ON THE CALCULATION OF RADIATION RESISTANCE OF ANTENNAS AND ANTENNA COMBINATIONS*

By

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Summary—*It is shown that there are two methods for calculating the radia-
tion of antennas and antenna systems. One depends on the consideration of the elec-
tromagnetic field produced by the radiating system, that is, on the integration of the
Poynting vector over a surface enclosing the system. The other is based on a con-
sideration of the electromagnetic phenomena on the conductor itself. The identity of
the two methods is demonstrated. The second method is much simpler to treat formally
and gives clearer results. The calculation of the radiated power is especially simple,
using a law that provides a connection between the radiated power and the Hertzian
vector for the system under consideration. This hitherto unknown law is derived. The
radiated power of any arbitrarily loaded antenna and of short-wave antenna sys-
tems with parallel elements is calculated by means of this law.*

THERE are two methods for calculating the radiation or radia-
tion resistance of antennas and antenna combinations. One de-
pends on the integration of the Poynting vector produced by the
radiating system in question, over a surface enclosing this system. The
other is based on direct integration along the conductor, and specifi-
cally on the calculation of the power which in turn is determined by the
phase displacement between the electric and magnetic field around the
conductors. This phase displacement is a result of the finite propaga-
tional velocity of the electromagnetic field and is generally specified by
the retarded expressions for all fields. Both methods are equivalent and
one can be converted into the other by means of Gauss' law. A.
Pistolkors[1] has mentioned the formal and essentially simpler second
method for the calculation of the radiation of antenna combinations
whose elements are several half waves long.

The derivation used by A. Pistolkors can be considerably simplified
by applying a law which relates the radiation to the Hertzian vector
for the system. We shall show in the following that a simple relation
exists between the radiating properties of a conductor and its Hertizan
vector and that the radiation resistance may then be obtained without
further integration. The Hertzian vector for any oscillating linear con-
ductor can be readily determined. This has been done by the author[2]

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[1] A. Pistolkors, Proc. I.R.E., **17**, 562, 1929.

[2] R. Bechmann, Proc. I.R.E., **19**, 461, 1931; *Jahrbuch für drahtl. Telegr.*, **36**, 182 and 201, 1930.

in a general manner. The electric fields from antennas and anter systems may then be obtained in a straightforward manner from ı Hertzian vector. The resulting expressions are of simple form as w previously shown.[2]

The two above-mentioned methods for determining the radia from antennas and antenna systems are identical. This is a result c Poynting energy law of electrodynamics. In the following we shall c sider briefly the connection between the two methods and derive above-mentioned relation between the radiation and the Hertzian tor.

The radiation from an oscillating electric system such as an ante or an antenna array is given by the Poynting vector $\overline{P}$. This is def in the usual manner by

$$\overline{P} = \frac{c}{4\pi} [\overline{E}, \overline{H}]$$

in which $\overline{E}$ and $\overline{H}$ are the electric and magnetic fields produced by t radiators considered. The total radiation S of the system is obtain by integrating the normal components of $\overline{P}$ over surface σ enclosing This is given by

$$S = \int_\sigma \overline{P}_n d\sigma$$

This expression (2) for the total radiation can be transformed e by means of Green's theorem.

$$S = \int_\sigma \overline{P}_n d\sigma = \int_\tau \operatorname{div} \overline{P} d\tau.$$

where $d\tau$ is an element of volume in the space enclosed by σ. evident that only elements of the conductor under consideration c tribute to the value of the integral. But according to (1):

$$\operatorname{div} \overline{P} = \frac{c}{4\pi} \operatorname{div} [\overline{E}, \overline{H}] = \frac{c}{4\pi} \{(\overline{H}, \operatorname{curl} \overline{E}) - (\overline{E}, \operatorname{curl} \overline{H})\}$$

In addition we have the two Maxwell equations for $\overline{E}$ and $\overline{H}$. If $\overline{I}$ is current density, these read

$$c \operatorname{curl} \overline{H} = \epsilon \frac{\partial \overline{E}}{\partial t} + 4\pi \overline{I}$$

$$c \operatorname{curl} \overline{E} = -\mu \frac{\partial \overline{H}}{\partial t}.$$

[2] Loc. cit.

