# Field line motion in classical electromagnetism

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We consider the concept of field line motion in classical electromagnetism for crossed electromagnetic fields and suggest definitions for this motion that are physically meaningful but not unique. Our choice has the attractive feature that the local motion of the field lines is in the direction of the Poynting vector. The animation of the field line motion using our approach reinforces Faraday's insights into the connection between the shape of the electromagnetic field lines and their dynamical effects. We give examples of these animations, which are available on the Web. © *2003 American Association of Physics Teachers.*
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## I. INTRODUCTION

Classical electromagnetism is a difficult subject for beginning students. This difficulty is due in part to the complexity of the underlying mathematics which obscures the physics.[1] The standard introductory approach also does little to connect the dynamics of electromagnetism to the everyday experience of students. Because much of our learning is done by analogy,[2] students have a difficult time constructing conceptual models of the ways in which electromagnetic fields mediate the interactions of the charged objects that generate them. However, there is a way to make this connection for many situations in electromagnetism.

This approach has been known since the time of Faraday who originated the concept of fields. He was also the first to understand that the geometry of electromagnetic field lines is a guide to their dynamical effects. By trial and error, Faraday deduced that the electromagnetic field lines transmit a tension along the field lines and a pressure perpendicular to the field lines. From his empirical knowledge of the shape of the field lines, he was able to understand the dynamical effects of those fields based on simple analogies to strings and ropes.[3–5]

As we demonstrate in this paper, the animation of the motion of field lines in dynamical situations reinforces Faraday's insight into the connection between shape and dynamics.[6] Animation allows the student to gain insight into the way in which fields transmit stresses by watching how the motion of material objects evolve in response to those stresses.[7] Such animations enable the student to better make the intuitive connection between the stresses transmitted by electromagnetic fields and the forces transmitted by more prosaic means, for example, by rubber bands and strings.

We emphasize that we consider here only the mathematics of how to animate field line motion and not the pedagogical results of using these animations in instruction. We are currently using these animations in an introductory studio physics course at MIT that most closely resembles the SCALE-UP course at NCSU.[8] The instruction includes pre- and post-tests in electromagnetism, with comparisons to control groups taught in the traditional lecture/recitation format. Although some of the results of this study have appeared in preliminary form,[9] the full study is still in progress, and the results will be reported elsewhere.

We note that the idea of moving field lines has long been considered suspect because of a perceived lack of physical meaning.[10,11] This skepticism is in part due to the fact that there is no unique way to define the motion of field lines. Nevertheless, the concept has become an accepted and useful one in space plasma physics.[12–14] In this paper we focus on a particular definition of the velocity of electric and magnetic field lines that is useful in quasi-static situations in which the **E** and **B** fields are mutually perpendicular. Although this definition can be identified with physically meaningful quantities in appropriate limits, it is not unique. We choose here a particular subset of the infinite range of possible field line motions.

Previous work in the animation of electromagnetic fields includes film loops of the electric field lines of accelerating charges[15] and of electric dipole radiation.[11] Computers have been used to illustrate the time evolution of quasi-static electro- and magneto-static fields,[16,17] although pessimism has been expressed about their educational utility for the average student.[18] *Electric and Magnetic Interactions* is a collection of three-dimensional movies of electric and magnetic fields.[19] *The Mechanical Universe* uses a number of three-dimensional animations of the electromagnetic field.[20] *Maxwell World* is a real-time virtual reality interface that allows users to interact with three-dimensional electromagnetic fields.[21] References 11 and 15 use essentially the method we define here for the animation, although they do not explicitly define the velocity field that we will introduce. The other citations on the animation of field lines are not explicit about the method they use. None of these treatments focuses on relating the dynamical effects of fields to the shape of the field lines. Our emphasis on dynamics and shape is similar to other pedagogical approaches for understanding forces in electromagnetism.[4,5]

## II. NONUNIQUENESS OF FIELD LINE MOTION

To demonstrate explicitly the ambiguities inherent in defining the evolution of field lines, consider a point charge with charge $q$ and mass $m$, initially moving upward along the negative $z$-axis in a constant background field $\mathbf{E} = -E_0\hat{\mathbf{z}}$. The position of the charge as a function of time is given by

$$\mathbf{X}_{\text{charge}}(t) = (\tfrac{1}{2}t^2 - 3t + \tfrac{9}{2})\hat{\mathbf{z}} \tag{1}$$

