3057	1.3686	1.4777	1.6347	1.7134	1.8501
12% annual pre-tax rate of return (k)						
Tax rate = 0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Tax rate = 0.10	1.1354	1.1581	1.1902	1.2645	1.2898	1.3256
Tax rate = 0.20	1.2911	1.3446	1.4238	1.6034	1.6698	1.7682
Tax rate = 0.30	1.4707	1.5652	1.7125	2.0390	2.1700	2.3742

for fully deductible and non-deductible IRAs. For any given tax rate, the after-tax value of a Roth IRA dollar increases with the expected length of time until withdrawal and with the expected annual pre-tax rate of return earned on the account. Thus, Roth dollars are worth most to those individuals who have considerable time before retirement and who are capable of earning higher average rates of return. Also, current Roth dollars are worth relatively more to individuals in higher tax brackets.

7. Valuation factors and investment analysis

The valuation models developed in this paper permit an individual to measure total wealth accumulation in a more accurate and consistent manner. While this is a worthy end in itself, these models also facilitate the analysis of choices faced by an individual in the process of retirement planning. In this section, we demonstrate their usefulness in calculating familiar measures of investment performance and argue that capital investment theory has many potential applications to retirement planning.

An investor always has alternatives. Money can be invested in tax-advantaged retirement accounts, ordinary taxable mutual funds, collectables, real estate, small businesses, and a host of other assets, real or financial, tax-advantaged or not. An investment in any given type of tax-advantaged retirement account should compete against all other alternative investments, whether tax-advantaged or not. A common method for comparing alternative investments is to calculate a net present value (NPV), internal rate of return (IRR), or profitability index (PI). Decision rules using these measures have been developed for a wide range of circumstances and assumptions. The valuation models developed here facilitate the calculation of such measures for investments in tax-advantaged accounts and thus permit the application of the well-developed theory of capital investment analysis to a variety of retirement planning problems. If nothing else, this should prompt a new perspective for old problems and perhaps a deeper understanding of the underlying issues.

Consider an investment of \$X in a Roth IRA. If the investor expects to liquidate the account at the end of n years, the present value of the future withdrawal can be calculated using Eq. (18). The NPV of the \$X investment is given by the present value of the future withdrawal less the initial investment. For our \$X investment in a Roth IRA, the NPV is given by

$$NPV = \frac{\$X(1 + k)^n}{[1 + k(1 - \tau)]^n} - \$X. \tag{20}$$

Theoretically, the NPV measures the immediate increase in the wealth of the investor as a result of undertaking the investment. For an investment to be judged favorable, the NPV must be positive.

The NPV is an absolute measure of investment performance. That is, it reflects the size or scale of the investment. When choosing among mutually exclusive investments in the absence of capital rationing, choosing the investment with the highest NPV maximizes the wealth of the investor. See Van Horne (1998) or any basic text in financial management for a discussion of capital investment decision rules.

When attempting to choose from among a large group of favorable investments under conditions of capital rationing, it is sometimes desirable to rank the alternatives using a *relative* measure of investment performance. The profitability index (PI), which is sometimes called a benefit/cost ratio, is often used for this purpose. The PI is calculated as the present value of future after-tax cash flows divided by the initial investment (measured in after-tax dollars). For the investment described above, the PI is given by

$$PI = \frac{\$X(1 + k)^n/[1 + k(1 - \tau)]^n}{\$X} = \frac{(1 + k)^n}{[1 + k(1 - \tau)]^n}. \tag{21}$$

For an investment to be favorable, it is necessary that PI > 1. Given the mathematical relationship between the NPV and the PI, both give the same answer in simple accept-reject investment decisions.

Note that the valuation factors contained in Tables (5) and (6) are essentially profitability indices for investments in a Roth IRA. As such, they can be used to compare Roth IRA investments with a wide range of alternatives.

