The Geometry of Pure Spinors, with Applications

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I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

(Signed) ________________________________

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Contents

Abstract

We consider spinors carrying a representation of the Clifford algebra generated by a complex $2r$-dimensional vector space V. Certain spinors, called *pure*, admit a remarkable geometrical interpretation since they may be placed in correspondence with maximal totally isotropic subspaces of dimension r. This correspondence has allowed the orbit structure of pure spinors under the action of the Spin group to be classified. An 'impure' spinor u may be put into correspondence with an isotropic subspace of dimension $\nu < r$. Except for low dimensions, the classification of the impure spinor orbits is incomplete. In the first part of the thesis, we present a coarse classification of impure spinors based on (1) the value of ν and (2) the decomposition of u into pure components.

In later chapters we examine certain equations for spinor fields on manifolds. On spacetime, we show that the conformal Killing equation for a null vector field is equivalent to a generalisation of the twistor equation. From it we derive Sommers' equation for a spinor corresponding to a null shear-free geodesic. We also find equations for a Dirac spinor equivalent to the conformal Killing and shear-free conditions for a timelike vector field.

Finally, it is shown that a pair of shear-free Weyl spinors may be used to construct a 2-form satisfying a generalised conformal Killing-Yano equation. Via spinors, the conformal Killing-Yano 2-form is used to construct symmetry operators for the massless Dirac equation and the vacuum Maxwell equation. In the case of the Dirac equation, we are able to construct a symmetry operator from a conformal Killing-Yano tensor of any degree in arbitrary dimensions.

Chapter 1

Introduction

The discovery of spinors may be attributed to Cartan [Car13]. For $n > 2$, the group $\mathrm{SO}(n)$ is not simply-connected. As a consequence, Cartan observed that the Lie algebra of $\mathrm{SO}(n)$ has representations which do not lift to representations of $\mathrm{SO}(n)$. Rather, they lift to representations of $\mathrm{Spin}(n)$, the simply-connected double covering of $\mathrm{SO}(n)$. Spinors may be thought of as vectors which carry an irreducible representation of $\mathrm{Spin}(p,q)$, that is, the Spin group of an orthogonal space with signature (p,q). In the case where the orthogonal space has maximal Witt index, Cartan discovered a remarkable geometrical interpretation of certain spinor directions, called *pure* [Car66]. The pure spinor directions are in one-to-one correspondence with the maximal isotropic subspaces of the underlying orthogonal space. In even dimensions, such spinors determine (projectively) a null $(n/2)$-form given by the product of a basis for the corresponding maximal isotropic subspace. Since a complex orthogonal space has maximal Witt index, the notion of pure spinors can be extended to a real orthogonal space with any signature by complexification.

Spinors may also be defined as vectors carrying an irreducible representation of the Clifford algebra generated by an orthogonal space. As a vector space, this algebra is isomorphic to the space of exterior forms. Since all Clifford algebras are either simple or semi-simple, the regular representation on a minimal left ideal is irreducible. The space of spinors may be identified with a minimal left ideal of the Clifford algebra. Spinors of a non-simple Clifford algebra are usually referred to as *semi-spinors*. Since the Spin group is contained in the Clifford algebra, the spinor representation of the Clifford algebra induces a spinor representation of the Spin group. In Chapter 2 we establish our conventions and notation for Clifford algebras, spinors and exterior forms.

For $n \leq 6$, the dimension of the space of maximal isotropic subspaces is equal to that of the space of spinor directions, so in these cases all spinors are pure. In higher dimensions a spinor must satisfy a set of quadratic constraint equations in order to be pure. Another characterisation is given by considering the action of the Spin group on the projective spinor space. An orbit of the Spin group forms a submanifold in the space of spinor directions, whose structure is determined by the isotropy group of a representative spinor. Pure spinors of a given parity form a single orbit, characterised by having the least dimension amongst the orbits of projective spinors. Thus the codimension of an orbit provides a 'measure of purity' for spinors. In general, the problem of classifying spinor orbits is difficult, and little is known for $n > 14$ [Igu70, Pop80]. In

Chapter 3 we examine other measures of the purity of a spinor. Using our technique, we are able to determine the isotropy groups of certain impure spinors. This in turn leads to a reduction in holonomy of a manifold admitting a parallel impure spinor.

Our motivation for studying spinors in higher dimensions comes from recent trends in mathematical physics. Spinors first came to the attention of physicists with Dirac's quantum theory of the electron [Dir28]. For the most part, the study of spinors has been restricted to those occurring in spacetime, where they have proven particularly useful in general relativity [PR86a]. Since all spinors are pure in this dimension, there has been no need to introduce pure spinors as a separate concept. They are implicit in Penrose's notion of 'flag planes', which correspond to the maximal isotropic spaces of a complex 4-dimensional orthogonal space. As the underlying real space has Lorentzian signature, pure spinors also exhibit a real structure: they determine a real null direction, or 'flagpole', in Penrose's terminology. This correspondence means that pure spinors are useful for studying the properties of null congruences. It is well-known that a vector field tangent to a congruence of null shear-free geodesics (NSFG) corresponds to a spinor field satisfying Sommers' equation [Som76]. This equation is a generalisation of the twistor equation involving an additional vector field. In Chapter 4 we derive Sommers' equation and interpret the additional terms as gauge terms of the covariant derivative. By replacing the standard covariant derivative with a $\mathrm{GL}(1, \mathbb{C})$-gauged covariant derivative, we obtain an equation of the same form as the twistor equation. We refer to spinors satisfying this equation as being *shear-free*. As a special case, we show that a null conformal Killing vector corresponds to a $\mathrm{U}(1)$-gauged twistor equation. Since a non-null vector can be written as a sum of two null vectors, a non-null vector corresponds to a pair of spinors, which may be thought of as a single *Dirac spinor*. We examine the condition that the vector field corresponding to a Dirac spinor is shear-free or conformal Killing. As in the null case, this may be written as an equation for a Dirac spinor of the same form as the twistor equation.

Currently, spinors play a prominent role in theories requiring higher dimensions. For this reason it is likely that pure spinors will become more relevant to physics, a view which has been advocated most notably by Budinich and Trautman ([BT86, BT88] and references therein). A significant result utilising pure spinors is due to Hughston and Mason, who discovered a generalisation of Robinson's theorem to all dimensions [HM88]. In four dimensions, Robinson's theorem can be stated as the following: a null self-dual 2-form is proportional to a solution of Maxwell's equations if and only if it admits a real null eigenvector tangent to a NSFG congruence [Rob61]. More generally, a spin-$(s/2)$ field is a solution of the massless field equation iff its s-fold principal spinor is shear-free. In higher (even) dimensions, it is natural to consider pure spinors, since these can be correlated with a distribution of null $(n/2)$-planes. Hughston and Mason have generalised Robinson's theorem to all even dimensions, provided that the s-fold principal spinor is pure. The shear-free condition is replaced by what they refer to as the *Frobenius-Cartan integrability condition*. This is the equation a pure spinor must satisfy if the corresponding null distribution is to be integrable. In four dimensions the Frobenius-Cartan condition is equivalent to Sommers' equation. As a special case, it is shown that a null self-dual $(n/2)$-form is exact iff its 2-fold principal spinor satisfies the integrability condition.

An alternative way of stating Robinson's theorem in four dimensions comes from a generalisation of the conformal Killing-Yano equation. Killing-Yano tensors were first introduced as a generalisation of Killing's equation to totally antisymmetric tensors [Yan52]. The conformal extension of the Killing-Yano (CKY) equation was found by Tachibana [Tac69]. The CKY equation can be expressed very compactly using exterior calculus [BCK97]. In the same paper, it is shown that by replacing the ordinary covariant derivative with a $GL(1, \mathbb{C})$-gauged covariant derivative, the CKY equation for a self-dual 2-form is equivalent to the condition that the eigenvectors of the 2-form be NSFG. For this reason, we refer to forms satisfying the gauged CKY equation as 'shear-free'. The shear-free 2-form equation was first studied by Dietz and Rüdiger [DR80], although they did not interpret the additional terms as arising from a gauged covariant derivative. A non-null self-dual 2-form admits two real eigenvectors, while a null 2-form admits only one. Thus by Robinson's theorem a null self-dual gauged CKY 2-form is proportional to a solution of Maxwell's equation. In Chapter 5 we review the properties of the gauged CKY equation and its relationship to shear-free spinors. We also calculate integrability conditions for the gauged CKY equation and the shear-free spinor equation, which will be used extensively in Chapter 6.

In Chapter 6 we consider an application of shear-free spinors to solving the vacuum Maxwell and Dirac equations. In conformally flat spacetime, Penrose has shown that a solution of the massless field equation may be constructed by 'raising' a Debye potential with a twistor [Pen75, PR86b]. Conversely, a Debye potential may be generated by 'lowering' a solution of the massless field equation with a twistor. More generally, a massless field can be used to generate another massless field of a different spin by repeated raising and lowering. In algebraically special spacetime, we show that solutions of the Dirac and Maxwell equations can be obtained from a Debye potential by raising with a shear-free spinor if the spinor is aligned with a repeated principal null direction. Under the same conditions, a Debye potential can be obtained from a Maxwell or Dirac field by lowering with a shear-free spinor. From this we obtain symmetry operators for the Maxwell and Dirac equations. Using the correspondence between shear-free spinors and CKY tensors, these operators may be written in terms of CKY tensors only. We then show how a CKY p-form may be used to construct a symmetry operator for the Dirac equation on a manifold of arbitrary dimension and signature [BC97].

Chapter 2

Clifford Algebras and Spinors

In this chapter we provide our conventions and notation for exterior forms, Clifford algebras, spinors, and the related differentiable structures on manifolds. For ease of reference, we have attempted to include here all of the standard results that will be used in this thesis. We make no attempt to prove them, except as examples in cases where our notation differs substantially from the norm. For the most part, these results can be found in Benn and Tucker [BT87], whose conventions we follow. The reader familiar with this notation may comfortably move on to later chapters, returning to check conventions as necessary.

2.1 The exterior algebra

Let V be an n-dimensional vector space over a field $F = \mathbb{R}$ or $\mathbb{C}$. The *dual space* V^* is the n-dimensional space of linear functions on V. Given a basis $\{e^a\}$ for V where $a \in \{1, 2, \ldots, n\}$, the dual basis $\{X_a\}$ for V^* is defined in terms of the Kronecker delta symbol by

$$X_a(e^b) \quad = \quad \delta^b{}_a \,. \tag{2.1.1}$$

Equivalently, we may regard elements of V as linear functions on V^* by defining

$$x(X) \equiv X(x) \quad \forall\, x \in V,\ X \in V^* \,. \tag{2.1.2}$$

We normally take this view, since in this chapter we will take V to be the space of covectors. With this notation, a *tensor* N of *type* (p, q) is a multilinear mapping

$$N : \underbrace{V^* \times V^* \times \ldots \times V^*}_{p} \times \underbrace{V \times V \times \ldots \times V}_{q} \longrightarrow F \,.$$

The space of such tensors is denoted $T_p{}^q(V)$. When q is zero, we say that N has *degree* p, and the index q will be omitted.

The space of totally antisymmetric tensors of degree p is denoted by $\Lambda_p(V)$, and its elements are referred to as *p-forms*. The space of p-forms has dimension $\binom{n}{p}$. It is convenient to identify the field F with $\Lambda_0(V)$, and V itself with $\Lambda_1(V)$. Because of antisymmetry, forms of degree greater than n are zero. The space of exterior forms

$\Lambda(V)$ is formed by taking the direct sum of the p-form subspaces,

$$\Lambda(V) \;=\; \sum_{p=0}^{n} \oplus \Lambda_p(V) \tag{2.1.3}$$

hence $\dim \Lambda(V) = 2^n$. This decomposition of $\Lambda(V)$ induces a set of *projection operators* $\mathscr{S}_p$ which map an arbitrary form to its p-form component. Thus a form ω may be written as

$$\omega \;=\; \sum_{p=0}^{n} \mathscr{S}_p(\omega)\,. \tag{2.1.4}$$

A form which lies completely in one of the p-form subspaces is called *homogeneous.*

The *exterior algebra* is an associative algebra formed from $\Lambda(V)$ and the *exterior product* on p-forms. We define the exterior product using the operator **Alt**, which maps a tensor of degree p to a totally antisymmetric tensor. For $N \in T_p(V)$,

$$\begin{aligned} &\mathbf{Alt}\, N(X_1, X_2, \ldots, X_p) \\ &\quad = \frac{1}{p\,!} \sum_{\sigma} \epsilon(\sigma) N(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(p)}) \quad \forall\; X_i \in V^*\,, \end{aligned} \tag{2.1.5}$$

where the sum is over all permutations σ, and $\epsilon(\sigma)$ is $+1$ on even permutations, -1 otherwise. Then the exterior product of two homogeneous forms is given by

$$\begin{aligned} \wedge : \Lambda_p(V) \times \Lambda_q(V) &\longrightarrow \Lambda_{p+q}(V) \\ \omega,\ \phi &\longmapsto \omega \wedge \phi = \mathbf{Alt}(\omega \otimes \phi)\,. \end{aligned} \tag{2.1.6}$$

The exterior product on inhomogeneous forms is defined by extending **Alt** to be distributive over addition. A simple calculation shows that

$$\omega \wedge \phi \;=\; (-1)^{pq}\, \phi \wedge \omega \qquad \omega \in \Lambda_p(V),\ \phi \in \Lambda_q(V)\,. \tag{2.1.7}$$

Similary, the symmetrising operator is defined by

$$\begin{aligned} &\mathbf{Sym}\, N(X_1, X_2, \ldots, X_p) \\ &\quad = \frac{1}{p\,!} \sum_{\sigma} N(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(p)}) \quad \forall\; X_i \in V^*\,. \end{aligned} \tag{2.1.8}$$

The direct sum decomposition (2.1.3) gives $\Lambda(V)$ the structure of a $\mathbb{Z}$-graded algebra. This induces two canonical isomorphisms. The automorphism η is defined on homogeneous forms by

$$\eta\omega \;=\; (-1)^p \omega \qquad \omega \in \Lambda_p(V)\,. \tag{2.1.9}$$

As an automorphism, it is a linear mapping such that

$$\eta(\omega \wedge \phi) \;=\; \eta\omega \wedge \eta\phi \qquad \omega,\ \phi \in \Lambda(V)\,. \tag{2.1.10}$$

It is *involutory*, meaning that $\eta^2 = 1$. Since the eigenvalues of η are ± 1, the automorphism induces a $\mathbb{Z}_2$-grading on $\Lambda(V)$. Those eigenvectors with eigenvalue $+1$ will be called *even*, and the space of even forms will be denoted $\Lambda^+(V)$. Likewise, eigenvectors with eigenvalue -1 will be called *odd*, and belong to the subspace $\Lambda^-(V)$. The isomorphism ξ is defined as the involutory *anti-automorphism* which fixes each 1-form,

$$x^\xi \;=\; x \qquad \forall\, x \in V\,. \tag{2.1.11}$$

To say that ξ is an anti-automorphism means that it is a linear mapping which reverses the order of exterior multiplication. That is,

$$(\omega \wedge \phi)^\xi \;=\; \phi^\xi \wedge \omega^\xi \quad \forall\, \phi,\ \omega \in \Lambda(V)\,. \tag{2.1.12}$$

Since any exterior form can be written as a linear combination of homogeneous forms, these two properties are sufficient to define ξ on $\Lambda(V)$. For a homogeneous form,

$$\begin{aligned} \omega^\xi &= (-1)^{p(p-1)/2}\omega \\ &= (-1)^{\lfloor p/2 \rfloor}\omega \qquad \omega \in \Lambda_p(V)\,, \end{aligned} \tag{2.1.13}$$

where $\lfloor p/2 \rfloor$ means the integer part of $p/2$, that is, the 'floor' function.

If X is in V^* then the *interior derivative* with respect to X is denoted by $X \rfloor$. It is an *anti-derivation* with respect to η, meaning a linear mapping such that

$$X \rfloor (\omega \wedge \phi) \;=\; X \rfloor \omega \wedge \phi + \eta\omega \wedge X \rfloor \phi \qquad \omega,\ \phi \in \Lambda(V)\,. \tag{2.1.14}$$

To complete the definition, we specify the action of $X \rfloor$ on 1-forms and scalars as

$$\begin{aligned} X \rfloor x &= x(X) \qquad \forall\, x \in V \\ X \rfloor \lambda &= 0 \qquad\quad\;\; \forall\, \lambda \in F\,. \end{aligned}$$

By considering its action on decomposable forms, it can be shown that if ω is a p-form then $X \rfloor \omega$ is a $(p-1)$-form. Then $X \rfloor \eta = -\eta X \rfloor$, from which it follows that

$$Y \rfloor X \rfloor \omega \;=\; -X \rfloor Y \rfloor \omega \qquad X,\ Y \in V^*,\ \omega \in \Lambda(V)\,. \tag{2.1.15}$$

Clearly $X \rfloor X \rfloor \omega = 0$. With our conventions,

$$e^a \wedge X_a \rfloor \omega \;=\; p\,\omega \qquad \forall\, \omega \in \Lambda_p(V)\,. \tag{2.1.16}$$

As usual, a summation convention is employed on matching upper and lower indices.

A *metric* g on V is a symmetric, non-degenerate $(0, 2)$ tensor. To say that g is non-degenerate means that for $x \in V$, $g(x, y) = 0$ for all $y \in V$ if and only if $x = 0$. The combination (V, g) is called an *orthogonal space*. Occasionally, it will be convenient to use the components of g with respect to some basis. We assign $g^{ab} = g(e^a, e^b)$, while the scalars g_{ab} are defined as satisfying the condition

$$g^{ac}g_{cb} \;=\; \delta^a{}_b\,. \tag{2.1.17}$$

This provides a convention for the raising and lowering of indices. We define $e_a \equiv g_{ab}e^b$

and $X^a \equiv g^{ab} X_b$. Similarly, indices of more general objects can be raised and lowered with the components of g. Note that this convention can be applied to any indexed objects, be they components of a tensor, sets of indexed forms, or more general tensors. Care is needed in the ordering of mixed indices, however. We will always stagger our indices so that lower indices have a place to which they can be raised, and vice-versa.

If g has *signature* (p, q) then there exists a basis such that

$$g(e^a, e^b) = \begin{cases} +1 & \text{for } a = b = 1, 2, \ldots, p \\ -1 & \text{for } a = b = p+1, p+2, \ldots, p+q = n \\ 0 & \text{for } a \neq b\,. \end{cases}$$

Such a basis is called *g-orthonormal.* The quantity $\min\{p, q\}$ is the *Witt index* of g. From any orthonormal basis we obtain a *volume n-form z*,

$$z \;=\; e^1 \wedge e^2 \wedge \ldots \wedge e^n\,. \tag{2.1.18}$$

Apart from a sign, the choice of z is independent of the choice of orthonormal basis. Choosing a sign amounts to choosing an orientation for V. We will frequently use the abbreviation e^{ab} for $e^a \wedge e^b$, and similarly for forms of higher degree, so $z = e^{12\ldots n}$.

The existence of a metric provides a canonical isomorphism between V and V^*. The mapping $\sharp : V \to V^*$ is defined by

$$y(x^\sharp) \;=\; g(x, y) \quad \forall\, y \in V\,. \tag{2.1.19}$$

If x is written in components as $x = x_a e^a$, then $x^\sharp$ is the vector obtained by raising the components of x with the metric tensor. That is,

$$\begin{aligned} x^\sharp &= g^{ab} x_b X_a \\ &= x^a X_a\,. \end{aligned} \tag{2.1.20}$$

The inverse of $\sharp$ is the mapping $\flat : V^* \to V$, where $Y^\flat$ is defined implicitly as the 1-form such that

$$y(Y) \;=\; g(y, Y^\flat) \quad \forall\, y \in V\,. \tag{2.1.21}$$

Likewise, if Y is written in components as $Y = Y^a X_a$, then $Y^\flat$ is the 1-form obtained by lowering the components of Y with g,

$$\begin{aligned} Y^\flat &= g_{ab} Y^b e^a \\ &= Y_a e^a\,. \end{aligned} \tag{2.1.22}$$

We then obtain a metric on V^* by defining

$$g^*(X, Y) \;=\; g(X^\flat, Y^\flat) \quad \forall\, X,\ Y \in V^*\,. \tag{2.1.23}$$

From now on, we will use the same symbol g for the metric on V^*, since there is little scope for confusion. More generally, the 'musical isomorphisms' allow us to change the type of any tensor.

The metric on V induces a linear isomorphism

$$\begin{array}{rcl} * : \Lambda_p(V) & \longrightarrow & \Lambda_{n-p}(V) \\ \omega & \longmapsto & *\omega \end{array}$$

called the *Hodge dual.* It may be defined recursively by requiring that

$$*\,(\omega \wedge x) \;=\; x^\sharp \rfloor *\omega \quad \forall\, x \in V,\ \omega \in \Lambda_p(V)\,. \tag{2.1.24}$$

This formula can be applied to a decomposable p-form to produce

$$*(x_1 \wedge x_2 \wedge \ldots \wedge x_p) \;=\; {x_p}^\sharp \rfloor {x_{p-1}}^\sharp \rfloor \ldots \rfloor {x_1}^\sharp \rfloor *1 \tag{2.1.25}$$

where we define the Hodge dual of a 0-form by $*1 = z$. If g has signature (s,t) then the inverse $*^{-1}$ is given by

$$**\omega \;=\; (-1)^{p(n-p)+t}\,\omega \quad \forall\, \omega \in \Lambda_p(V)\,. \tag{2.1.26}$$

The Hodge dual and its inverse are extended to $\Lambda(V)$ by linearity.

If ω and ϕ are p-forms then $\omega \wedge *\phi$ is an n-form, hence it is proportional to $*1$. It follows from the symmetry of g that

$$\omega \wedge *\phi \;=\; \phi \wedge *\omega \qquad \omega,\ \phi \in \Lambda_p(V)\,. \tag{2.1.27}$$

This defines an inner product on p-forms given by

$$\omega \wedge *\phi \;=\; (\omega \cdot \phi)\,*1 \qquad \omega,\ \phi \in \Lambda_p(V)\,, \tag{2.1.28}$$

which may be expressed using the basis vectors as

$$\omega \cdot \phi \;=\; \frac{1}{p\,!} X_{a_1} \rfloor X_{a_2} \rfloor \ldots X_{a_p} \rfloor \omega\; X^{a_1} \rfloor X^{a_2} \rfloor \ldots X^{a_p} \rfloor \phi\,. \tag{2.1.29}$$

2.2 The Clifford algebra

The *Clifford algebra* $\mathbf{C}(V,g)$ consists of the vector space $\Lambda(V)$ together with the *Clifford product*, denoted by juxtaposition. The identity element is the unit element of F. The Clifford product of a 1-form and an arbitrary form is given by

$$x\omega \;=\; x \wedge \omega + x^\sharp \rfloor \omega \tag{2.2.1}$$

$$\omega x \;=\; x \wedge \eta\omega - x^\sharp \rfloor \eta\omega \qquad x \in V,\ \omega \in \Lambda(V)\,. \tag{2.2.2}$$

From these relations, the Clifford product of two arbitrary forms can be determined. For a pair of 1-forms we have the familiar equation

$$xy + yx \;=\; 2g(x,y) \quad \forall\, x,\ y \in V\,. \tag{2.2.3}$$

The exterior product and interior derivative may be expressed as

$$x \wedge \omega = \frac{1}{2}\left(x\omega + \eta\omega\, x\right) \qquad (2.2.4)$$

$$x^\sharp \lrcorner\, \omega = \frac{1}{2}\left(x\omega - \eta\omega\, x\right) . \qquad (2.2.5)$$

From (2.1.16) and the Clifford relations we have the useful identity

$$e^a \omega e_a = (-1)^p (n-2p)\omega \quad \forall\, \omega \in \Lambda_p(V) . \qquad (2.2.6)$$

Since elements of the Clifford algebra are simply exterior forms, all of the operations defined in §2.1 can be applied to the Clifford algebra. For this reason, we will often refer to elements of the Clifford algebra as *Clifford forms.* From equations (2.2.1) and (2.2.2) it is clear that η is an automorphism of $\mathbf{C}(V,g)$ and ξ is an anti-automorphism of $\mathbf{C}(V,g)$, hence $\eta(\omega\phi) = \eta\omega\,\eta\phi$ and $(\omega\phi)^\xi = \phi^\xi\omega^\xi$. Similarly, $X \lrcorner$ is an anti-derivation over the Clifford product with respect to η. Since Clifford multiplication does not preserve homogeneity, $\mathbf{C}(V,g)$ is not a $\mathbb{Z}$-graded algebra. However, the action of η does induce a $\mathbb{Z}_2$-grading in the same way as for $\Lambda(V)$, and we classify elements of $\mathbf{C}(V,g)$ as being odd or even accordingly. The even elements of the Clifford algebra form the *even subalgebra* $\mathbf{C}^+(V,g)$.

Equations (2.1.25) and (2.2.1) show that the Hodge dual on a Clifford form satisfies

$$*\omega = \omega^\xi z \qquad \omega \in \mathbf{C}(V,g) . \qquad (2.2.7)$$

The metric on homogeneous forms may be written as

$$\omega \cdot \phi = \mathscr{S}_0(\omega^\xi \phi) \qquad \omega,\ \phi \in \Lambda_p(V) . \qquad (2.2.8)$$

Since the dot product is symmetric, it follows that for arbitrary Clifford forms

$$\mathscr{S}_0(\omega\phi) = \mathscr{S}_0(\phi\omega) \qquad \omega,\ \phi \in \mathbf{C}(V,g) . \qquad (2.2.9)$$

This fact will be useful in calculations.

Certain elements of $\mathbf{C}(V,g)$ are invertible. For example, from (2.2.3) it follows that $x^2 = g(x,x)$ for $x \in V$. If $g(x,x) \neq 0$ then

$$x^{-1} = \frac{1}{g(x,x)}\, x . \qquad (2.2.10)$$

Denoting the group of invertible elements by $\mathbf{C}^*(V,g)$, the *Clifford group* is the subgroup

$$\Gamma = \left\{ s \in \mathbf{C}^*(V,g) : sVs^{-1} = V \right\} . \qquad (2.2.11)$$

The *vector representation* χ is a mapping $\chi : \Gamma \to \mathrm{Aut}\, \mathbf{C}(V,g)$ such that

$$\chi(s)\omega = s\omega s^{-1} \quad \forall\, \omega \in \mathbf{C}(V,g) . \qquad (2.2.12)$$

Since

$$\begin{aligned} 2g(\chi(s)x, \chi(s)y) &= sxs^{-1}\, sys^{-1} + sys^{-1}\, sxs^{-1} \\ &= 2g(x,y) \qquad x,\ y \in V\,, \end{aligned}$$

clearly $\chi(s) \in \text{Ø}(V, g)$, the group of orthogonal transformations on (V, g). The range of χ depends on the parity of n. From now on, we will only consider the Clifford algebra generated by an even-dimensional space where $n = 2r$. In even dimensions, $\chi(\Gamma) = \text{Ø}(V, g)$. For a non-null 1-form, the operator $\rho_x : V \to V$ given by

$$\rho_x(y) = y - \frac{2g(x,y)}{g(x,x)}x \quad \forall\, y \in V \tag{2.2.13}$$

is the reflection of V in the the plane orthogonal to x. Any orthogonal transformation on V extends to an automorphism on $\mathbf{C}(V, g)$, hence any non-null 1-form x is in Γ with $\chi(x) = \eta\rho_x$. Since any orthogonal transformation can be written as a product of reflections, it follows that elements of Γ are of the form $\lambda x^1 x^2 \ldots x^h$, where each x^i is a non-null 1-form and λ is in the center of $\mathbf{C}(V, g)$, which is F for n even. Even elements of Γ form the subgroup Γ^+.

In even dimensions, the kernel of χ consists of the non-zero elements of F, denoted by F^* . We can find a subgroup of Γ with the same range under χ by imposing a normalising condition. We will treat the real and complex cases separately. When $F = \mathbb{C}$, the metric is not characterised by any signature, so the structure of the Clifford algebra depends only on the dimension. Accordingly, we will use the notation $\mathbf{C}_{2r}(\mathbb{C})$ for the Clifford algebra of an $2r$-dimensional complex space. For the subgroup

$$\mathrm{Pin}(2r, \mathbb{C}) = \left\{ s \in \Gamma : s^\xi s = 1 \right\} \tag{2.2.14}$$

we have $\chi(\mathrm{Pin}(2r, \mathbb{C})) = \mathrm{O}(2r, \mathbb{C})$. Since the orthogonal transformations on V formed from an even number of reflections have determinant $+1$, for the subgroup

$$\mathrm{Spin}(2r, \mathbb{C}) = \left\{ s \in \Gamma^+ : s^\xi s = 1 \right\} \tag{2.2.15}$$

we have

$$\chi(\mathrm{Spin}(2r, \mathbb{C})) = \mathrm{SO}(2r, \mathbb{C})\,. \tag{2.2.16}$$

When $F = \mathbb{R}$, the structure of $\mathbf{C}(V, g)$ depends only on the signature of g, so we will use the notation $\mathbf{C}_{p,q}(\mathbb{R})$. The Pin and Spin groups are defined slightly differently in the real case,

$$\mathrm{Pin}(p, q) = \left\{ s \in \Gamma : s^\xi s = \pm 1 \right\} \tag{2.2.17}$$

$$\mathrm{Spin}(p, q) = \left\{ s \in \Gamma^+ : s^\xi s = \pm 1 \right\}\,. \tag{2.2.18}$$

The images under χ are

$$\chi(\mathrm{Pin}(p, q)) = \mathrm{O}(p, q) \quad (n \text{ even}) \tag{2.2.19}$$

$$\chi(\mathrm{Spin}(p,q)) \quad = \quad \mathrm{SO}(p,q)\,. \tag{2.2.20}$$

In (2.2.16) and (2.2.20), the kernel of χ is the multiplicative group $\mathbb{Z}_2 = \{1,\ -1\}$, thus the Spin group is a *double covering* of the special orthogonal group. For a 2-form ϕ and an arbitrary form ω, the Clifford commutator is

$$[\phi, \omega] \quad = \quad -2\, X_a \,\lrcorner\, \phi \wedge X^a \,\lrcorner\, \omega \qquad \phi \in \Lambda_2(V),\ \omega \in \Lambda(V)\,. \tag{2.2.21}$$

With this commutator, the space $\Lambda_2(V)$ forms the Lie algebra of the Spin group. The exponential mapping given by

$$\exp(\omega) \quad = \quad \sum_{k=0}^{\infty} \frac{\omega^k}{k\,!} \qquad \omega \in \mathbf{C}(V,g) \tag{2.2.22}$$

sends this Lie algebra into the component of the Spin group connected to the identity.

2.3 Spinors

In this section, we will only consider the complexified Clifford algebra of an even-dimensional orthogonal space. From a real orthogonal space (V, g) we obtain a complex orthogonal space $(V^{\mathbb{C}}, g)$, where $V^{\mathbb{C}}$ is the complexification of V and g is extended to a metric on $V^{\mathbb{C}}$ by complex linearity. The complex Clifford algebra generated by $V^{\mathbb{C}}$ is isomorphic to the complexification of $\mathbf{C}(V, g)$,

$$\mathbf{C}(V^{\mathbb{C}}, g) \quad \simeq \quad \mathbb{C} \otimes \mathbf{C}(V, g)\,. \tag{2.3.1}$$

If $w \in V^{\mathbb{C}}$, then $w = x + iy$ for some $x,\ y \in V$. The *complex conjugate* of w is given by $\bar{w} = x - iy$. This operation extends to an algebra automorphism of $\mathbf{C}(V^{\mathbb{C}}, g)$, from which we can recover $\mathbf{C}(V, g)$,

$$\Re e\mathbf{C}(V^{\mathbb{C}}, g) \quad = \quad \{\omega \in \mathbf{C}(V^{\mathbb{C}}, g) : \bar{\omega} = \omega\} \tag{2.3.2}$$

$$\quad = \quad \mathbf{C}(V, g)\,. \tag{2.3.3}$$

As we are working over the complex field, the structure of $\mathbf{C}(V^{\mathbb{C}}, g)$ will depend only on n. However, the presence of a natural complex conjugate induces certain real structures which have properties related to the signature of g. On occasions, we will write $\mathbf{C}_{p,q}(\mathbb{C})$ to mean the complexified Clifford algebra generated by an orthogonal space with signature (p, q). It is understood that $\mathbf{C}_{p,q}(\mathbb{C}) \simeq \mathbf{C}_{p+q}(\mathbb{C})$. We will take the standard volume form z to be the volume form of V so that it is real. The square of z then depends on the signature, with

$$z^2 \quad = \quad (-1)^{(n^2+p-q)/2} \quad \forall\ n \tag{2.3.4}$$

$$\quad = \quad (-1)^{(p-q)/2} \quad (n \text{ even}), \tag{2.3.5}$$

although we can always choose $\check{z}$ such that $\check{z}^2 = 1$ by taking $\check{z} = z$ or $\check{z} = iz$ as appropriate. In even dimensions, $\check{z}$ commutes with even elements of the Clifford algebra and anti-commutes with odd elements. We can use this fact to write the action of η on a Clifford form as

$$\eta\omega = \check{z}\omega\check{z} \quad \forall\, \omega \in \mathbf{C}_{2r}(\mathbb{C})\,. \tag{2.3.6}$$

Note that in odd dimensions, the volume form is in the center of the Clifford algebra.

For n even, the Clifford algebra is isomorphic to a total matrix algebra,

$$\mathbf{C}_{2r}(\mathbb{C}) \simeq \mathcal{M}_{2^r}(\mathbb{C}) \tag{2.3.7}$$

and is thus simple. A *left ideal* $\mathcal{L}$ of an algebra $\mathcal{A}$ is a subalgebra of $\mathcal{A}$ such that $\mathcal{A}\mathcal{L} \subseteq \mathcal{L}$. It is *minimal* if it contains no left ideals apart from itself and zero. If $\mathcal{A}$ is simple, the regular representation given by the left action of $\mathcal{A}$ on $\mathcal{L}$ is faithful. It is also *irreducible*, since the only subspaces of $\mathcal{L}$ fixed by the representation are the trivial ones. The *spinor representation* of $\mathbf{C}_{2r}(\mathbb{C})$ is the regular representation on a minimal left ideal $S \subset \mathbf{C}_{2r}(\mathbb{C})$. The minimal left ideal is called the *spinor space*, and its elements are *spinors*. These are usually referred to in the physics literature as *Dirac spinors*. If $\mathbf{C}_{2r}(\mathbb{C})$ is thought of as a matrix algebra, an example of a minimal left ideal is the subalgebra of matrices with all columns but the first being zero. However, it is a fact that all irreducible representations of a simple algebra are equivalent [Alb41]. We will therefore regard *any* space carrying an irreducible representation of $\mathbf{C}_{2r}(\mathbb{C})$ as a spinor space, although for many calculations it will be convenient to use a minimal left ideal as the spinor space. A spinor space must therefore be a complex vector space of dimension 2^r. In any case, the action of a Clifford form ω on a spinor ψ will be denoted by the juxtaposition $\omega\psi$.

The spinor representation of $\mathbf{C}_{2r}(\mathbb{C})$ induces a *reducible* representation of the even subalgebra. This subalgebra has the structure

$$\mathbf{C}^+_{2r}(\mathbb{C}) \simeq \mathcal{M}_{2^{r-1}}(\mathbb{C}) \oplus \mathcal{M}_{2^{r-1}}(\mathbb{C})\,. \tag{2.3.8}$$

An algebra that is either simple or a direct sum of simple components is called *semi-simple*. Since $\check{z}^2 = 1$, the action of $\check{z}$ on S gives a decomposition

$$S = S^+ \oplus S^- \tag{2.3.9}$$

where S^+ and S^- are eigenspaces of $\check{z}$ with eigenvalues $+1$ and -1, respectively. These subspaces are preserved under the action of the even subalgebra, since if ψ is an eigenspinor of $\check{z}$ with eigenvalue $\epsilon = \pm 1$, then

$$\begin{aligned} \check{z}\omega\psi &= \omega\check{z}\psi \\ &= \epsilon\omega\psi \quad \forall\, \omega \in \mathbf{C}^+_{2r}(\mathbb{C})\,. \end{aligned} \tag{2.3.10}$$

Thus the representation of $\mathbf{C}^+_{2r}$ on S is reducible, while the representation of $\mathbf{C}^+_{2r}(\mathbb{C})$ on S^+ or S^- is irreducible. These representations are called the even and odd *semi-spinor representations*, with elements of S^+ and S^- being even or odd *semi-spinors*. They are often referred to as *Weyl spinors*. Each semi-spinor space is a vector space of complex dimension 2^{r-1}. Spinors lying in the same eigenspace are said to have the same *parity*. By restriction, the semi-spinor representations of $\mathbf{C}^+_{2r}(\mathbb{C})$ induce inequivalent

$p-q$ (mod 8)	z^2	c	c^2
0	$z^2=+1$	preserves S^+ and S^-	$(\psi^{\mathrm{c}})^{\mathrm{c}}=+\psi$
2	$z^2=-1$	swaps S^+ and S^-	$(\psi^{\mathrm{c}})^{\mathrm{c}}=+\psi$
4	$z^2=+1$	preserves S^+ and S^-	$(\psi^{\mathrm{c}})^{\mathrm{c}}=-\psi$
6	$z^2=-1$	swaps S^+ and S^-	$(\psi^{\mathrm{c}})^{\mathrm{c}}=-\psi$

Table 2.1: Properties of the charge conjugate.

irreducible representations of $\mathrm{Spin}(2r,\mathbb{C})$.

Spinor space admits an isomorphism $\mathrm{c}: S \to S$ called the *charge conjugate*, with the property that

$$(\omega\psi)^{\mathrm{c}} \;=\; \bar{\omega}\psi^{\mathrm{c}} \quad \forall\, \omega\in\mathbf{C}_{2r}(\mathbb{C})\,. \tag{2.3.11}$$

The precise definition of the charge conjugate depends subtly on the dimension. For our purposes we only require certain properties, which are summarised in Table 2.1. The spinor representation of $\mathbf{C}_{2r}(\mathbb{C})$ induces real representations of the real subalgebra, which may or may not be reducible. When $p-q\equiv 4$ or 6 (mod 8), the complex spinors regarded as a real vector space carry an irreducible representation of $\mathbf{C}_{p,q}(\mathbb{R})$. When $p-q\equiv 0$ or 2 (mod 8), Table 2.1 shows that the charge conjugate has real eigenvalues ± 1. Equation (2.3.11) shows that the real subalgebra preserves the eigenspaces of c, therefore the complex spinors induce a reducible representation of $\mathbf{C}_{p,q}(\mathbb{R})$. Spinors which satisfy

$$\psi^{\mathrm{c}} \;=\; \pm\psi \tag{2.3.12}$$

are called *Majorana spinors*. Each space of Majorana spinors carries an irreducible representation of $\mathbf{C}_{p,q}(\mathbb{R})$. For $p-q\equiv 0$ (mod 8), the eigenvalues of z are also real, so the Majorana spinors carry a reducible representation of the real even subalgebra. Spinors satisfying both (2.3.12) and $z\psi=\pm\psi$ are called *Majorana-Weyl spinors*, and carry irreducible representations of the real even subalgebra.

The spinor space possesses a non-degenerate, $\mathbb{C}$-bilinear inner product $(\ ,\ ): S\times S\to\mathbb{C}$ with ξ as adjoint. For $\omega\in\mathbf{C}_{2r}(\mathbb{C})$ we have

$$(\omega\psi,\phi) \;=\; (\psi,\omega^{\xi}\phi) \quad \forall\,\psi,\ \phi\in S\,, \tag{2.3.13}$$

showing that $(\ ,\ )$ is Spin-invariant. The symmetries of $(\ ,\ )$ depend only on the dimension n, and are summarised in Table 2.2. Where the inner product is block diagonal on $S^+\oplus S^-$, we mean that the inner product of an odd spinor and an even spinor is zero.

The inner product induces an isomorphism between S and its dual space S^*. If

n (mod 8)	$(\ ,\ )$
0	symmetric, block diagonal on $S^+ \oplus S^-$
2	symmetric
4	antisymmetric, block diagonal on $S^+ \oplus S^-$
6	antisymmetric

Table 2.2: Properties of the inner product on S.

$\psi \in S$, then $\bar{\psi} \in S^*$ is defined such that

$$\bar{\psi}(\phi) \quad = \quad (\psi, \phi) \quad \forall\ \phi \in S\,. \tag{2.3.14}$$

This should not be confused with the Dirac adjoint, which is the adjoint with respect to a Hermitian inner product. The space $S \otimes S^*$ may be regarded as a set of linear transformations on S by taking

$$(\psi \otimes \bar{\phi})\alpha \quad = \quad (\phi, \alpha)\psi \quad \forall\ \alpha \in S\,. \tag{2.3.15}$$

When the Clifford algebra is simple, it is isomorphic as an algebra to $S \otimes S^*$. The Clifford product of two tensors is given by

$$(\phi \otimes \bar{\psi})(\alpha \otimes \bar{\beta}) \quad = \quad (\psi, \alpha)\phi \otimes \bar{\beta}\,. \tag{2.3.16}$$

Thus we can always think of a Clifford form as either an exterior form or an element of $S \otimes S^*$. From the Clifford action on spinors, we can deduce the product of a Clifford form and a tensor, since if $\omega \in \mathbf{C}_{2r}(\mathbb{C})$,

$$\begin{aligned} \omega(\psi \otimes \bar{\phi})\alpha \quad &= \quad (\phi, \alpha)\omega\psi \\ &= \quad (\omega\psi \otimes \bar{\phi})\alpha \quad \forall\ \alpha \in S\,. \end{aligned} \tag{2.3.17}$$

Thus

$$\omega(\psi \otimes \bar{\phi}) \quad = \quad \omega\psi \otimes \bar{\phi}\,. \tag{2.3.18}$$

Similarly,

$$(\psi \otimes \bar{\phi})\omega \quad = \quad \psi\ \otimes \overline{\omega^{\xi}\phi}\,. \tag{2.3.19}$$

Elements of $S \otimes S^*$ may be classified as odd or even under η using (2.3.6), since

$$\eta(\psi \otimes \bar{\phi}) \quad = \quad (-1)^r \check{z}\psi\ \otimes \overline{\check{z}\phi}\,. \tag{2.3.20}$$

We can also determine the action of ξ, using the inner product. Table 2.2 shows that

$$(\psi, \phi) = (-1)^{\lfloor r/2 \rfloor}(\phi, \psi) \quad \forall\ \psi,\ \phi \in S\,. \tag{2.3.21}$$

From (2.3.13), we have

$$\begin{aligned}(\alpha,(\psi\otimes\bar{\phi})^{\xi}\beta) &= ((\psi\otimes\bar{\phi})\alpha,\beta)\\ &= (\phi,\alpha)(\psi,\beta)\\ &= (-1)^{\lfloor r/2\rfloor}(\alpha,\phi)(\psi,\beta)\\ &= (-1)^{\lfloor r/2\rfloor}(\alpha,(\phi\otimes\bar{\psi})\beta)\quad\forall\ \alpha,\ \beta\in S\end{aligned}\tag{2.3.22}$$

therefore

$$(\psi\otimes\bar{\phi})^{\xi} = (-1)^{\lfloor r/2\rfloor}\phi\otimes\bar{\psi}\quad\forall\ \psi,\ \phi\in S\,.\tag{2.3.23}$$

Since the Clifford algebra is isomorphic to a matrix algebra, it has a well-defined trace given by

$$\mathrm{Tr}(\omega) = 2^{r}\mathscr{S}_0(\omega)\quad\forall\ \omega\in\mathbf{C}_{2r}(\mathbb{C})\,.\tag{2.3.24}$$

Using the trace, we are able to express a given Clifford form in terms of a basis for forms $\{e^A\}$, where A is a multi-index over the set of naturally ordered sequences of distinct indices. Then $\{e_A\}$ is a set of forms chosen so that

$$\omega = \mathrm{Tr}(\omega e_A)e^A\quad\forall\ \omega\in\mathbf{C}_{2r}(\mathbb{C})\,.\tag{2.3.25}$$

It follows that the trace on $S\otimes S^*$ is given by

$$\mathrm{Tr}(\psi\otimes\bar{\phi}) = (\phi,\psi)\,.\tag{2.3.26}$$

Then an element of $S\otimes S^*$ can be expanded in the basis $\{e^A\}$ as

$$\psi\otimes\bar{\phi} = (\phi,e_A\psi)e^A\,.\tag{2.3.27}$$

The slight abuse of the equals sign is justified by the fact that $S\otimes S^*$ and $\mathbf{C}_{2r}(\mathbb{C})$ are canonically isomorphic. We can then use the projection operators to obtain homogeneous components of $\psi\otimes\bar{\phi}$. The components obey the duality condition

$$\mathscr{S}_{2r-p}\,(\psi\otimes\bar{\phi}) = (-1)^{r}\mathscr{S}_p\left(\psi\otimes\overline{\check{z}\phi}\right)\check{z}\,.\tag{2.3.28}$$

With these conventions, the *Fierz rearrangement formula* is found in the following way. For Clifford forms M and N, equation (2.3.14) shows that

$$\begin{aligned}(\phi,(M\psi\otimes\overline{N^{\xi}\alpha})\beta) &= (\phi,M\psi)(N^{\xi}\alpha,\beta)\\ &= (\phi,M\psi)(\alpha,N\beta)\end{aligned}\tag{2.3.29}$$

for spinors ψ, ϕ, α and β. From (2.3.19) and (2.3.27) we also have

$$(\phi,(M\psi\otimes\overline{N^{\xi}\alpha})\beta) = (\phi,e^A\beta)(\alpha,Ne_AM\psi)\tag{2.3.30}$$

from which we obtain the Fierz formula,

$$(\phi, M\psi)(\alpha, N\beta) \quad = \quad (\phi, e^A\beta)(\alpha, Ne_A M\psi)\,. \tag{2.3.31}$$

We are now in a position to introduce the notion of a pure spinor. A *totally isotropic subspace* X is a subspace of $V^{\mathbb{C}}$ satisfying

$$g(x,y) \quad = \quad 0 \quad \forall\; x,\; y \in X\,.$$

If X has dimension h, then z_X denotes the r-form product of some basis for X. Since g is degenerate on X, z_X is determined only up to complex scalings. X is *maximal* if it is of the highest possible dimension, given by the Witt index of g. As we are working over the complex field, a maximal totally isotropic subspace (MTIS) has dimension r. A spinor ψ may be correlated with an isotropic subspace in the following way. The *null space* T_ψ of ψ is the subspace of $V^{\mathbb{C}}$ given by

$$T_\psi = \left\{x \in V^{\mathbb{C}} : x\psi = 0\right\}\,. \tag{2.3.32}$$

For non-zero ψ, T_ψ is clearly an isotropic space, since for $x,\; y \in T_\psi$,

$$\begin{aligned} 2g(x,y)\psi \quad &= \quad (xy + yx)\psi \\ &= \quad 0\,. \end{aligned}$$

From the definition of χ it follows that

$$T_{s\psi} \quad = \quad \chi(s)T_\psi \quad \forall\; s \in \Gamma\,, \tag{2.3.33}$$

while from (2.3.11) we have

$$T_{\psi^{\mathrm{c}}} \quad = \quad \overline{T_\psi}\,. \tag{2.3.34}$$

The *nullity* of ψ is the (complex) dimension of T_ψ, denoted by $N(\psi)$. This term appears to have been used first by Trautman [TT94]. If the metric on the real space V has indefinite sign then V admits real null vectors. The *real index* of an isotropic subspace is $\dim_{\mathbb{C}}(X \cap \bar{X})$. The real index of ψ is the real index of T_ψ.