and its velocity is given by

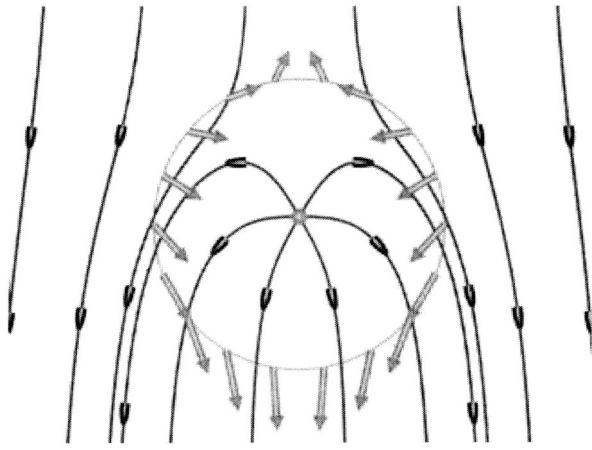$$\mathbf{V}_{\text{charge}}(t) = (t-3)\hat{\mathbf{z}}. \tag{2}$$

Fig. 1. Electric field lines for a point charge in a constant electric field. We also show the Maxwell stress vectors $\mathbf{dF} = \mathbf{T} \cdot \mathbf{n} dA$ on a spherical surface surrounding the charge. Because the magnitude of the stress vector varies greatly around the surface, the length of the displayed vectors is proportional to the square root of the length of the stress vector, rather than to length of the stress vector itself.

The charge comes to rest at the origin at $t = 3$ time units, and then moves back down the negative $z$-axis.

Figure 1 shows the charge at the origin when it comes to rest. The strength of the background field is such that the total field is zero at a distance of one unit above the charge. Figure 1 also shows a set of electric field lines at this time for the total field (that is, the field of the charge plus that of the constant field). Note that in Fig. 1 and in our other figures, we make no attempt to have the density of the field lines represent the strength of the field (there are difficulties associated with such a representation in any case[22]). We discuss other features of Fig. 1 below.

Now suppose we want to make a movie of the scenario in Fig. 1, showing both the moving charge and the time-changing electric field lines. How would we do this? Of course, there is no problem in animating the charge motion, but there are an infinite number of ways of animating the field line motion. We show one of these ways in Fig. 2, which gives two snapshots of the evolution of six different field lines. The animation method that we use here is as follows (this method is *not* the method we use later). At any time (frame), we start drawing each of our six field lines at six different points, with each of the six initial points fixed in time and space for a given field line. These six points are indicated by the spheres in Fig. 2. Figure 2(a) shows the field lines computed in this way when the charge is still out of sight on the negative $z$-axis at $t = 0$, moving upward. Figure 2(b) shows these "same" field lines when the charge has arrived at the origin at $t = 3$.

Although we have a perfectly well-defined animation of the field line motion using this method, the method has many undesirable features. To name two, the innermost four field lines in Fig. 2, which were originally connected to the charges producing the constant field in Fig. 2(a), are connected to the moving charge at the time shown in Fig. 2(b). Moreover, the motion of all of the field lines in the horizontal direction is opposite to the direction of the flow of energy whenever the charge is in motion. A much more satisfying way to animate the field lines for this problem is outlined in Sec. IV.
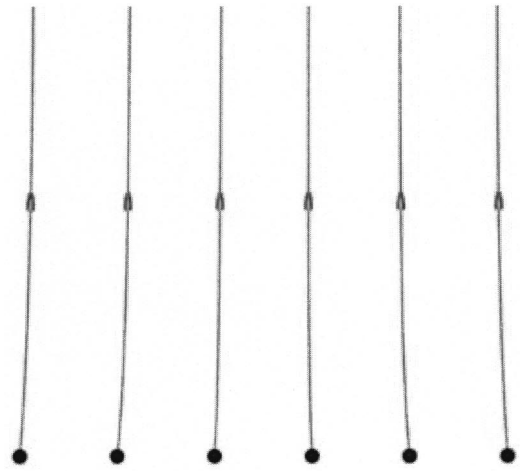


(a)
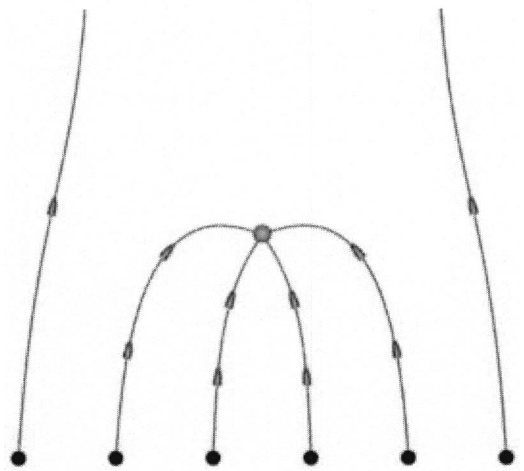


(b)

Fig. 2. Two frames of a method for animating field lines for a charge moving in a constant field. At any instant of time, we begin drawing the field lines from the six spheres, which are fixed in space. In (a), the particle is still out of sight on the negative $z$-axis, but the influence of its field can be seen. In (b), the particle is at the origin and has come to rest before beginning to move back down the negative $z$-axis.