For the space outside the conductor we take the dielectric constant $\epsilon = 1$ and the permeability $\mu = 1$. By using the above relations (5), we can change (4) to the form:

$$\operatorname{div} \overline{P} = -(\overline{E}\overline{I}) - \frac{1}{8\pi} \frac{\partial}{\partial t} (\overline{E}^2 + \overline{H}^2). \tag{6}$$

The second member on the right side of (6) represents the rate of change in total energy. We now make the assumptions that all observed oscillations of the conductor are represented as the product of a space and a time, in which the time factor is harmonic, and that all conductors are excited with the same frequency. Under these conditions the second member in (6) disappears. Consequently, according to (3) and (6), we get for the total radiation:

$$S = -\int_\tau (\overline{E}\overline{I}) d\tau. \tag{7}$$

Let us now consider a system of N conductors. For such a case the integration indicated by (3) is extended over all radiators of elements $d\tau$ in equation (7), which together produce the resulting field from the system. It follows from the above that if we designate the electric field strength produced by the $\bar{r}$-th radiator by $\overline{E}_r$, we get for (7):

$$S = -\sum_{s=1}^{N} \int_{\tau_s} \left(\sum_{r=1}^{N} \overline{E}_r, \overline{I}_s \right) d\tau_s. \tag{8}$$

where $\overline{I}_s$ is the current distribution on the s-th conductor. In view of the above assumption we use, for the current distribution on the s-th conductor:

$$\overline{I}_s = \overline{J}_s e^{-i\nu t}. \tag{9}$$

Further, we may express the complex electric field $\overline{E}_r$ by

$$\overline{E}_r = E_r e^{-i\nu t}. \tag{10}$$

Taking (9) and (10) into consideration we obtain for the time average of the total radiation $\tilde{S}$ the complex expression:[3]

$$\tilde{S} = -\tfrac{1}{2} Re \left\{ \sum_{s=1}^{N} \int_{\tau_s} (E_r \overline{J}_s^*) d\tau_s \right\} \tag{11}$$

in which $\overline{J}_s^*$ is the conjugated complex value of $\overline{J}_s$. If we remove the complex current amplitudes and observe that the field E_r produced by the r-th conductor is proportional to the current amplitude A_r of this conductor, we set:

[3] Re means that the real part of this expression is to be taken.

$$J_s = A_s J_s' \text{ and } E_r = A_r E_r' \tag{12}$$

ıd if we make a further simplification by introducing the coefficients $_{rs}$ by means of the equation

$$U_{rs} = -\int_{\tau_s} (E_r' J_s') d\tau_s \tag{13}$$

e get, for (11), the general law

$$\tilde{S} = \overline{Re}\left\{ \sum_{r=1}^{N} \sum_{s=1}^{N} \frac{A_r A_s^*}{2} U_{rs} \right\}. \tag{14}$$

'his is a generalization of the expression given by A. Pistolkors[4] for he radiation of an antenna system. In this expression the coefficient J_{rs} represents the coupling coefficients between the r-th and s-th conluctors, and the coefficient U_{ss} is the radiation resistance of the s-th onductor if it is present alone. In particular if all current amplitudes ıre assumed to be equal in amplitude and phase, and designated by A, ve get for the radiation,

$$\tilde{S} = \frac{|A|^2}{2} Re\left\{ \sum_{r=1}^{N} \sum_{s=1}^{N} U_{rs} \right\}. \tag{15}$$

Assuming equal current amplitudes, this gives us the resultant radiation resistance of the system:

$$R_{res} = \overline{Re}\left\{ \sum_{r=1}^{N} \sum_{s=1}^{N} U_{rs} \right\}. \tag{15a}$$