In general, there may be many spinors correlated with a given null space. We say that a non-zero spinor ψ is *pure* if T_ψ is maximal. Up to scalings, there is a one-to-one correspondence between ψ and T_ψ, thus ψ is said to *represent* T_ψ. Pure spinors are necessarily semi-spinors, hence we may classify a MTIS as odd or even according to the parity of its representative spinor. In the following lemma, we summarise some standard results. The proof may be found in [BT87].

Lemma 2.3.35 *Let u and v be pure spinors representing T_u and T_v.*

(1) u and v have the same parity iff $\dim_{\mathbb{C}}(T_u \cap T_v) \equiv r \pmod 2$.

(2) For $\lambda,\; \mu \in \mathbb{C}^*$, $\lambda u + \mu v$ is pure iff $\dim_{\mathbb{C}}(T_u \cap T_v) = r$ *or* $r - 2$.

(3) $(u, v) = 0$ iff $T_u \cap T_v \neq \{0\}$.

(4) If $\dim_{\mathbb{C}}(T_u \cap T_v) = h$ *then* $\mathscr{S}_p(u \otimes \bar{v}) = 0$ *for all* $p < h$ *and* $p > 2r - h$*, while*

$$\mathscr{S}_h(u \otimes \bar{v}) = z_{T_u \cap T_v} \,. \tag{2.3.36}$$

Part (2) of the lemma shows that all semi-spinors are pure for $r \leq 3$. In higher dimensions, part (4) shows that a semi-spinor u is pure if and only if

$$\mathscr{S}_p(u \otimes \bar{u}) \quad = \quad 0 \quad \forall\, p \neq r \,. \tag{2.3.37}$$

This is Cartan's characterisation of pure spinors. Then $u \otimes \bar{u}$ is a decomposable form of degree r, which we may identify with z_{T_u}. This form is an eigenvector of the Hodge dual, since

$$*\,(u \otimes \bar{u}) \quad = \quad (-1)^{r(r+1)/2}\, u \otimes \overline{zu} \tag{2.3.38}$$

where $zu = \pm u$ or $\pm iu$, depending on the signature of g and the parity of u. The Hodge dual decomposes the space of r-forms into two eigenspaces, one consisting of *self-dual* forms and the other of *anti self-dual* forms. Since the choice of which space is 'self-dual' is purely conventional, we will choose our space of self-dual forms so that the tensor product of an *even* pure spinor with itself is self-dual.

2.4 Calculus on manifolds

Until now we have considered purely algebraic properties of Clifford algebras. We now consider Clifford algebras and spinors on manifolds. Let $(\mathcal{M}, g)$ be an n-dimensional pseudo-Riemannian manifold. We use the standard notation $T_p\mathcal{M}$ for the space of tangent vectors at $p \in \mathcal{M}$. The collection of tangent spaces is the tangent bundle $T\mathcal{M}$. Similarly, the dual space of cotangent vectors at $p \in \mathcal{M}$ is $T_p^*\mathcal{M}$ and the cotangent bundle is $T^*\mathcal{M}$. The bundle of differential forms is denoted by $\Lambda\mathcal{M}$. A subscript is used to indicate a bundle of homogeneous forms, thus we can identify $\Lambda_1\mathcal{M}$ with $T^*\mathcal{M}$ and $\Lambda_0\mathcal{M}$ with the space of functions $\mathcal{F}(\mathcal{M})$. We use the prefix Γ to indicate the space of *sections* of a bundle. For example, a vector field on $\mathcal{M}$ is an element of $\Gamma\, T\mathcal{M}$.

From g we obtain the unique g-compatible torsion-free covariant derivative ∇ on tensor fields. The covariant derivative has certain fundamental tensors associated with it. The *curvature operator* is a derivation on tensor fields given by

$$R(X,Y) \quad = \quad \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \qquad X,\ Y \in \Gamma\, T\mathcal{M} \,. \tag{2.4.1}$$

This operator is $\mathcal{F}$-linear in X, Y and its operand, thus we may use it to define the $(3,1)$ *curvature tensor* **Curv**, given by

$$\mathbf{Curv}(X,Y,Z,\omega) \quad = \quad \omega(R(X,Y)Z) \tag{2.4.2}$$

where X, Y and Z are arbitrary vector fields and ω is an arbitrary 1-form. Since $R(X,Y)$ is antisymmetric in X and Y, the curvature tensor may be written in terms of a set of *curvature 2-forms* $R^a{}_b$ as

$$\mathbf{Curv} \;=\; 2\, R^a{}_b \otimes e^b \otimes X_a\,. \tag{2.4.3}$$

By contracting **Curv** we obtain the $(2,0)$ *Ricci tensor*,

$$\mathbf{Ric}(X,Y) \;=\; \mathbf{Curv}(X_a, X, Y, e^a)\,. \tag{2.4.4}$$

The Ricci tensor may be written using a set of *Ricci 1-forms* P_a as $\mathbf{Ric} = P_a \otimes e^a$ where $P_a = X_b \,\lrcorner\, R^b{}_a$. Contracting once more we obtain the *curvature scalar* $\mathscr{R}$,

$$\begin{aligned} \mathscr{R} &= X^a \,\lrcorner\, P_a \\ &= \mathbf{Ric}(X_a, X^a)\,. \end{aligned} \tag{2.4.5}$$

We will sometimes make use of the first *Bianchi identity*,

$$R^a{}_b \wedge e^b \;=\; 0 \tag{2.4.6}$$

and the contracted Bianchi identities

$$X_d \,\lrcorner\, X_c \,\lrcorner\, R_{ab} - X_b \,\lrcorner\, X_a \,\lrcorner\, R_{cd} \;=\; 0 \tag{2.4.7}$$

$$X_b \,\lrcorner\, P_a - X_a \,\lrcorner\, P_b \;=\; 0 \tag{2.4.8}$$

$$P_a \wedge e^a \;=\; 0\,. \tag{2.4.9}$$

For $n > 2$, the $(3,1)$ *conformal tensor* is constructed from the curvature tensor in such a way that it is invariant under conformal rescalings of the metric. We write it using a set of conformally invariant *conformal 2-forms* $C^a{}_b$ as

$$C \;=\; 2\, C^a{}_b \otimes e^b \otimes X_a \tag{2.4.10}$$

where

$$C_{ab} \;=\; R_{ab} - \frac{1}{n-2}\,(P_a \wedge e_b - P_b \wedge e_a) + \frac{1}{(n-2)(n-1)}\,\mathscr{R} e_{ab}\,. \tag{2.4.11}$$

The conformal tensor has the same symmetries as the curvature tensor, while the conformal 2-forms satisfy a ‘pairwise symmetry’ condition similar to (2.4.7).

In the absence of torsion, the *exterior derivative* $d : \Gamma\Lambda_p\mathcal{M} \to \Gamma\Lambda_{p+1}\mathcal{M}$ may be expressed in a basis as

$$d \;=\; e^a \wedge \nabla_{X_a}\,. \tag{2.4.12}$$

It is *nilpotent* with $d^2 \equiv 0$, and is an anti-derivation over the exterior product with respect to η. Its adjoint operator, the *co-derivative* $d^* : \Gamma\Lambda_p\mathcal{M} \to \Gamma\Lambda_{p-1}\mathcal{M}$ is given by

$$d^* \;=\; *^{-1} d * \eta\,. \tag{2.4.13}$$

The co-derivative is not an anti-derivation, but it does satisfy $(d^*)^2 \equiv 0$. From the metric-compatibility of ∇, it follows that

$$\nabla_X * \;=\; *\nabla_X \qquad \forall\; X \in \Gamma\, T\mathcal{M} \tag{2.4.14}$$

hence we may write the co-derivative as

$$d^* \;=\; -X^a \rfloor \nabla_{X_a}\,. \tag{2.4.15}$$

The complexified *Clifford bundle* $\mathbf{C}^{\mathbb{C}}(\mathcal{M}, g)$ is identified with the complexification of the exterior bundle,

$$\Lambda^{\mathbb{C}}\mathcal{M} \;=\; \bigcup_{p\in\mathcal{M}} \Lambda(T^*_p\mathcal{M}^{\mathbb{C}})\,. \tag{2.4.16}$$

The Clifford product and exterior operations are defined on each fibre of this bundle as in §2.2. Since ∇ commutes with contractions, it follows that it is a derivation on the tensor and exterior products. If we allow the covariant derivative to act on complex forms, the Clifford relations (2.2.1) and (2.2.2) show that ∇ is also a derivation over Clifford products. The curvature operator on a Clifford form is related to the curvature 2-forms and the Clifford commutator by

$$R(X,Y)\omega \;=\; \frac{1}{2}e^a(X)e^b(Y)\,[R_{ab},\omega] \quad \forall\; \omega \in \Gamma\Lambda^{\mathbb{C}}\mathcal{M}\,. \tag{2.4.17}$$

Since we regard Clifford forms as complex exterior forms, we can differentiate a Clifford form with the exterior derivative or co-derivative. In addition, we have the *Hodge-de Rham operator* $\not{d}$ defined in terms of the Clifford action by

$$\begin{aligned} \not{d} \;&=\; e^a\nabla_{X_a} \\ &=\; d - d^*. \end{aligned} \tag{2.4.18}$$

Clearly $\not{d}$ maps a homogeneous form to an inhomogeneous form. Its square, the *Laplace-Beltrami operator* $\triangle$ given by

$$\triangle \;=\; -(dd^* + d^*d) \tag{2.4.19}$$

preserves the degree of a form. On a Clifford form ω we have

$$\triangle\omega \;=\; \nabla^2\omega - \frac{1}{4}\mathscr{R}\omega - \frac{1}{4}R_{ab}\omega e^{ab} \tag{2.4.20}$$

where ∇^2 is the trace of the Hessian,

$$\nabla^2 \;=\; \nabla_{X_a}\nabla_{X^a} - \nabla_{\nabla_a X^a}\,. \tag{2.4.21}$$

The fibre $\mathbf{C}^{\mathbb{C}}_p(\mathcal{M}, g)$ at $p \in \mathcal{M}$ is isomorphic to $\mathbf{C}(T^*_p\mathcal{M}^{\mathbb{C}}, g) \simeq \mathbf{C}_n(\mathbb{C})$. Although we can always find a spinor representation for the Clifford algebra at each $p \in \mathcal{M}$, there are topological obstructions to forming a bundle of such representations. If $\mathcal{M}$ admits a *spinor bundle* $S(\mathcal{M})$, it is called a *spin manifold*. Locally, we may always consider the spinor bundle to be isomorphic to a sub-bundle of the Clifford bundle, with each fibre being a minimal left ideal of the Clifford algebra.

A spinor field on $\mathcal{M}$ is a section of the spinor bundle. The covariant derivative on tensors naturally induces a covariant derivative on spinor fields, for which we will use the same symbol ∇. The covariant derivative is a derivation with respect to the Clifford action, so for $\omega \in \Gamma\Lambda^{\mathbb{C}}\mathcal{M}$ and $\psi \in \Gamma S(\mathcal{M})$ the derivative of the spinor $\omega\psi$ obeys the Leibniz rule,

$$\nabla_X(\omega\psi) \quad = \quad \nabla_X\omega\,\psi + \omega\nabla_X\psi \quad \forall\; X \in \Gamma\, T\mathcal{M}\,. \tag{2.4.22}$$

It is also compatible with the inner product,

$$X(\psi, \phi) \quad = \quad (\nabla_X\psi, \phi) + (\psi, \nabla_X\phi) \quad \forall\; \psi,\; \phi \in \Gamma S(\mathcal{M}) \tag{2.4.23}$$

and the charge conjugate,

$$(\nabla_X\psi)^{\mathrm{c}} \quad = \quad \nabla_X\psi^{\mathrm{c}} \quad \forall\; \psi \in \Gamma S(\mathcal{M})\,. \tag{2.4.24}$$

The action of the curvature operator on spinor fields is given by

$$R(X,Y)\psi \quad = \quad \frac{1}{2}e^a(X)e^b(Y)\,R_{ab}\psi \quad \forall\; \psi \in \Gamma S(\mathcal{M})\,. \tag{2.4.25}$$

An important operator on spinor fields is the *Dirac operator* D, given by

$$\mathrm{D} \quad = \quad e^a\nabla_{X_a}\,. \tag{2.4.26}$$

Note that the Dirac operator formally resembles the Hodge-de Rham operator. Analogously to the Laplace-Beltrami operator on forms, the *spinor Laplacian* is the square of the Dirac operator. It is related to the Hessian and curvature scalar by

$$\mathrm{D}^2\psi \quad = \quad \nabla^2\psi - \frac{1}{4}\mathscr{R}\psi \quad \forall\; \psi \in \Gamma S(\mathcal{M})\,. \tag{2.4.27}$$

Chapter 3

Classification of Spinors

The problem of classifying spinors is usually formulated as

(1) determining the structure of the spinor orbits under the action of the Spin group;

(2) calculating the isotropy group (stabilizer) of each orbit; and

(3) describing the algebra of invariants of the spinor space.

The orbit of a spinor direction under the Spin group forms a manifold whose structure is determined by the isotropy group of the spinor. Classification of spinors was first studied by Chevalley, who examined the orbit of pure spinor directions [Che54]. He found that the Clifford group acts transitively on the space of pure spinor directions, and that the orbit of pure spinors is the orbit of least dimension. Chevalley's analysis classifies spinors in all dimensions up to six, since in those cases all spinors are pure. More recently, Igusa has classified spinors in dimensions up to twelve [Igu70]. In Igusa's formulation, spinors carry a representation of the Clifford algebra generated by a vector space W equipped with a non-degenerate quadratic form f. The base field F is of characteristic different from 2, and it is assumed that f has maximal index over F. A representative of each orbit is presented, together with its isotropy group as a subgroup of the Spin group, for all dimensions n of W up to twelve. Using similar techniques, full classifications of spinors have been found in thirteen dimensions by Kac and Vinberg [KV78] and in fourteen dimensions by Popov [Pop80] and Zhu [Zhu92]. Popov notes that the case of fourteen dimensions

> "...is one of the last where the problem of classifying spinors [in the sense of Igusa] has a reasonable meaning and can be conclusively solved (the cases of Spin(15) and Spin(16) can, apparently, be completely decomposed, but in higher dimensions difficulties in principal arise)."

The case of sixteen dimensions has been settled by Antonyan and Èlashvili [AE82].

The subspaces of W on which f is identically zero are called isotropic. The condition that f has maximal index means that the maximal totally isotropic subspaces (MTIS) of W are of dimension $r = \lfloor n/2 \rfloor$. There is a one-to-one correspondence between MTIS's and pure spinor directions. A metric g on W can be constructed by taking $g(x, y) = f(x + y) - f(x) - f(y)$ for all $x,\ y \in W$. When W is a real vector space,

g must have signature (p, q) where $|p - q| = 0$ or 1, and $r = \min\{p, q\}$. If W is a complex vector space then g is not characterised by any signature. However, if W is the complexification of a real vector space V, and g is a metric on V extended to W by complex linearity, then W exhibits certain real structures which depend on the signature of g. Little is known about the classification of spinors of a complexified space under the action of the real Spin group, however Kopczyński and Trautman have show that $\mathrm{Pin}(p, q)$ acts transitively on the space of pure spinor directions with a given real index, and that $\mathrm{Spin}(p, q)$ acts transitively on the space of pure spinor directions with a given real index and a given parity [KT92].

In this chapter we consider a 'coarse' classification of semi-spinors using two properties which are invariant under the action of the Spin group (in the following we will usually refer to semi-spinors simply as spinors). The motivation for the first of these is the following. A spinor basis consisting only of pure spinors can always be found, so any spinor can be expressed as a sum of pure spinors. However, the dimension of the space of spinors grows exponentially with n, since it is $2^{\lfloor n/2 \rfloor}$. We first pose the question: what is the *minimum* number of pure spinors required in order to express a given spinor as a sum of pure spinors? Since pure spinors are the 'simplest' spinors, many calculations are easier if the spinor is expressed in this minimal form.

Secondly, we consider the possible values for the nullity of a spinor, which for an impure spinor is necessarily less than r. This property has also been studied by Trautman and Trautman [TT94]. They have calculated the dimension of the space of spinors of a given nullity. In particular, for $n = 8$ and $n > 10$, they show that a 'generic' spinor has nullity 0. Trautman and Trautman also found that there are no spinors of nullity ν such that $r - 4 < \nu < r$ or $\nu = r - 5$. The classification of spinors given in this chapter is coarser than that given by Igusa in the sense that there are many distinct spinor orbits for which these two characteristics are the same.

In the following we will only consider spinors of the complexified Clifford algebra generated by a real $2r$-dimensional vector space equipped with a positive-definite metric. Although the base field is the complex numbers, the choice of metric provides applications to the geometry of Riemannian manifolds. The existence of a globally parallel spinor field on such a manifold leads to a reduction in holonomy, which can be calculated if the isotropy group of the spinor as a subgroup of the real Spin group is known. It is well known that the existence of a parallel pure spinor field implies a reduction of holonomy to $\mathrm{SU}(r)$ [LM89]. Using our classification, we are are able to determine the isotropy group of an impure spinor in some instances.

3.1 Pure spinors

The complexification of a real $2r$-dimensional orthogonal space (V, g) generates the Clifford algebra $\mathbf{C}_{2r}(\mathbb{C})$, which is the complexification of $\mathbf{C}_{2r}(\mathbb{R})$. Since the dimension of V is even, $\mathbf{C}_{2r}(\mathbb{C})$ is isomorphic to the algebra of $2^r \times 2^r$ complex matrices. The spinor representation of $\mathbf{C}_{2r}(\mathbb{C})$ induces a pair of inequivalent irreducible semi-spinor representations of the complex Spin group. The real Spin group is a subgroup of $\mathrm{Spin}(2r, \mathbb{C})$, defined as in §2.2. Note that we must have $s^\xi s = +1$ for $s \in \mathrm{Spin}(2r)$ since the metric is positive-definite. When extended by complex linearity to a metric on $V^{\mathbb{C}}$, g

has maximal Witt index, and so pure spinors correspond to MTIS's of dimension r. We will be considering a classification of semi-spinors based on some of their geometrical properties. We begin by reviewing some aspects of the geometry of pure spinors.

Given $G \subseteq \Gamma$, a pair of spinors ψ and ϕ will be called *G-equivalent* if there exists $s \in G$ such that $\psi = s\phi$. If this is the case we will write $\psi \overset{G}{\sim} \phi$. This relation decomposes S into equivalence classes or *G-orbits*. If a G-orbit is represented by ψ, then the structure of the orbit is determined by the subgroup of G which fixes ψ. When the base field is the complex numbers, Igusa has classified spinors where G is the complex Spin group. For our purposes, however, it is the action of the *real* Spin group on complex spinors that is of interest. We therefore define the *isotropy group* G_ψ as

$$G_\psi \quad = \quad \{s \in \mathrm{Spin}(2r) : s\psi = \psi\} \; . \tag{3.1.1}$$

Note that Igusa's classification is valid for any field of characteristic different from 2, provided that the signature of g is maximal. So if the base field is the real numbers, we have a classification of real spinors under $\mathrm{Spin}(r, r)$, but not under $\mathrm{Spin}(2r)$.

The correspondence between pure spinors and MTIS's allows the isotropy group of a pure spinor to be found. Firstly, we show that given a pure spinor, $V^{\mathbb{C}}$ can be decomposed into the direct sum of a MTIS and its complex conjugate. Let ψ be a pure spinor representing T_ψ. If $x \in T_\psi \cap \overline{T_\psi}$ then $x\psi = 0$ and $\bar{x}\psi = 0$. The vector $i(x - \bar{x})$ is real, and we have

$$\begin{aligned} g(i(x-\bar{x}), i(x-\bar{x}))\psi &= -(x-\bar{x})^2\psi \\ &= 0 \end{aligned}$$

therefore $g(i(x - \bar{x}), i(x - \bar{x})) = 0$. Then $i(x - \bar{x}) = 0$, since g is positive-definite, and thus $\bar{x} = x$. But x is null, and the only real null vector in $V^{\mathbb{C}}$ is the zero vector, so we have $T_\psi \cap \overline{T_\psi} = \{0\}$. Since $\dim_{\mathbb{C}} T_\psi = r$, $V^{\mathbb{C}}$ can be decomposed as the direct sum

$$V^{\mathbb{C}} \quad = \quad T_\psi \oplus \overline{T_\psi} \, . \tag{3.1.2}$$

The decomposition (3.1.2) induces an orthogonal *complex structure* on (V, g), that is, an orthogonal transformation $J : V \to V$ such that $J^2 = -1$. Let J be a $\mathbb{C}$-linear mapping on $V^{\mathbb{C}}$ defined by

$$\begin{aligned} Jx &= ix \quad \forall\, x \in T_\psi \, , \\ \overline{Jx} &= J\bar{x} \quad \forall\, x \in V^{\mathbb{C}} \, . \end{aligned} \tag{3.1.3}$$

Then $\overline{T_\psi}$ is also an eigenspace of J with eigenvalue $-i$. Certainly $J^2 = -1$, so the restriction of J to V is a complex structure on V. We can regard V as a complex vector space of dimension r by defining the action of $\mathbb{C}$ on V by

$$(\lambda + i\mu)v \quad = \quad \lambda v + \mu J v \quad \forall\ \lambda,\ \mu \in \mathbb{R},\ v \in V \, . \tag{3.1.4}$$

We now show that J is an orthogonal transformation on $(V^{\mathbb{C}}, g)$. By (3.1.2), we can always write $v \in V^{\mathbb{C}}$ as $v = x + y$ where $x \in T_\psi$ and $y \in \overline{T_\psi}$. Then

$$\begin{aligned} g(Jv, Jv) &= g(ix - iy, ix - iy) \\ &= -g(x,x) - g(y,y) + 2g(x,y) \\ &= g(x,x) + g(y,y) + 2g(x,y)\,, \quad \text{since } x \text{ and } y \text{ are null} \\ &= g(x+y, x+y) \\ &= g(v,v)\,. \end{aligned} \tag{3.1.5}$$

Since g is symmetric, this implies that $g(Ju, Jv) = g(u, v)$ for all $u,\ v \in V^{\mathbb{C}}$. Then the restriction of J to V is an orthogonal complex structure on (V, g). Regarding V as a complex vector space via (3.1.4), an Hermitian form $\langle\ ,\ \rangle$ on V can be constructed from g and J by taking

$$\langle x, y\rangle = g(x,y) + ig(x, Jy) \quad \forall\ x,\ y \in V\,. \tag{3.1.6}$$

Now we show that J commutes with $\chi(G_\psi)$. If $s \in G_\psi$ and $x \in T_\psi$ then

$$\begin{aligned} \chi(s)x\psi &= sxs^{-1}\psi \\ &= sx\psi \end{aligned}$$

so $\chi(s)x\psi = 0$. Then $\chi(s)x \in T_\psi$, and we have

$$\chi(G_\psi)T_\psi = T_\psi \tag{3.1.7}$$

and

$$\chi(G_\psi)\overline{T_\psi} = \overline{T_\psi} \tag{3.1.8}$$

since G_ψ is real. This implies that for $s \in G_\psi$ and $v = x + y$ where $x \in T_\psi$ and $y \in \overline{T_\psi}$ we have

$$\begin{aligned} J\chi(s)v &= J\chi(s)(x+y) \\ &= Jsxs^{-1} + Jsys^{-1} \\ &= isxs^{-1} - isys^{-1} \quad \text{by (3.1.8)} \\ &= sJxs^{-1} + sJys^{-1} \\ &= \chi(s)Jv\,. \end{aligned} \tag{3.1.9}$$

Clearly, $\chi(s)$ is an isometry of the Hermitian form $\langle\ ,\ \rangle$ for each $s \in G_\psi$. These properties of pure spinors give rise to the following well-known result. The proof is a modification of that found in Lawson and Michelsohn [LM89].

Theorem 3.1.10 *For a pure spinor ψ in $2r$ dimensions, $G_\psi \simeq \mathrm{SU}(r)$.*

Proof. If $s \in \mathrm{Spin}(2r)$ and $s\psi = \lambda\psi,\ \lambda \in \mathbb{C}$, then $\chi(s)T_\psi = T_\psi$. By (3.1.9) we have $\chi(s) \in \mathrm{U}(r)$ where

$$\mathrm{U}(r) = \{\sigma \in \mathrm{SO}(2r) : \sigma J = J\sigma\}\,. \tag{3.1.11}$$

Conversely, if $\sigma \in \mathrm{U}(r)$ then there exists $s \in \mathrm{Spin}(2r)$ such that $\chi(s) = \sigma$. Since $\chi(s)$ commutes with J, it must leave T_ψ and $\overline{T_\psi}$ fixed. Now

$$\begin{aligned} sx\psi &= (sxs^{-1})s\psi \\ &= 0 \quad \forall\, x \in T_\psi \end{aligned} \tag{3.1.12}$$

Since $sxs^{-1} \in T_\psi$, this implies that $s\psi = \lambda\psi,\ \lambda \in \mathbb{C}$. Thus

$$\mathrm{U}(r) \quad = \quad \chi\left(\{s \in \mathrm{Spin}(2r) : s\psi = \lambda\psi,\ \lambda \in \mathbb{C}\}\right). \tag{3.1.13}$$

To find the determinant of $\chi(s)$ we need the fact that for an n-form z in an n-dimensional vector space, we have $\tau z = \det(\tau) z$ for $\tau \in \mathrm{GL}(n, \mathbb{C})$. Consider an Hermitian form $\langle\ ,\ \rangle'$ on T_ψ defined by

$$\langle x, y\rangle' \quad = \quad g(\bar{x}, y)\,. \tag{3.1.14}$$

Now $\mathrm{U}(r)$ is isomorphic to the group

$$\{\tau \in \mathrm{Aut}\ T_\psi : \langle \tau x, \tau y\rangle' = \langle x, y\rangle'\ \forall\ x,\ y \in T_\psi\}$$

and $\mathrm{Aut}\ T_\psi \simeq \mathrm{GL}(r, \mathbb{C})$. The r-form z_{T_ψ} may be identified with $\psi \otimes \bar{\psi}$, and so

$$\begin{aligned} \chi(s)(\psi \otimes \bar{\psi}) &= s(\psi \otimes \bar{\psi})s^{-1} \\ &= s\psi \otimes \overline{s\psi} \\ &= \lambda^2 \psi \otimes \bar{\psi} \end{aligned} \tag{3.1.15}$$

hence $\det \chi(s) = \lambda^2$. Thus

$$\mathrm{SU}(r) \quad = \quad \chi\left(\{s \in \mathrm{Spin}(2r) : s\psi = \pm\psi\}\right)\,. \tag{3.1.16}$$

Since $\mathrm{SU}(r)$ is simply connected, $\chi^{-1}(\mathrm{SU}(r))$ consists of two connected components, one containing 1 and the other containing -1. The component containing -1 cannot fix ψ and so G_ψ is isomorphic to $\mathrm{SU}(r)$. ∎

We now consider an application of spinor geometry to the geometry of Riemannian manifolds. Let $\mathcal{M}$ be a connected, n-dimensional Riemannian manifold. Given a closed curve γ based at $p \in \mathcal{M}$, let $\sigma_\gamma : T_p\mathcal{M} \to T_p\mathcal{M}$ be the transformation given by parallel translation around γ. The set of all such transformations forms the group $\mathcal{H}_x \subseteq \mathrm{O}(T_p\mathcal{M}, g) \simeq \mathrm{O}(n)$. The conjugacy class of $\mathcal{H}_p$ as a subgroup of $\mathrm{O}(n)$ is independent of p. The *holonomy group* $\mathcal{H}(\mathcal{M})$ of $\mathcal{M}$ is the conjugacy class of $\mathcal{H}_p$ in $\mathrm{O}(n)$. A spinor ψ is *parallel* if $\nabla_X\psi = 0$ for all $X \in \Gamma\, T\mathcal{M}$. The following theorem of Lawson and Michelsohn [LM89] shows how the holonomy group of $\mathcal{M}$ is restricted by the presence of a parallel spinor field.

Theorem 3.1.17 *Let $\mathcal{M}$ be an n-dimensional Riemannian spin manifold admitting a globally parallel spinor field ψ. Then*

$$\mathcal{H}(\mathcal{M}) \subseteq G_\psi \tag{3.1.18}$$

where G_ψ is the conjugacy class of G_{ψ_p} in $\mathrm{Spin}(n)$ *at $p \in \mathcal{M}$. Conversely, if $\mathcal{H}_p \subseteq G_{\psi_p}$ for ψ_p, then ψ_p may be extended to a globally parallel spinor field on $\mathcal{M}$.*

Theorem 3.1.10 and Theorem 3.1.17 imply that if a $2r$-dimensional manifold $\mathcal{M}$ admits a parallel pure spinor field ψ then $\mathcal{H}(\mathcal{M}) \subseteq \mathrm{SU}(r)$. The existence of a parallel spinor also imposes strict integrability conditions. We have

$$R(X,Y)\psi \;=\; 0 \quad \forall\; X,\; Y \in \Gamma\, T\mathcal{M}\,. \tag{3.1.19}$$

Then from (2.4.25) we have

$$R_{ab}\psi \;=\; 0\,. \tag{3.1.20}$$

Multiplying on the left with e^b shows that

$$P_a\psi \;=\; 0 \tag{3.1.21}$$

since $e^b R_{ab} = -P_a$ by (2.4.6). Since each P_a annihilates ψ it is null, but it is also real so we must have $P_a = 0$ for each a. Thus an even-dimensional Riemannian manifold admitting a parallel spinor is Ricci-flat.

Now ψ determines a collection of null covectors $T_\psi(\mathcal{M})$ given by

$$T_\psi(\mathcal{M}) \;=\; \left\{x \in \Gamma\, T^*\mathcal{M}^{\mathbb{C}} : x\psi = 0\right\}\,. \tag{3.1.22}$$

The cotangent vectors of $T_\psi(\mathcal{M})$ at a point $p \in \mathcal{M}$ form a MTIS of $T_p^*\mathcal{M}^{\mathbb{C}}$, and so $T_\psi(\mathcal{M})$ determines a distribution of null r-planes contained in the complexified cotangent bundle. Since ψ is pure, in the same way as (3.1.3) we have a tensor field $J \in \Gamma\, {T_1}^1\mathcal{M}$ such that $J^2 x = -x$ for all $x \in \Gamma\, T^*\mathcal{M}$. Such a tensor field is an *almost complex structure* for $\mathcal{M}$. In this case, (3.1.5) shows that J is also an orthogonal transformation. For $x \in \Gamma\, T^*\mathcal{M}^{\mathbb{C}}$ and $X \in \Gamma\, T\mathcal{M}$ we have

$$\begin{aligned}\nabla_X\,(x\psi) &= \nabla_X x\,\psi + x\nabla_X\psi \\ &= \nabla_X x\,\psi\,.\end{aligned} \tag{3.1.23}$$

If $x \in T_\psi(\mathcal{M})$, this shows that $\nabla_X x\,\psi = 0$, hence $\nabla_X x \in T_\psi(\mathcal{M})$. Since ∇ commutes with complex conjugation, it is also true that if $y \in \overline{T_\psi(\mathcal{M})}$ then $\nabla_X y \in \overline{T_\psi(\mathcal{M})}$. Any $v \in \Gamma\, T^*\mathcal{M}$ can be written as $v = x + y$ where $x \in T_\psi(\mathcal{M})$ and $y \in \overline{T_\psi(\mathcal{M})}$. Then

$$\nabla_X(Jv) \;=\; i\nabla_X x - i\nabla_X y \tag{3.1.24}$$

but also

$$\begin{aligned}\nabla_X(Jv) &= \nabla_X J\,v + J\nabla_X v \\ &= \nabla_X J\,v + i\nabla_X x - i\nabla_X y\end{aligned} \tag{3.1.25}$$

and so J is parallel. A Riemannian manifold admitting a parallel orthogonal almost complex structure is called a *Kähler manifold.* This result together with (3.1.21) shows that an even-dimensional Riemannian manifold admitting a parallel pure spinor is Kähler and Ricci-flat.

3.2 Invariants of spinor space

In general, finding the isotropy group of an impure spinor is a difficult problem. Our approach is to investigate some properties of impure spinors which are invariant under the action of the Clifford group Γ. Given an equivalence relation $\sim$ on S, an *invariant* is a function f on S such that $f(\psi) = f(\phi)$ if $\psi \sim \phi$. A set of invariants is *complete* if it contains an element f such that $f(\psi) \neq f(\phi)$ for any pair of spinors such that $\psi \not\sim \phi$. A complete set of invariants is said to *separate* the orbits. In this section, we introduce a pair of invariants which in some sense indicate the amount by which a spinor deviates from being pure. For some spinors, knowledge of these invariants allows us to place restrictions on the corresponding isotropy group, and hence on the holonomy group of a manifold admitting a parallel impure spinor.

In $2r$ dimensions, we can choose a basis of pure spinors for the (2^{r-1})-dimensional space of semi-spinors, hence any spinor can be written as a sum of at most 2^{r-1} pure spinors. Obviously, not all linear combinations of pure spinors will be pure, but it is possible to find *pure subspaces* where each element (excluding the zero spinor) is pure.

Lemma 3.2.1 *Let $\{u_1, \ldots, u_m\}$ be a set of linearly independent pure spinors such that $u_i + u_j$ is pure for all $i,\ j \in \{1, \ldots, m\}$. Then every non-zero spinor $\psi \in \mathrm{sp}_{\mathbb{C}}\{u_1, \ldots, u_m\}$ is pure.*

Proof. Adopting the summation convention, let ψ be a non-zero spinor such that $\psi = \lambda^i u_i$, $\lambda^i \in \mathbb{C}$. In $2r$ dimensions, ψ is pure if and only if $\mathscr{S}_p(\psi \otimes \bar{\psi}) = 0$ for all $p \neq r$. Now

$$\mathscr{S}_p(\psi \otimes \bar{\psi}) \;=\; \lambda^i \lambda^j \mathscr{S}_p(u_i \otimes \bar{u}_j)\,. \tag{3.2.2}$$

Symmetrising, we see that

$$\mathscr{S}_p(\psi \otimes \bar{\psi}) \;=\; \frac{1}{2}\lambda^i \lambda^j \mathscr{S}_p(u_i \otimes \bar{u}_j + u_j \otimes \bar{u}_i)\,. \tag{3.2.3}$$

But each $u_i + u_j$ is pure, therefore

$$\begin{aligned} 0 \;&=\; \mathscr{S}_p\Big((u_i + u_j) \otimes \overline{(u_i + u_j)}\Big) \quad \forall\, p \neq r \\ &=\; \mathscr{S}_p(u_i \otimes \bar{u}_i) + \mathscr{S}_p(u_j \otimes \bar{u}_j) \\ &\qquad + \mathscr{S}_p(u_i \otimes \bar{u}_j + u_j \otimes \bar{u}_i) \end{aligned} \tag{3.2.4}$$

Since each u_i is pure, $\mathscr{S}_p(u_i \otimes \bar{u}_i) = 0$ for all $p \neq r$ hence

$$\mathscr{S}_p(u_i \otimes \bar{u}_j + u_j \otimes \bar{u}_i) \;=\; 0 \quad \forall\, p \neq r \tag{3.2.5}$$

so ψ is pure. ∎

We can always find pure subspaces of dimension r. To illustrate this, we will construct a spinor representation using a minimal left ideal of $\mathbf{C}_{2r}(\mathbb{C})$. Suppose that Y is a maximal totally isotropic subspace of $V^{\mathbb{C}}$, and let z_Y be an r-form product of some basis for Y. Then the subspace $\mathbf{C}_{2r}(\mathbb{C})z_Y$ is a minimal left ideal, hence we may identify it with the space of spinors S. Given a second MTIS X complementary to Y, we have the decomposition $V^{\mathbb{C}} = X \oplus Y$. Bases $\{x^i\}$ and $\{y^i\}$ for X and Y may be chosen so that

$$x^i y^j + y^j x^j \;=\; \delta^{ij} \quad \forall\, i,\ j \in \{1, \ldots, r\}\,. \tag{3.2.6}$$

Such a basis is known as a Witt basis (in the following, the indices i and j run from 1 to r). In this basis, we can fix $z_Y = y^1 \wedge y^2 \wedge \ldots \wedge y^r$. Now any element of $\mathbf{C}_{2r}(\mathbb{C})$ can be written in terms of the x^i's and y^i's. Using (3.2.6), an element of $\mathbf{C}_{2r}(\mathbb{C})$ may be written with all the y^i's occurring on the right-hand side. Since vectors in Y annihilate z_Y it is clear that $\mathbf{C}_{2r}(\mathbb{C})z_Y = \mathbf{C}(X, g)z_Y$. As X is isotropic, its Clifford algebra is identical to its exterior algebra, thus $S = \Lambda(X)z_Y$. The odd and even semi-spinor spaces $S^{\pm}$ may be identified with the corresponding spaces of odd and even forms, $\Lambda^{\pm}(X)z_Y$. Each of these carries an irreducible representation of the even subalgebra, and hence of Spin$(2r, \mathbb{C})$. This identification of the space of spinors will be used throughout the chapter. A basis for S^+ is given by

$$\Big\{ z_Y,\ x^{ij} z_Y,\ x^{ijkl} z_Y, \ldots,\ x^{12\ldots r} z_Y \Big\}$$

where we have used the abbreviation $x^i \wedge x^j = x^{ij}$. Each element of this basis is a pure spinor. For example, z_Y represents Y, while $x^{12} z_Y$ represents the space spanned by $\{x^1, x^2, y^3, y^4, \ldots, y^r\}$. Note that since g is positive-definite on real vectors, we can always choose $X = \overline{Y}$ and $x^i = \bar{y}^i$, however any MTIS complementary to Y may be used, a fact which will be useful later on.

A pure subspace must necessarily consist of spinors with the same parity. It can easily be verified that the set $\mathcal{P}$ of linearly independent even spinors

$$\mathcal{P} \;=\; \Big\{ z_Y,\ x^{12} z_Y,\ x^{13} z_Y, \ldots,\ x^{1r} z_Y \Big\} \tag{3.2.7}$$

has the property that the MTIS's corresponding to any pair of distinct elements intersect in $r-2$ dimensions. This shows that the sum of any two elements of $\mathcal{P}$ is pure, and so by Lemma 3.2.1, $\mathcal{P}$ is a basis for a pure subspace of dimension r. For $r > 3$, extending $\mathcal{P}$ by adding another element of the form $x^{ij} z_Y$, $2 \leq i < j \leq r$ introduces impure spinors, since the MTIS of such an element would intersect with the MTIS of at least one element of $\mathcal{P}$ in less that $r-2$ dimensions. In addition, extending $\mathcal{P}$ by adding adding an element of the form $x^{i_1 \cdots i_{2p}} z_Y$, $p \geq 2$ also introduces impure spinors, since the MTIS of such an element would intersect with the MTIS of z_Y in $r - 2p$ dimensions. Thus it seems that r is the maximal dimension of a pure subspace for $r > 3$. Note that for $r = 3$ the set

$$\mathcal{P}_3 \;=\; \Big\{ z_Y,\ x^{12} z_Y,\ x^{13} z_Y,\ x^{23} z_Y \Big\} \tag{3.2.8}$$

is the basis of a pure subspace, which shows that all semi-spinors are pure for $r \leq 3$. In view of this we might suppose that any semi-spinor can be written as a sum of at most $\lceil 2^{r-1}/r \rceil$ pure spinors. In fact, in low dimensions it is possible to do much better than this. For a spinor ψ, we define the *pure index* $P(\psi)$ to be the least number of pure spinors $\{u_i\}$ such that $\psi = \sum_i u_i$. It is easy to show that $P(\psi)$ is an invariant of the orbit. Let $s \in \Gamma$ and suppose that $P(\psi) = m$ and $P(s\psi) = n$. Then we can write

$$s\psi \quad = \quad u_1 + u_2 + \cdots + u_n \tag{3.2.9}$$

where u_i is pure for each $i \in \{1, 2, \ldots, n\}$. So

$$\psi \quad = \quad s^{-1}u_1 + s^{-1}u_2 + \cdots + s^{-1}u_n. \tag{3.2.10}$$

Now if u is pure, then so is su, and so we have $P(\psi) = m \leq n$. By the same argument, $n \leq m$ and so $P(\psi) = P(s\psi)$.

The second invariant is the nullity of a spinor, as discussed in §2.3. Pure spinors are characterised by the fact that they correspond to maximal totally isotropic subspaces, and this correspondence is one-to-one up to scalings. Impure spinors also induce totally isotropic subspaces via (2.3.32), but these are not maximal, nor is the correspondence unique. From (2.3.33), it is clear that the nullity of a spinor is invariant under the action of the Clifford group, and hence of the Spin group. A pure spinor always has $P(\psi) = 1$ and $N(\psi) = r$. The following lemma places an upper bound on the nullity of an impure spinor. A similar result has been found independently by Trautman and Trautman [TT94]. Details are given in §3.5.

Lemma 3.2.11 *For an impure semi-spinor ψ in $2r$ dimensions, $N(\psi) \leq r - 4$.*

Proof. Suppose that the spinor space of $\mathbf{C}_{2r}(\mathbb{C})$ is identified with $\Lambda(X)z_Y$ as above. Now the Clifford group acts transitively on the space of pure spinors, so for any pair of pure spinors u and v there exists $s \in \Gamma$ such that $su = v$ and hence $\chi(s)T_u = T_v$. The null space of an impure semi-spinor ψ is contained in some MTIS represented by a pure spinor, which we can map to z_Y using an element of Γ. That is, there exists $s \in \Gamma$ such that $s\psi$ is even and $sT_\psi s^{-1} \subseteq Y$, thus $ys\psi = 0$ for all $y \in sT_\psi s^{-1}$. Suppose that $N(\psi) = h$. We can choose a Witt basis $\{x^i, y^i\}$ for $V^{\mathbb{C}}$ such that $\{x^i\}$ is a basis for X, $\{y^i\}$ is a basis for Y and $\{y^{r-h+1}, \ldots, y^r\}$ is a basis for $sT_\psi s^{-1}$. Now $s\psi$ can be written as

$$s\psi \quad = \quad \omega z_Y \tag{3.2.12}$$

where $\omega \in \Lambda^+(X)$. For each basis vector y^i,

$$\begin{aligned} y^i \omega z_Y &= \left([y^i, \omega] + \omega y^i\right) z_Y \\ &= [y^i, \omega] z_Y \quad \text{since } y^i z_Y = 0, \\ &= 2(y^i)^\sharp \rfloor \omega z_Y \quad \text{since } \omega \text{ is even.} \end{aligned} \tag{3.2.13}$$

Now $y^i s\psi = 0$ for $i \in \{r - h + 1, \ldots, r\}$, hence

$$(y^i)^\sharp \rfloor \omega z_Y \quad = \quad 0 \quad \text{for } i \in \{r - h + 1, \ldots, r\}. \tag{3.2.14}$$

Since $\{x^i, y^i\}$ is a Witt basis, $(y^i)^\sharp \rfloor \omega \in \Lambda^-(X)$ and so (3.2.14) implies that

$$(y^i)^\sharp \rfloor \omega \quad = \quad 0 \quad \text{for } i \in \{r - h + 1, \ldots, r\}. \tag{3.2.15}$$

This can only be true if $\omega \in \Lambda^+(\hat{X})$ where $\hat{X} = \mathrm{sp}_{\mathbb{C}}\{x^1, \ldots, x^{r-h}\}$. Now we examine some cases. If $h = r-1$ then $\Lambda^+(\hat{X})$ is 1-dimensional, thus $s\psi$ is proportional to z_Y, which is pure. But $s\psi$ is pure if and only if ψ is pure. This is a contradiction since $h < r$. If $h = r-2$ then

$$s\psi \quad = \quad \left(\lambda_1 + \lambda_2 x^{12}\right) z_Y, \quad \lambda_i \in \mathbb{C}. \tag{3.2.16}$$

Thus ψ is a linear combination of two pure spinors with corresponding MTIS's Y and $\mathrm{sp}_{\mathbb{C}}\{x^1, x^2, y^3, \ldots, y^r\}$. Since the dimension of their intersection is $r-2$, $s\psi$ is pure, which is a contradiction. If $h = r-3$ then

$$s\psi \quad = \quad \left(\lambda_1 + \lambda_2 x^{12} + \lambda_3 x^{13} + \lambda_4 x^{23}\right) z_Y, \quad \lambda_i \in \mathbb{C}. \tag{3.2.17}$$

As noted in (3.2.8), any linear combination of these spinors is pure, which is a contradiction, so $N(\psi) \leq r-4$. ■

In the case of a spinor ψ with pure index 2, Lemma 3.2.11 can be used to show that the null space of ψ is precisely the intersection of the MTIS's of its pure components.

Lemma 3.2.18 *Let ψ be an impure semi-spinor such that $\psi = u_1 + u_2$, where u_1 and u_2 are pure. Then $T_\psi = T_{u_1} \cap T_{u_2}$.*

Proof. It is clear that $T_{u_1} \cap T_{u_2} \subseteq T_\psi$. Consider $w \in T_\psi$. Now

$$(xw + wx)\psi \quad = \quad 0 \quad \forall\, x \in T_{u_1} \cap T_{u_2} \tag{3.2.19}$$

so $g(x, w) = 0$ for all $x \in T_{u_1} \cap T_{u_2}$, that is, $w \in (T_{u_1} \cap T_{u_2})^\perp$. Suppose that $\dim_{\mathbb{C}} T_{u_1} \cap T_{u_2} = h$. Since $T_{u_1} \cap T_{u_2} \subseteq T_\psi$, in $2r$ dimensions we must have $h \leq r-4$ by Lemma 3.2.11. We may choose a Witt basis $\{x^i, y^i\}$ such that $T_{u_1} = \mathrm{sp}_{\mathbb{C}}\{x^1, \ldots, x^r\}$, $T_{u_1} \cap T_{u_2} = \mathrm{sp}_{\mathbb{C}}\{x^1, \ldots, x^h\}$ and $T_{u_2} = \mathrm{sp}_{\mathbb{C}}\{x^1, \ldots, x^h, y^{h+1}, \ldots, y^r\}$. Let $U_1 = \mathrm{sp}_{\mathbb{C}}\{x^{h+1}, \ldots, x^r\}$ and $U_2 = \mathrm{sp}_{\mathbb{C}}\{y^{h+1}, \ldots, y^r\}$. Then $T_{u_1} \cap T_{u_2}$ is orthogonal to U_1, U_2 and itself. These subspaces are pairwise-disjoint, and the sum of their dimensions is $2r - h$ so we see that

$$(T_{u_1} \cap T_{u_2})^\perp \quad = \quad (T_{u_1} \cap T_{u_2}) \oplus U_1 \oplus U_2\,. \tag{3.2.20}$$

Hence we can write w as

$$w \quad = \quad w_0 + w_1 + w_2 \tag{3.2.21}$$

where

$$w_0 \in T_{u_1} \cap T_{u_2},\ w_1 \in U_1 \text{ and } w_2 \in U_2\,.$$

Suppose that $w_1 = 0$. Then since w, $w_0 \in T_\psi$ and $w_2 \in T_{u_2}$ we have $w_2 u_1 = 0$. But then $w_2 \in T_{u_1}$ and so $w_2 = 0$. That is, $w \in T_{u_1} \cap T_{u_2}$. Similarly, if $w_2 = 0$ then $w \in T_{u_1} \cap T_{u_2}$.