We again emphasize that there are an infinite number of ways to animate electric field line motions for the problem at hand. We have shown one way in Fig. 2. Another way would be to start out with the same six points shown in Fig. 2(a), but now let their positions oscillate sinusoidally in the horizontal direction with some amplitude and frequency. At any instant of time, we could then construct the field lines that pass through those six points. This construction would also give us an animation of the field line motion for these six field lines which would be continuous and well-behaved. However, other than the fact that the field lines would be valid field lines at that instant, the manner in which they evolve has little to do with the physics of the problem.

### III. DRIFT VELOCITIES OF ELECTRIC AND MAGNETIC MONOPOLES

Before we present our preferred method of animation of field lines, we first discuss the drifts of electric and magnetic

monopoles in crossed electric and magnetic fields. This discussion will help motivate our definitions of field line motion in the following sections. For an electric charge with velocity **v**, mass $m$, and electric charge $q$, the nonrelativistic equation of motion in constant **E** and **B** fields is

$$\frac{d}{dt} m\mathbf{v} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{3}$$

If we define the **E**×**B** drift velocity for electric monopoles to be

$$\mathbf{V}_{d,E} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}, \tag{4}$$

and make the substitution

$$\mathbf{v} = \mathbf{v}' + \mathbf{V}_{d,E}, \tag{5}$$

then Eq. (3) becomes (assuming **E** and **B** are perpendicular and constant):

$$\frac{d}{dt} m\mathbf{v}' = q\mathbf{v}' \times \mathbf{B}. \tag{6}$$

The motion of the electric charge thus reduces to a gyration about the magnetic field line superimposed on the steady drift velocity given by Eq. (4). This expression for the drift velocity is only physically meaningful if the right-hand side is less than the speed of light. This assumption is equivalent to the requirement that the energy density in the electric field be less than that in the magnetic field.

For a hypothetical magnetic monopole of velocity **v**, mass $m$, and magnetic charge $q_m$, the nonrelativistic equation of motion is[23]

$$\frac{d}{dt} m\mathbf{v} = q_m(\mathbf{B} - \mathbf{v} \times \mathbf{E}/c^2). \tag{7}$$

If we define the **E**×**B** drift velocity for magnetic monopoles to be

$$\mathbf{V}_{d,B} = c^2 \frac{\mathbf{E} \times \mathbf{B}}{E^2}, \tag{8}$$

and make the substitution analogous to Eq. (5), then we recover Eq. (6) with **B** replaced by $-\mathbf{E}/c^2$. That is, the motion of the hypothetical magnetic monopole reduces to a gyration about the electric field line superimposed on a steady drift velocity given by Eq. (8). This expression for the drift velocity is only physically meaningful if it is less than the speed of light. This assumption is equivalent to the requirement that the energy density in the magnetic field be less than that in the electric field. Note that these drift velocities are independent of both the charge and the mass of the monopoles.

In situations where **E** and **B** are not independent of space and time, the drift velocities given above are still approximate solutions to the full motion of the monopoles as long as the radius and period of gyration are small compared to the characteristic length and time scales of the variation in **E** and **B**. There are other drift velocities that depend on both the sign of the charge and the magnitude of its gyroradius, but these can be made arbitrarily small if the gyroradius of the monopole is made arbitrarily small.[24] The gyroradius depends on the kinetic energy of the charge as seen in a frame moving with the drift velocities. When we say that we are considering "low energy" test monopoles in what follows,

we mean that we take the kinetic energy (and thus the gyroradii) of the monopoles in a frame moving with the drift velocity to be as small as we desire.

The definition we will use to construct our electric field line motions in Sec. IV is equivalent to taking the local velocity of an electric field line in electro-quasi-statics to be the drift velocity of low energy test magnetic monopoles spread along that field line. Similarly, the definition we will use to construct our magnetic field line motions in Sec. V is equivalent to taking the local velocity of a magnetic field line in magneto-quasi-statics to be the drift velocity of low energy test electric charges spread along that field line. These choices are thus physically based in terms of test particle motion and have the advantage that the local motion of the field lines is in the direction of the Poynting vector.

## IV. A PHYSICALLY-BASED EXAMPLE OF ELECTRIC FIELD LINE MOTION

We now introduce our preferred way of defining the electric field line motion in the problem described in Sec. II. In this section and in Secs. V–VI we concentrate on explaining our calculational technique. In Sec. VII, we return to the question of the physical motivation for our choice of animation algorithms.