In order to be able to calculate the radiated power, we must form the coefficients U_{rs} in accordance with (13). For this purpose we may carry out the integration given by (13). The coefficient U_{rs} is obtained in a much simpler manner if we start from the well-known Hertizan vector for the radiation. This is taken from a previous paper.[2] Without restricting the generality we assume that all radiators are parallel to each other, and that the common direction of the radiators is along the z axis of a cylindrical coördinate system. The electric field strength $\overline{E}$ and the Hertzian vector $\overline{Z}$ are related according to the equation

$$\overline{E} = \text{grad div } \overline{Z} - \frac{1}{c^2} \frac{\partial^2 \overline{Z}}{\partial t^2}. \tag{16}$$

[1] *Loc. cit.*
[2] *Loc. cit.*

If the Hertzian vector produced by the s-th conductor, (whose z component is other than zero according to the above assumption) is

$$\overline{Z}_s = (\overline{Z}_z)_s = \Pi_s e^{-i\nu t} \tag{17}$$

and we again remove the current amplitude A_s as before, by setting

$$\Pi_s = A_s \Pi_s'. \tag{18}$$

We get from (16) and (17) the z component of the electric field produced by the s-th radiator, thus:

$$(\overline{E}_z)_s = \frac{\partial^2 \Pi_s}{\partial Z^2} + k^2 \Pi_s \tag{19}$$

in which $k = 2\pi/\lambda = \nu/c$, where λ is the wavelength of the exciting oscillation, and c is the velocity of light. Taking (19) into consideration and observing that J' is a real magnitude and designating the element of the conductor $d\tau_s$ by dz_s we get in place of (13):

$$U_{rs} = -\int_{z_s} \left(\frac{\partial^2 \Pi_r'}{\partial z^2} + k^2 \Pi_r' \right) J_s' dz_s. \tag{20}$$

If we consider the case of sinusoidal current distribution for the radiator, we get for J_s the differential equation:

$$\frac{d^2 J_s}{dz^2} + k^2 J_s = 0. \tag{21}$$

By double partial integration of (20) and taking into consideration (21), we may readily obtain the following expression

$$U_{rs} = \left[\Pi_r' \frac{\partial J_s'}{\partial z} - \frac{\partial \Pi_r'}{\partial z} J_s' \right]_{z_{s1}}^{z_{s2}} \tag{22}$$

which must be taken between the upper and lower extremities of the s-th conductor designated respectively by z_{s1} and z_{s2}. The law given by (22) represents the above-mentioned simple combination of the radiation resistance with the Hertzian vector. In particular if the conductors considered here have a length that is a multiple of a half wavelength, that is, if there are current nodes at the ends of the conductors, so that $J_s(z_{s2}) = J_s(z_{s1}) = 0$, then (22) simplifies to:

$$U_{rs} = \left(\Pi_r' \frac{\partial J_s'}{\partial z} \right)_{z_{s2}} - \left(\Pi_r' \frac{\partial J_s'}{\partial z} \right)_{z_{s1}}. \tag{23}$$

As an example we now shall calculate the coefficient U_{rs} of two conductors each of whose lengths is a multiple of a half wavelength of the exciting oscillation, and which are parallel but at different heights. We thus get a general expression. All further expressions for coupling coefficients of special arrangements and radiation resistances of conductors whose lengths are a multiple of a half wavelength, are obtained from this merely by specialization.