Calculating the NPV and PI of an investment in a Roth IRA is straightforward since both the investment and the withdrawal are in after-tax dollars. The calculation of the NPV and PI for investments in deductible and non-deductible tax-deferred accounts must insure that all cash flows are expressed in after-tax dollars. Consider an investment of \$X pre-tax dollars in a fully deductible IRA that is to be liquidated through a single withdrawal. The after-tax dollar investment is given by $\$X(1\text{-}\tau)$. The NPV of the investment is given by

$$NPV = \frac{\$X(1 + k)^n(1 - \tau)}{[1 + k(1 - \tau)]^n} - \$X(1 - \tau) \tag{22}$$

and the PI is calculated as:

$$PI = \frac{\$X(1+k)^n(1-\tau)/[1+k(1-\tau)]^n}{\$X(1-\tau)} = \frac{(1+k)^n}{[1+k(1-\tau)]^n}. \tag{23}$$

Similar performance measures can be calculated for a $X investment in a non-deductible IRA that is to be liquidated through a single future withdrawal. If the investment is made entirely with after-tax dollars, the NPV is calculated as

$$NPV = \frac{\$X(1+k)^n(1-\tau) + \$X\tau}{[1+k(1-\tau)]^n} - \$X \tag{24}$$

and the PI is given by

$$PI = \frac{\$X[(1+k)^n(1-\tau)+\tau]/[1+k(1-\tau)]^n}{\$X} = \frac{(1+k)^n(1-\tau)+\tau}{[1+k(1-\tau)]^n}. \tag{25}$$

Note that the after-tax rate of return available on ordinary taxable investments, $k(1-\tau)$, is effectively used as the opportunity cost (discount rate) for calculating the current after-tax value of tax-advantaged retirement accounts. The use of this same opportunity cost to evaluate an ordinary taxable investment earning a pre-tax rate of k must result in a NPV = 0 and a PI = 1. Thus, an investment in a tax-advantaged retirement account will dominate a regular taxable investment in the same assets if and only if its NPV > 0 and PI > 1. It is obvious from Eq. (21) and (23) that Roth IRAs and fully deductible retirement accounts satisfy this requirement as long as the tax rate is positive. Although less apparent on the surface, it can also be shown that non-deductible IRAs will satisfy the requirement for multiyear investments under the assumption of a constant tax rate. Randolph (1994) reaches the same basic conclusion using a different approach.

One of the advantages of viewing the process of retirement planning as a capital investment problem is the presence of a large and well-developed body of theory. Over the past forty years, the dimensions of the capital investment problem have become clearer, normative theory has grown and matured, and decision rules have been developed for a wide range of circumstances and assumptions. An important result of this process is a clearer understanding of the factors that should be considered when structuring an appropriate method of analysis.

8. Discussion

The main usefulness of valuation models derived in this paper is to allow individuals to more accurately measure their wealth on a consistent basis. By converting different types of tax-advantaged and outright taxable investment accounts into a single unit of measurement (ordinary after-tax dollars) a prospective retiree is better able to answer the question "How much money do I really have?" As wealth accumulation is an important variable influencing retirement decisions for many individuals (Hatcher, 1997), it is reasonable to argue that better measures of wealth accumulation will lead to better retirement decisions.

The models developed in this paper for the valuation of tax-advantaged retirement accounts are based upon a number of simplifying assumptions, none of which are likely to be perfectly satisfied by any single retiree. Two of the most limiting assumptions are that future annual rates of return and future tax rates are known and constant. While these assumptions were useful in developing the reasonably simple models presented, they are *not* necessary for the general valuation approach describe here. If a prospective retiree has specific expectations of future changes in either rates of return or tax rates, the general valuation process can be applied using an Excel spreadsheet that permits separate estimates for rate of return and tax rate for each future year. Of course, the process of forecasting rates of return and tax rates many years into the future is difficult at best. The assumption of constancy may be a good first approximation.

The valuation models described here (like virtually all valuation models) require estimates of variables that cannot be known with certainty until the future reveals itself. Consequently, the resulting estimates are virtually always imperfect. In an uncertain world, the best that we can hope to do is make decisions that are consistent with our current set of expectations for the future, even as we know that those expectations are likely to change with the passage of time. Hopefully, the models presented here afford prospective retirees and their financial planners a way to achieve greater internal consistency (and perhaps improved effectiveness) in the financial planning process.

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