First, we need to calculate the time-dependent electric and magnetic fields for this problem in the electro-quasi-static approximation, assuming nonrelativistic motion. By electro-quasi-statics, we imply that there is unbalanced charge, and also that the system is constrained to a region of size $D$ such that if $T$ is the characteristic time scale for variations in the sources, then $D \ll cT$. By using Ampere's law including the displacement current, we can argue on dimensional grounds that $cB \approx DE/cT = (V/c)E$, where $V = D/T$. With this estimate of $B$, it is straightforward to show that if we neglect terms of order $(V/c)^2$, then the curl of the electric field is zero. In this situation, our solution for $\mathbf{E}(\mathbf{x}, t)$ as a function of time is just a series of electrostatic solutions appropriate to the source strength and location at any particular time. Thus our time-dependent electric field due to the charge in this problem is given by

$$\mathbf{E}_{\text{charge}}(\mathbf{x}, t) = \frac{q}{4\pi\varepsilon_0} \frac{\mathbf{x} - \mathbf{x}_{\text{charge}}(t)}{|\mathbf{x} - \mathbf{x}_{\text{charge}}(t)|^3}. \tag{9}$$

The time-dependent magnetic field in the same approximation can be found by using the Lorentz transformation properties of the electromagnetic fields, and is given by

$$\mathbf{B}(\mathbf{x}, t) = \frac{1}{c^2} \mathbf{V}_{\text{charge}}(t) \times \mathbf{E}_{\text{charge}}(\mathbf{x}, t). \tag{10}$$

To verify directly that Eq. (10) is the appropriate solution, simply take the curl of this magnetic field. The total electric field is then the sum of the expression in Eq. (9) and the background electric field, and the total magnetic field is just given by Eq. (10). Note that the electric and magnetic fields are everywhere perpendicular to one another.

Given these explicit solutions for the electric and magnetic field, we can calculate at every point in space and time the magnetic monopole drift velocity $\mathbf{V}_{d,B}(\mathbf{x}, t)$ given in Eq. (8). We call this velocity field the velocity of the time-dependent electric field lines in electro-quasi-statics. Note that this velocity is parallel to the Poynting vector, so that an observed motion of a field line in our animations indicates the direc-
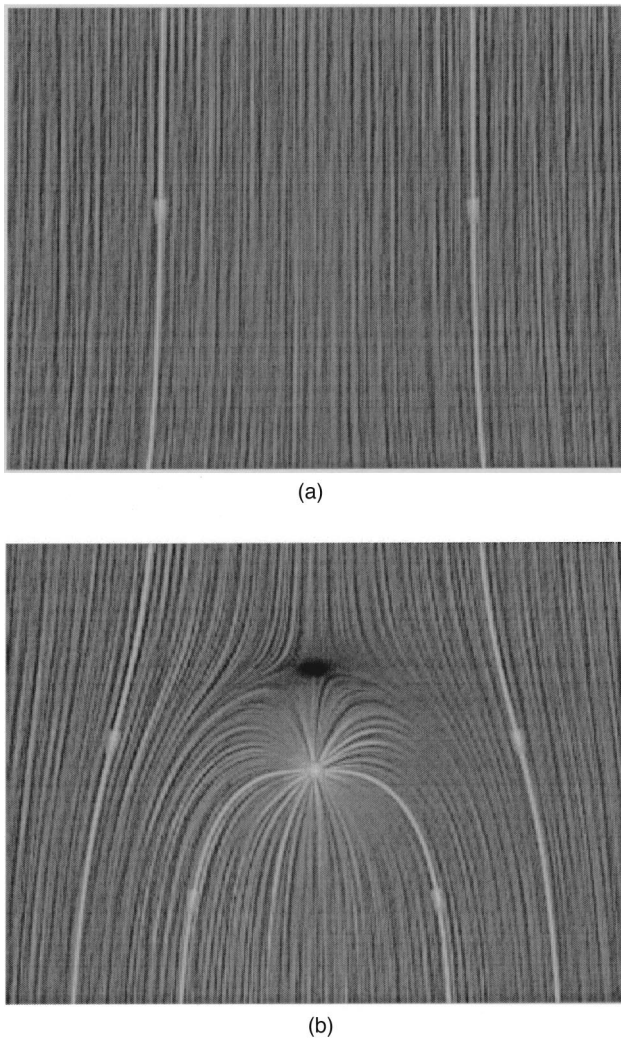
Fig. 3. Two frames of a physically-based method for animating field lines for a charge moving in a constant field. We show both a field line representation and a method of representing the electric field in which the direction of the field at any point is indicated by the orientation of the correlations in the texture near that point. The texture is color-coded so that black represents low electric field magnitudes and white represents high electric field magnitudes.

tion of energy flow at that point. This definition is valid only when we have a set of discrete sources (point charges, point dipoles, etc.). In other situations, for example, a continuous charge density, it is not appropriate (see Sec. VII C).