Conversely, suppose that $w_1 \neq 0$ and $w_2 \neq 0$. Since $w \in T_\psi$, it is null, therefore $g(w_1, w_2) = 0$ and we may choose a new Witt basis so that

$$x^{h+1} = w_1 \text{ and } y^{h+2} = w_2\,.$$

Then since $w\psi = 0$ we have

$$x^{h+1}u_2 \;=\; -y^{h+2}u_1\,. \tag{3.2.22}$$

Now $h+4 \le r$, so we can multiply both sides of (3.2.22) by x^{h+3} to obtain

$$\begin{aligned} x^{h+3}x^{h+1}u_2 &= -x^{h+3}y^{h+2}u_1 \\ &= y^{h+2}x^{h+3}u_1 \\ &= 0\,. \end{aligned} \tag{3.2.23}$$

But also

$$\begin{aligned} y^{h+3}x^{h+3}x^{h+1}u_2 &= \left(1 - x^{h+3}y^{h+3}\right)x^{h+1}u_2 \\ &= x^{h+1}u_2 \end{aligned} \tag{3.2.24}$$

so $x^{h+1}u_2 = 0$, which is a contradiction. Thus the only possibility is that $w_1 = w_2 = 0$, and $w \in T_{u_1} \cap T_{u_2}$. ∎

Since the MTIS's corresponding to a pair of pure spinors with the same parity must intersect in r (mod 2) dimensions, an immediate consequence of Lemma 3.2.18 is that a spinor ψ with $P(\psi) = 2$ has $N(\psi) \equiv r - 4$ (mod 2). This is not necessarily the case if $P(\psi) > 2$. For example, in fourteen dimensions ($r = 7$), Table 3.2 in §3.5 shows that there are spinors of nullity 0. As we shall see in the next sections, for dimensions eight and ten the value of $N(\psi)$ determines $P(\psi)$, and vice-versa.

While these two functions are invariants of the spinor space, in general they do not separate the orbits. In the next sections we will determine the pure index and nullity for all spinors up to twelve dimensions, and partially in fourteen dimensions. While this relies partly on the classification of spinors in twelve and fourteen dimensions, since our aims are modest we are able to use more elementary techniques than those used in [Igu70] and [Pop80] for the full classification.

3.3 Spinors in eight dimensions and triality

Eight dimensions ($r = 4$) is the lowest even dimension which admits impure spinors. Since the semi-spinor spaces are 8-dimensional and admit 4-dimensional pure subspaces we can determine the pure index and nullity of an impure spinor.

Theorem 3.3.1 *Let ψ be an impure semi-spinor in eight dimensions. Then $N(\psi) = 0$ and $P(\psi) = 2$.*

Proof. The space of even semi-spinors S^+ can be decomposed into two pure subspaces S_1^+ and S_2^+ where

$$\begin{aligned} S_1^+ &= \mathrm{sp}_{\mathbb{C}}\left\{z_Y,\; x^{12}z_Y,\; x^{13}z_Y,\; x^{14}z_Y\right\}, \\ S_2^+ &= \mathrm{sp}_{\mathbb{C}}\left\{x^{23}z_Y,\; x^{24}z_Y,\; x^{34}z_Y,\; x^{1234}z_Y\right\} \end{aligned}$$

and $S^+ = S_1^+ \oplus S_2^+$. A similar decomposition is possible for the space of odd semi-spinors S^-. Hence ψ can always be written as a sum of two pure spinors, one lying in S_1^+ and the other in S_2^+. The intersection of the MTIS's represented by the two pure spinors must have dimension 4 (mod 2), but it cannot be 4 or 2 since then ψ would be pure. By Lemma 3.2.18 we must have $N(\psi) = 0$. ∎

The spinor representation of $\mathrm{Spin}(8, \mathbb{C})$ has the interesting property of *triality*. The three spaces $V^{\mathbb{C}}$, S^+ and S^- each have complex dimension eight, and the Spin-invariant inner product $(\ , \ )$ with ξ as adjoint is symmetric, $\mathbb{C}$-bilinear and non-degenerate on each of the semi-spinor spaces. We may take the inner product to be scaled so that

$$\overline{(\psi, \phi)} \quad = \quad (\psi^{\mathrm{c}}, \phi^{\mathrm{c}}) \quad \forall\ \psi,\ \phi \in S\,. \tag{3.3.2}$$

Then the three spaces $(V^{\mathbb{C}}, g)$, S^+ and S^- with $(\ , \ )$ as metric are isomorphic as orthogonal spaces. Cartan's 'principle of triality' is the existence of an isometry which cyclically permutes these three spaces. For this dimension and signature, the complex semi-spinors are simply the complexifications of the real semi-spinors, and the charge conjugation operator preserves the semi-spinor spaces. From Table 2.1, we can see that this only occurs when the signature satisfies $p - q \equiv 0 \pmod 8$. It then makes sense to talk about the real semi-spinor spaces given by $\Re e S^{\pm} = \{\psi \in S^{\pm} : \psi^{\mathrm{c}} = \psi\}$. On $\Re e S^{\pm}$, the inner product is positive-definite, so V, $\Re e S^+$ and $\Re e S^-$ are also isomorphic as real orthogonal spaces. For a semi-spinor u, equation (2.3.20) shows that $u \otimes \bar{u}$ is an even form, while (2.3.23) shows that it is also even under ξ. In eight dimensions, the only forms which satisfy both conditions are 0-forms, 4-forms and 8-forms. The 0-form component of $u \otimes \bar{u}$ is proportional to (u, u), and dual to the 8-form component by (2.3.28), so u is pure if and only if $(u, u) = 0$. Since the inner product is positive-definite, pure spinors are necessarily complex.

Theorem 3.1.10 shows that the isotropy group of a pure spinor in eight dimensions is SU(4). In general, the converse is not true. The following procedure due to Benn shows how a pure spinor can be constructed from a complex impure semi-spinor of a certain type [Ben90]. We then show that the isotropy group of the impure spinor is also SU(4). Consider an impure semi-spinor u. Since $(u, u) \neq 0$ we can scale u to obtain a unit spinor $\hat{u}$. The charge conjugate $\hat{u}^{\mathrm{c}}$ has the same parity as $\hat{u}$ and is also a unit spinor by (3.3.2). Now $(\hat{u}, \hat{u}^{\mathrm{c}})$ is real, and we have the inequality

$$(\hat{u}, \hat{u}^{\mathrm{c}}) \geq 1 \tag{3.3.3}$$

with equality if and only if $\hat{u} = \hat{u}^{\mathrm{c}}$. To see this, we observe that $i(\hat{u} - \hat{u}^{\mathrm{c}})$ is a real spinor. The inner product is positive-definite on real spinors, thus

$$-((\hat{u} - \hat{u}^{\mathrm{c}}), (\hat{u} - \hat{u}^{\mathrm{c}})) \geq 0 \tag{3.3.4}$$

from which (3.3.3) is immediate.

Now it is clear that if $(\hat{u}, \hat{u}^{\mathrm{c}}) = 1$ then u is proportional to a real spinor. On the other hand, if we suppose that u is proportional to a real spinor, say u', then the only possible normalisations of u are $\hat{u} = \pm 1/\sqrt{(u', u')}\, u'$. Thus $\hat{u}$ is real and so $(\hat{u}, \hat{u}^{\mathrm{c}}) = 1$. That is, $(\hat{u}, \hat{u}^{\mathrm{c}}) = 1$ if and only if u is proportional to a real spinor.

Suppose that u is not proportional to a real spinor. Then $(\hat{u}, \hat{u}^{\mathrm{c}}) > 1$, and with $\lambda = (\hat{u}, \hat{u}^{\mathrm{c}})$ the spinor $\hat{v}$ given by

$$\hat{v} \;=\; \frac{\lambda \hat{u} - \hat{u}^{\mathrm{c}}}{\sqrt{\lambda^2 - 1}} \tag{3.3.5}$$

is orthogonal to $\hat{u}$ with $(\hat{v}, \hat{v}) = -1$. So if ψ is given by

$$\psi = \hat{u} + \hat{v} \tag{3.3.6}$$

then ψ is null and hence pure. This argument can be used to show that the isotropy group of u is SU(4), as was noted by Lawson and Michelsohn [LM89].

Theorem 3.3.7 *Let u be a semi-spinor in eight dimensions such that u is not proportional to a real spinor. Then $G_u \simeq \mathrm{SU}(4)$.*

Proof. If u is pure then the result follows from Theorem 3.1.10. If u is impure then we can construct a pure spinor ψ from u as in (3.3.6). Clearly $G_u = G_{\hat{u}}$ and $G_{\hat{u}} = G_{\hat{u}^{\mathrm{c}}}$ since the isotropy group is a real subgroup, hence $G_u \subseteq G_\psi$.

Consider $s \in G_\psi$. Expressing ψ in terms of $\hat{u}$ and $\hat{u}^{\mathrm{c}}$, the equation $s\psi = \psi$ can be written as

$$\left(\sqrt{\lambda^2 - 1} + \lambda\right) s\hat{u} - s\hat{u}^{\mathrm{c}} \;=\; \left(\sqrt{\lambda^2 - 1} + \lambda\right) \hat{u} - \hat{u}^{\mathrm{c}}\,. \tag{3.3.8}$$

Taking the charge conjugate of (3.3.8) we have

$$s\hat{u} - \left(\sqrt{\lambda^2 - 1} + \lambda\right) s\hat{u}^{\mathrm{c}} \;=\; \hat{u} - \left(\sqrt{\lambda^2 - 1} + \lambda\right) \hat{u}^{\mathrm{c}} \tag{3.3.9}$$

since s and λ are real, and $\lambda > 1$. As $\sqrt{\lambda^2 - 1} + \lambda > 1$, we can solve the system of equations (3.3.8) and (3.3.9) to show that $s\hat{u} = \hat{u}$, hence $G_u = G_\psi \simeq \mathrm{SU}(4)$. ∎

If we now consider u to be a spinor field on an 8-dimensional Riemannian spin manifold $\mathcal{M}$, then ψ is a pure spinor field on $\mathcal{M}$. Furthermore, if u is globally parallel then Theorem 3.1.17 shows that $\mathcal{H}(\mathcal{M}) \subseteq \mathrm{SU}(4)$. The spinor ψ is also parallel, since the covariant derivative is compatible with the spinor inner product and commutes with charge conjugation. Thus the only parallel semi-spinors which do not immediately imply a reduction of $\mathcal{H}(\mathcal{M})$ to SU(4) are those proportional to a real spinor, which are necessarily impure. It is well-known that the principle of triality may be used to determine the isotropy group of a real semi-spinor as follows. Let

$$E \;=\; V \oplus \Re e S^+ \oplus \Re e S^-\,. \tag{3.3.10}$$

Then we have a faithful representation $\rho : \mathrm{Spin}(8) \to \mathrm{Aut}\, E$ induced from χ and the spinor representation by

$$\rho(s)(x + u + v) \;=\; \chi(s)x + su + sv \tag{3.3.11}$$

where $x \in V$, $u \in \Re e S^+$ and $v \in \Re e S^-$. With the metric G defined by

$$G(\Phi_1, \Phi_2) \;=\; g(x_1, x_2) + (u_1, u_2) + (v_1, v_2) \tag{3.3.12}$$

for $\Phi_i = x_i + u_i + v_i$, we have an orthogonal space (E, G). Given a mapping $\sigma \in \mathrm{SO}(E, G)$ which preserves each of the subspaces V, $\Re eS^+$ and $\Re eS^-$, there exists $s \in \mathrm{Spin}(8)$ such that $\sigma = \rho(s)$.

Because of the isometry between V, $\Re eS^+$ and $\Re eS^-$, it is possible to construct an orthogonal transformation on (E, G) which permutes these three spaces, the *triality map* T. The construction of T is given in [BT87]. The triality map is an isometric isomorphism which sends

$$T : V \longrightarrow \Re eS^+ \longrightarrow \Re eS^- \longrightarrow V\,.$$

Associated with T is the *triality automorphism* τ of Spin(8). Given $s \in \mathrm{Spin}(8)$ we have $T\rho(s)T^{-1} \in \mathrm{SO}(E, G)$, and $T\rho(s)T^{-1}$ preserves V, $\Re eS^+$ and $\Re eS^-$. Thus $T\rho(s)T^{-1} = \rho(t)$ for some $t \in \mathrm{Spin}(8)$. Since ρ is faithful we can define τ by

$$T\rho(s)T^{-1} \quad = \quad \rho(\tau s)\,. \tag{3.3.13}$$

We wish to find the subgroup of Spin(8) which leaves a spinor in $\Re eS^+$ fixed, and then see how V transforms under this subgroup. Because of the triality map, this is equivalent to finding the group which leaves a vector in V fixed, and seeing how real semi-spinors transform under it. The subgroup of SO(8) that leaves a vector x in V fixed is isomorphic to SO(7). Denoting the pre-image of this group under χ by G_x, we have $G_x \simeq \mathrm{Spin}(7) \subset \mathrm{Spin}(8)$. The spinor space of Spin(7) has real dimension 8, so the real semi-spinor spaces $\Re eS^+$ and $\Re eS^-$ each carry an irreducible representation of G_x. Now if $\rho(s)x = x$ then $\rho(\tau s)Tx = Tx$. Since G_x leaves x fixed under the vector representation, this shows that $\tau G_x \simeq \mathrm{Spin}(7)$ leaves $Tx \in \Re eS^+$ fixed under the spinor representation. Since G_x acts irreducibly (under ρ) on $\Re eS^+$ and $\Re eS^-$, τG_x acts irreducibly on $\Re eS^-$ and V. Thus the isotropy group of a real spinor is Spin(7).

It is worth pointing out that the orbit structure is substantially different if semi-spinors are classified under the complex Spin group. We have shown that there are at least three distinct orbit types for spinors in eight dimensions, consisting of either pure spinors, impure spinors with isotropy group SU(4), or impure real spinors with isotropy group Spin(7). Under $\mathrm{Spin}(8, \mathbb{C})$, Igusa has shown that there are only two orbit types: the orbit of pure spinors, and a collection of impure spinor orbits. The isotropy group of an impure spinor under $\mathrm{Spin}(8, \mathbb{C})$ is $\mathrm{Spin}(7, \mathbb{C})$, while a pure spinor has isotropy group $\mathrm{SL}(4, \mathbb{C}) \cdot (\mathbb{C}_a)^6$. Here, the symbol $\cdot$ denotes the semi-direct product, and $\mathbb{C}_a$ is the additive group of complex numbers.

3.4 Spinors in ten dimensions

In ten dimensions ($r = 5$) we shall once again represent the spinor space by a minimal left ideal of the Clifford algebra. Later, we will utilise the fact that S admits pure subspaces. The 1-dimensional subspaces spanned by $x^{12345}z_Y$ and z_Y are pure, representing X and Y respectively. The 5-dimensional subspaces $\Lambda_1(X)z_Y$ and $\Lambda_4(X)z_Y$ have bases

$$\left\{x^1 z_Y,\ x^2 z_Y,\ x^3 z_Y,\ x^4 z_Y,\ x^5 z_Y\right\}$$

and

$$\left\{x^{2345}z_Y,\; x^{1345}z_Y,\; x^{1245}z_Y,\; x^{1235}z_Y,\; x^{1234}z_Y\right\}.$$

In each basis, the pairwise-intersection of the MTIS's corresponding to each basis vector intersect in three dimensions, so $\Lambda_1(X)z_Y$ and $\Lambda_4(X)z_Y$ are pure subspaces. From this we can determine the pure index and nullity of an impure spinor in ten dimensions.

Theorem 3.4.1 *Let ψ be an impure semi-spinor in ten dimensions. Then $N(\psi) = 1$ and $P(\psi) = 2$.*

Proof. Since Γ acts transitively on the space of pure spinors, there exists $s \in \Gamma$ which maps some pure component of ψ to z_Y, hence $s\psi = (1+\omega)z_Y$ for some $\omega \in \Lambda^+(X)$ with $\mathscr{S}_0(\omega) = 0$. If $\mathscr{S}_2(\omega) = 0$ then $\omega \in \Lambda_4(X)$, so ωz_Y is pure. In that case, clearly $P(\psi) = 2$. Suppose that $\mathscr{S}_2(\omega) \neq 0$. The Lie algebra of $\mathrm{Spin}(2r, \mathbb{C})$ is the space of two forms $\Lambda_2(V^{\mathbb{C}})$ with the Clifford commutator as Lie bracket. Since $\mathrm{Spin}(2r, \mathbb{C})$ is connected, exponentiation maps this Lie algebra onto $\mathrm{Spin}(2r, \mathbb{C})$, thus $\exp(-\mathscr{S}_2(\omega)) \in \mathrm{Spin}(2r, \mathbb{C}) \cap \Lambda^+(X)$. Since X is isotropic, its Clifford algebra is identical to its exterior algebra. Using the definition of the exponential we have

$$\exp(-\mathscr{S}_2(\omega)) \;=\; 1 - \mathscr{S}_2(\omega) + \frac{1}{2}\mathscr{S}_2(\omega)^2\,, \tag{3.4.2}$$

noting that forms of degree higher than five must vanish. Then it is clear that

$$\exp(-\mathscr{S}_2(\omega))\, s\psi \;=\; (1+\omega')z_Y \quad \text{where } \omega' \in \Lambda_4(X)\,. \tag{3.4.3}$$

Thus ψ can be written as

$$\psi \;=\; s^{-1}\exp(\mathscr{S}_2(\omega))\, z_Y + s^{-1}\exp(\mathscr{S}_2(\omega))\, \omega' z_Y\,. \tag{3.4.4}$$

Each of the components in (3.4.4) is pure, hence $P(\psi) = 2$.

The two spinors in (3.4.4) have the same parity, so the corresponding MTIS's must intersect in 1, 3 or 5 dimensions. They cannot intersect in 3 or 5 dimensions, since then ψ would be pure. So by Lemma 3.2.18, $N(\psi) = 1$. ∎

Knowing the pure index and nullity of an impure spinor in ten dimensions allows us to relate it to a spinor in eight dimensions. Previously, we have used the fact that the (complex) Clifford group acts transitively on the space of pure spinors. In the following we show how the isotropy group of a spinor in ten dimensions is related to that of a spinor in eight dimensions. Since the isotropy group is a subgroup of the *real* Spin group, we must first show that in some instances two spinors are in the same orbit under $\mathrm{Spin}(2r)$ rather than $\mathrm{Spin}(2r, \mathbb{C})$. Only in the former case can we guarantee that their isotropy groups are isomorphic.

Lemma 3.4.5 *Let (V, g) be a real $2r$-dimensional orthogonal space with $r > 1$, and let Y be a maximal totally isotropic subspace of $V^{\mathbb{C}}$. Given a null vector v there exists $s \in \mathrm{Spin}(2r)$ such that $\chi(s)v \in Y$.*

Proof. Consider a null vector $v \in V^{\mathbb{C}}$. Since g is positive-definite, we can put $X = \overline{Y}$ and write $V^{\mathbb{C}} = X \oplus Y$. Then v may be written as

$$v = x + y \quad \text{for } x \in X,\ y \in Y \tag{3.4.6}$$

where x and y satisfy the inequalities $g(x, \bar{x}) \geq 0$ and $g(y, \bar{y}) \geq 0$, with equality only when $x = 0$ or $y = 0$. Since v is null we have $g(x, y) = 0$.

Now for any non-zero null vector $u \in V^{\mathbb{C}}$, the vector σ_u given by

$$\sigma_u = \frac{1}{\sqrt{2g(u, \bar{u})}}\,(u + \bar{u}) \tag{3.4.7}$$

is a real unit vector. Its image under χ is a reflection in the plane orthogonal to $u + \bar{u}$ followed by η, hence $\sigma_u \in \mathrm{Pin}(2r)$ with ${\sigma_u}^{-1} = \sigma_u$. The action of σ_u on an arbitrary vector $w \in V^{\mathbb{C}}$ is given by

$$\begin{aligned}\chi(\sigma_u)w &= \sigma_u w {\sigma_u}^{-1} \\ &= (2g(w, \sigma_u) - w\sigma_u)\,\sigma_u \\ &= 2g(w, \sigma_u)\sigma_u - w\end{aligned} \tag{3.4.8}$$

since ${\sigma_u}^2 = 1$. Furthermore, σ_u sends u to $\bar{u}$ since

$$\begin{aligned}\chi(\sigma_u)u &= \frac{2g(u, u + \bar{u})}{2g(u, \bar{u})}\,(u + \bar{u}) - u \\ &= \bar{u}\,.\end{aligned} \tag{3.4.9}$$

Suppose that $y \neq 0$. Then

$$\begin{aligned}\chi(\sigma_y)v &= \sigma_y v\, {\sigma_y}^{-1} \\ &= -x + \bar{y}\end{aligned} \tag{3.4.10}$$

and clearly $\chi(\sigma_y)v \in X$. By (3.4.9), we can map $w = \chi(\sigma_y)v$ into Y by acting on it with σ_w. That is,

$$\begin{aligned}\chi(\sigma_w\sigma_y)v &= \chi(\sigma_w)\chi(\sigma_y)v \\ &= \chi(\sigma_w)w \\ &= -\bar{x} + y\,.\end{aligned} \tag{3.4.11}$$

Thus $s = \sigma_w\sigma_y$ is in $\mathrm{Spin}(2r)$ and $\chi(s)v \in Y$.

Now suppose that $y = 0$. Since $r > 1$ there is at least one vector $x_0 \in X$ which is independent from x. From x_0 we can construct x_1 such that $g(x, x_1) = g(x, \bar{x}_1) = 0$ by taking

$$x_1 = x_0 - \frac{g(x, \bar{x}_0)}{g(x, \bar{x})}\,x\,. \tag{3.4.12}$$

Then

$$\chi(\sigma_{x_1})x = -x \tag{3.4.13}$$

and so

$$\begin{aligned}\chi(\sigma_x\sigma_{x_1})x &= \chi(\sigma_x)\chi(\sigma_{x_1})x \\ &= -\chi(\sigma_x)x \\ &= -\bar{x}\,. \end{aligned}\tag{3.4.14}$$

Thus $s = \sigma_x\sigma_{x_1}$ is in $\mathrm{Spin}(2r)$ and $\chi(s)v \in Y$. ∎

Note that in all even dimensions (not just $r > 1$) we can map v into Y using an element of $\mathrm{Pin}(2r)$, either the identity if $v \in Y$ (in which case $x = 0$) or σ_x if $x \neq 0$, since then

$$\chi(\sigma_x)v = \bar{x} - y\,. \tag{3.4.15}$$

Let ψ be an impure even spinor in ten dimensions. By Theorem 3.4.1, we can write

$$\psi = u_1 + u_2 \tag{3.4.16}$$

where u_1 and u_2 are pure. Now $N(\psi) = 1$, so Lemma 3.2.18 shows that the intersection of the MTIS's represented by u_1 and u_2 is 1-dimensional. In fact, the vector $\mathscr{S}_1(u_1\otimes\bar{u}_2)$ is in the intersection of $T_{u_1} \cap T_{u_2}$ and thus spans T_ψ. For this dimension, the inner product is symmetric, and it follows from (2.3.23) that

$$\mathscr{S}_1\left(u_1 \otimes \bar{u}_2\right) = \mathscr{S}_1\left(u_2 \otimes \bar{u}_1\right)\,. \tag{3.4.17}$$

Then

$$\mathscr{S}_1\left(\psi \otimes \bar{\psi}\right) = \mathscr{S}_1(u_1 \otimes \bar{u}_1) + \mathscr{S}_1(u_2 \otimes \bar{u}_2) + 2\mathscr{S}_1(u_1 \otimes \bar{u}_2)\,. \tag{3.4.18}$$

Now u_1 and u_2 are pure, so $\mathscr{S}_p(u_i \otimes \bar{u}_i) = 0$ for $p \neq 5$. Thus the vector v given by

$$v = \mathscr{S}_1(\psi \otimes \bar{\psi}) \tag{3.4.19}$$

also spans T_ψ.

By Lemma 3.4.5, there exists $s \in \mathrm{Spin}(10)$ such that $\chi(s)v \in Y$, thus acting on ψ with s produces a spinor which is annihilated by $\chi(s)v \in Y$. Putting $\phi = s\psi$, we have $T_\phi = \chi(s)T_\psi \subset Y$. Furthermore, since $s \in \mathrm{Spin}(10)$ it follows that $G_\phi \simeq G_\psi$ (note that it is important that s be in the *real* Spin group: this would not necessarily be the case if s were in $\mathrm{Spin}(10,\mathbb{C})$). Putting $X = \overline{Y}$, we can choose a Witt basis $\{x^i, y^i\}$ for $V^{\mathbb{C}}$ such that

$$\begin{aligned} y^5 &= \chi(s)v \\ &= s\mathscr{S}_1(\psi \otimes \bar{\psi})s^{-1} \\ &= \mathscr{S}_1(s\psi \otimes \overline{s\psi}) \end{aligned}$$

$$= \quad \mathscr{S}_1(\phi \otimes \bar{\phi}) \tag{3.4.20}$$

and

$$x^i \quad = \quad \bar{y}^i \quad \forall\, i \in \{1, \ldots, 5\}\,. \tag{3.4.21}$$

Now $y^5\phi = 0$, and it follows that ϕ can be written in the form

$$\phi \quad = \quad \omega z_Y, \quad \omega \in \Lambda^+(\hat{X}) \tag{3.4.22}$$

where $\hat{X} = \mathrm{sp}_{\mathbb{C}}\{x^1, x^2, x^3, x^4\}$ (compare this with Lemma 3.2.11). Now $\Lambda^+(\hat{X})z_Y$ carries a spinor representation of the complexification of a real Clifford subalgebra. We can construct an orthonormal basis $\{e^a\}$ for V by putting

$$\begin{aligned} e^{2j-1} &= x^j + y^j \\ e^{2j} &= i(x^j - y^j) \quad \forall\, j \in \{1, \ldots, 5\}\,. \end{aligned} \tag{3.4.23}$$

Then we have an orthogonal decomposition of V given by

$$V \quad = \quad \hat{V} \oplus W \tag{3.4.24}$$

where

$$\begin{aligned} \hat{V} &= \mathrm{sp}\{e^a\}, \quad a \in \{1, \ldots, 8\} \\ W &= \mathrm{sp}\{e^9,\ e^{10}\}\,. \end{aligned} \tag{3.4.25}$$

Putting $\hat{Y} = \mathrm{sp}_{\mathbb{C}}\{y^1, y^2, y^3, y^4\}$ we have

$$\hat{V}^{\mathbb{C}} \quad = \quad \hat{X} \oplus \hat{Y}\,. \tag{3.4.26}$$

If $\hat{g}$ is the restriction of g to $\hat{V}^{\mathbb{C}}$, it is clear that $\Lambda^+(\hat{X})z_Y$ carries a spinor representation for $\mathbf{C}(\hat{V}^{\mathbb{C}}, \hat{g}) \simeq \mathbf{C}_8(\mathbb{C})$, that is, the complexified Clifford algebra generated by an 8-dimensional orthogonal space. With this in mind, it is reasonable to ask how the isotropy group of a spinor in ten dimensions is related to the isotropy group of a spinor in eight dimensions.

Since $\hat{V}$ is a subspace of V, it will be convenient to refer to the real Spin group of $\mathbf{C}(\hat{V}^{\mathbb{C}}, \hat{g})$ as $\mathrm{Spin}(\hat{V}, \hat{g})$, noting that $\mathrm{Spin}(\hat{V}, \hat{g}) \simeq \mathrm{Spin}(8)$ and $\mathrm{Spin}(\hat{V}, \hat{g}) \subset \mathrm{Spin}(10)$. Let

$$\widehat{G}_\phi \quad = \quad \left\{ s \in \mathrm{Spin}(\hat{V}, \hat{g}) : s\phi = \phi \right\}\,. \tag{3.4.27}$$

As shown in §3.3, $\widehat{G}_\phi$ is isomorphic to either SU(4) or Spin(7), and clearly $\widehat{G}_\phi \subseteq G_\phi$. Our aim is to show that $\widehat{G}_\phi = G_\phi$.

Consider $s \in G_\phi$. It is true that s is even with $s^\xi s = 1$, so in order to show that $s \in \widehat{G}_\phi$ we only need demonstrate that $s \in \mathbf{C}(\hat{V}, \hat{g})$ and $s\hat{V}s^{-1} = \hat{V}$. Since s fixes ϕ we have

$$\chi(s)y^5 \quad = \quad s\mathscr{S}_1(\phi \otimes \bar{\phi})s^{-1}$$

$$
\begin{aligned}
&= \mathscr{S}_1(s\phi \otimes \overline{s\phi}) \\
&= y^5 \,.
\end{aligned} \tag{3.4.28}
$$

From the reality of s it follows that $sx^5s^{-1} = x^5$. Now $e^9 = x^5 + y^5$ and $e^{10} = i(x^5 - y^5)$, thus $\chi(s)$ fixes each element of W. We have

$$
\begin{aligned}
(e^a)^\sharp \lrcorner\, s &= \frac{1}{2}\,(e^a s - s e^a) \\
&= 0 \quad \text{for } a = 9 \text{ or } a = 10
\end{aligned} \tag{3.4.29}
$$

which implies that s is generated by $\hat{V}$ only. That is, $s \in \mathbf{C}(\hat{V}, \hat{g})$.

Since $\chi(s) \in \mathrm{SO}(V, g)$ and $\hat{V}$ is orthogonal to W, we can easily show that $s\hat{V}s^{-1} = \hat{V}$. Consider $v \in \hat{V}$. Since $V = \hat{V} \oplus W$ we can write

$$
\chi(s)v \;=\; v_0 + w \quad \text{for some } v_0 \in \hat{V},\; w \in W \,. \tag{3.4.30}
$$

Now $\chi(s)$ fixes w, thus

$$
\begin{aligned}
g(\chi(s)v, \chi(s)w) &= g(v_0 + w, w) \\
&= g(w, w) \,.
\end{aligned} \tag{3.4.31}
$$

But v and w are orthogonal, so $g(w, w) = 0$ and so $w = 0$. We can conclude that $G_\phi = \widehat{G}_\phi$, and since $G_\psi \simeq G_\phi$, the isotropy group of an impure spinor in ten dimensions is either SU(4) or Spin(7). Again, this is different if spinors are classified under $\mathrm{Spin}(10, \mathbb{C})$, where it has been shown that there are only two distinct orbit types.

3.5 Spinors in twelve and higher dimensions

In the case when V is 12-dimensional ($r = 6$), it has proven difficult to extend our classification of spinors without appealing to Igusa's classification. This can be attributed to the scarcity of pure spinors in higher dimensions, since the semi-spinor spaces in twelve dimensions have dimension 32, while the pure subspaces have dimension at most 6. However, it is possible to put bounds on the pure index and nullity of an impure spinors by using a dimensional reduction argument similar to that used in §3.4. Let $\hat{X} = \mathrm{sp}_{\mathbb{C}}\{x^1, \ldots, x^5\}$ and $\hat{Y} = \mathrm{sp}_{\mathbb{C}}\{y^1, \ldots, y^5\}$. Then in a similar manner to (3.4.25), we have a real 10-dimensional subspace $\hat{V}$ of V such that $\hat{V}^{\mathbb{C}} = \hat{X} \oplus \hat{Y}$. The spaces $\Lambda(X)$ and $\Lambda^\pm(X)$ can be decomposed as

$$
\begin{aligned}
\Lambda(X) &= \Lambda(\hat{X}) \oplus \left(x^6 \wedge \Lambda(\hat{X})\right) \\
\Lambda^\pm(X) &= \Lambda(\hat{X})^\pm \oplus \left(x^6 \wedge \Lambda^\mp(\hat{X})\right)
\end{aligned} \tag{3.5.1}
$$

and the subspaces $\Lambda^\pm(\hat{X})z_Y$ of S each carry a semi-spinor representation of $\mathbf{C}(\hat{V}^{\mathbb{C}}, g)$, which is isomorphic to $\mathbf{C}_{10}(\mathbb{C})$. It is convenient to write $\hat{S}^\pm = \Lambda^\pm(\hat{X})z_Y$ and so $S^\pm = \hat{S}^\pm \oplus x^6\hat{S}^\mp$. Thus an even spinor ψ may be written as

$$
\psi \;=\; \omega_1 z_Y + x^6 \omega_2 z_Y \tag{3.5.2}
$$

where $\omega_1 \in \Lambda^+(\hat{X})$ and $\omega_2 \in \Lambda^-(\hat{X})$. Now $\omega_1 z_Y$ and $\omega_2 z_Y$ are in $\hat{S}^+$ and $\hat{S}^-$, respectively, so by Theorem (3.4.1) they can each be written as a sum of at most two pure spinors in ten dimensions. Note that by a "pure spinor in ten dimensions" we mean a spinor which is annihilated by a 5-dimensional totally isotropic subspace of $\hat{V}^{\mathbb{C}}$. Since each spinor in $\hat{S}^+$ is annihilated by y^6, and each spinor in $x^6\hat{S}^-$ is annihilated by x^6, it is clear that $\omega_1 z_Y$ and $x^6\omega_2 z_Y$ can each be written as a sum of at most two pure spinors in twelve dimensions. Thus $P(\psi) \leq 4$.

If ψ is impure then by Lemma 3.2.11 we have $N(\psi) = 0$, 1 or 2. We will see later that there exist spinors of nullity 0 and 2. Suppose that $N(\psi) = 1$. There exists $s \in \Gamma$ such that $sT_\psi s^{-1} \subseteq Y$, and we may choose a Witt basis such that y^6 spans $sT_\psi s^{-1}$. Then $s\psi$ has the form

$$s\psi \;=\; \omega z_Y, \quad \omega \in \Lambda^+(\hat{X})\,. \tag{3.5.3}$$

Since ωz_Y is an impure spinor in ten dimensions, it is annihilated by a 1-dimensional subspace of $\hat{V}^{\mathbb{C}}$, as well as by y^6. This is a contradiction, hence $N(\psi) = 0$ or 2.

We have established that there may be spinors with pure index at most four. However, given an impure spinor written as a sum of 3 or 4 pure spinors, it is difficult to say whether or not it could be written as a sum of only 2 or 3 pure spinors. In the following we state a necessary condition that an impure spinor must satisfy in order to have pure index 2, and consequently we are able to give an example of a spinor with pure index 3.

Proposition 3.5.4 *Let ψ be an impure semi-spinor in twelve dimensions with $P(\psi) = 2$. Then $\mathscr{S}_2(\psi \otimes \bar{\psi})$ is either decomposable or of maximal rank.*

Proof. Let $\psi = u_1 + u_2$ where u_1 and u_2 are pure. From (2.3.23) we have

$$\mathscr{S}_2(\psi \otimes \bar{\psi}) \;=\; 2\mathscr{S}_2(u_1 \otimes \bar{u}_2)\,. \tag{3.5.5}$$

Since ψ is impure, $N(\psi) = 0$ or 2. If $N(\psi) = 2$, then $\mathscr{S}_2(u_1 \otimes \bar{u}_2) = z_{T_{u_1} \cap T_{u_2}}$ by Lemma 2.3.35, and so it is decomposable.

Suppose that $N(\psi) = 0$. Now there exists $s \in \Gamma$ such that $su_1 = z_Y$, and since the MTIS represented by su_2 is complementary to Y, we may choose $X = T_{su_2}$. So we may assume without loss of generality that that u_1 represents Y and u_2 represents X. We can construct an orthonormal (but not real) basis $\{e^a\}$ for $V^{\mathbb{C}}$ by taking

$$\begin{aligned} e^{2j-1} &= x^j + y^j \\ e^{2j} &= i(x^j - y^j) \quad \forall\, j \in \{1, \ldots, 6\}. \end{aligned} \tag{3.5.6}$$

In this basis, the 2-form part of $u_1 \otimes \bar{u}_2$ can be written as

$$\begin{aligned} \mathscr{S}_2(u_1 \otimes \bar{u}_2) &= -\frac{1}{128}\mathrm{Tr}(u_1 \otimes \bar{u}_2 e_{ab})e^{ab} \\ &= -\frac{1}{128}(u_2, e_{ab}u_1)e^{ab}\,. \end{aligned} \tag{3.5.7}$$

(The factor of 128 is twice the trace of the identity, noting that for $r = 6$ the Clifford algebra is isomorphic to $\mathcal{M}_{64}(\mathbb{C})$.) Since the basis is orthonormal, $e_a = e^a$ for each a, so we can expand (3.5.7) to

$$\begin{aligned}
-128\mathscr{S}_2(u_1 \otimes \bar{u}_2) &= \sum_{j,k=1}^{r} (u_2, (e^{2j} \wedge e^{2k})u_1)e^{2j} \wedge e^{2k} \\
&\quad + \sum_{j,k=1}^{r} (u_2, (e^{2j-1} \wedge e^{2k-1})u_1)e^{2j-1} \wedge e^{2k-1} \\
&\quad + 2\sum_{j,k=1}^{r} (u_2, (e^{2j} \wedge e^{2k-1})u_1)e^{2j} \wedge e^{2k-1} \qquad (3.5.8)
\end{aligned}$$

where

$$\begin{aligned}
e^{2j} \wedge e^{2k} &= -x^{jk} - y^{jk} + x^j y^k - x^k y^j \\
e^{2j-1} \wedge e^{2k-1} &= x^{jk} + y^{jk} + x^j y^k - x^k y^j \\
e^{2j} \wedge e^{2k-1} &= i\left(x^{jk} - y^{jk} + x^j y^k + x^k y^j - \delta^{jk}\right) \\
&= i\left(x^{jk} - y^{jk} + x^j \wedge y^k + x^k \wedge y^j\right) . \qquad (3.5.9)
\end{aligned}$$

Since $y^j u_1 = 0$ and $x^j u_2 = 0$ for all j, the first two summations in (3.5.8) vanish, and we have

$$\begin{aligned}
-64\mathscr{S}_2(u_1 \otimes \bar{u}_2) &= (u_2, u_1) \sum_{j,k=1}^{r} \delta^{jk}\left(x^{jk} - y^{jk} + x^j \wedge y^k + x^k \wedge y^j\right) \\
&= 2(u_2, u_1)\left(x^1 \wedge y^1 + x^2 \wedge y^2 + \cdots + x^6 \wedge y^6\right) . \qquad (3.5.10)
\end{aligned}$$

Now $(u_2, u_1) \neq 0$ since $T_{u_1} \cap T_{u_2} = \{0\}$, so $\mathscr{S}_2(u_1 \otimes \bar{u}_2)$ is of maximal rank, in the sense that the 12-form

$$\underbrace{\mathscr{S}_2(u_1 \otimes \bar{u}_2) \wedge \ldots \wedge \mathscr{S}_2(u_1 \otimes \bar{u}_2)}_{6} \qquad (3.5.11)$$

is non-zero. ∎

Now that we have some necessary conditions for spinor in twelve dimensions to have pure index 2, we can construct a spinor which does not satisfy these conditions. Let

$$\begin{aligned}
u_1 &= x^{12} z_Y \\
u_2 &= x^{34} z_Y \\
u_3 &= x^{56} z_Y .
\end{aligned}$$

These spinors have been chosen so that they are each pure, with $\dim_{\mathbb{C}} T_{u_i} \cap T_{u_j} = 2$ for $i \neq j$, and so that the sum $\psi = u_1 + u_2 + u_3$ is impure. On calculating the 2-form part of $\psi \otimes \bar{\psi}$ we find that

$$\mathscr{S}_2(\psi \otimes \bar{\psi}) = \frac{1}{8}(z_Y, x^{123456} z_Y)\left(y^{12} + y^{34} + y^{56}\right) . \qquad (3.5.12)$$

	ω	$N(\omega z_Y)$	$P(\omega z_Y)$
(1)	1	6	1
(2)	$1 + x^{2356}$	2	2
(3)	$1 + x^{2356} + x^{1346}$	0	3
(4)	$1 + x^{2356} + x^{1346} + \lambda x^{1245}\,,\ \lambda \in \mathbb{C}^*$	0	2

Table 3.1: Representatives of the orbits of S^+ under $\mathrm{Spin}(12, \mathbb{C})$.

Clearly $\mathscr{S}_2(\psi \otimes \bar{\psi})$ is neither decomposable nor of maximal rank, hence ψ cannot be written as a sum of two pure spinors. So $P(\psi)$ must be 3.

Unfortunately, we have been unable to settle the question of whether or not there are spinors with pure index 4. We must appeal to Igusa's classification, which shows that such spinors do not exist in this dimension. Table 3.1 gives the representatives listed in [Igu70]. We have calculated the nullity and pure index of each representative.

By direct calculation, we can show that if $\psi = (1 + x^{2356} + x^{1346})\, z_Y$ then

$$\mathscr{S}_2(\psi \otimes \bar{\psi}) \;=\; -\frac{1}{8}(z_Y, x^{123456} z_Y) \left(x^{36} + y^{14} + y^{25}\right) , \qquad (3.5.13)$$

so we can conclude that spinor (3) has pure index 3. For each non-zero λ, a spinor of type (4) represents a distinct orbit. Although this spinor is written as a sum of four pure spinors, Igusa has shown that it is equivalent to a spinor of the form $(1 + \mu x^{123456})z_Y$, where $\mu \in \mathbb{C}^*$, so spinors of this type have pure index 2.

Table 3.2 lists the representatives of the orbits of S^+ in fourteen dimensions ($r = 7$), based on Popov's classification [Pop80]. Each spinor in the table represents a distinct orbit, with the exception of the spinors of type (5). Two spinors of type (5) with parameters λ_1 and λ_2 are equivalent if and only if $(\lambda_1)^8 = (\lambda_2)^8$. While it is a simple matter to determine the null space of a spinor, and hence the nullity, we have been unable to calculate the pure index of all the spinors in Table 3.2. Recall that in fourteen dimensions, a spinor with pure index 2 must have nullity 1 or 3. From this we can deduce that spinor (4) has pure index 3, spinors (5), (6) and (9) have pure index either 3 or 4, and the pure index of spinor (7) is 3, 4 or 5. In the case of spinor (8), we can refer to Table 3.1. Let ψ be spinor (8) and suppose that $P(\psi) = 2$. Then

$$\psi \;=\; (\omega_1 + \omega_2)\, z_Y \qquad (3.5.14)$$

where $\omega_1 z_Y$ and $\omega_2 z_Y$ are pure. But T_ψ is spanned by y^7, so by Lemma 3.2.18 we must have

$$\begin{aligned} y^7 \omega_1 z_Y &= 0 \quad \text{and} \\ y^7 \omega_2 z_Y &= 0\,, \end{aligned} \qquad (3.5.15)$$

which implies that ω_1 and ω_2 are in $\Lambda^+(\hat{X})$ where $\hat{X} = \mathrm{sp}_{\mathbb{C}}\{x^1, \ldots, x^6\}$. Thus they

	ω	$N(\omega z_Y)$	$P(\omega z_Y)$
(1)	1	7	1
(2)	$1+x^{4567}$	3	2
(3)	$1+x^{123456}$	1	2
(4)	$1+x^{1237}+x^{123456}$	0	3
(5)	$\lambda\left(1+x^{1237}+x^{4567}+x^{123456}\right),\ \lambda\in\mathbb{C}^*$	0	-
(6)	$1+x^{1237}+x^{3457}+x^{123456}$	0	-
(7)	$1+x^{1237}-x^{3457}-x^{2567}+x^{123456}$	0	-
(8)	$1+x^{2356}+x^{1346}$	1	3
(9)	$1+x^{1237}+x^{2356}+x^{1346}$	0	-

Table 3.2: Representatives of the orbits of S^+ under $\mathrm{Spin}(14,\mathbb{C})$.

are also pure spinors in twelve dimensions. This contradicts the fact that spinor (3) in Table 3.1 cannot be written as a sum of two pure spinors, hence $P(\psi)=3$.

Since the pure index of a spinor in fourteen dimensions is at most 5, dimensional reduction shows that a spinor in sixteen dimensions has pure index at most 10. According to [Pop80], sixteen is the highest number of dimensions in which it is reasonable to attempt a classification of spinors of the type described by Igusa. This has been done in [AE82]. Difficulties in principle arise in higher dimensions. Recently, a coarse classification of spinors using their nullity has been given by Trautman and Trautman for all dimensions [TT94]. In their notation, Σ_r^h is the space of semi-spinors of $2r$-dimensional space with nullity h. For each h, Σ_r^h is either empty or consists of a collection of spinor orbits. The key theorem of [TT94] is that

(1) $\dim\Sigma_r^r = 1+\frac{1}{2}r(r-1)$,

(2) Σ_r^{r-1}, Σ_r^{r-2}, Σ_r^{r-3} and Σ_r^{r-5} are empty, and

(3) $\dim\Sigma_r^h = h(2r-\frac{1}{2}(3h+1))+2^{r-h-1}$ for $h=r-4$ or $h<r-5$.

In particular, they show that Σ_r^0 is open and dense in the space of semi-spinors, so in sufficiently high dimensions, a generic semi-spinor has nullity 0.

Chapter 4

Spinor Equations for Shear-free Congruences

In the previous chapter we examined the properties of spinors in a purely algebraic context. In this chapter, we use the relationship between real null vector fields and spinors to analyse certain equations for vector fields. We restrict our attention to spacetime $(\mathcal{M}, g)$, where we take the Lorentzian signature to be $(3, 1)$. For this signature, the complexification of the tangent space at each $p \in \mathcal{M}$ admits 2-dimensional MTIS's with real index 1. That is to say, each MTIS has a 1-dimensional subspace spanned by a real null vector. In four dimensions all semi-spinors are pure, and so the correspondence between pure spinor directions and MTIS's gives rise to a correspondence between semi-spinors and real null directions. We can fix the scaling factor by using the isomorphism between $S \otimes S^*$ and $\mathbf{C}_{3,1}(\mathbb{C})$ described in §2.3. Then any equation for a null vector field has an equivalent spinorial form. This technique has been used with great success in the study of null shear-free congruences. In a basis $\{e^a\}$, the condition that the real null vector k determined by a semi-spinor u be tangent to a congruence of null shear-free geodesics (NSFG) is

$$(u, \nabla_{X_a} u) e^a u \quad = \quad 0\,. \tag{4.1}$$

We will refer to spinors satisfying (4.1) as *shear-free.*

In our notation, the *twistor equation* is

$$\nabla_X u - \frac{1}{4} X^\flat \mathrm{D} u \quad = \quad 0 \quad \forall\ X \in \Gamma\, T\mathcal{M}\,. \tag{4.2}$$

By putting $X = X_a$ and multiplying on the left by e^a, it can be seen that the numerical factor is chosen precisely so that we cannot conclude that u is parallel. Spinors satisfying (4.2) are called *twistors.* On spacetime, Penrose has defined the null vector field corresponding to a twistor as being tangent to a 'Robinson congruence' [PR86a]. A Robinson congruence is shear-free, however not all shear-free congruences are Robinson congruences. Sommers [Som76] has shown that the shear-free condition (4.1) is equivalent to a modification of the twistor equation, with terms involving a complex 1-form appearing on the right-hand side of (4.2). Sommers' equation is usually presented using the 2-component Newman-Penrose formalism. In this chapter we show

how Sommers' equation may be written in 'index-free' notation. We interpret the additional 1-form terms as arising from a $\mathrm{GL}(1, \mathbb{C})$-gauged covariant derivative, for which Sommers' equation is the corresponding 'gauged' twistor equation.