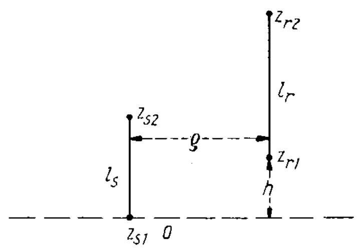

Fig. 1

Let us consider two conductors r and s (Fig. 1). The lengths of the r-th and s-th conductors are l_r and l_s respectively, where $l_r = m\lambda/2$ and $l_s = n\lambda/2$ with m and $n = 1, 2, 3 \cdots$. The coördinates of the endpoints of the conductors are x_r, y_r, $z_{r2} = l_r + h$, $z_{r1} = h$ and x_s, y_s, $z_{s2} = l_s$, $z_{s1} = 0$. Let the current function be given by $\bar{J}'_r = \sin\, k(z_r - h)$ and by $\bar{J}'_s = \sin\, kz_s$. The U_{rs} coefficient is given by (23). We take the Hertzian vector II for the arrangement under consideration from the earlier paper.[2] If A_r is the current amplitude of the r-th conductor, the above[2] becomes:

$$\Pi_r = \frac{A_r}{2kc}\left[e^{ik(z-h)}\{Ei(iku_2) - Ei(iku_1)\}\right.$$
$$\left. + e^{-ik(z-h)}\{Ei(iku_2') - Ei(iku_1')\}\right] \tag{24}$$

With the values

$$\begin{aligned} u_1 &= \sqrt{\rho^2 + (z - z_{r1})^2} - (z - z_{r1}); \\ u_1' &= \sqrt{\rho^2 + (z - z_{r1})^2} + (z - z_{r1}) \\ u_2 &= \sqrt{\rho^2 + (z - z_{r2})^2} - (z - z_{r2}); \\ u_2' &= \sqrt{\rho^2 + (z - z_{r2})^2} + (z - z_{r2}) \end{aligned} \tag{24a}$$

in which ρ is the distance to the base of the two conductors. The function $Ei(x)$ in (24) is the integral exponential function. Accordingly, for the coefficients U_{rs} defined by (23), we get

[2] *Loc. cit.*

$$U_{rs} = \frac{1}{2c}\left[e^{ikh}\{Ei(iku_{22}) + Ei(iku_{11}) - Ei(iku_{21}) - Ei(iku_{12})\}\right.$$
$$\left. + e^{-ikh}\{Ei(iku_{22}') + Ei(iku_{11}') - Ei(iku_{21}') - Ei(iku_{12}')\}\right] \tag{25}$$

and for the sake of brevity:

$$\begin{aligned} u_{22};\ u_{22}' &= \sqrt{\rho^2 + (h + l_r - l_s)^2} \mp (h + l_r - l_s) \\ u_{12};\ u_{12}' &= \sqrt{\rho^2 + (h + l_r)^2} \mp (h + l_r) \\ u_{21};\ u_{21}' &= \sqrt{\rho^2 + (h - l_s)^2} \mp (h - l_s) \\ u_{11};\ u_{11}' &= \sqrt{\rho^2 + h^2} \mp h. \end{aligned} \tag{25a}$$

The coefficients U_{rs} have the dimension of an ohmic resistance. We used the Gauss system in the above. In order to change to technical units we must remember that for the electrostatically measured unit of resistance, R' (electrostatic units) = 30 c.R.Ω. Thus we have found the most general expression for the radiation coupling of two parallel conductors whose lengths are a multiple of the half wavelength. All expressions given by A. Pistolkors[1] are special cases of (24). It is unnecessary to repeat the expressions obtained by specialization.

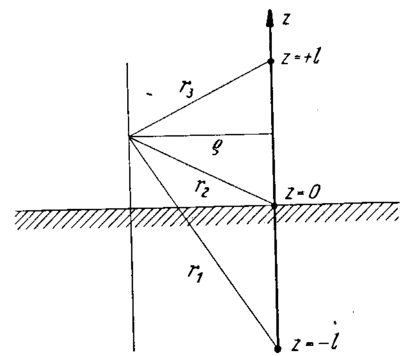

Fig. 2.

In the more general case, in which the current distribution has a finite value at the ends of the conductors, we return to (22). As an example we shall calculate with this expression the radiation resistance of a loaded antenna with a perfect reflecting ground. The radiation resistance for this arrangement has been derived by van der Pol[4] using the method of Poynting vectors.