We show in Fig. 3 two frames of an animation using this definition of the velocity of electric field lines to animate their motion. In addition to the field line representation, we also show in Fig. 3 a representation of the electric field in which the direction of the field at any point is indicated by the orientation of texture correlations near that point. This latler representation of the field structure is the line integral convolution technique of Cabral and Leedom.[25] Sundquist[26] has devised a novel way to animate such textures using the definitions of field motion contained in this paper. That is, the texture pattern in Fig. 3 evolves in time according to Eq. (8).[27]

This animation is constructed in the following way. Choose any point in space $\mathbf{x}_0$ at a given time $t$, and draw an electric field line through that point in the usual fashion (for example, construct the line that passes through $\mathbf{x}_0$ and is everywhere tangent to the electric field at that time). This procedure defines a particular field line at that instant of time $t$. Now increment the time by a small amount $\Delta t$. Move the starting point $\mathbf{x}_0$ to a new point at time $t+\Delta t$ given by $\mathbf{x}_0 + \Delta t\, \mathbf{V}_{d,B}(\mathbf{x}_0,t)$. Draw the field line that passes through this new point at time $t+\Delta t$. That field line is now the time-evolved field line at the time $t+\Delta t$. Using this method, the two outer field lines in Fig. 3(a) evolve into the two outer field lines in Fig. 3(b). A more elegant way to follow the evolution of the field lines is to follow contours on which the value of an electric flux function is constant, and we explain that method elsewhere.[28]

An immediate question arises as to why this method of construction yields a valid set of field lines at every instant of time. What if we had considered at time $t$ a different starting point $\mathbf{x}_1$ on the same field line as $\mathbf{x}_0$? For our method to make sense, the points $\mathbf{x}_0 + \Delta t \mathbf{V}_{d,B}(\mathbf{x}_0,t)$ and $\mathbf{x}_1 + \Delta t \mathbf{V}_{d,B}(\mathbf{x}_1,t)$ must both lie on the same field line at time $t+\Delta t$. This is in fact the case, but it is by no means obvious. We postpone the explanation as to why this is true to Sec. VII C.

What are the advantages to the student in viewing this kind of animation? To answer this question, let us first review how Faraday would have described the downward force on the charge in Figs. 1 and 3. First, surround the charge by an imaginary sphere, as in Fig. 1. The field lines piercing the lower half of the sphere transmit a tension that is parallel to the field. This is a stress pulling downward on the charge from below. The field lines draped over the top of the imaginary sphere transmit a pressure perpendicular to themselves. This is a stress pushing down on the charge from above. The total effect of these stresses is a net downward force on the charge. Over the course of the animation the displayed geometry simply translates upward or downward, so that it is also obvious from the animation that the force on the charge is constant in time as well.

Figure 1 also demonstrates how Maxwell would have explained this same situation quantitatively using his stress tensor. We show the Maxwell stress vectors $\mathbf{dF} = \overset{\leftrightarrow}{\mathbf{T}} \cdot \mathbf{n}\, dA$ on the imaginary spherical surface centered on the charge. Because the stresses vary so greatly in magnitude on the surface of the sphere, we show in Fig. 1 the proper direction of the stress vectors, but the stress vectors have a length that is proportional to the square root of their magnitude, to reduce the variation in the length of the vectors. The downward force on the charge is due both to a pressure transmitted perpendicularly to the electric field over the upper hemisphere in Fig. 1, and a tension transmitted along the electric field over the lower hemisphere in Fig. 1, as we (and Faraday) would expect given the overall field configuration.

We now argue that the animation of this scene greatly enhances Faraday's interpretation of the static image in Fig. 1. First of all, as the charge moves upward, it is apparent in the animation that the electric field lines are generally compressed above the charge and stretched below the charge.[29] This changing field configuration makes it intuitively plausible that the field enables the transmission of a downward force to the moving charge we see as well as an upward force to the charges that produce the constant field, which we cannot see. That is, it makes plausible the stress analysis of Fig. 1 that we carried out at one instant of time. Moreover, we know physically that as the particle moves upward, there is a continual transfer of energy from the kinetic energy of the

particle to the electrostatic energy of the total field. This is a difficult point to argue at an introductory level. However, it is easy to argue in the context of the animation, as follows.