As a preliminary to the study of null shear-free vector fields, we examine the spinor equation for a null conformal Killing vector k. Such vectors form a special case of NSFG's, however there appears to be no simple condition similar to (4.1) for the corresponding spinor. Usually, the conformal Killing (CK) equation is expressed in terms of the Lie derivative of g with respect to k. It may also be expressed as an equation for the 1-form $k^\flat$ using the covariant derivative, the exterior derivative and the co-derivative. In this form it is easy to see the relationship between CK vectors and NSFG vectors. Benn has shown that the shear-free equation can be interpreted as a 'gauged' version of the CK equation [Ben94]. From this form of the CK equation, we are able to show that a conformal Killing vector corresponds to a spinor satisfying a $\mathrm{U}(1)$-gauged twistor equation. It is then a simple matter to obtain the spinorial form of the shear-free equation by adding an extra term to the $\mathrm{U}(1)$-covariant derivative.

A vector field k satisfying Benn's shear-free equation is shear-free whether k is null or non-null, on a manifold of arbitrary dimension and signature. On spacetime, a real non-null vector corresponds (not uniquely) to a *pair* of semi-spinors, or equivalently, to a Dirac spinor. In §4.8, we determine a spinor equation for a Dirac spinor ψ which is equivalent to the condition that the vector corresponding to ψ is shear-free. For special choices of ψ and gauge terms, this reduces to the conformal Killing and shear-free equations in both the null and non-null cases.

4.1 Spinors of Lorentzian space

As the Clifford algebra of a vector space with Lorentzian signature has certain special properties not found in a general Clifford algebra, a brief review is worthwhile. The real Clifford algebra $\mathbf{C}_{3,1}(\mathbb{R})$ is isomorphic to the algebra of 4×4 real matrices. An arbitrary basis $\{e^a\}$ for covectors is indexed by elements of the set $\{0,\ 1,\ 2,\ 3\}$, where e^0 is timelike, that is, $g(e^0, e^0) < 0$. We take the set $\{e^A\} = \{1,\ e^a,\ e^{ab}\ (a < b),\ e^a z,\ z\}$ to be the 'standard' basis for forms, and hence for $\mathbf{C}_{3,1}(\mathbb{R})$. In an orthonormal basis the volume 4-form is $z = e^0 \wedge e^1 \wedge e^2 \wedge e^3$.

The Clifford algebra $\mathbf{C}_{3,1}(\mathbb{C})$ is simply the complexification of $\mathbf{C}_{3,1}(\mathbb{R})$, so it isomorphic to the algebra of 4×4 complex matrices. Then the space S of Dirac spinors has complex dimension four. Table 2.1 shows that the charge conjugate is involutory, thus S decomposes into two spaces of Majorana spinors (with real dimension four), each carrying an irreducible representation of the real subalgebra. As $z^2 = -1$, the charge conjugate interchanges the semi-spinor spaces. With $\check{z} = iz$, the semi-spinor spaces are the 2-dimensional eigenspaces of $\check{z}$ with eigenvalues ± 1. They each carry an irreducible representation of $\mathbf{C}^+_{3,1}(\mathbb{C})$, which induce irreducible representations of $\mathrm{Spin}(4, \mathbb{C}) \simeq \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$. The inner product on S is antisymmetric, and may be scaled so that

$$\overline{(\psi, \phi)} \quad = \quad (\psi^{\mathrm{c}}, \phi^{\mathrm{c}}) \quad \forall\ \psi,\ \phi \in S\,. \tag{4.1.1}$$

The inner product is block-diagonal on $S^+ \oplus S^-$, so the inner product of two semi-spinors of opposite parity is zero. However, the restriction of $(\ ,\ )$ to S^+ or S^- remains non-degenerate. Since each semi-spinor space is 2-dimensional, it follows from the antisymmetry of $(\ ,\ )$ that

$$(u,v)w + (v,w)u + (w,u)v \quad = \quad 0 \tag{4.1.2}$$

for u, v and w lying in the same semi-spinor space.

Using (2.3.27), an element of $S \otimes S^*$ can be expanded in the standard basis for $\mathbf{C}_{3,1}(\mathbb{C})$. So that (2.3.25) is satisfied, we choose $\{e_A\}$ so that

$$\{e_A\} \quad = \quad \left\{ \frac{1}{4},\ \frac{1}{4}e_a,\ -\frac{1}{4}e_{ab}\ (a<b),\ \frac{1}{4}e_a z,\ -\frac{1}{4}z \right\}. \tag{4.1.3}$$

Then for any spinors $\psi,\ \phi \in S$,

$$\psi \otimes \bar{\phi} \quad = \quad \frac{1}{4}(\phi,\psi) + \frac{1}{4}(\phi, e_a\psi)e^a - \frac{1}{8}(\phi, e_{ab}\psi)e^{ab} + \frac{1}{4}(\phi, e_a z\psi)e^a z - \frac{1}{4}(\phi, z\psi)z\,.$$

Note that the 2-form component picks up an extra factor of 1/2 due to double-counting. From the properties of the inner product, certain components of a symmetric or antisymmetric combination of two semi-spinors will vanish. Suppose that u and v are semi-spinors with the same parity. From (2.3.20) we can see that $u \otimes \bar{v}$ is an even form. We use the convention that a $(1,1)$ tensor is symmetric if the $(0,2)$ tensor related by to it by metric duality is symmetric. Then

$$\begin{aligned}
\mathbf{Sym}(u\otimes\bar{v}) &= \frac{1}{2}\left(u\otimes\bar{v} + v\otimes\bar{u}\right) \\
&= \frac{1}{2}\left(u\otimes\bar{v} - (u\otimes\bar{v})^{\xi}\right) \quad \text{by (2.3.23)} \\
&= \frac{1}{2}\left(u\otimes\bar{v} - \mathscr{S}_0(u\otimes\bar{v}) + \mathscr{S}_2(u\otimes\bar{v}) - \mathscr{S}_4(u\otimes\bar{v})\right) \\
&= \mathscr{S}_2(u\otimes\bar{v}) \\
&= -\frac{1}{8}(v, e_{ab}u)e^{ab} \qquad u,\ v\in S^+ \text{ or } S^-\,.
\end{aligned} \tag{4.1.4}$$

This 2-form is an eigenvector of the Hodge dual, since

$$\begin{aligned}
*(u\otimes\bar{v} + v\otimes\bar{u}) &= i\,(u\otimes\bar{v} + v\otimes\bar{u})\check{z} \quad \text{by (2.2.7)} \\
&= i\,(u\otimes\overline{\check{z}v} + v\otimes\overline{\check{z}u}) \\
&= \pm\, i\,(u\otimes\bar{v} + v\otimes\bar{u}) \qquad u,\ v\in S^{\pm}\,.
\end{aligned} \tag{4.1.5}$$

We will refer to eigenvectors of the Hodge dual with eigenvalue $+i$ or $-i$ as self-dual or anti self-dual, respectively. Similarly,

$$\begin{aligned}
\mathbf{Alt}(u\otimes\bar{v}) &= \frac{1}{2}\left(u\otimes\bar{v} - v\otimes\bar{u}\right) \\
&= \mathscr{S}_0(u\otimes\bar{v}) + \mathscr{S}_4(u\otimes\bar{v}) \\
&= \frac{1}{4}(v,u) + \frac{1}{4}(v,\check{z}u)\check{z}
\end{aligned}$$

$$= \frac{1}{4}(v,u) \pm \frac{1}{4}\, i{*}(v,u) \qquad u,\ v \in S^{\pm}\,. \tag{4.1.6}$$

If u and v have opposite parity then $u \otimes \bar{v}$ is an odd form, and we have

$$\begin{aligned}
\mathbf{Alt}(u\otimes\bar{v}) &= \mathscr{S}_1(u\otimes\bar{v}) \\
&= \frac{1}{4}(v, e_a u)e^a \quad u \in S^{\pm},\ v\in S^{\mp} && (4.1.7)\\
\mathbf{Sym}(u\otimes\bar{v}) &= \mathscr{S}_3(u\otimes\bar{v}) \\
&= -\frac{1}{4}(v, e_a \check{z} u)e^a \check{z} \\
&= \mp\, i{*}\mathscr{S}_1(u\otimes\bar{v}) \quad u \in S^{\pm},\ v\in S^{\mp}\,. && (4.1.8)
\end{aligned}$$

Using equation (4.1.7), we can associate a real null vector with a semi-spinor u. In four dimensions, u and u^{c} are pure spinors with opposite parity. By Lemma 2.3.35, the intersection of T_u and $T_{u^{\mathrm{c}}}$ is 1-dimensional. Suppose that x spans $T_u \cap T_{u^{\mathrm{c}}}$. Then $xu = 0$ and $xu^{\mathrm{c}} = 0$, but also $\bar{x}u^{\mathrm{c}} = 0$ and $\bar{x}u = 0$. Therefore $\bar{x} \in T_u \cap T_{u^{\mathrm{c}}}$, and we must have $\bar{x} = \mathrm{e}^{i\lambda}x$ for some $\lambda \in \mathbb{R}$. Then the covector $k^\flat = \mathrm{e}^{i\lambda/2}x$ is real, null and spans $T_u \cap T_{u^{\mathrm{c}}}$. Thus any semi-spinor uniquely determines a real null direction. Conversely, a real null vector uniquely determines a semi-spinor direction. Suppose that $k^\flat$ is contained in a MTIS X. Then there exists a semi-spinor u which represents X, hence $k^\flat u = 0$. Given another semi-spinor v independent from u, we have a basis $\{u, v\}$ for semi-spinors of a given parity. In four dimensions, Lemma 2.3.35 shows that the intersection of T_u and T_v can only be $\{0\}$ for independent semi-spinors. Now suppose that $k^\flat(\lambda u + \mu v) = 0$ for some $\lambda,\ \mu \in \mathbb{C}$. Then $\mu k^\flat v = 0$, but $k^\flat$ cannot annihilate v, hence $\mu = 0$. So u is unique up to complex scalings. From Lemma 2.3.35 it is clear that $T_u \cap T_{u^{\mathrm{c}}}$ is 1-dimensional, so in fact $k^\flat$ spans $T_u \cap T_{u^{\mathrm{c}}}$. By (4.1.1) and the properties of the inner product, we have $\overline{(u^{\mathrm{c}}, e_a u)} = -(u^{\mathrm{c}}, e_a u)$. So for a given u, we can fix k by taking

$$\begin{aligned}
k^\flat &= 4\mathscr{S}_1(iu \otimes \bar{u}^{\mathrm{c}}) \\
&= (iu^{\mathrm{c}}, e_a u)e^a\,. && (4.1.9)
\end{aligned}$$

For a given k there is still a U(1)-scaling freedom in the choice of u, since the transformation $u \mapsto \mathrm{e}^{i\lambda}u$, $\lambda \in \mathbb{R}$ leaves k fixed.

4.2 Vector fields associated with spinors

On a spacetime $(\mathcal{M}, g)$, the relations given in §4.1 show the correspondence between spinor fields and differential forms on $\mathcal{M}$. In particular, (4.1.9) shows the relationship between a semi-spinor and a real null vector field. In a similar fashion, we can obtain a real timelike vector field from a Dirac spinor field ψ by taking the 1-form part of $\psi \otimes \overline{z\psi^{\mathrm{c}}}$. Let

$$\begin{aligned}
x &= 4\mathscr{S}_1(\psi \otimes \overline{z\psi^{\mathrm{c}}}) \\
&= (z\psi^{\mathrm{c}}, e_a\psi)e^a\,. && (4.2.1)
\end{aligned}$$

Then x is real since

$$\begin{aligned} \bar{x} &= (z\psi, e_a\psi^{c})e^a \\ &= -(e_a\psi^{c}, z\psi)e^a \\ &= (z\psi^{c}, e_a\psi)e^a\,. \end{aligned} \tag{4.2.2}$$

Any Dirac spinor can be written as the sum of an odd semi-spinor and an even semi-spinor. Suppose that $\psi = u + v^{c}$ where $u,\ v \in S^+(\mathcal{M})$, with corresponding null vectors

$$k^\flat = (iu^{c}, e_a u)e^a \quad \text{and} \quad l^\flat = -(iv^{c}, e_a v)e^a\,.$$

Then

$$\begin{aligned} x &= (-i\check{z}\psi^{c}, e_a\psi)e^a \\ &= (iu^{c} - iv, e_a u + e_a v^{c})e^a \\ &= k^\flat - l^\flat\,. \end{aligned} \tag{4.2.3}$$

Then $g(x,x) = -2g(k^\flat, l^\flat)$. We can find $g(x,x)$ in terms of u and v using the Fierz rearrangement formula (2.3.31). Then

$$\begin{aligned} g(x,x) &= 2(u^{c}, e_a u)(v, e^a v^{c}) \\ &= 2(u^{c}, e^A v^{c})(v, e^a e_A e_a u) \quad \text{by (2.3.31)}\,. \end{aligned} \tag{4.2.4}$$

Since u and v have the same parity, the only contributions to the sum over A come from even basis elements. Furthermore, by (2.2.6) we have $e^a e_{bc} e_a = 0$. Thus

$$\begin{aligned} g(x,x) &= \frac{1}{2}(u^{c}, v^{c})(v, e^a e_a u) - \frac{1}{2}(u^{c}, zv^{c})(v, e^a z e_a u) \\ &= 2(u^{c}, v^{c})(v, u) + 2(u^{c}, zv^{c})(v, zu) \\ &= -4|(u,v)|^2\,. \end{aligned} \tag{4.2.5}$$

Hence x is timelike if and only if u and v are linearly independent. In particular, x is null if ψ is a semi-spinor, or if it is Majorana. Similarly, it can be shown that the vector field $(i\psi^{c}, e^a\psi)X_a$ is real and spacelike.

4.3 The conformal Killing equation

A vector field K is *conformal Killing* if

$$\mathscr{L}_K g = 2\lambda g \qquad \lambda \in \mathcal{F}(\mathcal{M})\,. \tag{4.3.1}$$

Geometrically, this means that the operation of Lie transporting vectors along the flows of a Killing field is a conformal isometry. Equation (4.3.1) can also be written in terms of the covariant derivative. For an arbitrary vector field, the *covariant derivative* $\nabla K^\flat$ of K is a $(2,0)$ tensor given by

$$\nabla K^\flat(X,Y) = g(\nabla_X K, Y) \quad \forall\ X,\ Y \in \Gamma T\mathcal{M}\,. \tag{4.3.2}$$

The covariant derivative can be decomposed into symmetric and antisymmetric components, and the symmetric component can be decomposed further into a trace-free component and the trace. For arbitrary K, the Lie derivative of g is related to ∇ by

$$\mathscr{L}_K g(X,Y) \quad = \quad g(\nabla_X K, Y) + g(\nabla_Y K, X)\,, \tag{4.3.3}$$

from which it is clear that $2\,\mathbf{Sym}(\nabla K^\flat) = \mathscr{L}_K g$. Then in four dimensions, the covariant derivative can be written as

$$\nabla_X K^\flat \quad = \quad \frac{1}{2} X \,\lrcorner\, dK^\flat - \frac{1}{4} d^* K^\flat\, X^\flat + \frac{1}{2}\left(\mathscr{L}_K g - \frac{1}{4}\mathrm{Tr}(\mathscr{L}_K g) g\right)(X)\,,$$

where the trace of a $(2,0)$ tensor T is $T(X_a, X^a)$. Note that for a symmetric $(2,0)$ tensor, the 1-form field $T(X)$ is defined unambiguously. Then (4.3.1) is equivalent to

$$\nabla_X K^\flat \quad = \quad \frac{1}{2} X \,\lrcorner\, dK^\flat - \frac{1}{4} d^* K^\flat\, X^\flat \quad \forall\ X \in \Gamma T\mathcal{M}\,. \tag{4.3.4}$$

Putting $X = X_a$ and wedging e^a onto both sides of (4.3.4) shows that the numerical coefficients are such that we cannot conclude that $dK^\flat = 0$. Similarly, by taking the interior derivative with respect to X^a, we cannot conclude that $d^*K^\flat = 0$. Contracting (4.3.3) on X^a and X_a, it is clear that when K satisfies (4.3.1) we have $\lambda = -\frac{1}{4} d^* K^\flat$. When $\lambda = 0$, K is called a *Killing vector*.

Now we will derive the spinorial version of the conformal Killing equation. If x is the null or timelike covector field corresponding to a Dirac spinor ψ as in (4.2.1), we have

$$\begin{aligned}
\nabla_{X_a} x \quad &= \quad X_a(z\psi^{\mathrm{c}}, e_b\psi)e^b + (z\psi^{\mathrm{c}}, e_b\psi)\nabla_{X_a} e^b \\
&= \quad (z\nabla_{X_a}\psi^{\mathrm{c}}, e_b\psi)e^b + (z\psi^{\mathrm{c}}, \nabla_{X_a} e_b\,\psi)e^b \\
&\qquad + (z\psi^{\mathrm{c}}, e_b\nabla_{X_a}\psi)e^b + (\psi^{\mathrm{c}}, e_b z\psi)\nabla_{X_a} e^b\,.
\end{aligned} \tag{4.3.5}$$

For a metric-compatible connection we have

$$X_c \,\lrcorner\, \nabla_{X_a} e^b + X^b \,\lrcorner\, \nabla_{X_a} e_c \quad = \quad 0 \tag{4.3.6}$$

so the terms in (4.3.5) involving derivatives of the basis covectors cancel, hence

$$\begin{aligned}
\nabla_{X_a} x \quad &= \quad (z\nabla_{X_a}\psi^{\mathrm{c}}, e_b\psi)e^b + (z\psi^{\mathrm{c}}, e_b\nabla_{X_a}\psi)e^b \\
&= \quad 2\,\Re e(z\psi^{\mathrm{c}}, e_b\nabla_{X_a}\psi)e^b\,.
\end{aligned} \tag{4.3.7}$$

The remaining terms of (4.3.4) are given by

$$\begin{aligned}
d^* x\, X_a^\flat \quad &= \quad -2\,\Re e(z\psi^{\mathrm{c}}, \mathrm{D}\psi)e_a \\
&= \quad -2\,\Re e(z\psi^{\mathrm{c}}, g_{ab}\mathrm{D}\psi)e^b
\end{aligned} \tag{4.3.8}$$

$$\begin{aligned}
X_a \,\lrcorner\, dx \quad &= \quad \nabla_{X_a} x - \nabla_{X_b} x(X_a)e^b \\
&= \quad 2\,\Re e(z\psi^{\mathrm{c}}, e_b\nabla_{X_a}\psi)e^b - 2\,\Re e(z\psi^{\mathrm{c}}, e_a\nabla_{X_b}\psi)e^b \\
&= \quad 2\,\Re e(z\psi^{\mathrm{c}}, e_b\nabla_{X_a}\psi - e_a\nabla_{X_b}\psi)e^b\,.
\end{aligned} \tag{4.3.9}$$

Thus $x^\sharp$ is a null or timelike conformal Killing vector if and only if ψ satisfies

$$\Re e(z\psi^{\mathrm{c}}, e_a\nabla_{X_b}\psi + e_b\nabla_{X_a}\psi - \frac{1}{2}g_{ab}\mathrm{D}\psi) \;=\; 0\,. \tag{4.3.10}$$

Since the conformal factor is related to ψ by $\lambda = \frac{1}{2}\Re e(z\psi^{\mathrm{c}}, \mathrm{D}\psi)$, $x^\sharp$ is a Killing vector if it satisfies $\Re e(z\psi^{\mathrm{c}}, \mathrm{D}\psi) = 0$ in addition to (4.3.10). We will refer to (4.3.10) as the *spinorial conformal Killing equation*. We can re-write (4.3.10) in a compact form using the *twistor operator* L_X, given by

$$L_X\psi \;=\; \nabla_X\psi - \frac{1}{4}X^\flat\mathrm{D}\psi \qquad \psi\in\Gamma S(\mathcal{M})\,. \tag{4.3.11}$$

Now

$$\begin{aligned} X^\flat L_Y\psi + Y^\flat L_X\psi \;&=\; X^\flat\nabla_Y\psi - \frac{1}{4}X^\flat Y^\flat\mathrm{D}\psi + Y^\flat\nabla_X\psi - \frac{1}{4}Y^\flat X^\flat\mathrm{D}\psi \\ &=\; X^\flat\nabla_Y\psi + Y^\flat\nabla_X\psi - \frac{1}{2}g(X,Y)\mathrm{D}\psi \end{aligned} \tag{4.3.12}$$

so (4.3.10) is equivalent to

$$\Re e(z\psi^{\mathrm{c}}, X^\flat L_Y\psi + Y^\flat L_X\psi) \;=\; 0 \quad \forall\ X,\ Y\in\Gamma T\mathcal{M}\,. \tag{4.3.13}$$

4.4 Null conformal Killing vectors

The simplest case of (4.3.13) is for a null vector field. Consider a semi-spinor u, with $k^\flat$ defined as in (4.1.9). Since the null space of u is 2-dimensional, there exists a complex vector m independent from k such that $m^\flat u = 0$. We can scale m so that $g(m,\bar m) = 1$. We can also find a real null vector l such that $g(k,l) = 1$, with all other pairs zero, so that $\{k,l,m,\bar m\}$ is a a null basis. The Clifford relations then imply that

$$\begin{aligned} k^\flat l^\flat + l^\flat k^\flat \;&=\; 2 \\ m^\flat\bar m^\flat + \bar m^\flat m^\flat \;&=\; 2 \end{aligned} \tag{4.4.1}$$

while all other pairs anti-commute. We can obtain a basis $\{u,v\}$ for semi-spinors by setting $v = \frac{1}{2}\bar m^\flat l^\flat u$. Clearly, the null space of v is spanned by $\{l^\flat, \bar m^\flat\}$. Multiplying v on the left by $k^\flat m^\flat$, it follows that $u = \frac{1}{2}k^\flat m^\flat v$. Now

$$\begin{aligned} (iu^{\mathrm{c}}, e_a v)e^a \;&=\; \frac{1}{2}(iu^{\mathrm{c}}, e_a\bar m^\flat l^\flat u)e^a \\ &=\; \frac{1}{2}(iu^{\mathrm{c}}, (2\bar m_a - \bar m^\flat e_a)l^\flat u)e^a \\ &=\; (iu^{\mathrm{c}}, l^\flat u)\bar m^\flat \quad \text{since } \bar m^\flat u^{\mathrm{c}} = 0 \\ &=\; \bar m^\flat \quad \text{since } g(k,l) = 1\,. \end{aligned} \tag{4.4.2}$$

We can find the remaining basis vectors in a similar fashion, giving

$$\begin{array}{rclcrcl} k^\flat & = & (iu^{\mathrm{c}}, e_a u)e^a & \qquad & m^\flat & = & (iv^{\mathrm{c}}, e_a u)e^a \\ l^\flat & = & -(iv^{\mathrm{c}}, e_a v)e^a & & \bar{m}^\flat & = & (iu^{\mathrm{c}}, e_a v)e^a\,. \end{array} \tag{4.4.3}$$

Our goal now is to find an equation for a semi-spinor corresponding to a null CK vector which has a form similar to that of the twistor equation. It is simplest to work with the components of the twistor operator. Since the twistor operator is linear in X and preserves the parity of u, there exist complex 1-forms α and β such that

$$L_X u = \alpha(X)u + \beta(X)v\,. \tag{4.4.4}$$

Since $e^a L_{X_a} = 0$, these 1-forms satisfy the relation $\alpha u + \beta v = 0$. We can expand a 1-form in the null basis as

$$\alpha \;=\; \alpha(l)k^\flat + \alpha(k)l^\flat + \alpha(\bar{m})m^\flat + \alpha(m)\bar{m}^\flat\,. \tag{4.4.5}$$

Then from $\alpha u + \beta v = 0$ and the relationship between u and v, the components of α and β are related by

$$\alpha(k) + \beta(\bar{m}) \;=\; 0 \quad \text{and} \quad \alpha(m) - \beta(l) \;=\; 0\,. \tag{4.4.6}$$

Since $\check{z}u = \pm u$, we have $zu = \mp iu$ and (4.3.13) is equivalent to

$$\Re e(iu^{\mathrm{c}}, X^\flat L_Y u + Y^\flat L_X u) \;=\; 0 \quad \forall\; X,\; Y \in \Gamma T\mathcal{M}\,. \tag{4.4.7}$$

Substituting the components of $L_X u$ into (4.4.7), we have

$$\Re e(iu^{\mathrm{c}}, \alpha(X)Y^\flat u + \alpha(Y)X^\flat u + \beta(X)Y^\flat v + \beta(Y)X^\flat v) \;=\; 0 \tag{4.4.8}$$

for all real vector fields X and Y. Clearly, (4.4.8) is not linear over complex vectors, however we can write it as a real tensor equation. First we define a complex symmetric $(2,0)$ tensor W by

$$W(X,Y) \;=\; (iu^{\mathrm{c}}, \alpha(X)Y^\flat u + \alpha(Y)X^\flat u + \beta(X)Y^\flat v + \beta(Y)X^\flat v)\,. \tag{4.4.9}$$

Then it is clear that (4.4.8) is equivalent to

$$\Re e\,[W(X,Y)] \;=\; 0 \tag{4.4.10}$$

for all real vector fields X and Y. If W acts on *complex* vector fields we have

$$\overline{W(X,Y)} \;=\; \bar{W}(\bar{X},\bar{Y})\,. \tag{4.4.11}$$

However, since X and Y are real (4.4.10) is equivalent to

$$\Re e\,[W]\,(X,Y) \;=\; 0\,. \tag{4.4.12}$$

Now for an arbitrary (real or complex) vector field X we have

$$(iu^{\mathrm{c}}, X^\flat u) \;=\; k^\flat(X) \quad \text{and} \quad (iu^{\mathrm{c}}, X^\flat v) \;=\; \bar{m}^\flat(X) \tag{4.4.13}$$

so from (4.4.9) we see that

$$W \;=\; 2\,\mathbf{Sym}\left(\alpha \otimes k^\flat + \beta \otimes \bar{m}^\flat\right) . \tag{4.4.14}$$

Then the real symmetric tensor $T = 2\,\Re e[W]$ is given by

$$T \;=\; 2\,\mathbf{Sym}\left((\alpha + \bar{\alpha}) \otimes k^\flat + \bar{\beta} \otimes m^\flat + \beta \otimes \bar{m}^\flat\right) .$$

It is clear that (4.4.8) is equivalent to $T = 0$. Since T is symmetric, it has ten independent components. Evaluating $T(k,k)$, $T(k,l)$,..., $T(\bar{m},\bar{m})$, we find that the equation $T = 0$ is equivalent to the following six equations for the components of $L_X u$:

$$\Re e[\alpha(k)] \;=\; 0 \tag{4.4.15}$$
$$\beta(k) \;=\; 0 \tag{4.4.16}$$
$$\Re e[\alpha(l)] \;=\; 0 \tag{4.4.17}$$
$$\alpha(m) + \bar{\alpha}(m) + \beta(l) \;=\; 0 \tag{4.4.18}$$
$$\beta(m) \;=\; 0 \tag{4.4.19}$$
$$\Re e[\beta(\bar{m})] \;=\; 0\,. \tag{4.4.20}$$

Using these equations together with (4.4.6), we can show that u satisfies the twistor equation of a U(1)-*gauged covariant derivative.*

Theorem 4.4.21 *Let u be a semi-spinor on a spacetime $(\mathcal{M}, g)$. If $k = (iu^{\mathrm{c}}, e^a u)X_a$ is conformal Killing then there exists a real 1-form $\mathcal{A}$ such that*

$$\widehat{\nabla}_X u - \frac{1}{4} X^\flat \widehat{\mathrm{D}} u \;=\; 0 \quad \forall\; X \in \Gamma\, T\mathcal{M} \tag{4.4.22}$$

where

$$\widehat{\nabla}_X u \;=\; \nabla_X u + i\mathcal{A}(X)u \tag{4.4.23}$$
$$\widehat{\mathrm{D}} u \;=\; \mathrm{D}u + i\mathcal{A}u\,. \tag{4.4.24}$$

Conversely, if u satisfies (4.4.22) for some real $\mathcal{A}$, then k is conformal Killing.

Proof. The gauged Dirac operator in (4.4.22) is simply the Dirac operator of $\widehat{\nabla}$, that is, $\widehat{\mathrm{D}} = e^a \widehat{\nabla}_{X_a}$. Exposing $\mathcal{A}$, it is clear that (4.4.22) is equivalent to

$$L_X u \;=\; -i\mathcal{A}(X)u + \frac{1}{4} i X^\flat \mathcal{A} u \quad \forall\; X \in \Gamma\, T\mathcal{M}. \tag{4.4.25}$$

Using the Clifford relations, we can rearrange this as

$$L_X u \;=\; -i\mathcal{A}(X)u + \frac{1}{4} i(2\mathcal{A}(X) - \mathcal{A}X^\flat)u$$

$$= -\frac{1}{2}i\mathcal{A}(X)u - \frac{1}{4}\mathcal{A}X^\flat u\,. \tag{4.4.26}$$

Equation (4.4.25) is clearly linear in X, so if it holds for real vectors then it also holds for complex vectors. We will show that there exists a real 1-form $\mathcal{A}$ such that (4.4.25) is satisfied for all complex vectors. If k is a conformal Killing vector then u satisfies (4.4.7), and hence the components of $L_X u$ satisfy the equations (4.4.15)–(4.4.20). Now we simply evaluate (4.4.26) on each basis vector to solve for the components of $\mathcal{A}$. For $X = k$ we have

$$\alpha(k)u = -\frac{1}{2}i\mathcal{A}(k)u\,,$$

since $\beta(k) = 0$ and $k^\flat u = 0$. Therefore

$$\mathcal{A}(k) = 2i\alpha(k)\,. \tag{4.4.27}$$

Since $\Re e[\alpha(k)] = 0$ we know that $\alpha(k)$ is pure imaginary, so this equation is consistent with $\mathcal{A}$ being real. Similarly, if $X = m$ then we find that

$$\alpha(m)u = -\frac{1}{2}i\mathcal{A}(m)u$$

and so

$$\mathcal{A}(m) = 2i\alpha(m)\,. \tag{4.4.28}$$

The calculations for the remaining vectors are only slightly more complicated. When $X = l$ we first expand $\mathcal{A}$ using the null basis. Then

$$\begin{aligned}\alpha(l)u + \beta(l)v &= -\frac{1}{2}i\mathcal{A}(l)u - \frac{1}{4}i\left(\mathcal{A}(l)k^\flat + \mathcal{A}(k)l^\flat + \mathcal{A}(\bar{m})m^\flat + \mathcal{A}(m)\bar{m}^\flat\right) l^\flat u \\ &= -i\mathcal{A}(l)u - \frac{1}{2}i\mathcal{A}(m)v\end{aligned}$$

hence

$$\mathcal{A}(l) = i\alpha(l) \tag{4.4.29}$$
$$\mathcal{A}(m) = 2i\beta(l)\,. \tag{4.4.30}$$

Once again, $\alpha(l)$ is pure imaginary, so this is consistent with the reality of $\mathcal{A}$. By (4.4.6), equation (4.4.30) is consistent with (4.4.28). Finally, when $X = \bar{m}$ we have

$$\begin{aligned}\alpha(\bar{m})u + \beta(\bar{m})v &= -\frac{1}{2}i\mathcal{A}(\bar{m})u - \frac{1}{4}i\left(\mathcal{A}(l)k^\flat + \mathcal{A}(k)l^\flat + \mathcal{A}(\bar{m})m^\flat + \mathcal{A}(m)\bar{m}^\flat\right)\bar{m}^\flat u \\ &= -i\mathcal{A}(\bar{m})u + \frac{1}{2}i\mathcal{A}(k)v\end{aligned}$$

hence

$$\mathcal{A}(k) = -2i\beta(\bar{m}) \tag{4.4.31}$$

$$\mathcal{A}(\bar{m}) \;=\; i\alpha(\bar{m})\,. \tag{4.4.32}$$

Equation (4.4.31) is consistent with (4.4.27) by (4.4.6). Since we require $\mathcal{A}$ to be real we must have $\overline{\mathcal{A}(m)} = \mathcal{A}(\bar{m})$. Taking the conjugate of (4.4.18), from (4.4.32) we have

$$\begin{aligned} \mathcal{A}(\bar{m}) &= -i\left(\overline{\alpha(m)} + \bar{\beta}(l)\right) \\ &= -2i\overline{\alpha(m)} \quad \text{by (4.4.6)} \\ &= \overline{\mathcal{A}(m)}\,. \end{aligned} \tag{4.4.33}$$

So if k is conformal Killing then u satisfies (4.4.25) where $\mathcal{A}$ is real. We can easily express the components of $\mathcal{A}$ in terms of the twistor operator by contracting on v:

$$\begin{array}{rclrcl} (u,v)\mathcal{A}(k) &=& 2i(L_k u, v) & (u,v)\mathcal{A}(l) &=& i(L_l u, v) \\ (u,v)\mathcal{A}(m) &=& 2i(L_m u, v) & (u,v)\mathcal{A}(\bar{m}) &=& i(L_{\bar{m}} u, v) \end{array} \tag{4.4.34}$$

Now suppose that u satisfies (4.4.25). We observe that for a p-form ω,

$$\begin{aligned} \overline{(iu^{\mathrm{c}}, \omega u)} &= (-iu, \bar{\omega} u^{\mathrm{c}}) \\ &= (i\bar{\omega} u^{\mathrm{c}}, u) \\ &= (iu^{\mathrm{c}}, \bar{\omega}^{\xi} u) \qquad \omega \in \Gamma\Lambda_p(\mathcal{M}) \end{aligned} \tag{4.4.35}$$

hence

$$\Re e(iu^{\mathrm{c}}, \omega u) \;=\; 0 \left\{ \begin{array}{l} \text{if } \omega \text{ real with } p = 2,\ 3 \text{ or} \\ \text{if } \omega \text{ pure imag. with } p = 0,\ 1 \text{ or } 4\,. \end{array} \right. \tag{4.4.36}$$

For real vector fields X and Y,

$$\begin{aligned} X^\flat L_Y u + Y^\flat L_X u &= -i\mathcal{A}(Y) X^\flat u + \frac{1}{4} i X^\flat Y^\flat \mathcal{A} u \\ &\qquad - i\mathcal{A}(X) Y^\flat u + \frac{1}{4} i Y^\flat X^\flat \mathcal{A} u \\ &= -i\mathcal{A}(Y) X^\flat u - i\mathcal{A}(X) Y^\flat u + \frac{1}{2} i g(X,Y)\mathcal{A} u\,. \end{aligned} \tag{4.4.37}$$

In each of the three terms above, the 1-form preceding u is pure imaginary, therefore by (4.4.36), equation (4.4.7) is satisfied for all real vector fields X and Y. It follows that k is conformal Killing. ∎

Equation (4.4.22) is the twistor equation for the covariant derivative $\widehat{\nabla}$. It is a U(1)-gauged covariant derivative in the sense that, for a real function λ,

$$\begin{aligned} \nabla_X(\mathrm{e}^{i\lambda} u) &= \mathrm{e}^{i\lambda}\left(\nabla_X u + i\, d\lambda(X) u\right) \\ &= \mathrm{e}^{i\lambda}\left(\widehat{\nabla}_X u - i(\mathcal{A} - d\lambda)(X) u\right). \end{aligned} \tag{4.4.38}$$

Then

$$\nabla_X(\mathrm{e}^{i\lambda} u) + i(\mathcal{A} - d\lambda)(X)\mathrm{e}^{i\lambda} u \;=\; \mathrm{e}^{i\lambda}\widehat{\nabla}_X u \tag{4.4.39}$$

and so using the transformations

$$\begin{array}{rcl} u & \longmapsto & \mathrm{e}^{i\lambda} u \\ \mathcal{A} & \longmapsto & \mathcal{A} - d\lambda\,, \quad \lambda \in \mathcal{F}(\mathcal{M}) \end{array} \tag{4.4.40}$$

we have

$$\widehat{\nabla}_X u \;\longmapsto\; \mathrm{e}^{i\lambda}\widehat{\nabla}_X u\,. \tag{4.4.41}$$

Of course, we could have anticipated that the spinorial conformal Killing equation for a null vector would have this covariance, since k only determines u up to a U(1)-scaling.

4.5 Shear-free vector fields

In this section we show that the condition that a vector field be shear-free may be interpreted as a generalisation of the conformal Killing equation (4.3.1). We then show that a semi-spinor corresponding to a null shear-vector field satisfies a generalisation of the twistor equation. In a similar way to the conformal Killing equation, we may interpret this generalised twistor equation as a 'gauged' twistor equation, with a $\mathrm{GL}(1,\mathbb{C})$ gauge term. The pure imaginary part of this gauge term arises from the U(1)-scaling covariance mentioned in the previous section, while the real part of the gauge term comes from the fact that any vector proportional to a shear-free vector is also shear-free.

On an n-dimensional pseudo-Riemannian manifold $(\mathcal{M}, g)$, it is usual to define the shear of a vector field K in one of two ways, depending on whether K is null or non-null. In addition, the shear of a null vector is only defined if K is tangent to a geodesic congruence, that is, if $\nabla_K K = fK$ for some $f \in \mathcal{F}(\mathcal{M})$. We will refer to such vector fields as 'geodesic', since their integral curves may be reparametrised so as to be geodesic. An alternative characterisation of the shear-free property is given by the observation that a shear-free vector field induces a conformal isometry on its orthogonal space. Thus we say that K is *shear-free* if

$$\mathscr{L}_K g(X,Y) \;=\; 2\lambda g(X,Y) \quad \forall\ X,\ Y \in K^\perp \tag{4.5.1}$$

for some $\lambda \in \mathcal{F}(\mathcal{M})$, where

$$K^\perp \;=\; \{X \in \Gamma\, T\mathcal{M} : g(K,X) = 0\}\,. \tag{4.5.2}$$

This definition may be used regardless of whether K is null or non-null, however note that if K is null then K is in $K^\perp$. In that case, (4.3.3) shows that $g(\nabla_K K, Y) = 0$ for all $Y \in K^\perp$. But the only vector fields orthogonal to all of $K^\perp$ are those proportional to K itself, hence K is geodesic. In any case, we may choose a vector field W such that $g(K,W) = 1$. Then the space of vector fields may be decomposed as $\mathrm{sp}\{W\} \oplus K^\perp$. If K is non-null then we may take W to be proportional to K.

If K satisfies (4.5.1), then there exists a real 1-form A such that

$$\mathscr{L}_K g - 2\lambda g \;=\; -2\left(K^\flat \otimes A + A \otimes K^\flat\right)\,. \tag{4.5.3}$$

To see this, let $T = \mathscr{L}_K g - 2\lambda g$. Then we define A by

$$A(X) = -\frac{1}{2}T(W,X) \quad \forall\ X \in K^{\perp} \tag{4.5.4}$$
$$A(W) = -\frac{1}{4}T(W,W)\,. \tag{4.5.5}$$

Since (4.5.3) is an equation for symmetric tensors, we need only show that it holds for all pairs (X,X). By the decomposition, X may be written as $X' + fW$ where $X' \in K^{\perp}$ and $f \in \mathcal{F}(\mathcal{M})$. Then

$$\begin{aligned} T(X,X) &= T(X',X') + 2fT(X',W) + f^2T(W,W) \\ &= 2fT(X',W) + f^2T(W,W) \quad \text{by (4.5.1)} \end{aligned} \tag{4.5.6}$$

while

$$\begin{aligned} \left(K^\flat \otimes A + A \otimes K^\flat\right)(X,X) &= 2g(K,X)A(X) \\ &= 2fg(K,W)\left(A(X') + fA(W)\right) \\ &= -fg(K,W)\left(T(W,X') + \frac{1}{2}fT(W,W)\right) \\ &= -\frac{1}{2}T(X,X)\,. \end{aligned} \tag{4.5.7}$$

Then we can write (4.5.3) in terms of the symmetrised covariant derivative as

$$\mathbf{Sym}(\nabla K^\flat) - \lambda g = -\left(K^\flat \otimes A + A \otimes K^\flat\right)\,. \tag{4.5.8}$$

Taking the trace shows that $n\lambda = -d^*K^\flat + 2A(K)$, hence

$$\mathbf{Sym}(\nabla K^\flat) + \frac{1}{n}d^*K^\flat\, g = -\left(K^\flat \otimes A + A \otimes K^\flat\right) + \frac{2}{n}A(K)\, g\,. \tag{4.5.9}$$

Equation (4.5.9) can be written in a compact form using a gauged covariant derivative $\widehat{\nabla}$, similar to that introduced in Theorem 4.4.21 for spinor fields. If we say that a p-form ω has *gauge term* $2A$ then this means that the action of $\widehat{\nabla}$ on ω is given by

$$\widehat{\nabla}_X\omega = \nabla_X\omega + 2A(X)\omega \quad \forall\ X \in \Gamma T\mathcal{M} \tag{4.5.10}$$

where A is a 1-form (the factor of 2 will be convenient later). This naturally induces a gauged exterior derivative $\hat{d}$ and co-derivative $\hat{d}^*$,

$$\begin{aligned} \hat{d}\omega &= e^a \wedge \widehat{\nabla}_{X_a}\omega \\ &= d\omega + 2A \wedge \omega \end{aligned} \tag{4.5.11}$$
$$\begin{aligned} \hat{d}^*\omega &= -X^a \lrcorner \widehat{\nabla}_{X_a}\omega \\ &= d^*\omega - 2A^\sharp \lrcorner\, \omega\,. \end{aligned} \tag{4.5.12}$$

Unlike their ungauged counterparts, these operators are not nilpotent, as

$$\hat{d}^2\omega = 2\,dA \wedge \omega \tag{4.5.13}$$
$$\hat{d}^{*2}\omega = -X_b \lrcorner X_a \lrcorner dA\, X^b \lrcorner X^a \lrcorner \omega\,. \tag{4.5.14}$$

It should be emphasised that, in general, the 'gauge term' $2A$ is determined by the object being differentiated, as it will often be convenient to associate different gauge terms with different tensors or spinors. We can develop a useful 'calculus' of gauged tensors and spinors by requiring that $\widehat{\nabla}$ be compatible with the tensor and Clifford products, so that the gauge term of any such product is simply the sum of the gauge terms of its constituents. For example, if a spinor u has gauge term $\mathcal{A}$ then the gauge term of the spinor ωu is $2A + \mathcal{A}$, since

$$\begin{aligned}\widehat{\nabla}_X(\omega u) &= \widehat{\nabla}_X\omega\, u + \omega\widehat{\nabla}_X u\\ &= \nabla_X(\omega u) + (2A + \mathcal{A})(X)\omega u\,.\end{aligned} \tag{4.5.15}$$

We will also require that $\widehat{\nabla}$ be compatible with the inner product on spinors and the charge conjugate.

Associating the gauge term $2A$ with K, equation (4.5.9) is equivalent to

$$\mathbf{Sym}(\widehat{\nabla}K^\flat) + \frac{1}{n}\hat{d}^*K^\flat g = 0\,. \tag{4.5.16}$$

Benn has shown that this equation is equivalent to the shear-free condition [Ben94]. Specifically, if $n > 1$ and K is a non-null vector field satisfying (4.5.16), then K is proportional to a unit shear-free vector. For $n > 2$, a null vector satisfying (4.5.16) is shear-free and geodesic.

Now we turn our attention to shear-free vectors on spacetime. Contracting (4.5.16) on an arbitrary vector X, the shear-free condition with $n = 4$ is equivalent to

$$\widehat{\nabla}_X K^\flat - \frac{1}{2}X \lrcorner \hat{d}K^\flat + \frac{1}{4}\hat{d}^*K^\flat X^\flat = 0 \quad \forall\, X \in \Gamma T\mathcal{M}\,. \tag{4.5.17}$$

Notice that this equation has the same form as the conformal Killing equation (4.3.4). Exposing the gauge term A we have

$$\begin{aligned}&\nabla_X K^\flat - \frac{1}{2}X \lrcorner dK^\flat + \frac{1}{4}d^*K^\flat X^\flat\\ &= -2A(X)K^\flat + X \lrcorner (A \wedge K^\flat) + \frac{1}{2}A(K)X^\flat \quad \forall\, X \in \Gamma T\mathcal{M}\,.\end{aligned} \tag{4.5.18}$$

Comparing this with the calculation following (4.3.4), it is clear that if x is the real covector obtained from a Dirac spinor ψ then $x^\sharp$ is shear-free if and only if

$$\begin{aligned}&\Re e(z\psi^{\mathrm{c}}, X^\flat L_Y\psi + Y^\flat L_X\psi\\ &\quad + A(Y)X^\flat\psi + A(X)Y^\flat\psi - \frac{1}{2}g(X,Y)A\psi) = 0\end{aligned} \tag{4.5.19}$$

for all real vectors X and Y. If we define a gauged covariant derivative on spinors by

$$\mathbb{\nabla}_X\psi = \nabla_X\psi + A(X)\psi \tag{4.5.20}$$

we obtain a gauged twistor operator,

$$\mathbb{L}_X\psi = \mathbb{\nabla}_X\psi - \frac{1}{4}X^\flat\mathbb{D}\psi \qquad \psi\in\Gamma S(\mathcal{M})\,. \tag{4.5.21}$$

(We reserve the 'hat' symbol for later use). Then (4.5.19) is equivalent to

$$\Re e(z\psi^{\mathrm{c}}, X^\flat\mathbb{L}_Y\psi + Y^\flat\mathbb{L}_X\psi) = 0 \tag{4.5.22}$$

for real vectors X and Y. Since (4.5.22) has precisely the same form as (4.3.10), we can use the proof of Theorem 4.4.21 to show that the shear-free condition for a null geodesic vector field is equivalent to a $\mathrm{GL}(1,\mathbb{C})$-gauged twistor equation for a semi-spinor.

Theorem 4.5.23 *Let u be a semi-spinor on a spacetime $(\mathcal{M},g)$. If $k=(iu^{\mathrm{c}}, e^a u)X_a$ is shear-free then there exists a complex 1-form $\mathcal{A}$ such that*

$$\widehat{\nabla}_X u - \frac{1}{4}X^\flat\widehat{\mathrm{D}}u = 0 \quad \forall\ X\in\Gamma T\mathcal{M} \tag{4.5.24}$$

where

$$\widehat{\nabla}_X u = \nabla_X u + \mathcal{A}(X)u \tag{4.5.25}$$

$$\widehat{\mathrm{D}}u = \mathrm{D}u + \mathcal{A}u\,. \tag{4.5.26}$$

Conversely, if u satisfies (4.5.24) for some complex $\mathcal{A}$, then k is shear-free.

Proof. If k is shear-free then there exists a real 1-form A such that (4.5.18) holds, hence u satisfies (4.5.22). Since this equation has the same form as (4.4.7), the components of $\mathbb{L}_X u$ satisfy the equations (4.4.6) and (4.4.15)–(4.4.20). So we can find a real 1-form $\mathcal{A}'$ such that

$$\mathbb{L}_X u = -i\mathcal{A}'u + \frac{1}{4}iX^\flat\mathcal{A}'u \tag{4.5.27}$$

in exactly the same way as for Theorem 4.4.21. Expanding (4.5.27) in terms of the ordinary covariant derivative ∇, it is clear that u is a solution of (4.5.24) with $\mathcal{A}=A+i\mathcal{A}'$, that is, $\Re e[\mathcal{A}]=A$.