Let us consider a loaded conductor of length l, placed along the z axis of a cylindrical coördinate system (ρ, z). The coördinates of the

[1] *Loc. cit.*
[4] van der Pol., *Jahrb. d. drahtl. Telegr.*, **13**, 217, 1918.

end points of the conductors are $z_{01}=0$, $z_{02}=l$ (Fig. 2). The current distribution along the conductor is given by the expression $J'_1=\cos(kz-\beta)$. For brevity, we introduce for the upper end of the conductor $kl=\Theta$ and $kl-\beta=\alpha$. A radiating conductor above an infinitely conducting earth gives rise to an image whose energy contribution must be added to that of the conductor. For this mirrored conductor the current distribution is $\bar{J}'_2=\cos(kz+\beta)$ with the limits $z_{01}=-l$ and $z_{02}=0$. The amplitudes of the currents in the antenna and its mirror counterpart are equal. The radiation resistance R of the latter system is calculated from the relation

$$R = -\int_{-l}^{+l} E_z'J'dz. \tag{26}$$

The total current function here is $J=J_1+J_2$. According to the current distribution on the conductor and its mirror image, E_z consists of two parts. Let the part associated with J_1 be E_1, and that with J_2 be E_2. Therefore, $E_z=E_1+E_2$. Consequently (26) may be resolved into four parts as follows:

$$R = -\left\{\int_0^l E_1'J_1'dz + \int_0^l E_2'J_1'dz + \int_{-l}^0 E_1'J_2'dz + \int_{-l}^0 E_2'J_2'dz\right\}. \tag{27}$$

For reasons of symmetry, the first and fourth and also the second and third members are respectively equal. Consequently taking (22) into consideration we obtain for the radiation resistance of the loaded conductor above an infinitely conducting earth, the expression:

$$R = \lim_{\rho=0} 2\left\{(\Pi_1'+\Pi_2')\frac{dJ_1'}{dz}\Big]_0^l - \frac{\partial}{\partial z}(\Pi_1'+\Pi_2')\cdot J_1'\Big]_0^l\right\} \tag{28}$$

in which Π_1 is the Hertzian vector of the loaded antenna, Π_2 is the Hertzian vector of its mirror image. The expressions Π_1 and Π_2 are taken from an earlier paper.[2] If A is the current amplitude for the conductor and its mirror image, we have:

$$\begin{aligned}\Pi_1 &= \frac{iA}{2kc}[e^{i(kz-\beta)}\{Ei(iku_{21}) - Ei(iku_{11})\} \\ &\quad - e^{-i(kz-\beta)}\{Ei(iku_{21}') - Ei(iku_{11}')\}] \\ \Pi_2 &= \frac{iA}{2kc}[e^{i(kz-\beta)}\{Ei(iku_{22}) - Ei(iku_{12})\} \\ &\quad - e^{-i(kz-\beta)}\{Ei(iku_{22}') - Ei(iku_{12}')\}]\end{aligned} \tag{29}$$

[2] R. Bechmann, *loc. cit.*

with the values

$$\begin{aligned}u_{21};\, u_{21}' &= r_3 \mp (z-l) \\ u_{11};\, u_{11}' &= r_2 \mp z \\ u_{22};\, u_{22}' &= r_2 \mp z \\ u_{12};\, u_{12}' &= r_1 \mp (z+l)\end{aligned} \tag{29a}$$

in which, for brevity, we make

$$r_3 = \sqrt{\rho^2+(z-l)^2}$$

$$r_2 = \sqrt{\rho^2+z^2}$$

$$r_1 = \sqrt{\rho^2+(z+l)^2}.$$

We next obtain (28) for finite values of ρ making use of (29) and (29a). For this we get