The overall appearance of the upward motion of the charge through the electric field is that of a point being forced into a resisting medium, with stresses arising in that medium as a result of that encroachment. Thus it is plausible to argue based on the animation that the energy of the upwardly moving charge is decreasing as more and more energy is stored in the compressed electrostatic field, and conversely when the charge is moving downward. Moreover, because the field line motion in the animation is in the direction of the Poynting vector, we can explicitly see the electromagnetic energy flow away from the immediate vicinity of the z-axis into the surrounding field when the charge is slowing. Conversely, we see the electromagnetic energy flows back toward the immediate vicinity of the z-axis from the surrounding field when the charge is accelerating back down the z-axis. All of these features make viewing the animation a much more informative experience than viewing a single static image.

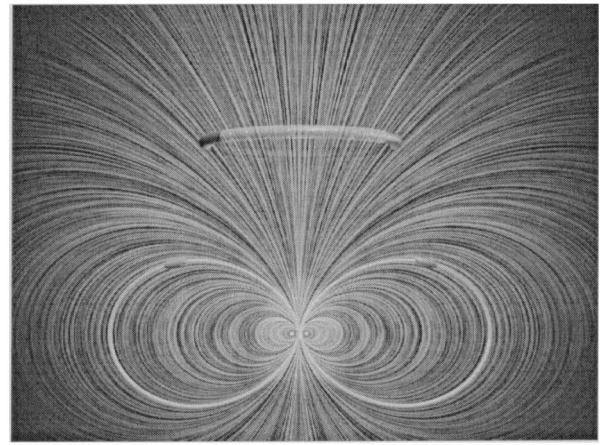## V. AN EXAMPLE OF MAGNETIC FIELD LINE MOTION

Let us turn from electro-quasi-statics to a magneto-quasi-static example. By magneto-quasi-statics, we assume that there is no unbalanced electric charge, and that the system is constrained to a region of size $D$ such that if $T$ is the characteristic time scale for variations in the sources, then $D \ll cT$. Then using Faraday's law, we can argue on dimensional grounds that $E \approx DB/T = VB$, where $V = D/T$. By using Ampere's law including the displacement current, it is straightforward to show that if we neglect terms of order $(V/c)^2$, then we can neglect the displacement current, and the **B** field is determined by

$$\nabla \times \mathbf{B}(X,t) = \mu_0 \mathbf{J}(\mathbf{X},t). \tag{11}$$

That is, as time increases our solution for $\mathbf{B}(\mathbf{X},t)$ as a function of time is just a series of magneto-static solutions appropriate to the source strength at any particular time. The electric field is then derived from this magnetic field using Faraday's law.

Now consider a particular situation. A permanent magnet is fixed at the origin with its dipole moment pointing upward. On the z-axis above the magnet, we have a co-axial, conducting, nonmagnetic ring with radius $a$, inductance $L$, and resistance $R$ [see Fig. 4(a)]. The center of the conducting ring is constrained to move along the vertical axis. The ring is released from rest at $t = 0$ and falls under gravity toward the stationary magnet. Eddy currents arise in the ring because of the changing magnetic flux as the ring falls toward the magnet, and the sense of these currents is to repel the ring.

This physical situation can be formulated mathematically in terms of three coupled ordinary differential equations for the position $\mathbf{X}_{ring}(t)$ of the ring, its velocity $\mathbf{V}_{ring}(t)$, and the current $I(t)$ in the ring.[28] We consider here the particular situation where the resistance of the ring (which in our model can have any value) is identically zero, and the mass of the ring is small enough (or the field of the magnet is large enough) so that the ring levitates above the magnet. We let the ring begin at rest a distance $2a$ above the magnet. The ring begins to fall under gravity. When the ring reaches a



(a)
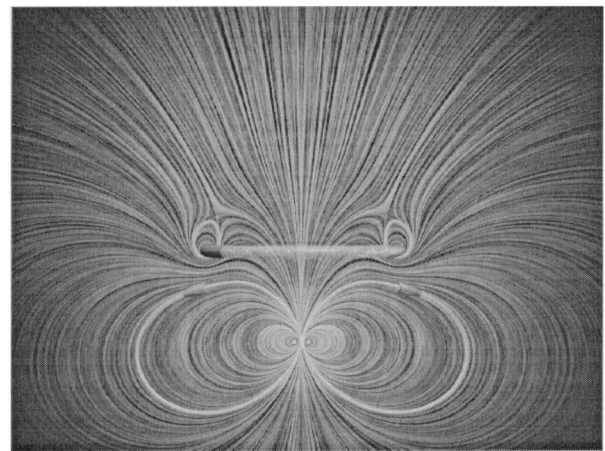


(b)