Conversely, if u satisfies (4.5.24) then

$$\begin{aligned}\Re e(iu^{\mathrm{c}}, X^\flat L_Y u + Y^\flat L_X u &\\ + \mathcal{A}(Y)X^\flat u + \mathcal{A}(X)Y^\flat u - \frac{1}{2}g(X,Y)\mathcal{A}u) = 0&\end{aligned} \tag{4.5.28}$$

for X, Y real. By (4.4.36), any pure imaginary 1-form acting on u in the above will vanish. Since $\Re e[\mathcal{A}(Y)X^\flat]=\Re e[\mathcal{A}](Y)X^\flat$, we have

$$\Re e(iu^{\mathrm{c}}, X^\flat L_Y u + Y^\flat L_X u$$

$$+ \Re e[\mathcal{A}](Y)X^\flat u + \Re e[\mathcal{A}](X)Y^\flat u - \frac{1}{2}g(X,Y)\Re e[\mathcal{A}]u) = 0\,. \qquad (4.5.29)$$

Comparing this with (4.5.19), it is clear that k is a solution of (4.5.18) with $A = \Re e[\mathcal{A}]$, hence it is shear-free. Note that this is consistent with our usage of $\widehat{\nabla}$, since if $k^\flat = 4\mathbf{Alt}(u \otimes \bar{u}^c)$ we would expect the gauge term of $k^\flat$ to be $\mathcal{A} + \bar{\mathcal{A}} = 2\Re e[\mathcal{A}]$. ∎

Equation (4.5.24) is equivalent to what is usually referred to as Sommers' equation, although it is normally written in 2-component form with the 1-form $\mathcal{A}$ exposed. From now on, it will be understood that a *shear-free spinor* is a semi-spinor which satisfies (4.5.24) for some complex 1-form $\mathcal{A}$. We can show directly that the shear-free spinor equation is equivalent to (4.1). Now

$$\begin{aligned}&(u, \nabla_{X_a}u - \frac{1}{4}e_a \mathrm{D}u + \mathcal{A}_a u - \frac{1}{4}e_a\mathcal{A}u)e^a u \\ &\quad = \quad (u, \nabla_{X_a}u)e^a u - \mathscr{S}_1(\mathrm{D}u \otimes \bar{u})u - \mathscr{S}_1(\mathcal{A}u \otimes \bar{u})u\,. \end{aligned} \qquad (4.5.30)$$

If u and v are semi-spinors with opposite parity, the intersection of their null spaces is a 1-dimensional space spanned by $\mathscr{S}_1(v \otimes \bar{u})$, hence $\mathscr{S}_1(v \otimes \bar{u})u = 0$. Since $\mathrm{D}u$ and $\mathcal{A}u$ have opposite parity to u, the last two terms in (4.5.30) vanish. So if u is shear-free then (4.1) holds. Conversely, if (4.1) holds then setting $\mathcal{A} = 0$ in (4.5.30) shows that $(u, L_{X_a}u)e^a u = 0$. This implies that $(u, L_{X_a}u)e^a$ is a null vector, so there exists a semi-spinor v with opposite parity to u such that $\mathscr{S}_1(v \otimes \bar{u}) = (u, L_{X_a}u)e^a$. Writing $\mathscr{S}_1(v \otimes \bar{u})$ in terms of the basis vectors we have

$$(u, L_{X_a}u - \frac{1}{4}e_a v) \quad = \quad 0\,. \qquad (4.5.31)$$

Since $(\ ,\ )$ is antisymmetric, and the space of semi-spinors is 2-dimensional, the only possibility is that $L_{X_a}u - \frac{1}{4}e_a v$ is proportional to u for each a. That is, there exists a set of complex functions $\mathcal{A}_a$ such that

$$L_{X_a}u - \frac{1}{4}e_a v \quad = \quad -\mathcal{A}_a u\,. \qquad (4.5.32)$$

Multiplying on the left by e^a, we see that $v = \mathcal{A}_a e^a u = \mathcal{A}u$, so

$$L_{X_a}u \quad = \quad -\mathcal{A}_a u + \frac{1}{4}e_a \mathcal{A}u \qquad (4.5.33)$$

which is clearly equivalent to (4.5.24).

4.6 Conformal properties of shear-free spinors

In this section we examine the behaviour of the the GL$(1,\mathbb{C})$-twistor equation under rescalings of the metric. First we recall the properties of the covariant derivative induced by conformal rescalings of the metric. If g is scaled according to $\tilde{g} = \mathrm{e}^{2\lambda}g$ for $\lambda \in \mathcal{F}(\mathcal{M})$, then the $\tilde{g}$-compatible torsion-free covariant derivative $\widetilde{\nabla}$ is related to ∇ by

$$\widetilde{\nabla}_X Y \quad = \quad \nabla_X Y + X\lambda\, Y + Y\lambda\, X - g(X,Y)d\lambda^\sharp \quad \forall\ X,\ Y \in \Gamma T\mathcal{M}\,. \qquad (4.6.1)$$

Let $\{e^a\}$ be a g-orthonormal co-frame with dual $\{X_a\}$, and let $\{\tilde{e}^a\}$ be a $\tilde{g}$-orthonormal co-frame with dual $\{\tilde{X}_a\}$ where $\tilde{e}^a = \mathrm{e}^{\lambda} e^a$ and $\tilde{X}_a = \mathrm{e}^{-\lambda} X_a$. Since the structure of the Clifford algebra depends only on the signature of g, at each point $p \in \mathcal{M}$ there is an algebra isomorphism $\rho : \mathbf{C}^{\mathbb{C}}(T_p^*\mathcal{M}, \tilde{g}) \to \mathbf{C}^{\mathbb{C}}(T_p^*\mathcal{M}, g)$, defined on the basis covectors by $\rho(\tilde{e}^a) = e^a$. Then for a 1-form A we have

$$\begin{aligned} \rho(A) &= A(\tilde{X}_a)\rho(\tilde{e}^a) \\ &= A(\tilde{X}_a)e^a \\ &= \mathrm{e}^{-\lambda} A\,. \end{aligned} \tag{4.6.2}$$

If the $\tilde{g}$-Clifford product of arbitrary forms ϕ and ω is denoted by $\vee$, then $\rho(\omega \vee \phi) = \rho(\omega)\rho(\phi)$ and $\rho(\lambda) = \lambda$ for $\lambda \in \mathbb{C}$. It follows that $\rho(\omega) = \mathrm{e}^{-p\lambda}\omega$ if ω is a p-form. This isomorphism may be extended to a bundle isomorphism between the Clifford bundles of $(\mathcal{M}, \tilde{g})$ and $(\mathcal{M}, g)$.

Now suppose that $(\mathcal{M}, g)$ is a spin manifold. Unlike the tangent and cotangent bundles, the spinor bundle depends specifically on the metric. In general, the spinor bundle with respect to one metric is unrelated to the spinor bundle with respect to a different metric. However, if the metrics are conformally related, the isomorphism ρ induces a spinor representation of the fibres of $\mathbf{C}^{\mathbb{C}}(\mathcal{M}, \tilde{g})$ on the fibres of $S(\mathcal{M})$. We denote the action of a $\tilde{g}$-Clifford form ω on a spinor field ψ by $\omega \circ \psi$, where $\omega \circ \psi = \rho(\omega)\psi$. Under this action, the covariant derivative on spinor fields induced by $\widetilde{\nabla}$ is related to the standard spinor covariant derivative by

$$\widetilde{\nabla}_Y \psi = \nabla_Y \psi + \frac{1}{2}\left(Y^\flat d\lambda - Y\lambda\right)\psi \quad \forall\, \psi \in \Gamma S(\mathcal{M})\,. \tag{4.6.3}$$

Then the Dirac operator of $\widetilde{\nabla}$ is given by

$$\begin{aligned} \widetilde{\mathrm{D}}\psi &= \tilde{e}^a \circ \widetilde{\nabla}_{\tilde{X}_a}\psi \\ &= \mathrm{e}^{-\lambda} e^a \left\{\nabla_{X_a}\psi + \frac{1}{2}\left(e_a d\lambda - X_a\lambda\right)\psi\right\} \\ &= \mathrm{e}^{-\lambda}\left\{\mathrm{D}\psi + \frac{3}{2} d\lambda\,\psi\right\} \quad \forall\, \psi \in \Gamma S(\mathcal{M})\,. \end{aligned} \tag{4.6.4}$$

We are now in a position to see how the $\mathrm{GL}(1,\mathbb{C})$-gauged twistor equation changes under conformal rescalings. The twistor operator of $\widetilde{\nabla}$ is

$$\widetilde{L}_Y \psi = \widetilde{\nabla}_Y \psi - \frac{1}{4} Y^{\tilde{\flat}} \circ \widetilde{\mathrm{D}}\psi \quad \forall\, \psi \in \Gamma S(\mathcal{M}) \tag{4.6.5}$$

where $Y^{\tilde{\flat}}$ is the metric dual of Y with respect to $\tilde{g}$. This is simply a rescaling of $Y^\flat$, as $Y^{\tilde{\flat}} = \mathrm{e}^{2\lambda} Y^\flat$. Then from (4.6.2) we have $Y^{\tilde{\flat}} \circ \psi = \mathrm{e}^{\lambda} Y^\flat \psi$. If $\widetilde{\psi} = \mathrm{e}^{\mu}\psi$ for some complex function μ, then

$$\begin{aligned} \widetilde{L}_Y \widetilde{\psi} &= \nabla_Y \widetilde{\psi} + \frac{1}{2}\left(Y^\flat d\lambda - Y\lambda\right)\widetilde{\psi} - \frac{1}{4} Y^\flat \left(\mathrm{D}\widetilde{\psi} + \frac{3}{2} d\lambda\,\widetilde{\psi}\right) \\ &= \mathrm{e}^{\mu}\nabla_Y \psi + \mathrm{e}^{\mu} Y\mu\,\psi + \frac{1}{2}\mathrm{e}^{\mu}\left(Y^\flat d\lambda - Y\lambda\right)\psi \end{aligned}$$

$$
\begin{aligned}
& - \frac{1}{4} \mathrm{e}^{\mu} Y^{\flat} \left(\mathrm{D}\psi + d\mu \, \psi + \frac{3}{2} d\lambda \, \psi \right) \\
= & \ \mathrm{e}^{\mu} \left\{ L_Y \psi + \left(Y\mu - \frac{1}{2} Y\lambda \right) \psi - \frac{1}{4} Y^{\flat} \left(d\mu - \frac{1}{2} d\lambda \right) \psi \right\} . \qquad (4.6.6)
\end{aligned}
$$

Since $Y\mu = d\mu(Y)$, if ψ satisfies (4.5.24) then the equation above becomes

$$
\begin{aligned}
\widetilde{L}_Y \widetilde{\psi} &= \left\{ -\mathcal{A}(Y) + \frac{1}{4} Y^{\flat} \mathcal{A} + \left(d\mu - \frac{1}{2} d\lambda \right) (Y) - \frac{1}{4} Y^{\flat} \left(d\mu - \frac{1}{2} d\lambda \right) \right\} \widetilde{\psi} \\
&= \left(-\mathcal{A} + d\mu - \frac{1}{2} d\lambda \right) (Y) \circ \widetilde{\psi} - \frac{1}{4} Y^{\tilde{\flat}} \vee \left(-\mathcal{A} + d\mu - \frac{1}{2} d\lambda \right) \circ \widetilde{\psi} . \qquad (4.6.7)
\end{aligned}
$$

So if ψ satisfies (4.5.24) with gauge term $\mathcal{A}$, then $\widetilde{\psi}$ satisfies the $\mathrm{GL}(1,\mathbb{C})$-twistor equation of $(\mathcal{M}, \tilde{g})$, with gauge term $\widetilde{\mathcal{A}} = \mathcal{A} - d\mu + \frac{1}{2} d\lambda$. In particular, (4.5.24) is conformally invariant if we take $\mu = \lambda/2$. While $\mathcal{A}$ is not preserved for arbitrary scalings of ψ and g, we observe that $\widetilde{\mathcal{A}}$ and $\mathcal{A}$ only differ by an exact form, so that $d\widetilde{\mathcal{A}} = d\mathcal{A}$. As was noted by Sommers [Som76], the 2-form $d\mathcal{A}$ is thus invariant under rescalings of ψ, and so is determined by the congruence rather than by ψ. If $d\mathcal{A}$ vanishes, then $\mathcal{A}$ is (locally) exact, and so ψ is proportional to a twistor and the congruence corresponding to ψ is a Robinson congruence.

4.7 Timelike conformal Killing vectors

In this section we generalise the previous results on semi-spinors to the Dirac spinor corresponding to a timelike vector field. We will show that a spinor corresponding to a timelike conformal Killing vector satisfies a gauged twistor equation. Unlike the null case, we have a great deal of freedom in the choice of Dirac spinor for a given timelike vector, thus the gauged covariant derivative has other terms in addition to a U(1)-component. We show how these additional gauge terms arise from the group of transformations which leave the timelike vector fixed.

Let ψ be a Dirac spinor, which we may write as $\psi = u + v^{\mathrm{c}}$ where u and v are linearly independent even semi-spinors. Then the covector $x = 4\mathscr{S}_1(z\psi \otimes \bar{\psi}^{\mathrm{c}})$ is real and timelike, with $g(x,x) = -4|(u,v)|^2$. Given u and v, the null basis vectors $\{k, l, m, \bar{m}\}$ are given by (4.4.3). Then $\{k^{\flat}, m^{\flat}\}$ spans the null space of u and $\{l^{\flat}, \bar{m}^{\flat}\}$ spans the null space of v. Since $x = k^{\flat} - l^{\flat}$ it is clear that $g(k,l) = 2|(u,v)|^2$. Using the Fierz rearrangement it can be shown that $g(m,\bar{m}) = 2|(u,v)|^2$ also, while all other pairs are zero. Putting $\mu = 2|(u,v)|^2$, the Clifford products of the basis covectors are

$$
\begin{aligned}
k^{\flat} l^{\flat} + l^{\flat} k^{\flat} &= 2\mu \\
m^{\flat} \bar{m}^{\flat} + \bar{m}^{\flat} m^{\flat} &= 2\mu \qquad (4.7.1)
\end{aligned}
$$

while all other basis elements anti-commute. We will take $\{u, v, u^{\mathrm{c}}, v^{\mathrm{c}}\}$ as a basis for Dirac spinors. Later we will need to know the action of a pair of basis covectors on each element of the spinor basis. For example, $\bar{m}^{\flat} u$ is a non-zero odd spinor. Since $\bar{m}^{\flat} \bar{m}^{\flat} u = 0$ and $k^{\flat} \bar{m}^{\flat} u = -\bar{m}^{\flat} k^{\flat} u = 0$, the null space of $\bar{m}^{\flat} u$ is spanned by $\{k^{\flat}, \bar{m}^{\flat}\}$. Thus $\bar{m}^{\flat} u$ must be proportional to u^{c}. Suppose that $\bar{m}^{\flat} u = \lambda u^{\mathrm{c}}$ for some complex function λ. Then

$$\begin{aligned}(iv^{\mathrm{c}}, \bar{m}^\flat u) &= g(m,\bar{m})\\ &= \mu\\ &= (iv^{\mathrm{c}}, \lambda u^{\mathrm{c}})\end{aligned} \tag{4.7.2}$$

therefore

$$\bar{m}^\flat u = -\frac{\mu}{(iu^{\mathrm{c}}, v^{\mathrm{c}})}\, u^{\mathrm{c}}\,. \tag{4.7.3}$$

Similarly,

$$l^\flat u^{\mathrm{c}} = -\frac{\mu}{(iu, v)}\, v$$

hence

$$\begin{aligned}l^\flat \bar{m}^\flat u &= -\frac{\mu}{(iu^{\mathrm{c}}, v^{\mathrm{c}})}\, l^\flat u^{\mathrm{c}}\\ &= -\frac{\mu^2}{|(u,v)|^2}\, v\\ &= -2\mu v\,.\end{aligned} \tag{4.7.4}$$

From the Clifford relations and the properties of the charge conjugate, we obtain the following table of multiples:

$$\begin{array}{rclcrcl} k^\flat l^\flat u &=& 2\mu u & & k^\flat l^\flat u^{\mathrm{c}} &=& 2\mu u^{\mathrm{c}}\\ \bar{m}^\flat l^\flat u &=& 2\mu v & & m^\flat l^\flat u^{\mathrm{c}} &=& 2\mu v^{\mathrm{c}}\\ m^\flat \bar{m}^\flat u &=& 2\mu u & & \bar{m}^\flat m^\flat u^{\mathrm{c}} &=& 2\mu u^{\mathrm{c}}\\ k^\flat m^\flat v &=& 2\mu u & & k^\flat \bar{m}^\flat v^{\mathrm{c}} &=& 2\mu u^{\mathrm{c}}\,. \end{array} \tag{4.7.5}$$

In §4.4 we found that the spinorial conformal Killing equation for a semi-spinor was equivalent to a U(1)-gauged twistor equation. In this section we hope to find a gauged twistor equation (possibly with more general gauge terms) for the Dirac spinor ψ which is equivalent to (4.3.10). The main problem is to determine the form of the gauge terms. In the case of a null conformal Killing vector, the pure imaginary 1-form $i\mathcal{A}$ was chosen because of the U(1)-scaling freedom in the choice of semi-spinor. For a given timelike vector x, however, we have a much greater freedom in the choice of Dirac spinor ψ. We would anticipate that any equation that ψ must satisfy in order for x to be conformal Killing must also be covariant under transformations of ψ which fix x. We begin by examining the Lie algebra of real 2-forms which annihilate x under the Clifford commutator. At a fixed point $p \in \mathcal{M}$, let

$$\mathfrak{g}_x = \Big\{\sigma \in \Lambda_2(T_p^*\mathcal{M}) : [\sigma, x] = 0\Big\} \tag{4.7.6}$$

with the Clifford commutator as Lie bracket. Now exponentiation maps $\mathfrak{g}_x$ into the group $G_x \subseteq \mathrm{Spin}(3,1)$ which fixes x under the vector representation χ. It follows that any transformation

$$\psi \quad \longmapsto \quad \exp(\sigma)\psi\,, \qquad \sigma \in \mathfrak{g}_x \tag{4.7.7}$$

fixes x. Since x is timelike, we must have $G_x \simeq \mathrm{Spin}(3)$. In fact, we can show that $\exp(\mathfrak{g}_x) = G_x$. By evaluating the commutator of x with each element of a basis for the real 2-forms, we find that $\mathfrak{g}_x$ is generated by the 2-forms

$$\begin{aligned}
\sigma_1 &= \frac{1}{4\mu}(k+l)(\bar{m}+m) \\
&= \frac{1}{4\mu}(k+l)\wedge(\bar{m}+m) && (4.7.8) \\
\sigma_2 &= \frac{1}{4\mu}i(k+l)(\bar{m}-m) \\
&= \frac{1}{4\mu}i(k+l)\wedge(\bar{m}-m) && (4.7.9) \\
\sigma_3 &= \frac{1}{2\mu}i(\mu - m\bar{m}) \\
&= -\frac{1}{2\mu}i\, m\wedge\bar{m} && (4.7.10)
\end{aligned}$$

with commutation relations

$$[\sigma_1,\sigma_2] \;=\; \sigma_3 \qquad [\sigma_1,\sigma_3] \;=\; -\sigma_2 \qquad [\sigma_2,\sigma_3] \;=\; \sigma_1\,.$$

This can be recognised as the Lie algebra $\mathfrak{su}(2)$. Exponentiation sends $\mathfrak{su}(2)$ onto the simply-connected group $\mathrm{SU}(2) \simeq \mathrm{Spin}(3)$, hence the only elements of $\mathrm{Spin}(3,1)$ which fix x under χ are exponentials of elements of $\mathfrak{g}_x$.

Now for any real 2-form σ at $p \in \mathcal{M}$ we have

$$\begin{aligned}
[\sigma, x] &= (z\psi^{\mathrm{c}}, e_a\psi)[\sigma, e^a] \\
&= -(z\psi^{\mathrm{c}}, [\sigma, e_a]\psi)e^a \\
&= (z\psi^{\mathrm{c}}, e_a\sigma\psi)e^a - (z\psi^{\mathrm{c}}, \sigma e_a\psi)e^a \\
&= (z\psi^{\mathrm{c}}, e_a\sigma\psi)e^a + (z\psi, \sigma e_a\psi^{\mathrm{c}})e^a \\
&= 2\Re e(z\psi^{\mathrm{c}}, e_a\sigma\psi)e^a\,. && (4.7.11)
\end{aligned}$$

Thus if $\sigma \in \mathfrak{g}_x$ then

$$\Re e(z\psi^{\mathrm{c}}, e_a\sigma\psi) \;=\; 0\,. \tag{4.7.12}$$

Now since $\psi = u + v^{\mathrm{c}}$, using the action of the basis covectors on u and v^{c} given in (4.7.5) we see that

$$\sigma_1\psi \;=\; -\frac{1}{2}\check{z}\psi^{\mathrm{c}} \qquad \sigma_2\psi \;=\; -\frac{1}{2}i\check{z}\psi^{\mathrm{c}} \qquad \sigma_3\psi \;=\; -\frac{1}{2}i\psi\,.$$

Expressing σ as the linear combination $\sigma = \alpha^i\sigma_i$ for $\alpha^i \in \mathbb{R}$, equation (4.7.11) is equivalent to

$$\Re e(z\psi^{\mathrm{c}}, i\alpha^3 e_a\psi + (\alpha^1 + i\alpha^2)e_a\check{z}\psi^{\mathrm{c}}) \quad = \quad 0 \quad \forall\, \alpha^i \in \mathbb{R}\,. \tag{4.7.13}$$

While this analysis is at a fixed point $p \in \mathcal{M}$, it is clear that we can replace the constants α^i by real functions, hence

$$\Re e(z\psi^{\mathrm{c}}, i\lambda e_a\psi + \mu e_a\check{z}\psi^{\mathrm{c}}) \quad = \quad 0 \tag{4.7.14}$$

for all real functions λ and complex functions μ. Returning to the spinorial conformal Killing equation (4.3.10), the above equation shows that if we define a gauged covariant derivative $\widehat{\nabla}$ with gauge terms of the form

$$\widehat{\nabla}_X\psi \quad = \quad \nabla_X\psi - 2A^i(X)\sigma_i\psi \tag{4.7.15}$$

for real 1-forms A^i, then (4.3.10) is equivalent to

$$\Re e(z\psi^{\mathrm{c}}, e_a\widehat{\nabla}_{X_b}\psi + e_b\widehat{\nabla}_{X_a}\psi - \frac{1}{2}g_{ab}\widehat{\mathrm{D}}\psi) \quad = \quad 0\,. \tag{4.7.16}$$

Expanding the σ_i terms in (4.7.15) shows that

$$\widehat{\nabla}_X\psi \quad = \quad \nabla_X\psi + \Big(A^1(X) + iA^2(X)\Big)\,\check{z}\psi^{\mathrm{c}} + iA^3(X)\psi\,. \tag{4.7.17}$$

So far, we have shown how the the covariant derivative may be modified by gauge terms arising from the isotropy group of x. However, there is another transformation of ψ which fixes x, the *chiral transformation* given by

$$\psi \quad \longmapsto \quad \exp(i\lambda\check{z})\psi\,, \quad \lambda \in \mathcal{F}(\mathcal{M}). \tag{4.7.18}$$

Noting that $\exp(i\lambda\check{z}) = \cos\lambda + i\sin\lambda\,\check{z}$, it is straightforward to show that this transformation preserves x. We could not have expected to find it from the isotropy group of x because it does not lie in the Spin group, or even in the Clifford group, since it does not preserve the space of 1-forms under the vector representation. The chiral transformation gives rise to a 'chiral' U(1)-gauge term of the form $i\lambda\check{z}\psi$. It is easily seen that

$$\Re e(z\psi^{\mathrm{c}}, i\lambda e_a\check{z}\psi) \quad = \quad 0 \quad \forall\, \lambda \in \mathcal{F}(\mathcal{M})\,. \tag{4.7.19}$$

Inserting a term of the form $i\mathcal{B}(X)\check{z}\psi$ into (4.7.17) and replacing $A^3(X)$ by $\mathcal{A}(X)$ and $(A^1(X)+iA^2(X))$ by $\mathcal{C}(X)$, it can be seen that the spinorial conformal Killing equation (4.3.10) is equivalent to (4.7.16), where this time $\widehat{\nabla}_X\psi$ is given by

$$\widehat{\nabla}_X\psi \quad = \quad \nabla_X\psi + i\,\mathcal{A}(X)\psi + i\,\mathcal{B}(X)\check{z}\psi + i\,\mathcal{C}(X)\check{z}\psi^{\mathrm{c}} \tag{4.7.20}$$

for real 1-forms $\mathcal{A}$, $\mathcal{B}$ and complex 1-form $\mathcal{C}$. We will suppose that ψ satisfies a 'gauged' twistor equation with gauge terms of this form.

We will write $L_X\psi$ in components as

$$L_X\psi \;=\; \alpha(X)u + \beta(X)v + \gamma(X)u^{\mathrm{c}} + \delta(X)v^{\mathrm{c}}\,. \tag{4.7.21}$$

Since $e^a L_{X_a}\psi = 0$, we must have $\alpha u + \beta v + \gamma u^{\mathrm{c}} + \delta v^{\mathrm{c}} = 0$. This gives a relationship between certain components of $L_X\psi$. An arbitrary 1-form α may be written in terms of the basis covectors as

$$\alpha \;=\; \frac{1}{\mu}\left(\alpha(l)k^\flat + \alpha(k)l^\flat + \alpha(\bar{m})m^\flat + \alpha(m)\bar{m}^\flat\right)\,. \tag{4.7.22}$$

Expressing $\alpha u + \beta u + \gamma u + \delta u = 0$ in components, it can be shown that

$$\begin{array}{rclcrcl} \alpha(k)+\beta(\bar{m}) & = & 0 & \qquad & \alpha(m)-\beta(l) & = & 0 \\ \gamma(k)+\delta(m) & = & 0 & & \gamma(\bar{m})-\delta(l) & = & 0\,. \end{array} \tag{4.7.23}$$

Now the spinorial conformal Killing equation for ψ is (4.3.13). We may write this equation using the components of ψ and $L_X\psi$, in order to express it as a tensor equation for the real part of a complex tensor. First we define a symmetric $(2,0)$ tensor W by

$$\begin{aligned} W(X,Y) \;=\; & (iu^{\mathrm{c}} - iv, \alpha(X)Y^\flat u + \alpha(Y)X^\flat u + \beta(X)Y^\flat v + \beta(Y)X^\flat v \\ & + \gamma(X)Y^\flat u^{\mathrm{c}} + \gamma(Y)X^\flat u^{\mathrm{c}} + \delta(X)Y^\flat v^{\mathrm{c}} + \delta(Y)X^\flat v^{\mathrm{c}})\,. \end{aligned}$$

Then (4.3.13) is equivalent to

$$\Re e[W](X,Y) \;=\; 0 \tag{4.7.24}$$

for all real vector fields X and Y. Now for an arbitrary vector field X we have

$$\begin{array}{rclcrcl} (iu^{\mathrm{c}}, X^\flat u) & = & k^\flat(X) & \qquad & (iu^{\mathrm{c}}, X^\flat v) & = & \bar{m}^\flat(X) \\ (-iv, X^\flat v^{\mathrm{c}}) & = & -l^\flat(X) & & (-iv, X^\flat u^{\mathrm{c}}) & = & \bar{m}^\flat(X)\,. \end{array}$$

Noting that the inner product of an odd semi-spinor and an even semi-spinor is zero, it is clear that W may be written as

$$W \;=\; 2\,\mathbf{Sym}\left(\alpha\otimes k^\flat - \delta\otimes l^\flat + \beta\otimes\bar{m}^\flat + \gamma\otimes\bar{m}^\flat\right)\,. \tag{4.7.25}$$

Then the real tensor $T = 2\,\Re e[W]$ is given by

$$T \;=\; 2\,\mathbf{Sym}\left((\alpha+\bar{\alpha})\otimes k^\flat - (\delta+\bar{\delta})\otimes l^\flat + (\bar{\beta}+\bar{\gamma})\otimes m^\flat + (\beta+\gamma)\otimes\bar{m}^\flat)\right)\,.$$

from which we can see that (4.3.10) is equivalent to the tensor equation $T = 0$. Evaluating $T(k,k)$, $T(k,l)$, $\ldots$,$T(\bar{m},\bar{m})$ we find that the equation $T = 0$ is equivalent to the following equations for the components of $L_X\psi$:

$$\Re e[\delta(k)] \;=\; 0 \tag{4.7.26}$$

$$\Re e[\alpha(k)] - \Re e[\delta(l)] \;=\; 0 \tag{4.7.27}$$

$$\beta(k) + \gamma(k) - \delta(m) - \bar{\delta}(m) \;=\; 0 \tag{4.7.28}$$

$$\begin{aligned}
\Re e[\alpha(l)] &= 0 && (4.7.29)\\
\alpha(m)+\bar{\alpha}(m)+\beta(l)+\gamma(l) &= 0 && (4.7.30)\\
\beta(m)+\gamma(m) &= 0 && (4.7.31)\\
\Re e[\beta(\bar{m})]+\Re e[\gamma(\bar{m})] &= 0\,. && (4.7.32)
\end{aligned}$$

The last equation is actually the same as the first equation by (4.7.23). Given these constraints and (4.7.23), we now try to solve for gauge terms such that ψ satisfies a gauged twistor equation.

Theorem 4.7.33 *Let ψ be a Dirac spinor on a spacetime $(\mathcal{M},g)$. If $x^\sharp = (z\psi^{\mathrm{c}}, e^a\psi)X_a$ is conformal Killing then there exist real 1-forms $\mathcal{A}$ and $\mathcal{B}$ and a complex 1-form $\mathcal{C}$ such that*

$$\widehat{\nabla}_X\psi - \frac{1}{4}X^\flat\widehat{\mathrm{D}}\psi = 0 \quad \forall\, X \in \Gamma\, T\mathcal{M} \qquad (4.7.34)$$

where

$$\begin{aligned}
\widehat{\nabla}_X\psi &= \nabla_X\psi + i\mathcal{A}(X)\psi + i\mathcal{B}(X)\check{z}\psi + i\mathcal{C}(X)\check{z}\psi^{\mathrm{c}}\\
\widehat{\mathrm{D}}\psi &= \mathrm{D}\psi + i\mathcal{A}\psi + i\mathcal{B}\check{z}\psi + i\mathcal{C}\check{z}\psi^{\mathrm{c}}\,.
\end{aligned}$$

Conversely, if ψ satisfies (4.7.34) for some real $\mathcal{A}$, $\mathcal{B}$ and complex $\mathcal{C}$ then $x^\sharp$ is a conformal Killing vector.

Proof. If $x^\sharp$ is conformal Killing then equations (4.7.26)–(4.7.32) hold. We will show that it is possible to find $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ such that

$$\begin{aligned}
L_X\psi &= -i\mathcal{A}(X)\psi + \frac{1}{4}iX^\flat\mathcal{A}\psi - i\mathcal{B}(X)\check{z}\psi + \frac{1}{4}iX^\flat\mathcal{B}\check{z}\psi\\
&\qquad - i\,\mathcal{C}(X)\check{z}\psi^{\mathrm{c}} + \frac{1}{4}iX^\flat\mathcal{C}\check{z}\psi^{\mathrm{c}} \qquad (4.7.35)
\end{aligned}$$

where $\mathcal{A}$ and $\mathcal{B}$ are real and $\mathcal{C}$ is complex. Using the Clifford relations, this equation can be written as

$$\begin{aligned}
L_X\psi &= -\frac{1}{2}i\mathcal{A}(X)\psi - \frac{1}{4}i\mathcal{A}X^\flat\psi - \frac{1}{2}i\mathcal{B}(X)\check{z}\psi - \frac{1}{4}i\mathcal{B}X^\flat\check{z}\psi\\
&\qquad - \frac{1}{2}i\,\mathcal{C}(X)\check{z}\psi^{\mathrm{c}} - \frac{1}{4}i\,\mathcal{C}X^\flat\check{z}\psi^{\mathrm{c}}\,. \qquad (4.7.36)
\end{aligned}$$

For $\psi = u + v^{\mathrm{c}}$ we have $\check{z}\psi = u - v^{\mathrm{c}}$ and $\check{z}\psi^{\mathrm{c}} = -u^{\mathrm{c}} + v$. Putting these components into the above, we have

$$\begin{aligned}
&\alpha(X)u + \beta(X)v + \gamma(X)u^{\mathrm{c}} + \delta(X)v^{\mathrm{c}}\\
&= -\frac{1}{2}i\left(\mathcal{A}(X)+\mathcal{B}(X)\right)u - \frac{1}{4}i\left(\mathcal{A}+\mathcal{B}\right)X^\flat u\\
&\qquad - \frac{1}{2}i\,\mathcal{C}(X)v - \frac{1}{4}i\,\mathcal{C}X^\flat v + \frac{1}{2}i\,\mathcal{C}(X)u^{\mathrm{c}} + \frac{1}{4}i\,\mathcal{C}X^\flat u^{\mathrm{c}}\\
&\qquad - \frac{1}{2}i\left(\mathcal{A}(X)-\mathcal{B}(X)\right)v^{\mathrm{c}} - \frac{1}{4}i\left(\mathcal{A}-\mathcal{B}\right)X^\flat v^{\mathrm{c}}\,. \qquad (4.7.37)
\end{aligned}$$

Setting $\mathcal{F} = \mathcal{A} + \mathcal{B}$ and $\mathcal{G} = \mathcal{A} - \mathcal{B}$, this becomes

$$\begin{aligned} &\alpha(X)u + \beta(X)v + \gamma(X)u^{\mathrm{c}} + \delta(X)v^{\mathrm{c}} \\ = &-\frac{1}{2}i\mathcal{F}(X)u - \frac{1}{4}i\mathcal{F}X^\flat u - \frac{1}{2}i\,\mathcal{C}(X)v - \frac{1}{4}i\,\mathcal{C}X^\flat v \\ &+\frac{1}{2}i\,\mathcal{C}(X)u^{\mathrm{c}} + \frac{1}{4}i\,\mathcal{C}X^\flat u^{\mathrm{c}} - \frac{1}{2}i\mathcal{G}(X)v^{\mathrm{c}} - \frac{1}{4}i\mathcal{G}X^\flat v^{\mathrm{c}}\,. \end{aligned} \tag{4.7.38}$$

The procedure used in solving for $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{C}$ is similar to that used in the null case. We simply evaluate (4.7.38) on each basis vector in turn to solve for the components of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{C}$, subject to the constraints given above.

To illustrate the calculation, we will compute (4.7.38) with $X = k$ in detail. Since $k^\flat u = k^\flat u^{\mathrm{c}} = 0$, we have

$$\begin{aligned} &\alpha(k)u + \beta(k)v + \gamma(k)u^{\mathrm{c}} + \delta(k)v^{\mathrm{c}} \\ = &-\frac{1}{2}i\mathcal{F}(k)u - \frac{1}{2}i\,\mathcal{C}(k)v - \frac{1}{4}i\,\mathcal{C}k^\flat v \\ &+\frac{1}{2}i\,\mathcal{C}(k)u^{\mathrm{c}} - \frac{1}{2}i\mathcal{G}(k)v^{\mathrm{c}} - \frac{1}{4}i\mathcal{G}k^\flat v^{\mathrm{c}}\,. \end{aligned} \tag{4.7.39}$$

Now

$$\begin{aligned} \mathcal{C}k^\flat v &= \frac{1}{\mu}\left(\mathcal{C}(l)k^\flat + \mathcal{C}(k)l^\flat + \mathcal{C}(\bar{m})m^\flat + \mathcal{C}(m)\bar{m}^\flat\right)k^\flat v \\ &= \frac{1}{\mu}\mathcal{C}(k)l^\flat k^\flat v + \frac{1}{\mu}\mathcal{C}(\bar{m})m^\flat k^\flat v \quad \text{since } k^\flat k^\flat v = \bar{m}^\flat k^\flat v = 0 \\ &= \frac{1}{\mu}\mathcal{C}(k)(2\mu - k^\flat l^\flat)v - \frac{1}{\mu}\mathcal{C}(\bar{m})k^\flat m^\flat v \\ &= 2\,\mathcal{C}(k)v - 2\,\mathcal{C}(\bar{m})u\,. \end{aligned} \tag{4.7.40}$$

Similarly,

$$\mathcal{G}k^\flat v^{\mathrm{c}} = 2\,\mathcal{G}(k)v^{\mathrm{c}} - 2\,\mathcal{G}(m)u^{\mathrm{c}}\,. \tag{4.7.41}$$

Substituting into (4.7.39) we have

$$\begin{aligned} &\alpha(k)u + \beta(k)v + \gamma(k)u^{\mathrm{c}} + \delta(k)v^{\mathrm{c}} \\ = &-\frac{1}{2}i\left(\mathcal{F}(k) - \mathcal{C}(\bar{m})\right)u - i\,\mathcal{C}(k)v + \frac{1}{2}i\left(\mathcal{G}(m) + \mathcal{C}(k)\right)u^{\mathrm{c}} - i\,\mathcal{G}(k)v^{\mathrm{c}}\,. \end{aligned}$$

From this we obtain expressions for the components of the gauge terms in terms of the components of $L_k\psi$. In the same way, we can substitute each basis vector for X in turn, from which we obtain the following equations:

$$\begin{array}{rclrcl}
\mathcal{F}(k)-\mathcal{C}(\bar{m}) &=& 2i\alpha(k) & \mathcal{F}(l) &=& i\alpha(l)\\
\mathcal{C}(k) &=& i\beta(k) & \mathcal{F}(m)+\mathcal{C}(l) &=& 2i\beta(l)\\
\mathcal{G}(m)+\mathcal{C}(k) &=& -2i\gamma(k) & \mathcal{C}(l) &=& -i\gamma(l)\\
\mathcal{G}(k) &=& i\delta(k) & \mathcal{G}(l)-\mathcal{C}(\bar{m}) &=& 2i\delta(l)
\end{array}$$

$$\begin{array}{rclrcl}
\mathcal{F}(m)+\mathcal{C}(l) &=& 2i\alpha(m) & \mathcal{F}(\bar{m}) &=& i\alpha(\bar{m})\\
\mathcal{C}(m) &=& i\beta(m) & \mathcal{F}(k)-\mathcal{C}(\bar{m}) &=& -2i\beta(\bar{m})\\
\mathcal{C}(m) &=& -i\gamma(m) & \mathcal{G}(l)-\mathcal{C}(\bar{m}) &=& 2i\gamma(\bar{m})\\
\mathcal{G}(m)+\mathcal{C}(k) &=& 2i\delta(m) & \mathcal{G}(\bar{m}) &=& i\delta(\bar{m})
\end{array}$$

When (4.7.26)–(4.7.32) hold we can find a solution for this set of equations. Clearly $\mathcal{C}(k) = i\beta(k)$, $\mathcal{C}(l) = -i\gamma(l)$ and $\mathcal{C}(m) = i\beta(m)$, which is consistent with $\mathcal{C}(m) = -i\gamma(m)$ by (4.7.31). Now $\mathcal{F}(k) = 2i\alpha(k) + \mathcal{C}(\bar{m})$, so for $\mathcal{F}$ to be real we require that $\Im m[2i\alpha(k) + \mathcal{C}(\bar{m})] = 0$. It follows that we must impose the condition $\Im m[\mathcal{C}(\bar{m})] = -2\Re e[\alpha(k)]$. Continuing, $\mathcal{F}(l) = i\alpha(l)$ is real since $\Re e[\alpha(l)] = 0$, and

$$\begin{aligned}
\mathcal{F}(m) &= 2i\alpha(m) - \mathcal{C}(l)\\
&= 2i\alpha(m) + i\gamma(l)\\
&= 2i\alpha(m) - i\alpha(m) - i\bar{\alpha}(m) - i\beta(l) \quad \text{by (4.7.30)}\\
&= -i\bar{\alpha}(m) \quad \text{by (4.7.23)}\,.
\end{aligned} \tag{4.7.42}$$

This is consistent with the reality of $\mathcal{F}$ since we also have $\mathcal{F}(\bar{m}) = i\alpha(\bar{m}) = \overline{\mathcal{F}(m)}$. For $\mathcal{G}$ we have $\mathcal{G}(k) = i\delta(k)$, which is real since $\Re e[\delta(k)] = 0$, and $\mathcal{G}(l) = 2i\delta(l) + \mathcal{C}(\bar{m})$, for which

$$\begin{aligned}
\Im m[\mathcal{G}(l)] &= 2\Re e[\delta(l)] + \Im m[\mathcal{C}(\bar{m})]\\
&= 2\Re e[\alpha(k)] + \Im m[\mathcal{C}(\bar{m})] \quad \text{by (4.7.27)}\\
&= 0\,.
\end{aligned} \tag{4.7.43}$$

Finally,

$$\begin{aligned}
\mathcal{G}(m) &= 2i\delta(m) - \mathcal{C}(k)\\
&= 2i\delta(m) + i\gamma(k) - i\delta(m) - i\bar{\delta}(m) \quad \text{by (4.7.28)}\\
&= -i\bar{\delta}(m)
\end{aligned} \tag{4.7.44}$$

while $\mathcal{G}(\bar{m}) = i\delta(\bar{m}) = \overline{\mathcal{G}(m)}$, so $\mathcal{G}$ is real. Then the components of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are

$$\begin{aligned}
\mathcal{A}(k) &= -\Im m[\alpha(k)] + \frac{1}{2}i\delta(k) + \frac{1}{2}\Re e[\mathcal{C}(\bar{m})] & \mathcal{A}(m) &= -\frac{1}{2}i\left(\bar{\alpha}(m) + \bar{\delta}(m)\right) \\
\mathcal{A}(l) &= \frac{1}{2}i\alpha(l) - \Im m[\delta(l)] + \frac{1}{2}\Re e[\mathcal{C}(\bar{m})] & \mathcal{A}(\bar{m}) &= \frac{1}{2}i\left(\alpha(\bar{m}) + \delta(\bar{m})\right) \\
\mathcal{B}(k) &= -\Im m[\alpha(k)] - \frac{1}{2}i\delta(k) + \frac{1}{2}\Re e[\mathcal{C}(\bar{m})] & \mathcal{B}(m) &= -\frac{1}{2}i\left(\bar{\alpha}(m) - \bar{\delta}(m)\right) \\
\mathcal{B}(l) &= \frac{1}{2}i\alpha(l) + \Im m[\delta(l)] - \frac{1}{2}\Re e[\mathcal{C}(\bar{m})] & \mathcal{B}(\bar{m}) &= \frac{1}{2}i\left(\alpha(\bar{m}) - \delta(\bar{m})\right) \\
\mathcal{C}(k) &= i\beta(k) & \mathcal{C}(m) &= i\beta(m) \\
\mathcal{C}(l) &= -i\gamma(l) & \Im m[\mathcal{C}(\bar{m})] &= -2\,\Re e[\alpha(k)]\,.
\end{aligned}$$

We are free to choose the real part of $\mathcal{C}(\bar{m})$.

Conversely, if (4.7.35) holds then

$$\begin{aligned}
0 = \Re e(z\psi^{\mathrm{c}}, & X^\flat L_Y\psi + Y^\flat L_X\psi \\
& + i\mathcal{A}(Y)X^\flat\psi + i\mathcal{A}(X)Y^\flat\psi - \frac{1}{2}ig(X,Y)\mathcal{A}\psi \\
& + i\mathcal{B}(Y)X^\flat\check{z}\psi + i\mathcal{B}(X)Y^\flat\check{z}\psi - \frac{1}{2}ig(X,Y)\mathcal{B}\check{z}\psi \\
& + i\mathcal{C}(Y)X^\flat\check{z}\psi^{\mathrm{c}} + i\mathcal{C}(X)Y^\flat\check{z}\psi^{\mathrm{c}} - \frac{1}{2}ig(X,Y)\mathcal{C}\check{z}\psi^{\mathrm{c}})\,.
\end{aligned} \tag{4.7.45}$$

From (4.7.14) and (4.7.19) it can then be seen that all the terms in (4.7.45) vanish except for the twistor operator terms, thus ψ satisfies the spinorial conformal Killing equation (4.3.13). ∎

4.8 Timelike shear-free vector fields

As in Theorem 4.5.23, once the calculations for the conformal Killing vector field have been done, the equation for the shear-free condition is obtained by simply adding another gauge term to the covariant derivative.

Theorem 4.8.1 *Let ψ be a Dirac spinor on a spacetime $(\mathcal{M}, g)$. If $x^\sharp = (z\psi^{\mathrm{c}}, e^a\psi)X_a$ is shear-free then there exists a real 1-form $\mathcal{B}$ and complex 1-forms $\mathcal{A}$ and $\mathcal{C}$ such that*

$$\widehat{\nabla}_X\psi - \frac{1}{4}X^\flat\widehat{\mathrm{D}}\psi = 0 \quad \forall\ X \in \Gamma T\mathcal{M} \tag{4.8.2}$$

where

$$\begin{aligned}
\widehat{\nabla}_X\psi &= \nabla_X\psi + \mathcal{A}(X)\psi + i\mathcal{B}(X)\check{z}\psi + i\mathcal{C}(X)\check{z}\psi^{\mathrm{c}} \\
\widehat{\mathrm{D}}\psi &= \mathrm{D}\psi + \mathcal{A}\psi + i\mathcal{B}\check{z}\psi + i\mathcal{C}\check{z}\psi^{\mathrm{c}}\,.
\end{aligned}$$

Conversely, if ψ satisfies (4.8.2) for some real $\mathcal{B}$ and complex $\mathcal{A}$ and $\mathcal{C}$ then $x^\sharp$ is shear-free.

Proof. If $x^\sharp$ is shear-free then there exists a real 1-form A such that ψ satisfies (4.5.22), where $\mathbb{L}_X$ is the twistor operator of $\mathbb{V}$, and $\mathbb{V}_X\psi = \nabla_X\psi + A(X)\psi$. Since (4.5.22) has the same form as (4.3.10), the components of $\mathbb{L}_X\psi$ satisfy equations (4.7.23) and (4.7.26)–(4.7.32). Then we can find real 1-forms $\mathcal{A}'$, $\mathcal{B}$ and $\mathcal{C}$ such that ψ satisfies (4.7.35), where $L_X\psi$ is replaced by $\mathbb{L}_X\psi$. Since $\mathbb{L}_X\psi = L_X\psi + A(X)\psi - \frac{1}{4}X^\flat A\psi$, it is clear that ψ satisfies (4.8.2) with $\mathcal{A} = A + i\mathcal{A}'$. Conversely, if ψ satisfies (4.8.2) then it is clear from (4.7.14) and (4.7.19) that x satisfies (4.5.18) where $A = \Re e[\mathcal{A}]$. ∎

Chapter 5

Conformal Killing-Yano Tensors and Shear-free Spinors

In this chapter we consider generalisations of the conformal Killing equation and the shear-free equation to tensors of higher degree. Killing tensors were first introduced as solutions of a generalisation of the Killing vector equation. One possibility is to replace the vector field by a totally symmetric tensor, whilst Yano extended Killing's equation to a totally antisymmetric tensor, or differential p-form [Yan52]. We refer to these as *Killing-Yano tensors*. The Killing-Yano (KY) equation is not invariant under conformal rescalings of the metric. The conformal generalisation of the KY equation has been given by Tachibana for the case of a 2-form [Tac69], and by Kashiwada and Tachibana for forms of higher degree [Kas68, TK69]. It then emerges that a Killing-Yano tensor is a co-closed conformal Killing-Yano (CKY) tensor. The original presentation of CKY tensors used tensor components. Since it is an equation for a p-form, we are able to write the CKY equation very compactly using exterior calculus. This chapter is primarily a review of certain properties of CKY tensors, which will be used extensively in Chapter 6 to construct symmetry operators for the Maxwell and Dirac equations.