$$\begin{aligned}U = \frac{1}{2c}&[4Ei(ik\rho) - 2Ei(ik(\sqrt{\rho^2+l^2}-l)) - 2Ei(ik(\sqrt{\rho^2+l^2}+l)) \\ &+ e^{2i\beta}\{2Ei(ik(\sqrt{\rho^2+l^2}-l)) - Ei(ik(\sqrt{\rho^2+(2l)^2}-2l)) \\ &- Ei(ik\rho)\} + e^{-2i\beta}\{2Ei(ik(\sqrt{\rho^2+l^2}+l)) \\ &- Ei(ik(\sqrt{\rho^2+2l)^2}+2l)) - Ei(ik\rho)\}] \\ &+ \frac{1}{c}\left(\frac{e^{ik\sqrt{\rho^2+(2l)^2}}}{ik\sqrt{\rho^2+(2l)^2}} - \frac{e^{ik\rho}}{ik\rho}\right)\cos^2\alpha.\end{aligned} \tag{30}$$

The real part of (30) reads:

$$\begin{aligned}Re(U) = \frac{1}{2c}&[4Ci(k\rho) - 2Ci(k(\sqrt{\rho^2+l^2}-l)) - 2Ci(k(\sqrt{\rho^2+l^2}+l)) \\ &+ \cos 2\beta\{2Ci(k(\sqrt{\rho^2+l^2}-l)) - Ci(k(\sqrt{\rho^2+(2l)^2}-2l)) \\ &- 2Ci(k\rho) + 2Ci(k(\sqrt{\rho^2+l^2}+l)) - Ci(k(\sqrt{\rho^2+(2l)^2}+2l))\} \\ &+ \sin 2\beta\{2Si(k(\sqrt{\rho^2+l^2}+l)) - Si(k(\sqrt{\rho^2+(2l)^2}+2l)) \\ &- 2Si(k(\sqrt{\rho^2+l^2}-l)) + Si(k(\sqrt{\rho^2+(2l)^2}-2l))\}] \\ &+ \frac{1}{c}\left\{\frac{\sin k\sqrt{\rho^2+(2l)^2}}{k\sqrt{\rho^2+(2l)^2}} - \frac{\sin k\rho}{k\rho}\right\}\cos^2\alpha.\end{aligned} \tag{31}$$

We still must make the limit transition to $\rho=0$ in (31) in accordance with (28). For this purpose we develop the members which approach infinity. The integral cosine $Ci(x)$ for small values of x has the development

$$Ci(x) = C + \log x - \frac{1}{2}\frac{x^2}{2!} + \cdots \tag{32}$$

in which $C = 0.577$, the Euler constant. Let us develop the infinite expressions for $\rho = 0$ in (31) in the vicinity of zero. We get

$$\begin{aligned} Ci(k(\sqrt{\rho^2 + l^2} - l)) &= Ci\left(\frac{k\rho^2}{2l}\right) = C + \log\frac{k}{2l} + 2\log\rho \\ Ci(k(\sqrt{\rho^2 + (2l)^2} - 2l)) &= Ci\left(\frac{k\rho^2}{4l}\right) = C + \log\frac{k}{4l} + 2\log\rho. \end{aligned} \tag{32a}$$

The limiting value in (31) remains finite, as the infinite members in $\log\rho$ drop out for $\rho = 0$. Changing from the Gauss to the technical system of units we get for the expression, for the radiation resistance R of the loaded conductor.

$$\begin{aligned} R &= 15\sin 2\beta\{2Si(2\Theta) - Si(4\Theta)\} \\ &\quad + 15\cos 2\beta\{2Ci(2\Theta) - Ci(4\Theta) - \log\Theta - C\} \\ &\quad + 30\left\{\cos^2\alpha\left(\frac{\sin 2\Theta}{2\Theta} - 1\right) - Ci(2\Theta) + 2\log 2\Theta + C\right\} \end{aligned} \tag{33}$$

This expression is identical with the law for the radiation resistance of a loaded conductor over an infinitely conducting earth previously calculated by van der Pol.[9]

It is found that the calculation of the radiation resistances by the direct method, shown in more detail here, is much simpler and leads to clearer results in the case of rather complicated antenna systems.

[9] van der Pol, *loc. cit.*