Fig. 4. Two frames from an example of a physically-based method of animating magnetic field lines for a conducting ring falling in the magnetic field of a permanent magnet. The spheres in the ring in the animation give the sense and the relative strength of the eddy current in the ring as it falls and rises.

distance of about $a$ above the ring, its acceleration slows because of the increasing current in the ring. As the current increases, energy is stored in the magnetic field, and when the ring comes to rest, all of the initial gravitational potential of the ring is stored in the magnetic field. That magnetic energy is then returned to the ring as it ''bounces'' and returns to its original position a distance $2a$ above the magnet. Because there is no dissipation in the system for our particular choice of $R$ in this example, this motion repeats indefinitely.

How do we compute the time evolution of the magnetic field lines in this situation? As before, we first find the total electric and magnetic fields for this problem. In the rest frame of the ring, the magnetic field of the ring can be derived from the vector potential $\mathbf{A}(\mathbf{x},t)$.[30] The electric field of the ring in its rest frame is given by the negative of the partial time derivative of $\mathbf{A}(\mathbf{x},t)$. We then calculate the electric field in the rest frame of the laboratory using the Lorentz transformation properties of the electric field [for example, we use the equation analogous to Eq. (10)]. The magnetic field in the laboratory is just the sum of the magnetic field of the magnet and the magnet field of the ring in its rest frame (assuming nonrelativistic motion). Given these explicit solu-

tions for the total electric and magnetic field, we can now calculate for all space and time the electric monopole drift velocity $\mathbf{V}_{d,E}(\mathbf{x},t)$ given in Eq. (4). This is the velocity field we call the velocity of the time-dependent magnetic field lines in magneto-quasi-statics. Note again that this velocity is parallel to the Poynting vector, so that the observed motion in our animations indicates the direction of energy flow at that point.

We show in Fig. 4 two frames of an animation using this definition of the velocity of magnetic field lines to animate the motion. The animation is constructed in the same manner as that described for Fig. 3, except that we use Eq. (4) for the velocity field. There are two zeroes in the total magnetic field when the ring is in motion [see Fig. 4(b)]. These zeroes appear just above the ring at the two places where the field configuration looks locally like an "X" tilted at about a 45 degree angle. Very near these two zeroes, our assumption that $E \ll cB$ is violated, which means that our magneto-quasi-static equation approximation is no longer valid. However, it can be shown[28] that the region where the drift velocity is very large, and thus where our magneto-quasi-static approximation is invalid, is a region that for all practical purposes is vanishingly small. Similar considerations apply to electro-quasi-statics.

What are the advantages to the student of presenting this kind of animation? One advantage is again the reinforcement of Faraday's insight into the connection between shape and dynamics. As the ring moves downward, it is apparent in the animation that the magnetic field lines are generally compressed below the ring. This makes it intuitively plausible that the compressed field enables the transmission of an upward force to the moving ring as well as a downward force to the magnet. Moreover, we know physically that as the ring moves downward, there is a continual transfer of energy from the kinetic energy of the ring to the magnetostatic energy of the total field. This point is difficult to argue at an introductory level. However, it is easy to argue in the context of the animation, as follows.

The overall appearance of the downward motion of the ring through the magnetic field is that of a ring being forced downward into a resisting physical medium, with stresses that develop due to this encroachment. Thus it is plausible to argue based on the animation that the energy of the downwardly moving ring is decreasing as more and more energy is stored in the magnetostatic field, and conversely when the ring is rising. Moreover, because the field line motion is in the direction of the Poynting vector, we can explicitly see electromagnetic energy flowing away from the immediate vicinity of the ring into the surrounding field when the ring is falling and flowing back out of the surrounding field toward the immediate vicinity of the ring when it is rising. All of these features make watching the animation a much more informative experience than viewing any single image of this situation.

## VI. AN EXAMPLE OF MOVING FIELD LINES IN DIPOLE RADIATION

Before turning to a physical justification of our choices for defining motion of the electromagnetic field, we consider one final example of field line animation, this time in a situation that is not quasi-static. Some of what we discuss here has
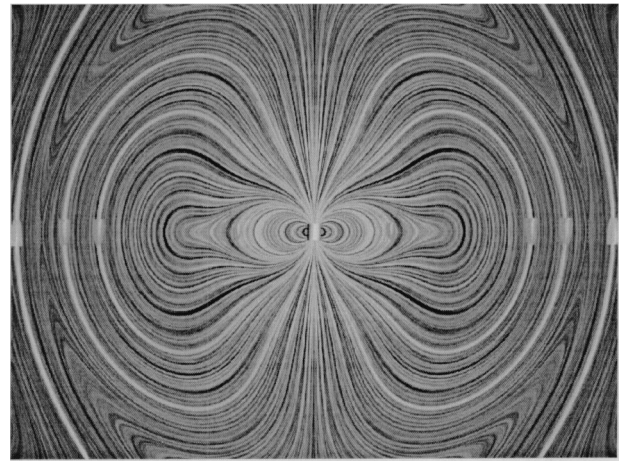