On spacetime, the only non-trivial CKY forms are 2-forms. In this case, there is a useful relationship between self-dual CKY 2-forms and shear-free spinors. Equation (4.1.4) shows how any self-dual 2-form ω may be related to a pair of even semi-spinors u and v. It can be shown that if ω is a CKY 2-form then u and v each satisfy the shear-free spinor equation (4.5.24). If the gauge term associated with u is $\mathcal{A}$ then v satisfies (4.5.24) with gauge term $-\mathcal{A}$, that is, the gauge terms of u and v sum to zero. More generally, Dietz and Rüdiger have shown that for ω non-null (that is, $\omega \cdot \omega \neq 0$), u and v are shear-free spinors with *independent* gauge terms if and only if ω satisfies a certain generalisation of the CKY equation [DR80]. Although they did not interpret it as such, this equation can be obtained from the CKY equation by replacing the ordinary covariant derivative with a $\mathrm{GL}(1, \mathbb{C})$-covariant derivative. This is analogous to the way we obtained the shear-free vector equation from the conformal Killing equation in §4.5. For this reason we refer to it as the *gauged CKY equation*. When written in this form, we may also consider *null* 2-forms which are solutions of the gauged CKY equation, a case which was not examined by Dietz and Rüdiger. We then have a single equation which characterises the semi-spinors determined by a 2-form as shear-free in both the null and non-null cases. In four dimensions we refer to 2-forms satisfying the gauged CKY equation as 'shear-free' also.

5.1 The conformal Killing-Yano equation

In §4.3 we made the observation that the numerical coefficients in the conformal Killing equation (4.3.4) are just such that we cannot conclude that $K^\flat$ is closed or co-closed. This suggests how the equation may be generalised to arbitrary forms in higher dimensions. Replacing $K^\flat$ in (4.3.4) by a complex p-form ω and inserting the appropriate coefficients, the conformal Killing-Yano equation on an n-dimensional pseudo-Riemannian manifold $(\mathcal{M}, g)$ is

$$\nabla_X \omega = \frac{1}{p+1} X \lrcorner\, d\omega - \frac{1}{n-p+1} X^\flat \wedge d^*\omega \quad \forall\ X \in \Gamma\, T\mathcal{M}\,. \tag{5.1.1}$$

With $X = X_a$, equation (5.1.1) implies that

$$\begin{aligned} e^a \wedge \nabla_{X_a}\omega &= \frac{1}{p+1} e^a \wedge X_a \lrcorner\, d\omega - \frac{1}{n-p+1} e^a \wedge e_a \wedge d^*\omega \\ &= d\omega \end{aligned}$$

and

$$\begin{aligned} -X^a \lrcorner\, \nabla_{X_a}\omega &= -\frac{1}{p+1} X^a \lrcorner\, X_a \lrcorner\, d\omega + \frac{1}{n-p+1} X^a \lrcorner\, (e_a \wedge d^*\omega) \\ &= \frac{1}{n-p+1} \left(n\, d^*\omega - e_a \wedge X^a \lrcorner\, d^*\omega\right) \\ &= d^*\omega \end{aligned}$$

as required. Taking the derivative of (5.1.1) we have

$$\begin{aligned} \nabla_Y \nabla_X \omega &= \frac{1}{p+1} (\nabla_Y X) \lrcorner\, d\omega + \frac{1}{p+1} X \lrcorner\, \nabla_Y d\omega \\ &\quad - \frac{1}{n-p+1} \nabla_Y X^\flat \wedge d^*\omega - \frac{1}{n-p+1} X^\flat \wedge \nabla_Y d^*\omega \\ &= \nabla_{\nabla_Y X}\omega + \frac{1}{p+1} X \lrcorner\, \nabla_Y d\omega - \frac{1}{n-p+1} X^\flat \wedge \nabla_Y d^*\omega\,. \end{aligned}$$

Contracting the above with $X = X^a$ and $Y = X_a$ gives the integrability condition

$$\nabla^2 \omega = -\frac{1}{p+1} d^* d\omega - \frac{1}{n-p+1} d d^*\omega \tag{5.1.2}$$

which we will use later. From the definition of the co-derivative, it is clear that

$$\begin{aligned} *(X^\flat \wedge d^*\omega) &= (-1)^{p-1} *(d^*\omega \wedge X^\flat) \\ &= (-1)^{p-1} X \lrcorner\, ** ^{-1} d * \eta\omega \\ &= -X \lrcorner\, d*\omega\,. \end{aligned} \tag{5.1.3}$$

Equivalently,

$$*(X \lrcorner d\omega) \quad = \quad -X^\flat \wedge d^* {*}\omega \,. \tag{5.1.4}$$

Since the Hodge dual commutes with ∇, it follows that the dual of a CKY p-form is a CKY $(n-p)$-form. Thus in even dimensions, say $n = 2r$, a CKY r-form decomposes into self-dual and anti self-dual r-forms which are also CKY tensors.

If $\mathcal{M}$ is 4-dimensional, it can be seen that the only non-trivial CKY tensors are 2-forms. A CKY 0-form or 4-form is constant (parallel), while comparing (5.1.1) with (4.3.4) we see that a CKY 1-form is simply the metric dual of a conformal Killing vector. Since the CKY equation is invariant under Hodge duality, a CKY 3-form is simply the Hodge dual of a CKY 1-form. For a 2-form in four dimensions, it has been shown in [BCK97] that (5.1.1) is equivalent to Tachibana's conformal generalisation of the Killing-Yano equation in the following way. If ω is a 2-form then

$$\omega(X,Y) \quad = \quad \frac{1}{2} Y \lrcorner X \lrcorner \omega \quad \forall\ X,\ Y \in \Gamma T\mathcal{M} \,. \tag{5.1.5}$$

When $p = 2$ and $n = 4$, equation (5.1.1) implies that

$$\begin{aligned} \nabla_Y \omega(X,Z) + \nabla_Z \omega(X,Y) \quad = \quad & \frac{1}{3} d^*\omega(X) g(Y,Z) - \frac{1}{6} d^*\omega(Y) g(Z,X) \\ & - \frac{1}{6} d^*\omega(Z) g(X,Y) \,. \end{aligned} \tag{5.1.6}$$

This is Tachibana's original equation [Tac69]. Putting $X = X_a$ and $Y = X_b$ and multiplying both sides of (5.1.6) by e^{ab}, we recover equation (5.1.1). Yano's equation is obtained by setting the left-hand side of (5.1.6) to zero, thus a Killing-Yano tensor is simply a co-closed conformal Killing-Yano tensor.

5.2 The gauged CKY equation

In §4.5 we showed that the shear-free vector equation may be considered a generalisation of the conformal Killing equation for a vector field, where the ordinary covariant is replaced by a gauged covariant derivative. Modifying (5.1.1) in the same way, we say that a complex p-form ω is a *gauged CKY tensor* if there exists a complex 1-form A such that

$$\widehat{\nabla}_X \omega \quad = \quad \frac{1}{p+1} X \lrcorner \hat{d}\omega - \frac{1}{n-p+1} X^\flat \wedge \hat{d}^*\omega \quad \forall\ X \in \Gamma T\mathcal{M} \tag{5.2.1}$$

where $\widehat{\nabla}_X \omega$ is given by (4.5.10). It can easily be shown that $\hat{d}^*\omega = *^{-1} \hat{d} * \eta\omega$, hence (5.2.1) is also invariant under Hodge duality. We emphasize again that the gauge term $2A$ is determined by ω. In general, the gauge terms of different gauged CKY tensors will be unrelated.

When ω is a 1-form, (5.2.1) is simply the shear-free vector equation (4.5.16) on $(\mathcal{M}, g)$. Now suppose that $(\mathcal{M}, g)$ is a spacetime. When ω is a 2-form, we can regard it as a linear mapping on vector fields. The gauged CKY equation can then be interpreted in terms of the real eigenvectors of ω. An eigenvector of ω is any (complex) vector field X such that

$$X \lrcorner \omega \;=\; \lambda X^\flat \tag{5.2.2}$$

for some complex function λ. Since 2-forms are 'middle forms' in spacetime, we may restrict our attention to self-dual gauged CKY 2-forms without loss of generality. From the definition of the inner product on forms (2.1.29), if $*\omega = i\omega$ then

$$\omega^2 *1 \;=\; i\,\omega \wedge \omega \quad \text{where } \omega^2 = \omega \cdot \omega\,. \tag{5.2.3}$$

Then for any vector field X,

$$\begin{aligned} \omega^2 *X^\flat &= i\,X \lrcorner (\omega \wedge \omega) \\ &= 2i\,X \lrcorner \omega \wedge \omega\,. \end{aligned} \tag{5.2.4}$$

For this dimension and signature, the Hodge dual has the property that

$$** \phi \;=\; \begin{cases} -\phi & \text{if } \phi \text{ is even} \\ \phi & \text{if } \phi \text{ is odd.} \end{cases} \tag{5.2.5}$$

Taking the Hodge dual of (5.2.4), we see that

$$\begin{aligned} \omega^2 X^\flat &= 2i *(X \lrcorner \omega \wedge \omega) \\ &= 2i\,(X \lrcorner \omega)^\sharp \lrcorner *\omega \\ &= -2\,(X \lrcorner \omega)^\sharp \lrcorner \omega\,. \end{aligned} \tag{5.2.6}$$

It follows from (5.2.2) that if X is an eigenvector of ω then its eigenvalue λ satisfies

$$\lambda^2 \;=\; -\frac{1}{2}\omega^2\,. \tag{5.2.7}$$

Furthermore, the eigenspaces of ω are isotropic. Suppose that X and Y are eigenvectors with non-zero eigenvalue λ. Contracting (5.2.2) on the left with $Y \lrcorner$ shows that

$$Y \lrcorner X \lrcorner \omega \;=\; \lambda g(X,Y)\,. \tag{5.2.8}$$

Since the left-hand side is antisymmetric and $\lambda \neq 0$ we must have $g(X,Y) = 0$. Now suppose that $\lambda = 0$. By duality, (5.2.2) is equivalent to

$$X^\flat \wedge \omega \;=\; 0\,. \tag{5.2.9}$$

Contracting the above with $Y \lrcorner$ shows that $g(X,Y)\omega = 0$.

If ω is null then it has a unique eigenvalue 0, and the eigenspace is a 2-dimensional (and hence maximal) isotropic space. Since the Lorentzian signature is $(3,1)$, a MTIS contains precisely one real null direction. If ω is non-null then it has two 2-dimensional eigenspaces, corresponding to the two distinct eigenvalues. Each of these is also a MTIS containing a real null direction. With $n = 4$ and $p = 2$, exposing the gauge terms in

(5.2.1) yields Dietz and Rüdiger's generalisation of the CKY equation [DR80],

$$\begin{aligned}\nabla_X\omega - \frac{1}{3}X\lrcorner\, d\omega + \frac{1}{3}X^\flat\wedge d^*\omega \\ = \quad -2A(X)\omega + \frac{2}{3}X\lrcorner\,(A\wedge\omega) + \frac{2}{3}X^\flat\wedge A^\sharp\lrcorner\,\omega \quad \forall\ X\in\Gamma T\mathcal{M}\,. \end{aligned}\qquad (5.2.10)$$

Taking the exterior product of (5.2.10) with $*\omega$ gives the equation

$$2\omega^2 A \;=\; \frac{3}{4}d\omega^2 - (d^*\omega)^\sharp\lrcorner\,\omega\,. \qquad (5.2.11)$$

If ω is non-null we may solve for A in terms of ω. On the other hand, if ω is null then this shows that $(d^*\omega)^\sharp$ is an eigenvector of ω. In that case we can solve for the components of A using a null tetrad adapted to the eigenvectors of ω. In either case, there is only one A such that ω satisfies (5.2.10).

Writing (5.2.10) in spinorial form, Dietz and Rüdiger have shown that a non-null 2-form ω satisfies (5.2.10) if and only if its real eigenvectors are shear-free [DR80]. For this reason we refer to self-dual solutions of (5.2.10) as *shear-free 2-forms.* As we shall show, Dietz and Rüdiger's result also holds if ω is null. In the notation of §4.1, we may relate ω to a pair of even spinors u_1 and u_2 by

$$\omega \;=\; \frac{1}{2}\left(u_1\otimes\bar{u}_2 + u_2\otimes\bar{u}_1\right), \qquad (5.2.12)$$

where u_1 and u_2 are determined up to complex factors e^{λ} and $\mathrm{e}^{-\lambda}$, respectively. The real eigenvectors of ω are the real null directions determined by u_1 and u_2 via (4.1.9), thus ω is null if and only if u_1 and u_2 are linearly independent. Then the spinorial form of (5.2.10) is

$$\begin{aligned}4\nabla_X u_1\otimes\bar{u}_2 - X^\flat \mathrm{D}u_1\otimes\bar{u}_2 + 4u_1\otimes\overline{\nabla_X u_2} - u_1\otimes\overline{X^\flat\mathrm{D}u_2} \\ + \nabla_{X_a}u_1\otimes\overline{e^aX^\flat u_2} + e^aX^\flat u_1\otimes\overline{\nabla_{X_a}u_2} + (u_1\leftrightarrow u_2) \\ = \quad -12A(X)u_1\otimes\bar{u}_2 + 2X^\flat A u_1\otimes\bar{u}_2 + 2u_1\otimes\overline{X^\flat A u_2} + (u_1\leftrightarrow u_2)\,.\end{aligned}\qquad (5.2.13)$$

We can write this more compactly using the twistor operator, however we also need terms such as $e^aX^\flat u_1\otimes\overline{e_a\mathrm{D}u_2}$. The identity (2.2.6) shows that $e^a\phi e_a = 0$ when ϕ is a middle form. Since the symmetric product of two odd (or even) spinors is a 2-form, we have

$$\begin{aligned}e^aX^\flat u_i\otimes\overline{e_a\mathrm{D}u_j} + e^a\mathrm{D}u_j\otimes\overline{e_aX^\flat u_i} \;&=\; e^a\left(X^\flat u_i\otimes\overline{\mathrm{D}u_j} + \mathrm{D}u_j\otimes\overline{X^\flat u_i}\right)e_a \\ &=\; 0\,.\end{aligned}\qquad (5.2.14)$$

Adding the appropriate terms, equation (5.2.13) is equivalent to

$$\begin{aligned}6L_Xu_1\otimes\bar{u}_2 + 6u_1\otimes\overline{L_Xu_2} \\ - L_{X_a}u_1\otimes\overline{X^\flat e^a u_2} - X^\flat e^a u_1\otimes\overline{L_{X_a}u_2} + (u_1\leftrightarrow u_2) \\ = \quad -12A(X)u_1\otimes\bar{u}_2 + 2X^\flat A u_1\otimes\bar{u}_2 + 2u_1\otimes\overline{X^\flat A u_2} + (u_1\leftrightarrow u_2)\,.\end{aligned}\qquad (5.2.15)$$

We can use this form of (5.2.10) to prove the theorem of Dietz and Rüdiger.

Theorem 5.2.16 *On spacetime, $\omega = \mathbf{Sym}\,(u_1 \otimes \bar{u}_2)$ is a shear-free 2-form if and only if u_1 and u_2 are shear-free spinors. The gauge terms satisfy*

$$\mathcal{A}_1 + \mathcal{A}_2 \;=\; 2A\,,$$

where $\mathcal{A}_1$, $\mathcal{A}_2$ and A are the gauge terms of u_1, u_2 and ω, respectively.

Proof. It is straightforward to show that if u_1 and u_2 satisfy

$$L_X u_i \;=\; -\mathcal{A}_i(X)u_i + \frac{1}{4}X^\flat \mathcal{A}_i u_i \quad \text{(no sum)}$$

then ω satisfies (5.2.15) with $2A = \mathcal{A}_1 + \mathcal{A}_2$ with the use of the relation

$$e^a X^\flat u_i \otimes \overline{e_a A u_j} + e^a A u_j \otimes \overline{e_a X^\flat u_i} \;=\; 0\,. \tag{5.2.17}$$

Now suppose that ω satisfies (5.2.15) for some A. As an abbreviation, we will write (5.2.15) as

$$\begin{aligned} &6\mathbb{L}_X u_1 \otimes \bar{u}_2 + 6u_1 \otimes \overline{\mathbb{L}_X u_2} \\ &\quad - \mathbb{L}_{X_a} u_1 \otimes \overline{X^\flat e^a u_2} - X^\flat e^a u_1 \otimes \overline{\mathbb{L}_{X_a} u_2} + (u_1 \leftrightarrow u_2) \;=\; 0\,, \end{aligned} \tag{5.2.18}$$

where $\mathbb{L}_X$ is the twistor operator of the gauged covariant derivative given by

$$\mathbb{\nabla}_X u \;=\; \nabla_X u + A(X)u\,. \tag{5.2.19}$$

Contracting (5.2.18) on, say, u_1 yields

$$\begin{aligned} &6(\mathbb{L}_X u_2, u_1)u_1 - 6(u_1, u_2)\mathbb{L}_X u_1 + 6(\mathbb{L}_X u_1, u_1)u_2 \\ &\quad - (X^\flat e^a u_2, u_1)\mathbb{L}_{X_a} u_1 - (X^\flat e^a u_1, u_1)\mathbb{L}_{X_a} u_2 \\ &\qquad - (\mathbb{L}_{X_a} u_2, u_1)X^\flat e^a u_1 - (\mathbb{L}_{X_a} u_1, u_1)X^\flat e^a u_2 \;=\; 0\,. \end{aligned} \tag{5.2.20}$$

Using (4.1.2), we can rearrange the inner products above so that

$$\begin{aligned} &2(\mathbb{L}_X u_2, u_1)u_1 - 4(\mathbb{L}_X u_1, u_2)u_1 + 6(\mathbb{L}_X u_1, u_1)u_2 \\ &\quad - (X^\flat e^a u_2, u_1)\mathbb{L}_{X_a} u_1 - (X^\flat e^a u_1, u_1)\mathbb{L}_{X_a} u_2 \;=\; 0 \end{aligned} \tag{5.2.21}$$

where we have also used the relation

$$\begin{aligned} (\mathbb{L}_{X_a} u_i, X^\flat e^a u_j) \;&=\; (e^a X^\flat \mathbb{L}_{X_a} u_i, u_j) \\ &=\; 2(\mathbb{L}_X u_i, u_j) - (X^\flat e^a \mathbb{L}_{X_a} u_i, u_j) \\ &=\; 2(\mathbb{L}_X u_i, u_j) \quad \text{since } e^a \mathbb{L}_{X_a} = 0\,. \end{aligned}$$

We will consider the null and non-null cases separately. If ω is non-null, then u_1 and u_2 are linearly independent. Writing $\mathbb{L}_X u_1$ and $\mathbb{L}_X u_2$ as

$$\mathbb{L}_X u_1 \;=\; \alpha(X)u_1 + \beta(X)u_2$$

$$\mathbb{L}_X u_2 \;=\; \gamma(X)u_1 + \delta(X)u_2$$

where α, β, γ and δ are complex 1-forms, we must have

$$\begin{aligned} \alpha u_1 + \beta u_2 &= 0 \\ \gamma u_1 + \delta u_2 &= 0 \end{aligned}$$

since $e^a \mathbb{L}_{X_a} = 0$. The coefficient of u_2 in (5.2.21) must vanish, hence

$$\begin{aligned} (6\mathbb{L}_X u_1 - X^\flat \beta u_2 - X^\flat \delta u_1, u_1) &= (6\mathbb{L}_X u_1 + X^\flat(\alpha - \delta)u_1, u_1) \\ &= 0\,. \end{aligned} \tag{5.2.22}$$

Since the only even semi-spinors orthogonal to u_1 are proportional to u_1, there exists a complex 1-form $\mathcal{A}_1'$ such that

$$6\mathbb{L}_X u_1 + X^\flat(\alpha - \delta)u_1 \;=\; -6\mathcal{A}_1'(X)u_1\,. \tag{5.2.23}$$

Putting $X = X_a$ and multiplying on the left by e^a shows that

$$4(\alpha - \delta)u_1 \;=\; -6\mathcal{A}_1' u_1 \tag{5.2.24}$$

and so

$$\mathbb{L}_X u_1 \;=\; -\mathcal{A}_1'(X)u_1 + \frac{1}{4}X^\flat \mathcal{A}_1' u_1\,. \tag{5.2.25}$$

Writing $\mathbb{L}_X$ in terms of A and the ordinary twistor operator, it is clear that u_1 is a shear-free spinor with gauge term $\mathcal{A}_1 = A + \mathcal{A}_1'$. Contracting (5.2.18) on u_2 instead shows that u_2 is also a shear-free spinor with gauge term, say, $\mathcal{A}_2$. In the first part of the proof it was shown that if this is the case then the gauge term of ω is given by $2A = \mathcal{A}_1 + \mathcal{A}_2$.

If ω is null then u_1 and u_2 are linearly dependent, and they may be scaled so that $\omega = u \otimes \bar{u}$ for some even semi-spinor u. Equation (5.2.21) shows that u must satisfy

$$2(\mathbb{L}_X u, u)u - (X^\flat e^a u, u)\mathbb{L}_{X_a} u \;=\; 0\,. \tag{5.2.26}$$

Given another spinor v independent from u, we have a basis $\{u,\, v\}$ for the spinor space. Writing $\mathbb{L}_X u = \alpha(X)u + \beta(X)v$ for complex 1-forms α and β, the coefficient of u in (5.2.26) satisfies

$$(2\mathbb{L}_X u - X^\flat \alpha u, u) \;=\; 0 \quad \forall\, X \in \Gamma T\mathcal{M}\,, \tag{5.2.27}$$

hence there exists a complex 1-form $\mathcal{A}'$ such that

$$2\mathbb{L}_X u - X^\flat \alpha u \;=\; -2\mathcal{A}'(X)u \quad \forall\, X \in \Gamma T\mathcal{M}\,. \tag{5.2.28}$$

Contracting on e^a and X_a, we have

$$\alpha u \;=\; \frac{1}{2}\mathcal{A}' u \tag{5.2.29}$$

and so

$$\mathbb{L}_X u = -\mathcal{A}'(X)u + \frac{1}{4}X^\flat \mathcal{A}' u\,. \tag{5.2.30}$$

That is, u is a shear-free spinor with gauge term $\mathcal{A} = A + \mathcal{A}'$. However, since we must also have $2A + 2\mathcal{A}' = 2A$, $\mathcal{A}'$ must be zero and the gauge term of u is simply A. ∎

If the gauge term of ω vanishes then (5.2.15) is the spinorial form of the CKY equation. This is usually known as the 2-*index Killing spinor equation* [WP70], which is a generalisation of the twistor equation to spin-1 fields. In that case the gauge terms of u_1 and u_2 must have opposite sign, say $\mathcal{A}$ and $-\mathcal{A}$. If in addition ω is null, the gauge term of u must vanish, hence u satisfies the twistor equation.

It is immediate from the theorem that if ω is shear-free then the null 2-forms

$$\phi_1 = u_1 \otimes \bar{u}_1\,, \qquad \phi_2 = u_2 \otimes \bar{u}_2$$

are also shear-free. By Robinson's theorem, a null shear-free 2-form is proportional to an exact form. A self-dual exact form is also co-exact, thus if ω is null it can be scaled to a solution of the vacuum Maxwell equations (this is not the case if ω is non-null). In higher (even) dimensions, Hughston and Mason have found a generalisation of Robinson's theorem [HM88]. On a $2r$-dimensional pseudo-Riemannian manifold $(\mathcal{M}, g)$, a self-dual decomposable r-form ω determines a collection of null 1-forms T_ω where

$$T_\omega(\mathcal{M}) = \{x \in \Gamma\, T^*\mathcal{M}^{\mathbb{C}} : x \wedge \omega = 0\}\,. \tag{5.2.31}$$

We will denote the collection of vector fields dual to $T_\omega(\mathcal{M})$ by $T^*_\omega(\mathcal{M})$. Since ω is self-dual we have

$$T^*_\omega(\mathcal{M}) = \{X \in \Gamma\, T\mathcal{M}^{\mathbb{C}} : X \,\lrcorner\, \omega = 0\}\,. \tag{5.2.32}$$

At any point $p \in \mathcal{M}$, $T^*_\omega(\mathcal{M})$ determines a MTIS of $T_p\mathcal{M}^{\mathbb{C}}$, and so $T^*_\omega(\mathcal{M})$ determines a distribution of null r-planes on $\mathcal{M}$. The distribution is said to be *integrable* if

$$[T^*_\omega(\mathcal{M}), T^*_\omega(\mathcal{M})] \subseteq T^*_\omega(\mathcal{M}) \tag{5.2.33}$$

where $[\ ,\ ]$ is the usual commutator on vector fields. If this is the case then $T^*_\omega(\mathcal{M})$ defines a *foliation* of $\mathcal{M}$. Now any decomposable self-dual r-form can be written in terms of a pure spinor (see §2.3). Recall that the collection of null spaces determined by a spinor field u is denoted by $T_u(\mathcal{M})$. Identifying ω with $u \otimes \bar{u}$ where u is a pure spinor field, it is clear that $T_u(\mathcal{M})$ and $T_\omega(\mathcal{M})$ are identical. Hughston and Mason gave a spinorial proof that the integrability condition (5.2.33) is equivalent to each of the following:

(1) There exists a complex 1-form α such that

$$d\omega + \alpha \wedge \omega = 0\,. \tag{5.2.34}$$

(2) For each $X^\flat \in T_u(\mathcal{M})$,

$$u \otimes \overline{\nabla_X u} - \nabla_X u \otimes \bar{u} \quad = \quad 0\,. \tag{5.2.35}$$

Furthermore, they showed that there exists a scaling $\hat{\omega}$ of ω such that $d\hat{\omega} = 0$. (In fact, Hughston and Mason gave a much more general result for spinor fields of arbitrary valence. The form in which we have presented these results is due to Trautman [Tra93]).

Returning to the 4-dimensional Lorentzian case where ω is a null self-dual 2-form, it is well-known that (5.2.34) is the condition that the real null eigenvector of ω is shear-free. It follows that (5.2.35) is equivalent to Sommers' equation for this dimension and signature. We can see this directly as follows. From (4.1.6) it is clear that (5.2.35) can be written equivalently as

$$(u, \nabla_X u) \quad = \quad 0 \quad \forall\ X^\flat \in T_u(\mathcal{M})\,. \tag{5.2.36}$$

In terms of the null basis given in §4.4, the shear-free spinor equation (4.1) is

$$(u, \nabla_k u) l^\flat u + (u, \nabla_m u)\bar{m}^\flat u \quad = \quad 0 \tag{5.2.37}$$

since $k^\flat u = m^\flat u = 0$. Clearly, if (5.2.36) holds then u is shear-free. Conversely, if u is shear-free then $(u, \nabla_k u) = (u, \nabla_m u) = 0$ since $l^\flat u$ and $\bar{m}^\flat u$ are linearly independent, thus (5.2.36) holds. From this argument it can also be seen that ω must also be a solution of the gauged CKY equation. That is, if ω satisfies (5.2.34) for some α, it also satisfies (5.2.10) for some A. The relationship between α and A can be found in terms of the null basis, although it should be noted that α is not unique since the transformation

$$\alpha \quad \longmapsto \quad \alpha + \lambda\, k^\flat + \mu\, m^\flat \tag{5.2.38}$$

for complex functions λ, μ leaves (5.2.34) invariant. Given that (5.2.34) and (5.2.35) are equivalent to the foliating condition (5.2.33) in all even dimensions, it seems worthwhile to investigate the connection between these equations and the gauged CKY equation (5.2.1) in higher dimensions. We hope to do this in future work.

5.3 Integrability conditions for shear-free spinors

Until now, we have not assumed any explicit restrictions on the form of the metric. However, the CKY 2-form equation imposes strict integrability conditions on the conformal tensor. The Petrov classification scheme characterises the Jordan canonical form of the conformal tensor regarded as a linear mapping on the space of self-dual 2-forms. In the following we briefly summarise the Petrov types in a form that will be useful in the study of CKY tensors. On spacetime, the Hodge dual squares to -1 when acting on 2-forms. With the Hodge dual as complex structure, the 6-dimensional space of real 2-forms may be regarded as a 3-dimensional complex space. The conformal tensor C

Type	Eigenvalues of C	PND's
I	Distinct μ_1, μ_2, μ_3 with $\Sigma\mu_i = 0$, $\dim C_{\mu_i} = 1$	Four 1-PND's
D	μ, -2μ with $\dim C_\mu = 2$, $\dim C_{-2\mu} = 1$	Two 2-PND's
II	μ, -2μ with $\dim C_\mu = 1$, $\dim C_{-2\mu} = 1$	One 2-PND, two 1-PND's
N	$\mu = 0$ with $\dim C_\mu = 2$	One 4-PND
III	$\mu = 0$ with $\dim C_\mu = 1$	One 3-PND, one 1-PND

Table 5.1: Petrov types

may be regarded as a trace-free complex linear operator on 2-forms by defining

$$C\phi \;=\; \frac{1}{2} X_b \,\lrcorner\, X_a \,\lrcorner\, \phi \, C^{ab} \qquad \phi \in \Gamma\Lambda_2^{\mathbb{C}}\mathcal{M}\,. \tag{5.3.1}$$

In an Einstein space, this operator is self-adjoint with respect to the dot product on 2-forms. The Petrov type of a spacetime is determined by the number of eigenforms and eigenvalues of the conformal tensor when acting in this way on the space of self-dual 2-forms. An algebraically general spacetime has three independent eigenforms with distinct eigenvalues. Since C is trace-free, the eigenvalues sum to zero. All other cases are algebraically special. Details may be found in Thorpe [Tho69], or the more recent book by O'Neill [O'N95].

Any null self-dual 2-form has one real eigenvector, in the sense of (5.2.2), with eigenvalue zero. The *principal null directions* (PND's) of the conformal tensor are the real directions determined by null self-dual 2-forms satisfying

$$\phi \cdot C\phi = 0\,. \tag{5.3.2}$$

An algebraically general spacetime admits four independent PND's, while the Petrov type of an algebraically special spacetime is determined by the way the PND's coincide as *repeated principal null directions* (RPND's). From (5.3.2) it is clear that a null eigenform of C determines a PND. In fact, any null eigenform of C determines a *repeated* principal null direction, and vice-versa. The *multiplicity* of a PND is 1 if ϕ is not an eigenform of C, 2 if $C\phi = \mu\phi$ with $\mu \neq 0$, 3 if $C\phi = 0$ and $\dim\ker C = 1$, or 4 if $C\phi = 0$ with $\dim\ker C = 2$. A PND with multiplicity m is called *m-principal*, while 1-principal null directions are said to be *simple*. Table 5.1 lists the properties of the eigenvalues and PND's of C for each Petrov type, where C_μ is the eigenspace of C with eigenvalue μ. If the conformal tensor vanishes, the spacetime is said to be conformally flat or Petrov type O.

The admissible Petrov types of a spacetime admitting a CKY 2-form may be determined using the integrability conditions of the shear-free spinor equation. Suppose that u is a shear-free spinor with gauge term $q\mathcal{A}$ (the constant q is inserted for later convenience). Differentiating (4.5.26) introduces the curvature operator $\widehat{R}(X,Y)$ of $\widehat{\nabla}$.

This is related to the $R(X,Y)$ by

$$\widehat{R}(X,Y)u \;=\; R(X,Y)u + qY \lrcorner X \lrcorner \mathcal{F}u\,, \tag{5.3.3}$$

where

$$\mathcal{F} \;=\; d\mathcal{A} \tag{5.3.4}$$

is the GL$(1,\mathbb{C})$-curvature. Differentiating the shear-free spinor equation (4.5.24) gives the integrability condition

$$R_{ab}u \;=\; \frac{1}{2}e_b\widehat{\nabla}_a\widehat{\mathrm{D}}u - \frac{1}{2}e_a\widehat{\nabla}_b\widehat{\mathrm{D}}u - 2qX_b \lrcorner X_a \lrcorner \mathcal{F}u\,, \tag{5.3.5}$$

noting that the action of the curvature operator on spinors is given by (2.4.25). For vanishing torsion, the Bianchi identity (2.4.6) shows that $e^aR_{ab} = P_b$. Then multiplying (5.3.5) on the left by e^a produces

$$P_bu \;=\; -\widehat{\nabla}_b\widehat{\mathrm{D}}u - \frac{1}{2}e_b\widehat{\mathrm{D}}^2u + 2qX_b \lrcorner \mathcal{F}u\,. \tag{5.3.6}$$

A further multiplication by e^b gives

$$\mathscr{R}u \;=\; 4q\mathcal{F}u - 3\widehat{\mathrm{D}}^2u \tag{5.3.7}$$

since $e^bP_b = \mathscr{R}$ by (2.4.9). Combining (5.3.5)–(5.3.7) with the definition of the conformal 2-forms (2.4.11) we have

$$C_{ab}u \;=\; q\left(\frac{1}{6}e_{ab}\mathcal{F} + \frac{1}{2}\mathcal{F}e_{ab}\right)u\,. \tag{5.3.8}$$

Now u determines a null self-dual 2-form $\phi = u \otimes \bar{u}$. In components, (4.1.4) shows that ϕ is given by

$$\phi \;=\; -\frac{1}{8}(u, e_{ab}u)e^{ab}\,. \tag{5.3.9}$$

From (5.3.1), the action of the conformal tensor on ϕ is

$$\begin{aligned} C\phi &= -\frac{1}{8}(u, e_{ab}u)C^{ab} \\ &= -\frac{1}{8}(u, C_{ab}u)e^{ab} \end{aligned} \tag{5.3.10}$$

by the 'pairwise symmetry' of the conformal tensor. Substituting the integrability condition (5.3.8) yields

$$\begin{aligned} C\phi &= -\frac{1}{48}q(u, e_{ab}\mathcal{F}u) - \frac{1}{16}q(u, \mathcal{F}e_{ab}u) \\ &= -\frac{1}{48}q(u, e_{ab}\mathcal{F}u)e^{ab} + \frac{1}{16}q(u, e_{ab}\mathcal{F}u)e^{ab} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{24} q(u, e_{ab}\mathcal{F}u)e^{ab} \\ &= -\frac{1}{48} q(u, [\mathcal{F}, e_{ab}]u)e^{ab} \end{aligned} \tag{5.3.11}$$

where we have used the Clifford commutator. It is easily verified that

$$X_q \lrcorner X_p \lrcorner [\mathcal{F}, e_{ab}] \;=\; -X_a \lrcorner X_b \lrcorner [\mathcal{F}, e_{qp}] \tag{5.3.12}$$

so we may write

$$\begin{aligned} C\phi &= \frac{1}{48} q(u, e_{ab}u)[\mathcal{F}, e^{ab}] \\ &= -\frac{1}{6} q[\mathcal{F}, \phi]\,. \end{aligned} \tag{5.3.13}$$

This is sufficient to show that the eigenvector of ϕ is a principal null direction, since

$$\begin{aligned} \phi \cdot C\phi &= \frac{1}{6} q\mathscr{S}_0(\phi[\mathcal{F}, \phi]) \\ &= \frac{1}{6} q\left(\mathscr{S}_0(\phi\mathcal{F}\phi) - \mathscr{S}_0(\phi\phi\mathcal{F})\right) \\ &= 0 \quad \text{by (2.2.9)}\,. \end{aligned} \tag{5.3.14}$$

In vacuum spacetime, the Goldberg-Sachs theorem states that a NSFG must be a *repeated* principal null direction [GS62]. In more general spacetimes, necessary and sufficient conditions for a NSFG to be a RPND are given by the generalised Goldberg-Sachs theorem [KT62, RS63]. The following lemma shows that the eigenvector of ϕ is a repeated principal null direction precisely when u is an eigenspinor of $q\mathcal{F}$.

Lemma 5.3.15 *Let u be a shear-free spinor with gauge term $q\mathcal{A}$, and let $\phi = u \otimes \bar{u}$. Then $C\phi = \mu\phi$ if and only if $q\mathcal{F}u = -3\mu u$, where $\mathcal{F} = d\mathcal{A}$.*

Proof. If $C\phi = \mu\phi$ then (5.3.13) is equivalent to

$$u \otimes \overline{q\mathcal{F}u} + q\mathcal{F}u \otimes \bar{u} \;=\; -6\mu u \otimes \bar{u}\,. \tag{5.3.16}$$

Contracting with u we have

$$(q\mathcal{F}u, u)u \;=\; 0 \tag{5.3.17}$$

and so $q\mathcal{F}u = \lambda u$ for some complex function λ. Substituting this back into equation (5.3.16) shows that $\lambda = -3\mu$. ∎

We are now in a position to determine the Petrov type of a spacetime admitting a CKY 2-form. First we consider the null case. Recall that a CKY 2-form is simply a shear-free 2-form with gauge term zero. From Theorem 5.2.16, it is clear that the spinor u determined by a null CKY 2-form ϕ is a shear-free spinor with vanishing gauge term — that is, a twistor. Since the gauge term is zero, we have $C_{ab}u = 0$ and $C\phi = 0$. Given a second spinor v independent from u, we may construct a basis $\{\phi, \phi', \omega\}$ for the space of self-dual 2-forms, where

$$\phi' \;=\; -\frac{1}{8}(v, e_{ab}v)e^{ab} \quad \text{and} \quad \omega \;=\; -\frac{1}{8}(v, e_{ab}u)e^{ab}\,.$$

Then we must also have $C\omega = 0$. If the spacetime is not conformally flat, then the kernel of C is 2-dimensional. Thus the eigenvector of ϕ is a 4-principal null direction, and the spacetime must be Petrov type N.

From a non-null CKY 2-form ω we obtain a pair of independent shear-free spinors u_1 and u_2 with gauge terms $\mathcal{A}$ and $-\mathcal{A}$. With $\mathcal{F} = d\mathcal{A}$ and $q = 1$, the integrability condition (5.3.8) for u_1 shows that

$$\begin{aligned} C\omega &= -\frac{1}{8}(u_2, C_{ab}u_1)e^{ab} \\ &= -\frac{1}{8}(u_2, \frac{1}{6}e_{ab}\mathcal{F}u_1 + \frac{1}{2}\mathcal{F}e_{ab}u_1)e^{ab} \end{aligned} \tag{5.3.18}$$

while that for u_2 with $q = -1$ gives

$$\begin{aligned} C\omega &= \frac{1}{8}(C_{ab}u_2, u_1)e^{ab} \\ &= -\frac{1}{8}(\frac{1}{6}e_{ab}\mathcal{F}u_2 + \frac{1}{2}\mathcal{F}e_{ab}u_2, u_1)e^{ab}\,. \end{aligned} \tag{5.3.19}$$

Taking the difference of these two equations shows that

$$[\mathcal{F}, \omega] \;=\; 0\,. \tag{5.3.20}$$

The spinorial equivalent of this expression is

$$\mathcal{F}u_1 \otimes \bar{u}_2 + \mathcal{F}u_2 \otimes \bar{u}_1 + u_1 \otimes \overline{\mathcal{F}u_2} + u_2 \otimes \overline{\mathcal{F}u_1} \;=\; 0\,. \tag{5.3.21}$$

Contracting on u_1, we have

$$(u_2, u_1)\mathcal{F}u_1 + (\mathcal{F}u_2, u_1)u_1 + (\mathcal{F}u_1, u_1)u_2 \;=\; 0\,. \tag{5.3.22}$$

Now by (4.1.2), the first term can be rearranged as

$$(u_2, u_1)\mathcal{F}u_1 \;=\; (\mathcal{F}u_1, u_1)u_2 + (u_2, \mathcal{F}u_1)u_1\,. \tag{5.3.23}$$

Substituting into (5.3.22) shows that $(\mathcal{F}u_1, u_1) = 0$, hence $\mathcal{F}u_1 = -3\mu u_1$ for some complex function μ (the factor of -3 is for convenience). Similarly, contracting (5.3.21) on u_2 shows that u_2 is also an eigenspinor of $\mathcal{F}$ with eigenvalue $-3\mu'$, say. However, we must also have

$$\begin{aligned} (\mathcal{F}u_2, u_1) &= (u_2, \mathcal{F}^{\xi}u_1) \\ &= -(u_2, \mathcal{F}u_1) \end{aligned} \tag{5.3.24}$$

thus $\mu' = -\mu$. Substituting this back into (5.3.18) shows that $C\omega = -2\mu\omega$. If ϕ_1 and ϕ_2 are the 2-forms corresponding to u_1 and u_2, then the set $\{\phi_1, \phi_2, \omega\}$ is an eigenbasis for the space of self-dual 2-forms, with eigenvalues μ, μ and -2μ respectively, hence the spacetime must be Petrov type D or else conformally flat.

Chapter 6

Debye Potentials and Symmetry Operators for Massless Fields

On spacetime, the exterior form of the vacuum Maxwell equations for a 2-form F are

$$\begin{aligned} dF &= 0 \\ d^*F &= 0\,. \end{aligned} \tag{6.1}$$

They may be written more concisely using the Hodge-de Rham operator (2.4.18) as

$$\not{d}F = 0\,. \tag{6.2}$$

In flat space, the vacuum Maxwell equation may be solved by means of a *Hertz potential*. The Hertz potential scheme, expressed in exterior form, has been generalised to arbitrary spacetimes by Cohen and Kegeles [CK74]. To summarise their treatment, a Hertz potential H is a 2-form chosen so that

$$\triangle H = dP + d^*Q \tag{6.3}$$

where P is an arbitrary 1-form and Q is an arbitrary 3-form. Since $\triangle = -(dd^* + d^*d)$ we may write (6.3) as

$$d\,(d^*H + P) = -d^*(dH + Q)\,. \tag{6.4}$$

Then the 2-form F given by

$$\begin{aligned} F &= d\,(d^*H + dP) \\ &= -d^*(dH + d^*Q) \end{aligned} \tag{6.5}$$

is closed and co-closed, thus it is a solution of the vacuum Maxwell equation. Unfortunately, the utility of the Hertz potential in arbitrary spacetimes is somewhat limited since (6.3) is no easier to solve than (6.2). As discussed by Nisbet [Nis55], in flat space the Hertz potential scheme reduces to the problem of finding a solution of Laplace's equation. Given a null shear-free 2-form ω and a complex function f satisfying $\triangle f = 0$, it can be shown that the 2-form $f\omega$ is a Hertz potential. The function f is referred to as a *Debye potential.* Cohen and Kegeles found that the Debye potential scheme may be extended to the generalised Goldberg-Sachs class of spacetimes [KT62, RS63], that is, all algebraically special spacetimes with a repeated principal null direction tangent to a congruence of null shear-free geodesics. For an appropriate choice of P and Q, a Hertz potential can be constructed by scaling the null 2-form corresponding to the shear-free RPND by a complex function f satisfying a generalisation of Laplace's equation.

The vacuum Maxwell equation (6.2) formally resembles the massless Dirac equation. In fact, it is the massless equation of a spin-1 field. In conformally flat spacetime, a complex function satisfying the conformally covariant wave equation may be used to generate a massless field of arbitrary spin by sufficient applications of Penrose's *spin raising* operator [Pen75, PR86b]. The spin-raising operator is an operator constructed from a twistor which maps a solution of the spin-s massless field equation to a solution of the spin-$(s + \frac{1}{2})$ massless field equation. The spin-0 massless equation for a complex function f is

$$\triangle f - \frac{1}{6}\mathscr{R}f \quad = \quad 0\,. \tag{6.6}$$

Functions satisfying this equation may be used to construct Hertz-like potentials in conformally flat spacetime.

The operation 'dual' to spin raising is *spin lowering.* The spin lowering operator maps a solution of the spin-s massless field equation to a solution of the spin-$(s - \frac{1}{2})$ massless field equation by contraction with a twistor. In conformally flat spacetime, repeated applications of spin lowering may be used to generate a solution of (6.6) from any massless field [PR86b].

Spin raising and lowering is of limited use in non-conformally flat spacetime, as the integrability conditions for the twistor equation are highly restrictive. In this chapter we show that in the generalised Goldberg-Sachs class of spacetimes, a generalisation of spin raising and lowering is possible using shear-free spinors, of which twistors are a special case. Since shear-free spinors satisfy the twistor equation of a gauged covariant derivative, when raising or lowering with a shear-free spinor we obtain solutions of a 'gauged' massless field equation, where the gauge term is dependent on the gauge terms of the original spinor field and the shear-free spinor. The solutions are generated from a function satisfying a generalisation of (6.6) using a 'gauged' Laplacian. We will refer to such functions as 'Debye potentials' also.

We will only consider the massless Dirac and vacuum Maxwell fields, since there are strong algebraic consistency conditions relating massless fields of higher spin to the conformal curvature [Buc58, Buc62]. In some particular spacetimes, however, it is possible to find consistent solutions to higher-spin equations. Torres del Castillo has considered Debye potentials for the spin-$\frac{3}{2}$ Rarita-Schwinger field in algebraically special vacuum spacetimes [TdC89b] and in spacetimes which are either algebraically special solutions of the vacuum Einstein equations or solutions of the Einstein-Maxwell equations where a principal null direction of the Maxwell field is NSFG [TdC89a]. Cohen and Kegeles have examined Debye potentials for the spin-2 field corresponding to linearized gravity in algebraically special vacuum spacetimes [CK75, KC79]. It is hoped that our generalisation of spin raising and lowering will be applicable to higher-spin fields in more general spacetimes, particularly when the spacetime is not Ricci-flat.

An important application of spin raising and lowering is the construction of *symmetry operators* which map one solution of the massless field equation to another. A solution of the spin-s massless field equation is 'lowered' with r shear-free spinors ($1 \leq r \leq 2s$) and then 'raised' with r shear-free spinors to obtain a new solution for the spin-s equation. In this fashion we are able to construct a first-order symmetry operator for the massless Dirac equation and a second-order symmetry operator for the vacuum Maxwell equation, by using two shear-free spinors whose gauge terms sum to zero. If the spinors are both even, they correspond to a self-dual conformal Killing-Yano 2-form. If the spinors have opposite parity then they can be used to construct a (complex) conformal Killing vector. In that case, the symmetry operator is related to the Lie derivative.

We show that the symmetry operator for the massless Dirac operator is *R-commuting*, that is, its commutator with the Dirac operator is of the form $R\mathrm{D}$, where R is another operator. Kamran and McLenaghan have shown that the most general first-order operator which R-commutes with D can be constructed from CKY tensors of all possible degrees [KM84b]. There is no dependency between the various tensors, so the operator can be split up into components depending only on one CKY tensor, each of which is R-commuting on its own. The 0- and 4-form operators are trivial, while the 1- and 3-form operators are related to the Lie derivatives with respect to the corresponding conformal Killing vectors. Thus the only non-trivial operator is constructed from a CKY 2-form. This is same as the operator we construct from a pair of even shear-free spinors.

For the most part, the authors previously mentioned have used the Newman-Penrose formalism and adapted null bases to facilitate their calculations. While these methods are very efficient, they are only suitable for the 4-dimensional Lorentzian case. We have endeavoured to make our calculations basis-independent, which results in an especially compact notation with minimal use of indices. The main result of this chapter is the presentation of a symmetry operator for the massless Dirac equation on a manifold of arbitrary dimension and signature, constructed from a CKY p-form. While the relationship between CKY 2-forms and Killing spinors in spacetime is well understood, it is difficult to envision what the spinor equation corresponding to a CKY p-form would be in higher dimensions.

6.1 Debye potentials for the massless Dirac equation

In this section we show how Debye potentials may be constructed on spacetime for solutions of the massless Dirac equation of the gauged covariant derivative,

$$\widehat{\mathrm{D}}\psi \;=\; 0 \tag{6.1.1}$$

where ψ is a Dirac spinor with an associated gauge term. If the gauge term of ψ is, say, $q\mathcal{A}$, then (6.1.1) may be written equivalently as $\mathrm{D}\psi = -q\mathcal{A}\psi$. In this chapter, we will most often use functions, tensors and spinors whose gauge terms are constant multiples of the same 1-form $\mathcal{A}$. This constant will be referred to as the *charge*. Solutions of (6.1.1) are constructed by raising a charged function f with a shear-free spinor u. The charge of ψ must be the sum of the charges of f and u, hence to obtain an uncharged solution we require that these charges cancel. When differentiating uncharged spinors, we will drop the 'hat' symbol.

Consider a semi-spinor u with gauge term $q\mathcal{A}$ and a function f with gauge term $q'\mathcal{A}'$. The corresponding $\mathrm{GL}(1,\mathbb{C})$-curvatures are $q\mathcal{F}$ and $q'\mathcal{F}'$, respectively, where $\mathcal{F} = d\mathcal{A}$ and $\mathcal{F}' = d\mathcal{A}'$. The spinor ψ constructed by 'raising' f with u is given by

$$\psi \;=\; \hat{d}f\, u + \frac{1}{2} f \widehat{\mathrm{D}} u\,. \tag{6.1.2}$$

It has gauge term $q\mathcal{A} + q'\mathcal{A}'$ and $\mathrm{GL}(1,\mathbb{C})$-curvature $q\mathcal{F} + q'\mathcal{F}'$. When the gauge terms of u and f vanish, (6.1.2) is the Penrose spin-raising operator which generates a spin-$\frac{1}{2}$ field from a spin-0 field. Acting on ψ with the Dirac operator we have

$$\begin{aligned}
\widehat{\mathrm{D}}\psi \;&=\; e^a \left((\widehat{\nabla}_{X_a} \hat{d}f) u + \hat{d}f \widehat{\nabla}_{X_a} u + \frac{1}{2} (\widehat{\nabla}_{X_a} f) \widehat{\mathrm{D}} u + \frac{1}{2} f \widehat{\nabla}_{X_a} \widehat{\mathrm{D}} u \right) \\
&=\; \hat{d}^2 f\, u - \hat{d}^* \hat{d}f\, u + e^a \hat{d}f \widehat{\nabla}_{X_a} u + \frac{1}{2} \hat{d}f \widehat{\mathrm{D}} u + \frac{1}{2} f \widehat{\mathrm{D}}^2 u\,.
\end{aligned} \tag{6.1.3}$$

Now the Laplace-Beltrami operator for $\widehat{\nabla}$ is $\hat{\triangle} = -(\hat{d}^*\hat{d} + \hat{d}\hat{d}^*)$, so we see that $\hat{\triangle} f = -\hat{d}^*\hat{d}f$, while $\hat{d}^2 f = f q' \mathcal{F}'$ by (4.5.14). As $e^a \hat{d}f = 2X^a \lrcorner \hat{d}f - \hat{d}f e^a$, equation (6.1.3) becomes

$$\widehat{\mathrm{D}}\psi \;=\; \hat{\triangle} f u + f q' \mathcal{F}' u + \frac{1}{2} f \widehat{\mathrm{D}}^2 u + 2 \left(\widehat{\nabla}_{\hat{d}f^\sharp} u - \frac{1}{4} \hat{d}f \widehat{\mathrm{D}} u \right). \tag{6.1.4}$$

Now suppose that u is shear-free. Then the bracketed term vanishes, and we may use the integrability condition (5.3.7) to rewrite $\widehat{\mathrm{D}}^2 u$ so that

$$\begin{aligned}
\widehat{\mathrm{D}}\psi \;&=\; \hat{\triangle} f u + f q' \mathcal{F}' u + \frac{1}{2} f \left(-\frac{1}{3} \mathscr{R} u + \frac{4}{3} q \mathcal{F} u \right) \\
&=\; \left(\hat{\triangle} f - \frac{1}{6} \mathscr{R} f \right) u + f \left(\frac{2}{3} q \mathcal{F} + q' \mathcal{F}' \right) u\,.
\end{aligned} \tag{6.1.5}$$

If $\widehat{\mathrm{D}}\psi$ is to be proportional to u, we require that u be an eigenspinor of $\frac{2}{3} q\mathcal{F} + q'\mathcal{F}'$. Let $\phi = u \otimes \bar{u}$ be the null 2-form corresponding to u. By Lemma 5.3.15, $q\mathcal{F}u = -3\mu u$ if and only if $C\phi = \mu\phi$, which is equivalent to the condition that the NSFG vector field $k = (iu, e^a u^c) X_a$ is a repeated principal null direction. However, we cannot conclude that u is also an eigenspinor of $q'\mathcal{F}'$ unless the gauge terms of u and f are related. We will suppose that $q'\mathcal{A}' = \epsilon q \mathcal{A}$, from which it follows that $q'\mathcal{F}' = \epsilon q \mathcal{F}$. The constant factor ϵ is inserted so that the gauge term of f vanishes when the gauge term of u does, since if we have $q = 0$ but $q' \neq 0$ we would know that $q\mathcal{F}u = 0$ but we could not conclude that $q'\mathcal{F}'u = 0$.

Theorem 6.1.6 *Given a shear-free spinor u with charge q, corresponding to a repeated principal null direction, and a complex function f with charge ϵq, the spinor*

$$\psi \;=\; \hat{d}f\, u + \frac{1}{2} f \widehat{\mathrm{D}} u \tag{6.1.7}$$

satisfies

$$\widehat{\mathrm{D}}\psi \quad = \quad \left(\hat{\triangle} f - \frac{1}{6}\mathscr{R} f - (3\epsilon + 2)\,\mu f\right) u \tag{6.1.8}$$

where μ is the eigenvalue of the 2-form corresponding to u.

Proof. With $q'\mathcal{F}' = \epsilon q\mathcal{F}$, equation (6.1.5) becomes

$$\widehat{\mathrm{D}}\psi \quad = \quad \left(\hat{\triangle} f - \frac{1}{6}\mathscr{R} f\right) u + \left(\epsilon + \frac{2}{3}\right) f q\mathcal{F} u\,. \tag{6.1.9}$$

Since the null shear-free geodesic $k = (iu, e^a u^c)X_a$ is aligned with a repeated principal null direction, the null 2-form $\phi = u \otimes \bar{u}$ is an eigenvector of the conformal tensor. If the eigenvalue of ϕ is μ, by Lemma 5.3.15 we have $q\mathcal{F}u = -3\mu u$, so

$$\widehat{\mathrm{D}}\psi \quad = \quad \left(\hat{\triangle} f - \frac{1}{6}\mathscr{R} f - (3\epsilon + 2)\,\mu f\right) u\,. \tag{6.1.10}$$

■

A spacetime is in the generalised Goldberg-Sachs class if it admits a congruence of null shear-free geodesics aligned with a repeated principal null direction. Theorem 6.1.6 shows that we can solve the massless Dirac equation (with charge $q(\epsilon+1)$) in such a spacetime by finding a solution of the complex scalar equation

$$\hat{\triangle} f - \frac{1}{6}\mathscr{R} f \quad = \quad (3\epsilon + 2)\,\mu f\,. \tag{6.1.11}$$

We will refer to a function satisfying (6.1.11) as a *generalised Debye potential.* Although the Laplacian $\hat{\triangle}$ contains additional gauge terms, they can be expressed using a suitably adapted basis, while μ is simply the eigenvalue of the null 2-form corresponding to the shear-free RPND. If we wish to find an *uncharged* solution of the massless Dirac equation (as would normally be the case) then we require that u and f have opposite charges, that is, $\epsilon = -1$. Finally, if $q = 0$, u is simply a twistor. In that case f also has zero charge, and the eigenvalue μ of ϕ is zero so (6.1.11) reduces to (6.6).

6.2 Spin lowering with a shear-free spinor

In this section we show how to construct a solution of (6.1.11) by contracting a solution of the massless Dirac equation with a shear-free spinor. Let u and ψ be spinors with gauge terms $q\mathcal{A}$ and $q'\mathcal{A}'$, respectively. Since we require that $\widehat{\nabla}$ be compatible with the inner product, we have

$$\begin{aligned}\widehat{\nabla}_X(u,\psi) \quad &= \quad (\widehat{\nabla}_X u, \psi) + (u, \widehat{\nabla}_X \psi) \\ &= \quad (\nabla_X u, \psi) + (u, \nabla_X \psi) + (q\mathcal{A} + q'\mathcal{A}')(X)(u,\psi)\end{aligned} \tag{6.2.1}$$

so the function $f = (u, \psi)$ has gauge term $q\mathcal{A} + q'\mathcal{A}'$. The action of the gauged Laplace-Beltrami operator on f is

$$\hat{\triangle} f \;=\; \widehat{\nabla}_{X_a}\widehat{\nabla}_{X^a} f - \widehat{\nabla}_{\nabla_a X^a} f \tag{6.2.2}$$

hence

$$\begin{aligned}\hat{\triangle} f &= (\widehat{\nabla}_{X_a}\widehat{\nabla}_{X^a} u - \widehat{\nabla}_{\nabla_a X^a} u, \psi) + 2(\widehat{\nabla}_{X^a} u, \widehat{\nabla}_{X_a}\psi) \\ &\quad + (u, \widehat{\nabla}_{X_a}\widehat{\nabla}_{X^a}\psi - \widehat{\nabla}_{\nabla_a X^a}\psi) \\ &= (\widehat{\nabla}^2 u, \psi) + 2(\widehat{\nabla}_{X^a} u, \widehat{\nabla}_{X_a}\psi) + (u, \widehat{\nabla}^2\psi)\end{aligned} \tag{6.2.3}$$

where $\widehat{\nabla}^2$ is the trace of the Hessian of $\widehat{\nabla}$. The square of the Dirac operator on a spinor with charge q is related to $\widehat{\nabla}^2$ by

$$\widehat{\mathrm{D}}^2\psi = \widehat{\nabla}^2\psi - \frac{1}{4}\mathscr{R}\psi + q\mathcal{F}\psi \tag{6.2.4}$$

thus

$$\begin{aligned}\hat{\triangle} f &= (\widehat{\mathrm{D}}^2 u + \frac{1}{4}\mathscr{R}u - q\mathcal{F}u, \psi) + 2(\widehat{\nabla}_{X^a} u, \widehat{\nabla}_{X_a}\psi) \\ &\quad + (u, \widehat{\mathrm{D}}^2\psi + \frac{1}{4}\mathscr{R}\psi - q'\mathcal{F}'\psi)\,.\end{aligned} \tag{6.2.5}$$

Finally, since $(u, \mathcal{F}'\psi) = -(\mathcal{F}'u, \psi)$ we have

$$\begin{aligned}\hat{\triangle} f &= \frac{1}{2}\mathscr{R}(u, \psi) + (\widehat{\mathrm{D}}^2 u, \psi) - ((q\mathcal{F} - q'\mathcal{F}')u, \psi) \\ &\quad + 2(\widehat{\nabla}_{X^a} u, \widehat{\nabla}_{X_a}\psi) + (u, \widehat{\mathrm{D}}^2\psi)\,.\end{aligned} \tag{6.2.6}$$

To go further we must suppose that u is shear-free. Note that in that case u is a semi-spinor, so only the component of ψ with the same parity as u contributes to f.

If u is shear-free then

$$\begin{aligned}(\widehat{\nabla}_{X^a} u, \widehat{\nabla}_{X_a}\psi) &= \frac{1}{4}(e^a\widehat{\mathrm{D}}u, \widehat{\nabla}_{X_a}\psi) \\ &= \frac{1}{4}(\widehat{\mathrm{D}}u, \widehat{\mathrm{D}}\psi).\end{aligned} \tag{6.2.7}$$

Using (6.2.7), and replacing $\widehat{\mathrm{D}}^2 u$ via (5.3.7), equation (6.2.6) becomes

$$\begin{aligned}\hat{\triangle} f &= \frac{1}{2}\mathscr{R}(u, \psi) + ((\tfrac{4}{3}q\mathcal{F} - \tfrac{1}{3}\mathscr{R})u, \psi) - ((q\mathcal{F} - q'\mathcal{F}')u, \psi) \\ &\quad + \frac{1}{2}(\widehat{\mathrm{D}}u, \widehat{\mathrm{D}}\psi) + (u, \widehat{\mathrm{D}}^2\psi) \\ &= \frac{1}{6}\mathscr{R}f + ((\tfrac{1}{3}q\mathcal{F} + q'\mathcal{F}')u, \psi) + \frac{1}{2}(\widehat{\mathrm{D}}u, \widehat{\mathrm{D}}\psi) + (u, \widehat{\mathrm{D}}^2\psi).\end{aligned} \tag{6.2.8}$$

For f to be a generalised Debye potential, we require that u be an eigenspinor of $\frac{1}{3}q\mathcal{F} + q'\mathcal{F}'$. As in §6.1, we will suppose that the gauge terms of ψ and u are related by $q'\mathcal{A}' = \epsilon q\mathcal{A}$, and that u corresponds to a repeated principal null direction.

Theorem 6.2.9 *Given a shear-free spinor u with charge q, corresponding to a repeated principal null direction, and a spinor ψ with charge ϵq, the function*

$$f = (u, \psi) \tag{6.2.10}$$

satisfies

$$\hat{\triangle} f - \frac{1}{6}\mathscr{R} f \;=\; -(3\epsilon + 1)\,\mu f + \frac{1}{2}(\widehat{\mathrm{D}}u, \widehat{\mathrm{D}}\psi) + (u, \widehat{\mathrm{D}}^2\psi) \tag{6.2.11}$$

where μ is the eigenvector of the 2-form corresponding to u.

Proof. Equation (6.2.8) with $q'\mathcal{F}' = \epsilon q\mathcal{F}$ becomes

$$\begin{aligned}\hat{\triangle} f \;=\; & \frac{1}{6}\mathscr{R} f + (\epsilon + \tfrac{1}{3})(q\mathcal{F}u, \psi) \\ & + \frac{1}{2}(\widehat{\mathrm{D}}u, \widehat{\mathrm{D}}\psi) + (u, \widehat{\mathrm{D}}^2\psi).\end{aligned} \tag{6.2.12}$$

Since u corresponds to a repeated principal null direction, the 2-form $\phi = u \otimes \bar{u}$ is an eigenvector of the conformal tensor. If $C\phi = \mu\phi$ then by Lemma 5.3.15 we have $q\mathcal{F}u = -3\mu u$. Substituting this into (6.2.12) shows that

$$\hat{\triangle} f - \frac{1}{6}\mathscr{R} f \;=\; -(3\epsilon + 1)\mu f + \frac{1}{2}(\widehat{\mathrm{D}}u, \widehat{\mathrm{D}}\psi) + (u, \widehat{\mathrm{D}}^2\psi). \tag{6.2.13}$$

■

So if ψ satisfies the massless Dirac equation we can construct a function satisfying an equation of the same form as (6.1.11). In order for f to satisfy (6.1.11) precisely, it must have the same gauge term, and the eigenvalue μ appearing in (6.2.11) must be the same as that in (6.1.8). For this we require a second shear-free spinor with gauge term $-q\mathcal{A}$, that is, opposite charge to u.

6.3 Symmetry operators for the massless Dirac equation

In this section we show how a symmetry operator for the massless Dirac equation (without charge) can be constructed from a pair of shear-free spinors with opposite charges, corresponding to a CKY 2-form. We also show how the symmetry operator can be written in terms of the CKY 2-form, and hence generalised to an operator constructed from a CKY p-form which R-commutes with the Dirac operator in all dimensions and signatures. Part of this work has appeared in [Cha96] and [BC97].

Consider a pair of even shear-free spinors u_1 and u_2 with charges $+1$ and -1 respectively (that is, gauge terms $\mathcal{A}$ and $-\mathcal{A}$). In Chapter 5 we showed that this is the case if and only if the 2-form ω given by

$$\omega \;=\; \frac{1}{2}\left(u_1 \otimes \bar{u}_2 + u_2 \otimes \bar{u}_1\right) \tag{6.3.1}$$

is a self-dual CKY 2-form. It follows that the null 2-forms $\phi_1 = u_1 \otimes \bar{u}_1$ and $\phi_2 = u_2 \otimes \bar{u}_2$ are eigenvectors of the conformal tensor with the same eigenvalue. Thus u_1 and u_2 each correspond to a repeated principal null direction. At this point, we need not assume that u_1 and u_2 are linearly independent, and so ω may be null or non-null. Recall that if u_1 and u_2 are dependent, then $\mathcal{A}$ vanishes and $C\phi_i = C\omega = 0$. Now let ψ be an uncharged spinor. If the eigenvalue of ϕ_1 (and ϕ_2) is μ, then by Theorem 6.2.9 with $q = 1$ and $\epsilon = 0$, the function $f = (u_1, \psi)$ has charge $+1$ and satisfies

$$\hat{\triangle} f - \frac{1}{6}\mathscr{R} f \quad = \quad -\mu f + \frac{1}{2}(\widehat{\mathrm{D}}u_1, \mathrm{D}\psi) + (u_1, \mathrm{D}^2\psi). \tag{6.3.2}$$

Naturally, if u_1 and u_2 are dependent then $\mu = 0$. Let ψ' be the spinor obtained by raising f with u_2, that is

$$\psi' \quad = \quad \hat{d}f u_2 + \frac{1}{2} f \widehat{\mathrm{D}} u_2 \tag{6.3.3}$$

by (6.1.2). Since f and u_2 have opposite charge, ψ' is uncharged. Then by Theorem 6.1.6 with $q = -1$ and $\epsilon = -1$,

$$\begin{aligned} \mathrm{D}\psi' \quad &= \quad \left(\hat{\triangle} f - \frac{1}{6}\mathscr{R} f + \mu f\right) u_2 \\ &= \quad \left(\frac{1}{2}(\widehat{\mathrm{D}}u_1, \mathrm{D}\psi) + (u_1, \mathrm{D}^2\psi)\right) u_2 \end{aligned} \tag{6.3.4}$$

by (6.3.2). Clearly if ψ is a solution of the massless Dirac equation then so is ψ', thus the existence of a self-dual CKY 2-form allows us to construct new solutions of the massless Dirac equation from old ones. We can express this construction as an operator on the space of spinors which may be written using the CKY tensor corresponding to u_1 and u_2. The covariant derivative $\widehat{\nabla}$ is compatible with the spinor inner product, so

$$\begin{aligned} \hat{d}f \quad &= \quad \left((\widehat{\nabla}_{X_a} u_1, \psi) + (u_1, \nabla_{X_a}\psi)\right) e^a \\ &= \quad \left(\frac{1}{4}(e_a \widehat{\mathrm{D}} u_1, \psi) + (u_1, \nabla_{X_a}\psi)\right) e^a \,. \end{aligned} \tag{6.3.5}$$

Thus ψ' may be written using u_1 and u_2 as

$$\psi' \quad = \quad (u_1, \nabla_{X_a}\psi) e^a u_2 + \frac{1}{4}(e_a \widehat{\mathrm{D}} u_1, \psi) e^a u_2 + \frac{1}{2}(u_1, \psi)\widehat{\mathrm{D}} u_2 \,. \tag{6.3.6}$$

Writing (6.3.6) using tensor notation, we have

$$\psi' \quad = \quad \left(e^a u_2 \otimes \bar{u}_1 \nabla_{X_a} + \frac{1}{4} e^a u_2 \otimes \overline{e_a \widehat{\mathrm{D}} u_1} + \frac{1}{2}\widehat{\mathrm{D}} u_2 \otimes \bar{u}_1\right)\psi \,. \tag{6.3.7}$$

As u_2 is even and $\widehat{\mathrm{D}}u_1$ is odd, $e^a u_2 \otimes \overline{e_a \widehat{\mathrm{D}} u_1} = 2\widehat{\mathrm{D}} u_1 \otimes \bar{u}_2$ and so finally

$$\psi' \quad = \quad \left(e^a u_2 \otimes \bar{u}_1 \nabla_{X_a} + \frac{1}{2}\widehat{\mathrm{D}} u_1 \otimes \bar{u}_2 + \frac{1}{2}\widehat{\mathrm{D}} u_2 \otimes \bar{u}_1\right)\psi \,. \tag{6.3.8}$$

Now we will write ψ' in terms of ω only.

The tensor product $u_2 \otimes \bar{u}_1$ is an even form which can be expressed as

$$u_2 \otimes \bar{u}_1 = \frac{1}{4}(u_1, u_2) + \omega + \frac{1}{4}(u_1, u_2)\check{z}\,. \tag{6.3.9}$$

The derivative terms in (6.3.8) can be recognised by acting on ω with the Hodge-de Rham operator,

$$\begin{aligned}
\not{d}\omega &= d\omega - d^*\omega \\
&= \frac{1}{2}e^a\left(\widehat{\nabla}_{X_a}u_1 \otimes \bar{u}_2 + u_1 \otimes \overline{\widehat{\nabla}_{X_a}u_2} + \widehat{\nabla}_{X_a}u_2 \otimes \bar{u}_1 + u_2 \otimes \overline{\widehat{\nabla}_{X_a}u_1}\right) \\
&= \frac{1}{2}\left(\widehat{\mathrm{D}}u_1 \otimes \bar{u}_2 + \frac{1}{4}e^a u_1 \otimes \overline{e_a\widehat{\mathrm{D}}u_2} + \widehat{\mathrm{D}}u_2 \otimes \bar{u}_1 + \frac{1}{4}e^a u_2 \otimes \overline{e_a\widehat{\mathrm{D}}u_1}\right) \\
&= \frac{1}{2}\left(\widehat{\mathrm{D}}u_1 \otimes \bar{u}_2 + \frac{1}{2}\widehat{\mathrm{D}}u_2 \otimes \bar{u}_1 + \widehat{\mathrm{D}}u_2 \otimes \bar{u}_1 + \frac{1}{2}\widehat{\mathrm{D}}u_1 \otimes \bar{u}_2\right) \\
&= \frac{3}{4}\left(\widehat{\mathrm{D}}u_1 \otimes \bar{u}_2 + \widehat{\mathrm{D}}u_2 \otimes \bar{u}_1\right).
\end{aligned} \tag{6.3.10}$$

Equations (6.3.9) and (6.3.10) allow us to write (6.3.8) as

$$\psi' = \left(e^a\omega\nabla_{X_a} + \frac{2}{3}d\omega - \frac{2}{3}d^*\omega + \frac{1}{4}(u_1, u_2)\mathrm{D} - \frac{1}{4}(u_1, u_2)\check{z}\mathrm{D}\right)\psi\,.$$

Now equation (6.3.4) shows that if $\mathrm{D}\psi = 0$ then $\mathrm{D}\psi' = 0$. Since in that case the last two terms in the above expression vanish we must have

$$\mathrm{D}\left(\left(e^a\omega\nabla_{X_a} + \frac{2}{3}d\omega - \frac{2}{3}d^*\omega\right)\psi\right) = 0 \tag{6.3.11}$$

whenever $\mathrm{D}\psi = 0$. So the operator K_ω on spinors defined by

$$K_\omega = e^a\omega\nabla_{X_a} + \frac{2}{3}d\omega - \frac{2}{3}d^*\omega \tag{6.3.12}$$

is a symmetry operator for the massless Dirac equation whenever ω is a self-dual CKY 2-form. Because of the self-duality of ω, K_ω annihilates odd spinors. However, we could just have easily used the charge conjugates of u_1 and u_2 to construct the complex conjugate $\bar{\omega}$ of ω, which is an anti self-dual CKY 2-form. Then $K_{\bar{\omega}}$ is a symmetry operator which annihilates even spinors.

While ω in (6.3.12) is specifically a self-dual CKY 2-form, it is possible to generalise K_ω to a symmetry operator for the massless Dirac equation in all dimensions and signatures provided only that ω is a p-form satisfying the conformal Killing-Yano equation (5.1.1).

Theorem 6.3.13 *Let ω be a conformal Killing-Yano p-form on an n-dimensional pseudo-Riemannian manifold. Then the operator K_ω on spinors defined by*

$$K_\omega = e^a\omega\nabla_{X_a} + \frac{p}{p+1}d\omega - \frac{n-p}{n-p+1}d^*\omega \tag{6.3.14}$$

is a symmetry operator for the equation $\mathrm{D}\psi = 0$*, with*

$$\mathrm{D}K_\omega \;=\; \left(\omega\mathrm{D} + \frac{(-1)^p}{p+1}d\omega + \frac{(-1)^p}{n-p+1}d^*\omega\right)\mathrm{D}\,. \tag{6.3.15}$$

Proof. Consider an arbitrary spinor ψ. The only non-trivial calculation to be carried out in $\mathrm{D}K_\omega\psi$ is the derivative of the first term in (6.3.14). Now

$$\mathrm{D}\left(e^b\omega\nabla_{X_b}\psi\right) \;=\; e^a\left(\nabla_{X_a}e^b\omega\nabla_{X_b}\psi + e^b\nabla_{X_a}\omega\nabla_{X_b}\psi + e^b\omega\nabla_{X_a}\nabla_{X_b}\psi\right).$$

By (4.3.6), the metric-compatibility of ∇ implies that

$$\nabla_{X_a}e^b\omega\nabla_{X_b}\psi \;=\; -e^b\omega\nabla_{\nabla_a X_b}\psi\,. \tag{6.3.16}$$

Thus

$$\begin{aligned}\mathrm{D}\left(e^b\omega\nabla_{X_b}\psi\right) \;&=\; e^ae^b\omega\left(\nabla_{X_a}\nabla_{X_b} - \nabla_{\nabla_a X_b}\right)\psi \\ &\quad + e^ae^b\nabla_{X_a}\omega\nabla_{X_b}\psi\,.\end{aligned} \tag{6.3.17}$$

As $e^ae^b = e^{ab} + g^{ab}$, the first term in (6.3.17) can be written using the curvature 2-forms and the trace of the Hessian, as

$$\begin{aligned}e^ae^b\omega\left(\nabla_{X_a}\nabla_{X_b} - \nabla_{\nabla_a X_b}\right)\psi \;&=\; e^{ab}\omega\left(\nabla_{X_a}\nabla_{X_b} - \nabla_{\nabla_a X_b}\right)\psi + \omega\nabla^2\psi \\ &=\; \frac{1}{2}e^{ab}\omega R(X_a, X_b)\psi + \omega\nabla^2\psi \\ &=\; \frac{1}{4}e^{ab}\omega R_{ab}\psi + \omega\nabla^2\psi\,.\end{aligned} \tag{6.3.18}$$

By the 'pairwise symmetry' (2.4.7) of the curvature forms and using (6.2.4) and (2.4.20) this expression becomes

$$\begin{aligned}e^ae^b\omega\left(\nabla_{X_a}\nabla_{X_b} - \nabla_{\nabla_a X_b}\right)\psi \;&=\; \frac{1}{4}R_{ab}\omega e^{ab}\psi + \omega\left(\mathrm{D}^2\psi + \frac{1}{4}\mathscr{R}\psi\right) \\ &=\; \left(\nabla^2\omega - \triangle\omega\right)\psi + \omega\mathrm{D}^2\psi\,.\end{aligned} \tag{6.3.19}$$

As ω is a CKY tensor, we can use the integrability condition (5.1.2) to write (6.3.19) using $d^*d\omega$ and $dd^*\omega$.

$$\begin{aligned}&e^ae^b\omega\left(\nabla_{X_a}\nabla_{X_b} - \nabla_{\nabla_a X_b}\right)\psi \\ &\quad= \left(d^*d\omega + dd^*\omega - \frac{1}{p+1}d^*d\omega - \frac{1}{n-p+1}dd^*\omega\right)\psi + \omega\mathrm{D}^2\psi \\ &\quad= \left(\frac{p}{p+1}d^*d\omega + \frac{n-p}{n-p+1}dd^*\omega\right)\psi + \omega\mathrm{D}^2\psi\,.\end{aligned} \tag{6.3.20}$$

With the Clifford relation $e^ae^b = 2g^{ab} - e^be^a$, the second term in (6.3.17) becomes

$$\begin{aligned}e^ae^b\nabla_{X_a}\omega\nabla_{X_b}\psi \;&=\; 2\nabla_{X^b}\omega\nabla_{X_b}\psi - e^b{d\!\!\!/}\omega\nabla_{X_b}\psi \\ &=\; 2\nabla_{X^b}\omega\nabla_{X_b}\psi - e^bd\omega\nabla_{X_b}\psi + e^bd^*\omega\nabla_{X_b}\psi\,.\end{aligned} \tag{6.3.21}$$

Now we apply the Clifford form of the CKY equation (5.1.1) to obtain

$$\begin{aligned} e^a e^b \nabla_{X_a}\omega\nabla_{X_b}\psi &= \left\{\frac{1}{p+1}\left(e^b d\omega + (-1)^p d\omega e^b\right)\right. \\ &\quad \left. -\frac{1}{n-p+1}\left(e^b d^*\omega - (-1)^p d^*\omega e^b\right)\right\}\nabla_{X_b}\psi \\ &\quad - e^b d\omega\nabla_{X_b}\psi + e^b d^*\omega\nabla_{X_b}\psi \\ &= -\frac{p}{p+1}e^b d\omega\nabla_{X_b}\psi + \frac{n-p}{n-p+1}e^b d^*\omega\nabla_{X_b}\psi \\ &\quad + \frac{(-1)^p}{p+1}d\omega \mathrm{D}\psi + \frac{(-1)^p}{n-p+1}d^*\omega\mathrm{D}\psi\,. \end{aligned} \tag{6.3.22}$$

Combining (6.3.20) and (6.3.22) we have

$$\begin{aligned} \mathrm{D}\left(e^b\omega\nabla_{X_b}\psi\right) &= \omega\mathrm{D}^2\psi + \frac{(-1)^p}{p+1}d\omega\mathrm{D}\psi + \frac{(-1)^p}{n-p+1}d^*\omega\mathrm{D}\psi \\ &\quad + \frac{p}{p+1}d^*d\omega\psi - \frac{p}{p+1}e^b d\omega\nabla_{X_b}\psi \\ &\quad + \frac{n-p}{n-p+1}dd^*\omega\psi + \frac{n-p}{n-p+1}e^b d^*\omega\nabla_{X_b}\psi\,. \end{aligned} \tag{6.3.23}$$

The derivatives of the remaining terms of K_ω are

$$\begin{aligned} \mathrm{D}\left(d\omega\psi\right) &= (\not{d}d\omega)\psi + e^b d\omega\nabla_{X_b}\psi \\ &= -d^*d\omega\psi + e^b d\omega\nabla_{X_b}\psi \end{aligned} \tag{6.3.24}$$

$$\begin{aligned} \mathrm{D}\left(d^*\omega\psi\right) &= (\not{d}d^*\omega)\psi + e^b d^*\omega\nabla_{X_b}\psi \\ &= dd^*\omega\psi + e^b d^*\omega\nabla_{X_b}\psi \end{aligned} \tag{6.3.25}$$

from which it is clear that (6.3.15) follows. ∎

Clearly K_ω is a symmetry operator for the massless Dirac operator. As the dual of a CKY tensor is a CKY tensor, the operator $K_{*\omega}$ must also be a symmetry operator. Since $*\omega$ has degree $(n-p)$,

$$K_{*\omega} = e^a{*\omega}\nabla_{X_a} + \frac{n-p}{n-p+1}d{*\omega} - \frac{p}{p+1}d^*{*\omega}\,. \tag{6.3.26}$$

By (2.2.7), the Hodge dual of a p-form can be expressed in Clifford form as

$$*\omega = (-1)^{\lfloor p/2\rfloor}\omega z\,. \tag{6.3.27}$$

It follows that the derivative and co-derivative of $*\omega$ are

$$d{*\omega} = -(-1)^{\lfloor p/2\rfloor}d^*\omega z \tag{6.3.28}$$

$$d^*{*\omega} = -(-1)^{\lfloor p/2\rfloor}d\omega z\,. \tag{6.3.29}$$

Since ∇ is metric-compatible, it commutes with z so $K_{*\omega}$ is related to K_ω by

$$K_{*\omega} = (-1)^{\lfloor p/2 \rfloor} K_\omega z\,. \tag{6.3.30}$$

In even dimensions the volume form z anti-commutes with odd forms, thus

$$\omega z = (-1)^p z\omega \quad (n \text{ even}). \tag{6.3.31}$$

Then

$$K_{*\omega} = -(-1)^{\lfloor p/2 \rfloor}(-1)^p z K_\omega \quad (n \text{ even}). \tag{6.3.32}$$

In odd dimensions the volume form commutes with everything, so

$$K_{*\omega} = (-1)^{\lfloor p/2 \rfloor} z K_\omega \quad (n \text{ odd}). \tag{6.3.33}$$

We can easily show that K_ω is an R-commuting operator for D, since by (6.3.15),

$$[\mathrm{D}, K_\omega] = \left(\omega\mathrm{D} - e^a\omega\nabla_{X_a} + \frac{(-1)^p - p}{p+1}d\omega + \frac{(-1)^p + n - p}{n-p+1}d^*\omega\right)\mathrm{D}\,. \tag{6.3.34}$$

With some additional constraints on ω, it is possible to construct operators similar to K_ω which commute or anti-commute with D, and so are symmetry operators for the *massive* Dirac equation. Now

$$\begin{aligned}[\mathrm{D}, \omega\mathrm{D}] &= (\not{d}\omega + e^a\omega\nabla_{X_a} - \omega\mathrm{D})\mathrm{D} \\ &= (d\omega - d^*\omega + e^a\omega\nabla_{X_a} - \omega\mathrm{D})\mathrm{D}\end{aligned} \tag{6.3.35}$$

so we can write (6.3.34) as

$$[\mathrm{D}, K_\omega] = \left(\frac{1+(-1)^p}{p+1}d\omega - \frac{1-(-1)^p}{n-p+1}d^*\omega\right)\mathrm{D} - [\mathrm{D}, \omega\mathrm{D}]\,. \tag{6.3.36}$$

Thus if we define a new operator L^+_ω by

$$L^+_\omega = \frac{1}{2}(K_\omega + \omega\mathrm{D}) \tag{6.3.37}$$

we have

$$[\mathrm{D}, L^+_\omega] = \begin{cases} \dfrac{1}{p+1}d\omega\mathrm{D} & \text{if } \omega \text{ even} \\[2ex] \dfrac{-1}{n-p+1}d^*\omega\mathrm{D} & \text{if } \omega \text{ odd.} \end{cases} \tag{6.3.38}$$

From (6.3.38) it is clear that L^+_ω commutes with D when either

(1) ω is an odd co-closed CKY tensor (an odd KY tensor); or

(2) ω is an even closed CKY tensor (the Hodge dual of a KY tensor).

An operator which commutes with D is a symmetry operator for the massive Dirac equation. Recall that if $d^*\omega = 0$ then ω is a Killing-Yano (KY) tensor. In even dimensions, we can always construct an operator commuting with D from a Killing-Yano tensor ω, either L^+_ω if ω is odd, or $L^+_{*\omega}$ if ω is even, since the degree of $*\omega$ is $(n-p)$. In odd dimensions we cannot construct a commuting operator from an even Killing-Yano tensor in this way since the dual of an even KY tensor is an odd closed CKY tensor. However, we will show that we can construct an anti-commuting operator from an even KY tensor.

We make the observation that for a 1-form ω the operator L^+_ω is related to the Lie derivative with respect to the (metric) dual of ω and an appropriate conformal weight. The Lie derivative of a spinor with respect to a conformal Killing vector K is defined by

$$\mathscr{L}_K\psi \;=\; \nabla_K\psi + \frac{1}{4}dK^\flat\psi\,. \qquad (6.3.39)$$

Putting $\omega = K^\flat$ and $p=1$ into (6.3.37),

$$\begin{aligned} L^+_{K^\flat}\psi &= \frac{1}{2}e^aK^\flat\nabla_{X_a}\psi + \frac{1}{4}dK^\flat\psi - \frac{n-1}{2n}d^*K^\flat\psi + \frac{1}{2}K^\flat\mathrm{D}\psi \\ &= \nabla_K\psi - \frac{1}{2}K^\flat\mathrm{D}\psi + \frac{1}{4}dK^\flat\psi - \frac{n-1}{2n}d^*K^\flat\psi + \frac{1}{2}K^\flat\mathrm{D}\psi \\ &= \mathscr{L}_K\psi - \frac{n-1}{2n}d^*K^\flat\psi\,. \end{aligned} \qquad (6.3.40)$$

Similarly, if $\omega = *K^\flat$ for a vector K then in odd dimensions (6.3.33) shows that

$$L^+_{*K^\flat} \;=\; z\left(\mathscr{L}_K - \frac{n-1}{2n}d^*K^\flat\right) \quad (n \text{ odd}) \qquad (6.3.41)$$

whereas in even dimensions (6.3.32) shows that

$$L^+_{*K^\flat} \;=\; z\left(\mathscr{L}_K - \frac{n-1}{2n}d^*K^\flat - K^\flat\mathrm{D}\right) \quad (n \text{ even}). \qquad (6.3.42)$$

When K is a conformal Killing vector (that is, $K^\flat$ is a CKY 1-form), by (6.3.38) we have the well-known result (see, for example, [BT87])

$$\begin{aligned} [\mathrm{D}, L^+_{K^\flat}] &= [\mathrm{D}, \mathscr{L}_K - (n-1)/(2n)d^*K^\flat] \\ &= -\frac{1}{n}d^*K^\flat\mathrm{D}\,. \end{aligned} \qquad (6.3.43)$$

A conformal Killing vector is usually defined as satisfying $\mathscr{L}_Kg = 2\lambda g$ for some function λ. It can then be shown that $d^*K^\flat = -n\lambda$ and so (6.3.43) may be written as

$$[\mathrm{D}, \mathscr{L}_K + \tfrac{1}{2}(n-1)\lambda] \;=\; \lambda\mathrm{D}\,. \qquad (6.3.44)$$

If $\lambda = 0$ (and hence $d^*K = 0$) then K is a Killing vector, in which case $L^+_{K^\flat}$ commutes with D and $L^+_{K^\flat} = \mathscr{L}_K$. Note that $L^+_{*K^\flat}$ only commutes with D in odd dimensions.

Let L^-_ω be an operator on spinors define d by

$$L^-_\omega = \frac{1}{2}\left(K_\omega - \omega \mathrm{D}\right). \tag{6.3.45}$$

A similar calculation to that carried out in equations (6.3.34) to (6.3.38) shows that the anti-commutator $\{\mathrm{D}, L^-_\omega\}$ is

$$\{\mathrm{D}, L^-_\omega\} = \begin{cases} \dfrac{1}{n-p+1} d^*\omega \mathrm{D} & \text{if } \omega \text{ even} \\ \dfrac{-1}{p+1} d\omega \mathrm{D} & \text{if } \omega \text{ odd.} \end{cases} \tag{6.3.46}$$

Then L^-_ω anti-commutes with D when either

(1) ω is an even co-closed CKY tensor (an even KY tensor); or

(2) ω is an odd closed CKY tensor (the Hodge dual of a KY tensor).

In odd dimensions we cannot construct an anti-commuting operator from an odd KY tensor ω since $*\omega$ is an even closed CKY tensor. However, we have shown that $L^+_{*\omega}$ commutes with D in that case.

In even dimensions we can always construct an anti-commuting operator from a KY tensor ω, since either ω is even or $*\omega$ is an odd closed CKY tensor. As the volume form z anti-commutes with D, the anti-commutator in (6.3.46) can be changed into a commutator if L^-_ω is multiplied by z, since

$$[\mathrm{D}, zL^-_\omega] = -z\{\mathrm{D}, L_\omega\}. \tag{6.3.47}$$

Then

$$[\mathrm{D}, zL^-_\omega] = \begin{cases} \dfrac{1}{n-p+1} d^*\omega\, z\mathrm{D} & \text{if } \omega \text{ even} \\ \dfrac{1}{p+1} d\omega\, z\mathrm{D} & \text{if } \omega \text{ odd.} \end{cases} \tag{6.3.48}$$

The operators $L^-_{*\omega}$ and L^+_ω are related by

$$\begin{aligned} L^-_{*\omega} &= -(-1)^{\lfloor p/2 \rfloor}(-1)^p z K_\omega - (-1)^{\lfloor p/2 \rfloor}\omega z \mathrm{D} \\ &= -(-1)^{\lfloor p/2 \rfloor}(-1)^p z K_\omega - (-1)^{\lfloor p/2 \rfloor}(-1)^p z\omega \mathrm{D} \\ &= -(-1)^{\lfloor p/2 \rfloor}(-1)^p z L^+_\omega\,. \end{aligned} \tag{6.3.49}$$

Thus for a vector K,

$$\begin{aligned} L^-_{*K^\flat} &= zL^+_{K^\flat} \\ &= z\left(\mathscr{L}_K - \frac{n-1}{2n} d^*K^\flat\right). \end{aligned} \tag{6.3.50}$$

As $*K^\flat$ is an odd form of degree $(n-1)$, the anti-commutator (6.3.46) becomes

$$\begin{aligned}\{\mathrm{D}, L^-_{*K^\flat}\} &= -\frac{1}{n} d{*}K^\flat \mathrm{D} \\ &= \frac{1}{n} d^* K^\flat z \mathrm{D} \end{aligned} \tag{6.3.51}$$

when K is conformal Killing. The commutator (6.3.48) becomes

$$[\mathrm{D}, zL^-_{*K^\flat}] = -\frac{1}{n} d^* K^\flat z^2 \mathrm{D} \tag{6.3.52}$$

where z^2 is given by (2.3.5).

In 4-dimensional Lorentzian space-time, Kamran and McLenaghan [KM84b] have found that the most general first-order operator $\mathbf{L}$ (up to a trivial symmetry operator of the form $R\mathrm{D}$) which R-commutes with D can be constructed from a conformal Killing vector, a CKY 2-form and a CKY 3-form. As a CKY 3-form is the Hodge dual of a CKY 1-form, $\mathbf{L}$ may be written using a pair of conformal Killing vectors K_1 and K_2 and a CKY 2-form ω as

$$\begin{aligned}\mathbf{L} &= c_1 + L^+_{K_1^\flat} + zL^-_\omega + L^+_{*K_2^\flat} + c_2 z \\ &= c_1 + \left(\mathscr{L}_{K_1} - \frac{3}{8} d^* K_1^\flat\right) + zL^-_\omega \\ &\quad + z\left(\mathscr{L}_{K_2} - \frac{3}{8} d^* K_2^\flat - K_2^\flat \mathrm{D}\right) + c_2 z \end{aligned} \tag{6.3.53}$$

where c_1 and c_2 are constants. Then

$$[\mathrm{D}, \mathbf{L}] = \left(-\frac{1}{4} d^* K_1^\flat + \frac{1}{3} d^* \omega\, z + \frac{1}{2} dK_2^\flat\, z\right)\mathrm{D}\,. \tag{6.3.54}$$

It would be interesting to know if the most general first-order symmetry operators for other dimensions and signatures are obtained in this way from CKY tensors.

We have chosen to write $\mathbf{L}$ in such a way that it commutes with D when K_1 is a Killing vector, ω is a KY 2-form, and $*K_2^\flat$ is a KY 3-form (note that K_2 is not a Killing vector). If these conditions hold then McLenaghan and Spindel [MS79] have shown that $\mathbf{L}$ is the most general first-order operator commuting with D. For a KY 2-form ω, the operator zL^-_ω commutes with the Dirac operator, and so is a symmetry operator for the *massive* Dirac equation. This was first observed by Carter and McLenaghan, who interpreted zL^-_ω as a *generalised total angular momentum operator* [CM79]. The existence of a KY 2-form accounts for the separability of the Dirac equation in Kerr spacetime. The separated solutions appear as eigenvectors of the symmetry operator, with separation constants given by the eigenvalues [KM84a]. A comprehensive review of separation of variables in general relativity may be found in [KMW92b]. The existence of a KY 2-form in the Kerr solution also gives rise to the conserved quantity known as Carter's constant, which has been called the 'total angular momentum' of a test particle in a geodesic orbit [Car68]. The spinorial equivalent of a KY 2-form is a Killing spinor satisfying an additional skew-Hermiticity condition [CM79]. Using a

Killing spinor, Torres del Castillo [TdC85] independently found the symmetry operator of Kamran and McLenaghan [KM84b], and also a second-order symmetry operator for the vacuum Maxwell equation. We will examine symmetry operators for the vacuum Maxwell equation in §6.6.

6.4 Debye potentials for vacuum Maxwell fields

In this section we show how solutions of the vacuum Maxwell equation (6.2) are obtained from solutions of the massless Dirac equation (6.1.1) and the generalised Debye scalar equation (6.1.11) by spin-raising with shear-free spinors.

Theorem 6.4.1 *For an even shear-free spinor u with charge q and an odd spinor ψ with charge $-q$, the form F given by*

$$F = e^a u \otimes \overline{\widehat{\nabla}_{X_a}\psi} + \frac{1}{2}\widehat{\mathrm{D}}u \otimes \bar{\psi} + \frac{1}{2}\psi \otimes \overline{\widehat{\mathrm{D}}u} \tag{6.4.2}$$

satisfies

$$\not{d}F = \frac{1}{2}\widehat{\mathrm{D}}\psi \otimes \overline{\widehat{\mathrm{D}}u} + u \otimes \overline{\widehat{\mathrm{D}}^2\psi}\,. \tag{6.4.3}$$

Proof. As the charges of u and ψ sum to zero, F is uncharged. Acting on F with the Hodge-de Rham operator we obtain

$$\begin{aligned}\not{d}F = e^a \Big(& \nabla_{X_a} e^b u \otimes \overline{\widehat{\nabla}_{X_b}\psi} + e^b \widehat{\nabla}_{X_a} u \otimes \overline{\widehat{\nabla}_{X_b}\psi} + e^b u \otimes \overline{\widehat{\nabla}_{X_a}\widehat{\nabla}_{X_b}\psi} + \frac{1}{2}\widehat{\nabla}_{X_a}\widehat{\mathrm{D}}u \otimes \bar{\psi} \\ & + \frac{1}{2}\widehat{\mathrm{D}}u \otimes \overline{\widehat{\nabla}_{X_a}\psi} + \frac{1}{2}\widehat{\nabla}_{X_a}\psi \otimes \overline{\widehat{\mathrm{D}}u} + \frac{1}{2}\psi \otimes \overline{\widehat{\nabla}_{X_a}\widehat{\mathrm{D}}u} \Big).\end{aligned} \tag{6.4.4}$$

Now by (4.3.6), the first term becomes

$$e^a \nabla_{X_a} e^b u \otimes \overline{\widehat{\nabla}_{X_b}\psi} = -e^a e^b u \otimes \overline{\widehat{\nabla}_{\nabla_a X_b}\psi}\,. \tag{6.4.5}$$

For the second term,

$$\begin{aligned} e^a e^b \widehat{\nabla}_{X_a} u \otimes \overline{\widehat{\nabla}_{X_b}\psi} &= \left(2g^{ab} - e^b e^a\right) \widehat{\nabla}_{X_a} u \otimes \overline{\widehat{\nabla}_{X_b}\psi} \\ &= 2\widehat{\nabla}_{X^a} u \otimes \overline{\widehat{\nabla}_{X_a}\psi} - e^a \widehat{\mathrm{D}}u \otimes \overline{\widehat{\nabla}_{X_a}\psi}\,. \end{aligned} \tag{6.4.6}$$

Thus

$$\begin{aligned} \not{d}F &= e^a e^b u \otimes \overline{\left(\widehat{\nabla}_{X_a}\widehat{\nabla}_{X_b}\psi - \widehat{\nabla}_{\nabla_a X_b}\psi\right)} + 2\left(\widehat{\nabla}_{X^a} u - \frac{1}{4}e^a \widehat{\mathrm{D}}u\right) \otimes \overline{\widehat{\nabla}_{X_a}\psi} \\ &\quad + \frac{1}{2}\widehat{\mathrm{D}}^2 u \otimes \bar{\psi} + \frac{1}{2}e^a \psi \otimes \overline{\widehat{\nabla}_{X_a}\widehat{\mathrm{D}}u} + \frac{1}{2}\widehat{\mathrm{D}}\psi \otimes \overline{\widehat{\mathrm{D}}u} \\ &= e^a e^b u \otimes \overline{\left(\widehat{\nabla}_{X_a}\widehat{\nabla}_{X_b}\psi - \widehat{\nabla}_{\nabla_a X_b}\psi\right)} \\ &\quad + \frac{1}{2}\widehat{\mathrm{D}}^2 u \otimes \bar{\psi} + \frac{1}{2}e^a \psi \otimes \overline{\widehat{\nabla}_{X_a}\widehat{\mathrm{D}}u} + \frac{1}{2}\widehat{\mathrm{D}}\psi \otimes \overline{\widehat{\mathrm{D}}u} \end{aligned} \tag{6.4.7}$$

since u is shear-free. As $e^a e^b = e^{ab} + g^{ab}$, the first term in (6.4.7) can be written using the curvature operator of $\widehat{\nabla}$ and the trace of the Hessian.

$$\begin{aligned}
&e^a e^b u \otimes \overline{\left(\widehat{\nabla}_{X_a}\widehat{\nabla}_{X_b}\psi - \widehat{\nabla}_{\nabla_a X_b}\psi\right)} \\
&\quad = e^{ab} u \otimes \overline{\left(\widehat{\nabla}_{X_a}\widehat{\nabla}_{X_b}\psi - \widehat{\nabla}_{\nabla_a X_b}\psi\right)} + u \otimes \overline{\left(\widehat{\nabla}_{X_a}\widehat{\nabla}_{X^a}\psi - \widehat{\nabla}_{\nabla_a X^a}\psi\right)} \\
&\quad = \frac{1}{2} e^{ab} u \otimes \overline{\widehat{R}(X_a, X_b)\psi} + u \otimes \overline{\widehat{\nabla}^2\psi}\,. && (6.4.8)
\end{aligned}$$

The curvature of $\widehat{\nabla}$ is related to the curvature forms of ∇ by (5.3.3), while $\widehat{\nabla}^2$ is related to the square of the Dirac operator by (6.2.4). The 'pairwise symmetry' of R_{ab} then allows us to take R_{ab} through the tensor product in order to act on u. So we have

$$\begin{aligned}
&e^a e^b u \otimes \overline{\left(\widehat{\nabla}_{X_a}\widehat{\nabla}_{X_b}\psi - \widehat{\nabla}_{\nabla_a X_b}\psi\right)} \\
&\quad = \frac{1}{4} e^{ab} u \otimes \overline{R_{ab}\psi} - \frac{1}{2} q e^{ab} u \otimes \overline{X_b \,\lrcorner\, X_a \,\lrcorner\, \mathcal{F}\psi} + u \otimes \overline{\widehat{\mathrm{D}}^2\psi} \\
&\qquad + \frac{1}{4}\mathscr{R} u \otimes \bar{\psi} + q u \otimes \overline{\mathcal{F}\psi} \\
&\quad = \frac{1}{4} R_{ab} u \otimes \overline{e^{ab}\psi} - q\mathcal{F} u \otimes \bar{\psi} + u \otimes \overline{\widehat{\mathrm{D}}^2\psi} + \frac{1}{4}\mathscr{R} u \otimes \bar{\psi} + q u \otimes \overline{\mathcal{F}\psi}\,. && (6.4.9)
\end{aligned}$$

Applying the integrability condition (5.3.5) for u eliminates the curvature forms,

$$\begin{aligned}
&e^a e^b u \otimes \overline{\left(\widehat{\nabla}_{X_a}\widehat{\nabla}_{X_b}\psi - \widehat{\nabla}_{\nabla_a X_b}\psi\right)} \\
&\quad = \frac{1}{8} e_b \widehat{\nabla}_{X_a}\widehat{\mathrm{D}} u \otimes \overline{e^{ab}\psi} - \frac{1}{8} e_a \widehat{\nabla}_{X_b}\widehat{\mathrm{D}} u \otimes \overline{e^{ab}\psi} - \frac{1}{2} q X_b \,\lrcorner\, X_a \,\lrcorner\, \mathcal{F} u \otimes \overline{e^{ab}\psi} \\
&\qquad - q\mathcal{F} u \otimes \bar{\psi} + u \otimes \overline{\widehat{\mathrm{D}}^2\psi} + \frac{1}{4}\mathscr{R} u \otimes \bar{\psi} + q u \otimes \overline{\mathcal{F}\psi} \\
&\quad = -\frac{1}{4} e_a \widehat{\nabla}_{X_b}\widehat{\mathrm{D}} u \otimes \overline{e^{ab}\psi} - q\mathcal{F} u \otimes \bar{\psi} + u \otimes \overline{\widehat{\mathrm{D}}^2\psi} + \frac{1}{4}\mathscr{R} u \otimes \bar{\psi}\,. && (6.4.10)
\end{aligned}$$

Replacing e^{ab} in the first term with $e^a e^b - g^{ab}$ produces

$$\begin{aligned}
&e^a e^b u \otimes \overline{\left(\widehat{\nabla}_{X_a}\widehat{\nabla}_{X_b}\psi - \widehat{\nabla}_{\nabla_a X_b}\psi\right)} \\
&\quad = \frac{1}{4}\widehat{\mathrm{D}}^2 u \otimes \bar{\psi} - \frac{1}{4} e_a \widehat{\nabla}_{X_b}\widehat{\mathrm{D}} u \otimes \overline{e^a e^b\psi} \\
&\qquad - q\mathcal{F} u \otimes \bar{\psi} + u \otimes \overline{\widehat{\mathrm{D}}^2\psi} + \frac{1}{4}\mathscr{R} u \otimes \bar{\psi}\,. && (6.4.11)
\end{aligned}$$

As $\widehat{\nabla}_{X_b}\widehat{\mathrm{D}} u$ is odd and $e^b\psi$ is even,

$$e_a \widehat{\nabla}_{X_b}\widehat{\mathrm{D}} u \otimes \overline{e^a e^b\psi} = 2 e^b\psi \otimes \overline{\widehat{\nabla}_{X_b}\widehat{\mathrm{D}} u} \qquad (6.4.12)$$

so finally

$$e^a e^b u \otimes \overline{\left(\widehat{\nabla}_{X_a}\widehat{\nabla}_{X_b}\psi - \widehat{\nabla}_{\nabla_a X_b}\psi\right)}$$

$$\begin{aligned} &= \frac{1}{4}\widehat{\mathrm{D}}^2 u \otimes \bar{\psi} - \frac{1}{2} e^a \psi \otimes \overline{\widehat{\nabla}_{X_a} \widehat{\mathrm{D}} u} - q\mathcal{F} u \otimes \bar{\psi} \\ &\quad + u \otimes \overline{\widehat{\mathrm{D}}^2 \psi} + \frac{1}{4} \mathscr{R} u \otimes \bar{\psi} \,. \end{aligned} \tag{6.4.13}$$

Then (6.4.7) and (6.4.13) give

$$\begin{aligned} \not{d}F &= \frac{3}{4}\widehat{\mathrm{D}}^2 u \otimes \bar{\psi} - q\mathcal{F} u \otimes \bar{\psi} + \frac{1}{4}\mathscr{R} u \otimes \bar{\psi} \\ &\quad + \frac{1}{2}\widehat{\mathrm{D}}\psi \otimes \overline{\widehat{\mathrm{D}} u} + u \otimes \overline{\widehat{\mathrm{D}}^2 \psi} \,. \end{aligned} \tag{6.4.14}$$

The first three terms vanish by the integrability condition (5.3.7), hence

$$\not{d}F = \frac{1}{2}\widehat{\mathrm{D}}\psi \otimes \overline{\widehat{\mathrm{D}} u} + u \otimes \overline{\widehat{\mathrm{D}}^2 \psi} \,. \tag{6.4.15}$$

■

When $q = 0$, u is a twistor and (6.4.2) is the Penrose spin-raising operator which generates a spin-1 field from a spin-$\frac{1}{2}$ field. Clearly F satisfies (6.1) if $\widehat{\mathrm{D}}\psi = 0$, but in general F is a mixture of 0-, 2- and 4-form components since it is not a symmetric combination of spinors. We can show that the 0- and 4-form components vanish when $\widehat{\mathrm{D}}\psi = 0$.

Corollary 6.4.16 *When $\widehat{\mathrm{D}}\psi = 0$, F is an anti self-dual solution of the vacuum Maxwell equation.*

Proof. We need only verify that F is a 2-form when $\widehat{\mathrm{D}}\psi = 0$, that is, that F is symmetric as a tensor product. The only non-symmetric term in F is $e^a u \otimes \overline{\widehat{\nabla}_{X_a}\psi}$. This may be expressed in exterior form as

$$\begin{aligned} e^a u \otimes \overline{\widehat{\nabla}_{X_a}\psi} &= -\frac{1}{4}(e^a u, \widehat{\nabla}_{X_a}\psi) - \frac{1}{8}(e^a u, e_{bc}\widehat{\nabla}_{X_a}\psi)e^{bc} + \frac{1}{4}(e^a u, \widehat{\nabla}_{X_a}\psi)\check{z} \\ &= -\frac{1}{4}(u, \widehat{\mathrm{D}}\psi) - \frac{1}{8}(e^a u, e_{bc}\widehat{\nabla}_{X_a}\psi)e^{bc} + \frac{1}{4}(u, \widehat{\mathrm{D}}\psi)\check{z} \end{aligned} \tag{6.4.17}$$

so F is a 2-form when $\widehat{\mathrm{D}}\psi = 0$. Then $F^\xi = -F$, hence

$$\begin{aligned} *F &= -Fz \\ &= i\left(e^a u \otimes \overline{\check{z}\widehat{\nabla}_{X_a}\psi} + \frac{1}{2}\widehat{\mathrm{D}} u \otimes \overline{\check{z}\psi} + \frac{1}{2}\psi \otimes \overline{\check{z}\widehat{\mathrm{D}} u}\right) \\ &= -iF \end{aligned} \tag{6.4.18}$$

since ψ and $\widehat{\mathrm{D}} u$ are odd. It follows that F is anti self-dual. Finally, Theorem 6.4.1 shows that F is a solution of the vacuum Maxwell equation. ■

In §6.1 we showed how a solution of the massless Dirac equation (6.1.1) can be constructed from a generalised Debye potential and a shear-free spinor. Given a second shear-free spinor (with appropriate gauge term), Theorem 6.4.1 tells us how to construct a solution of the vacuum Maxwell equation. Thus using a pair of shear-free spinors it is possible to construct a solution of the vacuum Maxwell equation from a generalised Debye potential. Such a pair of spinors corresponds to a shear-free 2-form.

Let u_1 and u_2 be a pair of even shear-free spinors with gauge terms $q_1\mathcal{A}_1$ and $q_2\mathcal{A}_2$, respectively, and let f be a function with gauge term $-(q_1\mathcal{A}_1 + q_2\mathcal{A}_2)$, so that the sum of the gauge terms of f, u_1 and u_2 vanishes. Let the corresponding GL(1,$\mathbb{C}$)-curvatures be $q_1\mathcal{F}_1$ and $q_2\mathcal{F}_2$, where $q_i\mathcal{F}_i = d\mathcal{A}_i$. The spinor ψ constructed from f and u_1 via (6.1.2) is

$$\psi \;=\; \hat{d}f\, u_1 + \frac{1}{2} f \widehat{\mathrm{D}} u_1 \,. \tag{6.4.19}$$

It is an odd spinor with gauge term $-q_2\mathcal{A}_2$, that is, opposite charge to u_2. Let F be the exterior form constructed from ψ and u_2 as in (6.4.2). Then by (6.4.19),

$$\begin{aligned} F \;=\;& e^a u_2 \otimes \overline{\widehat{\nabla}_{X_a} \hat{d}f u_1} + \frac{1}{2} f e^a u_2 \otimes \overline{\widehat{\nabla}_{X_a} \widehat{\mathrm{D}} u_1} \\ &+ \frac{1}{2}\hat{d}f u_1 \otimes \overline{\widehat{\mathrm{D}} u_2} + \frac{1}{2}\hat{d}f u_2 \otimes \overline{\widehat{\mathrm{D}} u_1} + \frac{1}{2}\widehat{\mathrm{D}} u_1 \otimes \overline{\hat{d}f u_2} + \frac{1}{2}\widehat{\mathrm{D}} u_2 \otimes \overline{\hat{d}f u_1} \\ &+ \frac{1}{4} f \left(\widehat{\mathrm{D}} u_1 \otimes \overline{\widehat{\mathrm{D}} u_2} + \widehat{\mathrm{D}} u_2 \otimes \overline{\widehat{\mathrm{D}} u_1}\right). \end{aligned} \tag{6.4.20}$$

Since ψ and u_2 have opposite charge, Theorem 6.4.1 shows that

$$\not{d}F \;=\; \frac{1}{2}\widehat{\mathrm{D}}\psi \otimes \overline{\widehat{\mathrm{D}} u_2} + u_2 \otimes \overline{\widehat{\mathrm{D}}^2 \psi} \,. \tag{6.4.21}$$

This can be written more compactly as

$$\not{d}F \;=\; \nabla_{X_a}\left(u_2 \otimes \overline{\widehat{\mathrm{D}}\psi}\right) e^a \,. \tag{6.4.22}$$

By (6.1.5) with $q'\mathcal{F}' = -(q_1\mathcal{F}_1 + q_2\mathcal{F}_2)$, the action of the Dirac operator on ψ is

$$\widehat{\mathrm{D}}\psi \;=\; \left(\hat{\triangle} f - \frac{1}{6}\mathscr{R} f\right) u_1 - f\left(\frac{1}{3} q_1\mathcal{F}_1 + q_2\mathcal{F}_2\right) u_1 \,. \tag{6.4.23}$$

We require that u_1 be an eigenspinor of $\frac{1}{3}q_1\mathcal{F}_1 + q_2\mathcal{F}_2$. Suppose that u_1 and u_2 both correspond to RPND's. If u_1 and u_2 are proportional then they correspond to the same RPND, and hence to the same null 2-form. If they are independent then the spacetime must be type D, and the independent null 2-forms corresponding to u_1 and u_2 must have the same eigenvalue. In either case, Lemma 5.3.15 shows that if

$$\phi_i \;=\; u_i \otimes \bar{u}_i \quad \text{(no sum)} \tag{6.4.24}$$

and

$$C\phi_i \;=\; \mu\phi_i \quad \text{(no sum)} \tag{6.4.25}$$

then

$$q_i\mathcal{F}_i u_i \;=\; -3\mu u_i \quad \text{(no sum)}. \tag{6.4.26}$$

It can also be shown that u_1 is an an eigenspinor of $q_2\mathcal{F}_2$, however the eigenvalue depends on whether or not u_1 and u_2 are proportional.

The self-dual shear-free 2-form ω corresponding to u_1 and u_2 is given by (6.3.1). This can be written in exterior form as

$$\omega = -\frac{1}{8}(u_1, e_{ab}u_2)e^{ab}\,. \tag{6.4.27}$$

If u_1 and u_2 are proportional, ω is null and has one real eigenvector. If they are independent, ω is non-null and has two independent real eigenvectors. We will consider these possibilities as separate cases.

Case 1: $\boldsymbol{u_2 = \mathrm{e}^{\kappa} u_1}$. If $u_2 = \mathrm{e}^{\kappa}u_1$ where κ is a complex function then $q_1\mathcal{A}_1$ and $q_2\mathcal{A}_2$ only differ by an exact form:

$$q_1\mathcal{A}_1 - q_2\mathcal{A}_2 = d\kappa\,, \tag{6.4.28}$$

so $q_1\mathcal{F}_1 = q_2\mathcal{F}_2$. Then we have

$$q_2\mathcal{F}_2 u_1 = -3\mu u_1 \tag{6.4.29}$$
$$q_1\mathcal{F}_1 u_2 = -3\mu u_2\,. \tag{6.4.30}$$

Also, $\omega = \mathrm{e}^{\kappa}\phi_1$ so ω is an eigenvector of the conformal tensor with eigenvalue μ. Let λ_ω denote the eigenvalue of ω. Then from (6.4.23) and (6.4.29) we have

$$\widehat{\mathrm{D}}\psi = \left(\hat{\triangle} f - \frac{1}{6}\mathscr{R} f + 4\lambda_\omega f\right)u_1\,. \tag{6.4.31}$$

Case 2: $\boldsymbol{u_2 \neq \mathrm{e}^{\kappa} u_1}$. If u_1 and u_2 are independent then the spacetime must be type D. Then $\{\phi_1, \phi_2, \omega\}$ is an eigenbasis for the space of self-dual 2-forms. Since ϕ_1 and ϕ_2 both have eigenvalue μ, the eigenvalue λ_ω of ω must be -2μ since the conformal tensor is trace-free. Now

$$C\omega = -\frac{1}{8}(u_1, C_{ab}u_2)e^{ab}\,. \tag{6.4.32}$$

Using (5.3.13) we may write this in terms of $q_2\mathcal{F}_2$.

$$\begin{aligned} C\omega &= -\frac{1}{8}\left(\frac{1}{6}(u_1, e_{ab}q_2\mathcal{F}_2 u_2) + \frac{1}{2}(u_1, q_2\mathcal{F}_2 e_{ab}u_2)\right)e^{ab} \\ &= -\frac{1}{8}\left(-\frac{1}{2}\mu(u_1, e_{ab}u_2) - \frac{1}{2}(q_2\mathcal{F}_2 u_1, e_{ab}u_2)\right)e^{ab}\,. \end{aligned} \tag{6.4.33}$$

Then since $C\omega = -2\mu\omega$ we must have

$$q_2\mathcal{F}_2 u_1 \otimes \bar{u}_2 + u_2 \otimes \overline{q_2\mathcal{F}_2 u_1} = 3\mu\,(u_1 \otimes \bar{u}_2 + u_2 \otimes \bar{u}_1)\,. \tag{6.4.34}$$

Since u_1 and u_2 are independent, their inner product is non-zero. Contracting with u_1 we have

$$(u_1, u_2) q_2 \mathcal{F}_2 u_1 - (q_2 \mathcal{F}_2 u_1, u_1) u_2 \quad = \quad 3\mu (u_1, u_2) u_1 \,. \tag{6.4.35}$$

Contracting again with u_1 shows that

$$(u_1, u_2)(q_2 \mathcal{F}_2 u_1, u_1) \quad = \quad 0 \tag{6.4.36}$$

hence $q_2 \mathcal{F}_2 u_1 = \lambda u_1$ for some complex function λ. Now

$$(q_2 \mathcal{F}_2 u_1, u_2) \quad = \quad -(u_1, q_2 \mathcal{F}_2 u_2) \tag{6.4.37}$$

therefore $\lambda = 3\mu$, that is,

$$q_2 \mathcal{F}_2 u_1 \quad = \quad 3\mu u_1 \,. \tag{6.4.38}$$

Now $C\omega$ can also be written as

$$C\omega \quad = \quad \frac{1}{8}(C_{ab} u_1, u_2) e^{ab} \,, \tag{6.4.39}$$

so we can also relate $C\omega$ to $q_1 \mathcal{F}_1$. A similar calculation to the above shows that

$$q_1 \mathcal{F}_1 u_2 \quad = \quad 3\mu u_2 \,. \tag{6.4.40}$$

With $\lambda_\omega = -2\mu$, from (6.4.23) and (6.4.38) we have

$$\widehat{\mathrm{D}}\psi \quad = \quad \Big(\hat{\triangle} f - \frac{1}{6}\mathscr{R} f + \lambda_\omega f\Big) u_1 \,. \tag{6.4.41}$$

Although the numerical factors in (6.4.31) and (6.4.41) are different, they are related to the number m_ω of independent real eigenvectors of ω. If u_1 and u_2 are proportional then $m_\omega = 1$, otherwise $m_\omega = 2$. Inserting a factor of $1/m_\omega^2$ into (6.4.31) and (6.4.41), we are able to summarise the result of applying (6.4.31) or (6.4.41) to (6.4.22) as

$$\not{d} F \quad = \quad \nabla_{X_a}\left(\Big(\hat{\triangle} f - \frac{1}{6}\mathscr{R} f + \frac{4\lambda_\omega}{m_\omega^2} f\Big) u_2 \otimes \bar{u}_1\right) e^a \,. \tag{6.4.42}$$

So if f satisfies

$$\hat{\triangle} f - \frac{1}{6}\mathscr{R} f \quad = \quad -\frac{4\lambda_\omega}{m_\omega^2} f \tag{6.4.43}$$

then $\not{d} F = 0$ and $\widehat{\mathrm{D}}\psi = 0$, and Corollary 6.4.16 shows that F is a solution of the vacuum Maxwell equation.

In [BCK97], the authors show that a Hertz potential may be found by scaling a self-dual shear-free 2-form by a generalised Debye potential having the opposite charge. It is also necessary that the eigenvectors of the shear-free 2-form be RPND's. By writing F in terms of ω we can see that the same objective is achieved by spin-raising with shear-free spinors. A lengthy calculation shows that F can be expressed in terms of ω as

$$\begin{aligned} F &= d\Big(d^*(f\omega) - \frac{2}{3}f\hat{d}^*\omega\Big) + d^*\Big(\frac{2}{3}f\hat{d}\omega - d(f\omega)\Big) \\ &\quad + \frac{1}{4}(u_1, u_2)\,(1-\check{z})\,\Big(\hat{\triangle}f - \frac{1}{6}\mathscr{R}f - 2\mu f\Big). \end{aligned} \tag{6.4.44}$$

Now in [BCK97] it is demonstrated that if f satisfies (6.4.43) then

$$d\Big(d^*(f\omega) - \frac{2}{3}f\hat{d}^*\omega\Big) = d^*\Big(\frac{2}{3}f\hat{d}\omega - d(f\omega)\Big). \tag{6.4.45}$$

Comparing this with (6.4), it is clear that $f\omega$ is a Hertz potential. Furthermore, the last term in (6.4.44) must vanish. To see this, first suppose that ω has only one real eigenvector. Then u_2 is proportional to u_1, in which case $(u_1, u_2) = 0$. On the other hand, if ω has two real eigenvectors then $-2\mu = 4\lambda_\omega/m_\omega^2$ so the last term vanishes by (6.4.43). Then

$$\begin{aligned} F &= 2d\Big(d^*(f\omega) - \frac{2}{3}f\hat{d}^*\omega\Big) \\ &= 2d^*\Big(\frac{2}{3}f\hat{d}\omega - d(f\omega)\Big). \end{aligned} \tag{6.4.46}$$

Since F is closed and co-closed, it is a solution of the vacuum Maxwell equation.

6.5 Debye potentials from vacuum Maxwell fields via spin-lowering

In §6.2 we showed that a Debye potential for the massless Dirac equation may be generated by lowering a solution of (6.1.1) with a shear-free spinor corresponding to a RPND. In a similar fashion, a solution of the vacuum Maxwell equation can be lowered to produce a solution of (6.1.1). Given another shear-free spinor, we can lower again to produce a generalised Debye potential satisfying (6.4.43).

Theorem 6.5.1 *Given a shear-free spinor u and a 2-form F, the spinor $\psi = Fu$ satisfies*

$$\widehat{\mathrm{D}}\psi = \not{d}Fu\,. \tag{6.5.2}$$

Proof. Acting on ψ with the Dirac operator gives

$$\begin{aligned} \widehat{\mathrm{D}}\psi &= e^a\nabla_{X_a}Fu + e^aF\widehat{\nabla}_{X_a}u \\ &= \not{d}Fu + \frac{1}{4}e^aFe_a\widehat{\mathrm{D}}u\,. \end{aligned} \tag{6.5.3}$$

Since F is a 2-form, $e^aFe_a = 0$ and the result follows. ∎

Theorem 6.5.1 shows that if F is a solution of the vacuum Maxwell equation then ψ is a solution of (6.1.1). Furthermore, ψ is uncharged if u is a twistor. In that case, the action of u on F corresponds to the Penrose spin-lowering operator which takes a spin-1 field to a spin-$\frac{1}{2}$ field.

Since u is a semi-spinor only the self-dual or anti self-dual part of F is relevant. To see this, let $\check{z}u = \epsilon u$ where $\epsilon = \pm 1$. If F^+ and F^- are the self-dual and anti self-dual parts of F, then we observe that

$$*Fu = i(F^+ - F^-)u \tag{6.5.4}$$

but also

$$\begin{aligned} *Fu &= -iF^{\xi}\check{z}u \\ &= i\epsilon Fu \\ &= i\epsilon(F^+ + F^-)u. \end{aligned} \tag{6.5.5}$$

So if $\epsilon = 1$ then $F^- u = 0$, or if $\epsilon = -1$ then $F^+ u = 0$.

Now suppose that we have a pair of even shear-free spinors u_1 and u_2 as in §6.4, both corresponding to RPND's. Let $\psi = Fu_1$. Now ψ has gauge term $q_1\mathcal{A}_1$, and by Theorem 6.5.1, $\widehat{\mathrm{D}}\psi = \not{d}Fu_1$. If we use u_2 to lower ψ, we obtain the function

$$\begin{aligned} f &= (u_2, \psi) \\ &= (u_2, Fu_1) \end{aligned} \tag{6.5.6}$$

which has gauge term $q_1\mathcal{A}_1 + q_2\mathcal{A}_2$. Then by (6.2.8) with $q\mathcal{F} = q_2\mathcal{F}_2$ and $q'\mathcal{F}' = q_1\mathcal{F}_1$,

$$\hat{\triangle} f - \frac{1}{6}\mathscr{R} f = ((\tfrac{1}{3}q_2\mathcal{F}_2 + q_1\mathcal{F}_1)u_2, \psi) + \frac{1}{2}(\widehat{\mathrm{D}}u_2, \widehat{\mathrm{D}}\psi) + (u_2, \widehat{\mathrm{D}}^2\psi). \tag{6.5.7}$$

Let ω be the shear-free 2-form corresponding to u_1 and u_2 as in (6.4.27). From equations (6.4.26), (6.4.30) and (6.4.40) it can be seen that

$$\left(\frac{1}{3}q_2\mathcal{F}_2 + q_1\mathcal{F}_1\right)u_2 = -\frac{4\lambda_\omega}{m_\omega^2}u_2 \tag{6.5.8}$$

where $C\omega = \lambda_\omega \omega$ and m_ω is the number of independent real eigenvectors of ω. Since $\widehat{\mathrm{D}}\psi = \not{d}Fu_1$,

$$\begin{aligned} \widehat{\mathrm{D}}^2\psi &= \not{d}^2 Fu_1 + e^a \not{d}F\widehat{\nabla}_{X_a}u_1 \\ &= \triangle Fu_1 + \frac{1}{4}e^a\not{d}Fe_a\widehat{\mathrm{D}}u_1 \\ &= \triangle Fu_1 + \frac{1}{2}dF\widehat{\mathrm{D}}u_1 + \frac{1}{2}d^*F\widehat{\mathrm{D}}u_1\,. \end{aligned} \tag{6.5.9}$$

Hence

$$\begin{aligned} \hat{\triangle} f - \frac{1}{6}\mathscr{R} f = &-\frac{4\lambda_\omega}{m_\omega^2}f + (u_2, \triangle Fu_1) \\ &+ \frac{1}{2}(\widehat{\mathrm{D}}u_2, dFu_1) + \frac{1}{2}(\widehat{\mathrm{D}}u_1, dFu_2) \end{aligned}$$

$$-\frac{1}{2}(\widehat{\mathrm{D}}u_2, d^*Fu_1) - \frac{1}{2}(\widehat{\mathrm{D}}u_1, d^*Fu_2). \tag{6.5.10}$$

Clearly f satisfies (6.4.43) when F is a vacuum Maxwell field. Note also that the function obtained by lowering first with u_2 and then with u_1 is the same as f since

$$\begin{aligned}(u_2, Fu_1) &= -(Fu_1, u_2) \\ &= (u_1, Fu_2).\end{aligned} \tag{6.5.11}$$

Since f is symmetric in u_1 and u_2, it may be written entirely in terms of ω. Now

$$\begin{aligned}\mathscr{S}_0(\omega F) &= \frac{1}{2}\mathscr{S}_0\left(u_1 \otimes \bar{u}_2 F + u_2 \otimes \bar{u}_1 F\right) \\ &= -\frac{1}{2}\mathscr{S}_0\left(u_1 \otimes \overline{Fu_2} + u_2 \otimes \overline{Fu_1}\right) \\ &= -\frac{1}{8}(Fu_2, u_1) - \frac{1}{8}(Fu_1, u_2) \\ &= \frac{1}{4}f\end{aligned} \tag{6.5.12}$$

and so

$$\begin{aligned}f &= 4\mathscr{S}_0(\omega F) \\ &= -4\,\omega \cdot F\,.\end{aligned} \tag{6.5.13}$$

Similarly, $(u_2, \triangle F u_1) = -4\,\omega \cdot \triangle F$. By (6.3.10),

$$\begin{aligned}(\widehat{\mathrm{D}}u_2, d^*Fu_1) + (\widehat{\mathrm{D}}u_1, d^*Fu_2) &= -4\mathscr{S}_0\left(\widehat{\mathrm{D}}u_1 \otimes \overline{d^*Fu_2} + \widehat{\mathrm{D}}u_2 \otimes \overline{d^*Fu_1}\right) \\ &= -\frac{16}{3}\mathscr{S}_0\left(\hat{d}\omega d^*F - \hat{d}^*\omega d^*F\right) \\ &= \frac{16}{3}g(\hat{d}^*\omega, d^*F)\end{aligned} \tag{6.5.14}$$

and

$$\begin{aligned}(\widehat{\mathrm{D}}u_2, dFu_1) + (\widehat{\mathrm{D}}u_1, dFu_2) &= -4\mathscr{S}_0\left(\widehat{\mathrm{D}}u_1 \otimes \overline{dFu_2} + \widehat{\mathrm{D}}u_2 \otimes \overline{dFu_1}\right) \\ &= \frac{16}{3}\mathscr{S}_0\left(\hat{d}\omega dF - \hat{d}^*\omega dF\right) \\ &= \frac{16}{3}g(*\hat{d}\omega, *dF).\end{aligned} \tag{6.5.15}$$

Then (6.5.10) becomes

$$\hat{\triangle} f - \frac{1}{6}\mathscr{R} f = -\frac{4\lambda_\omega}{m_\omega^2}f - 4\left(\omega \cdot \triangle F - \frac{2}{3}g(*\hat{d}\omega, *dF) + \frac{2}{3}g(\hat{d}^*\omega, d^*F)\right). \tag{6.5.16}$$

This can be verified directly by acting with the Laplace-Beltrami operator on f and using the shear-free 2-form (5.2.10) equation and its integrability conditions.

6.6 Symmetry operators for the vacuum Maxwell equation

In this section we show how to construct first- and second-order symmetry operators for the vacuum Maxwell equation by raising and lowering with various combinations of shear-free spinors. In a 4-dimensional spacetime with Lorentzian signature, Kalnins *et al* [KMW92a] have given the most general second-order symmetry operator for Maxwell's equation. This incidentally shows that the only non-trivial first-order operator for Maxwell's equation is the Lie derivative with respect to a conformal Killing vector, thus we expect any first-order symmetry operator constructed by raising and lowering to be related to the Lie derivative in some way.

A first-order symmetry operator can be constructed as follows. Let u_1 be an *odd* shear-free spinor, and let u_2 be an *even* shear-free spinor with opposite charge to u_1. For a 2-form F, the spinor $\psi = Fu_1$ satisfies $\widehat{\mathrm{D}}\psi = \not{d}Fu_1$ by Theorem 6.5.1. Note that since u_1 is odd, $\psi = F^- u_1$. Let F' be the 2-form constructed from ψ and u_2 as in (6.4.2). Then by Theorem 6.4.1 and Corollary 6.4.16 it is clear that F' is an anti self-dual vacuum Maxwell field whenever F is. Thus we have a first-order symmetry operator $\mathcal{S}$ which may be written in terms of u_1 and u_2 as

$$\begin{aligned}\mathcal{S}_{u_1u_2}F &= -e^a u_2 \otimes \bar{u}_1 \nabla_{X_a} F - \frac{1}{4} e^a u_2 \otimes \overline{e_a \widehat{\mathrm{D}} u_1} F \\ &\quad - \frac{1}{2}\widehat{\mathrm{D}} u_2 \otimes \bar{u}_1 F + \frac{1}{2} F u_1 \otimes \overline{\widehat{\mathrm{D}} u_2}\,. \end{aligned} \tag{6.6.1}$$

Since u_1 and u_2 have opposite parity, $u_2 \otimes \overline{\widehat{\mathrm{D}} u_1}$ must be an even form. The 2-form part of $e^a u_2 \otimes \overline{e_a \widehat{\mathrm{D}} u_1}$ vanishes, and we have the identity

$$e^a u_2 \otimes \overline{e_a \widehat{\mathrm{D}} u_1} + e^a \widehat{\mathrm{D}} u_1 \otimes \overline{e_a u_2} = 0\,. \tag{6.6.2}$$

Then

$$\begin{aligned}\mathcal{S}_{u_1u_2}F &= -e^a u_2 \otimes \bar{u}_1 \nabla_{X_a} F \\ &\quad - \frac{1}{2}\left(\widehat{\mathrm{D}} u_2 \otimes \bar{u}_1 + \frac{1}{4} e^a u_2 \otimes \overline{e_a \widehat{\mathrm{D}} u_1}\right) F \\ &\quad + \frac{1}{2} F \left(u_1 \otimes \overline{\widehat{\mathrm{D}} u_2} + \frac{1}{4} e^a \widehat{\mathrm{D}} u_1 \otimes \overline{e_a u_2}\right) \\ &= -e^a u_2 \otimes \bar{u}_1 \nabla_{X_a} F - \frac{1}{2} e^a \nabla_{X_a} \left(u_2 \otimes \bar{u}_1\right) F \\ &\quad + \frac{1}{2} F \nabla_{X_a} \left(u_1 \otimes \bar{u}_2\right) e^a\,. \end{aligned} \tag{6.6.3}$$

Now the tensors $u_2 \otimes \bar{u}_1$ and $u_1 \otimes \bar{u}_2$ are odd forms. Let K be the (complex) vector whose dual is the 1-form part of $u_2 \otimes \bar{u}_1$, that is,

$$\begin{aligned} K^\flat &= \mathscr{S}_1 \left(u_2 \otimes \bar{u}_1\right) \\ &= \frac{1}{2}\left(u_2 \otimes \bar{u}_1 - u_1 \otimes \bar{u}_2\right). \end{aligned} \tag{6.6.4}$$

In components, $K^\flat = \frac{1}{4}(u_1, e_a u_2)e^a$. In this form, it is easy to verify that $K^\flat$ satisfies the CKY 1-form equation (5.1.1), thus K is a conformal Killing vector.

The 3-form component of $u_2 \otimes \bar{u}_1$ is related to $K^\flat$ by duality, since

$$\begin{aligned} \mathscr{S}_3\left(u_2 \otimes \bar{u}_1\right) &= \frac{1}{2}\left(u_2 \otimes \bar{u}_1 + u_1 \otimes \bar{u}_2\right) \\ &= \frac{1}{2}\left(-u_2 \otimes \overline{\check{z}u_1} + u_1 \otimes \overline{\check{z}u_2}\right) \\ &= -K^\flat \check{z} \\ &= -i * K^\flat . \end{aligned} \tag{6.6.5}$$

Then

$$\begin{aligned} u_2 \otimes \bar{u}_1 &= K^\flat - i * K^\flat \\ &= K^\flat(1-\check{z}) . \end{aligned} \tag{6.6.6}$$

Since $u_1 \otimes \bar{u}_2 = -\left(u_2 \otimes \bar{u}_1\right)^\xi$,

$$\begin{aligned} u_1 \otimes \bar{u}_2 &= -K^\flat - i * K^\flat \\ &= -K^\flat(1+\check{z}) . \end{aligned} \tag{6.6.7}$$

Using (6.3.28) and (6.3.29),

$$\begin{aligned} e^a \nabla_{X_a}\left(u_2 \otimes \bar{u}_1\right) &= (d-d^*)\left(K^\flat - i * K^\flat\right) \\ &= \left(dK^\flat - d^* K^\flat\right)(1-\check{z}) \end{aligned} \tag{6.6.8}$$

and

$$\begin{aligned} \nabla_{X_a}\left(u_1 \otimes \bar{u}_2\right) e^a &= (d+d^*)\left(K^\flat + i * K^\flat\right) \\ &= \left(dK^\flat + d^* K^\flat\right)(1-\check{z}) . \end{aligned} \tag{6.6.9}$$

Now $\check{z}$ commutes with even forms, and

$$\begin{aligned} F(1-\check{z}) &= F + i * F \\ &= 2F^- . \end{aligned} \tag{6.6.10}$$

So we can write $\mathcal{S}$ in terms of K as

$$\begin{aligned} \mathcal{S}_K F &= -2e^a K^\flat \nabla_{X_a} F^- - [dK^\flat, F^-] + 2d^* K^\flat F^- \\ &= -4\left(\nabla_K F^- + \frac{1}{4}[dK^\flat, F^-] - \frac{1}{2} d^* K^\flat F^-\right) + 2K^\flat \not{d} F^- . \end{aligned} \tag{6.6.11}$$

Now the Lie derivative of a form Φ can be expressed as follows (see [BT87]).

$$\mathscr{L}_X \Phi = \nabla_X \Phi + \frac{1}{4}[dX^\flat, \Phi] + \frac{1}{4}\mathscr{L}_X g(X_a, X^a)\Phi$$

$$-\frac{1}{8}\mathscr{L}_X g(X_a, X_b)\left(e^b\eta\Phi e^a + e^a\eta\Phi e^b\right) . \tag{6.6.12}$$

Then since K is conformal Killing and F^- is a 2-form it is clear that

$$\mathcal{S}_K F = -4\mathscr{L}_K F^- + 2K^\flat \not{d} F^- . \tag{6.6.13}$$

As anticipated, we cannot construct a first-order operator distinct from the Lie derivative by lowering and raising. This may not be the case in higher dimensions, since there may exist CKY tensors of degrees other than 1, 2 or 3. Work in this area is continuing. Notice also that we do not require that the shear-free spinors correspond to RPND's, as we do when raising from, or lowering to, a generalised Debye potential.

In a spacetime admitting a CKY 2-form, it is possible to construct a second-order symmetry operator. Suppose that ϕ_{12} is a self-dual CKY 2-form related to a pair of shear-free spinors u_1 and u_2 by

$$\phi_{12} = \frac{1}{2}\left(u_1 \otimes \bar{u}_2 + u_2 \otimes \bar{u}_1\right) . \tag{6.6.14}$$

We have previously shown that the 2-forms given by (6.4.24) satisfy (6.4.25), and that the spinors satisfy (6.4.26) with $q_1 = 1$, $q_2 = -1$ and $\mathcal{F}_i = \mathcal{F}$.

Given a 2-form F, we may construct a function with charge $+2$ (that is, gauge term $2\mathcal{A}$) by lowering twice with u_1. Let $f = (u_1, F u_1)$. This is an example of spin-lowering with a pair of proportional spinors. Substituting u_1 for u_2 in equation (6.4.27) we see that in this case $\omega = \phi_1$, and so ω has only one real independent eigenvector. Then f satisfies (6.5.10), with u_2 replaced by u_1, ω by ϕ_1 and $m_{\phi_1} = 1$. So when F is a vacuum Maxwell field, f satisfies satisfies

$$\hat{\triangle} f - \frac{1}{6}\mathscr{R} f = -4\lambda_{\phi_1} f . \tag{6.6.15}$$

Now f has charge $+2$, and u_2 has charge -1, so the anti self-dual 2-form F' constructed by raising f twice with u_2 has zero charge. Replacing u_1 by u_2 in (6.4.20) we have

$$\begin{aligned} F' = {} & e^a u_2 \otimes \overline{\widehat{\nabla}_{X_a}\hat{d}f u_2} + \frac{1}{2} f e^a u_2 \otimes \overline{\widehat{\nabla}_{X_a}\widehat{\mathrm{D}}u_2} \\ & + \hat{d}f u_2 \otimes \overline{\widehat{\mathrm{D}}u_2} + \widehat{\mathrm{D}}u_2 \otimes \overline{\hat{d}f u_2} + \frac{1}{2} f \widehat{\mathrm{D}}u_2 \otimes \overline{\widehat{\mathrm{D}}u_2} . \end{aligned} \tag{6.6.16}$$

Substituting u_2 for u_1 in (6.4.27) we see that in this case $\omega = \phi_2$. Then by (6.4.42) we have

$$\not{d} F' = \nabla_{X_a}\left(\left(\hat{\triangle} f - \frac{1}{6}\mathscr{R} f + 4\lambda_{\phi_2} f\right) u_2 \otimes \bar{u}_2\right) e^a . \tag{6.6.17}$$

By (6.4.25), the eigenvalues of ϕ_1 and ϕ_2 are equal, so $\not{d}F' = 0$ since f satisfies (6.6.15). Thus we have an operation which maps a vacuum Maxwell field to an anti self-dual vacuum Maxwell field. In [BCK97], Benn and Kress show how a symmetry operator $\mathcal{L}_{\phi_1\phi_2}$ may be constructed from ϕ_1 and ϕ_2. Their operator is identical (up to a trivial factor) to the operator derived from lowering and raising. From (6.5.13), $f = -4\phi_1 \cdot F$,

$$
\begin{tikzcd}[column sep=small]
\begin{array}{c} F \\ \text{Charge: } 0 \\ \not{d}F = 0 \end{array} \arrow[dr, "u_1"] & & & & \begin{array}{c} F' = e^a u_2 \otimes \overline{\widehat{\nabla}_a \psi'} + \frac{1}{2}\widehat{\mathrm{D}}u_2 \otimes \bar{\psi}' + \frac{1}{2}\psi' \otimes \overline{\widehat{\mathrm{D}}u_2} \\ \text{Charge: } 0 \\ \not{d}F' = 0 \end{array} \\
 & \begin{array}{c} \psi = Fu_1 \\ \text{Charge: } +1 \\ \widehat{\mathrm{D}}\psi = 0 \end{array} \arrow[dr, "u_i"] & & \begin{array}{c} \psi' = \hat{d}f u_j + \frac{1}{2} f \widehat{\mathrm{D}}u_j \\ \text{Charge: } +1 \\ \widehat{\mathrm{D}}\psi' = 0 \end{array} \arrow[ur, "u_2"'] & \\
 & & \begin{array}{c} f = (u_2, \psi) \\ \text{Charge: } 1 + q_i \\ \hat{\triangle} f - \frac{1}{6}\mathscr{R} f = -\frac{4\lambda_\omega}{m_\omega^2} f \end{array} \arrow[ur, "u_j"'] & &
\end{tikzcd}
$$

Figure 6.1: Raising and lowering with shear-free spinors, where either $i = 1$, $j = 2$ or $i = 2$, $j = 1$.

while from (6.4.46) we see that

$$F' = 2d\Big(d^*(f\phi_2) - \frac{2}{3} f \hat{d}^* \phi_2 \Big) . \tag{6.6.18}$$

So the above operation can be expressed as an operator $\mathcal{L}_{\phi_1\phi_2}$ acting on 2-forms, where

$$\mathcal{L}_{\phi_1\phi_2} F \quad = \quad -8d\Big(d^*((\phi_1 \cdot F)\, \phi_2) - \frac{2}{3}\, (\phi_1 \cdot F)\, \hat{d}^* \phi_2 \Big) . \tag{6.6.19}$$

We could also have lowered twice with u_2 first, and then raised twice with u_1. This would result in another symmetry operator $\mathcal{L}_{\phi_2\phi_1}$ obtained by interchanging ϕ_1 and ϕ_2 in (6.6.19). However, a tedious calculation using the integrability conditions for u_1 and u_2 shows that $\mathcal{L}_{\phi_1\phi_2}$ and $\mathcal{L}_{\phi_2\phi_1}$ only differ by terms which vanish when they act on vacuum Maxwell fields. This was also pointed out by Torres del Castillo [TdC85], who presented these operators in terms of 2-index Killing spinors.

Another way to generate a symmetry operator is by lowering first with u_1 and then with u_2. The function $f = (u_2, Fu_1)$ has zero charge and satisfies (6.4.43) with $\omega = \phi_{12}$ when F is a vacuum Maxwell field. Note that the value of $m_{\phi_{12}}$ depends on whether or not u_1 and u_2 are independent. Also, since $f = -4\phi_{12} \cdot F$ it clearly makes no difference if we lower with u_2 first and then with u_1. Now let F' be the anti self-dual 2-form constructed from f by raising first with u_1 and then with u_2 via (6.4.20). By (6.4.42) and (6.4.43) we have $\not{d}F' = 0$ once again. Although the 2-form constructed by raising first with u_2 and then with u_1 will in general be different from F', equation (6.4.44) shows that they only differ by a term which vanishes when f satisfies (6.4.43), that is, when F is a vacuum Maxwell field. Thus we have another operation which maps a vacuum Maxwell field to an anti self-dual vacuum Maxwell field. Once again, this may be related to the operator constructed from ϕ_{12} as discussed in [BCK97]. By (6.5.13) and (6.4.46), this may be written as $\mathcal{L}_{\phi_{12}\phi_{12}}$ where

$$\mathcal{L}_{\phi_{12}\phi_{12}}F \quad = \quad -8d\bigg(d^*((\phi_{12}\cdot F)\,\phi_{12}) - \frac{2}{3}\,(\phi_{12}\cdot F)\,d^*\phi_{12}\bigg)\,. \tag{6.6.20}$$

Unfortunately, nothing new is gained from this operator. Writing (6.6.20) entirely in terms of u_1 and u_2, and using the integrability conditions (5.3.6) and (5.3.7), it can be shown that $\mathcal{L}_{\phi_{12}\phi_{12}}$ only differs from $\mathcal{L}_{\phi_1\phi_2}$ and $\mathcal{L}_{\phi_2\phi_1}$ by terms which vanish when they act on vacuum Maxwell fields.

Given that the operator K_ω given by (6.3.14) is a symmetry operator for the massless Dirac equation in all dimensions, it is reasonable to hope that a symmetry operator may exist for some generalisation of the vacuum Maxwell equation in all dimensions. Preliminary work suggests that in $2r$-dimensions it may be possible to construct an operator from a CKY r-form which maps exact r-forms to exact r-forms.

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