Fig. 5. One frame of an animation of the electric field lines around a time-varying electric dipole. The frame shows the near zone, the intermediate zone, and the far zone.

been treated elsewhere,[11,31] but without explicit reference to the connection between field line motion and the direction of the Poynting vector.

Consider an electric dipole whose dipole moment varies in time according to

$$\mathbf{p}(t) = p_0(1 + 0.1\cos(2\pi t/T))\hat{\mathbf{z}}. \quad (12)$$

The solutions for the electric and magnetic fields in this situation in the near, intermediate, and far zone are commonly quoted, and will not be given here. We take those fields and use Eq. (8) to calculate the electric field line velocity in this situation. Figure 5 shows one frame of an animation of these fields.

Close to the dipole, the field line motion and thus the Poynting vector is first outward and then inward, corresponding to energy flow outward as the quasi-static dipolar electric field energy is being built up, and energy flow inward as the quasi-static dipole electric field energy is being destroyed. This behavior is related to the "casual surface" discussed elsewhere.[31] Even though the energy flow direction changes sign in these regions, there is still a small time-averaged energy flow outward. This small energy flow outward represents the small amount of energy radiated away to infinity. The outer field lines detach from the dipole and move off to infinity. Outside of the point at which they neck off, the velocity of the field lines, and thus the direction of the electromagnetic energy flow, is always outward. This is the region dominated by radiation fields, which consistently carry energy outward to infinity.

## VII. THE PHYSICAL MOTIVATION FOR OUR DEFINITIONS

### A. Thought experiments in magneto-quasi-statics

To understand more fully what our animation procedures mean physically, consider a few simple situations. If we hold a magnet in our hand and move it at constant speed, do the field lines move with the magnet? Most readers would agree that the familiar dipole field line pattern should move with the magnet, and it is easy to see how to calculate this motion—we simple translate the familiar static dipolar field lines along with the magnet. But consider a more complicated situation. Suppose we have a stationary solenoid that

carries a current that is increasing with time. At any instant, we can easily draw the magnetic field lines of the solenoid. However, in contrast to the moving magnet case, there is now no intuitively obvious way to follow the motion of a given field line from one instant of time to the next.

The use of the motion of low-energy test electric charges, however, offers a plausible way to define this motion. By "test" we mean electric charges whose charge is vanishingly small, and that therefore do not modify the original current distribution or affect each other. To see how to approach the concept of field line motion using test electric charges, we first return to the case of translational motion, but consider a moving solenoid instead of a moving magnet. Consider the following thought experiment. We have a solenoid carrying a constant current provided by a battery. The axis of the solenoid is vertical. We place the entire apparatus on a cart, and move the cart horizontally at a constant velocity $\mathbf{V}$ as seen in the laboratory frame ($V \ll c$).

As above, our intuition is that the magnetic field lines associated with the currents in the solenoid should move with their source, for example, with the solenoid on the cart. How do we make this intuition quantitative in this case? First, we realize that in the laboratory frame there will be a "motional" electric field $\mathbf{E} = -\mathbf{V} \times \mathbf{B}$, even though there is zero electric field in the rest frame of the solenoid. Now, given the existence of this laboratory electric field, imagine placing a low-energy test electric charge in the center of the moving solenoid. The charge will gyrate about the magnetic field and the center of gyration will move in the laboratory frame because it drifts in the $-\mathbf{V} \times \mathbf{B}$ electric field with a drift velocity is given by Eq. (4). An $\mathbf{E} \times \mathbf{B}$ electric drift velocity in the $-\mathbf{V} \times \mathbf{B}$ electric field is just $\mathbf{V}$. That is, the test electric charge "hugs" the "moving" field line, moving at the velocity our intuition expects. The drift motion of this low energy test charge constitutes a convenient way to define the motion of the field line. This definition has the advantage that this motion is in the direction of the local Poynting vector. In the more general case (for example, a stationary solenoid with current increasing with time, or two sources of magnetic field moving at two different velocities), we simply extend this concept. That is, we define field line motion to be the motion that we would observe for hypothetical low-energy test electric charges initially spread along the given magnetic field line, drifting in the electric field that is associated with the time changing magnetic field. This is the definition of motion that we have used above.

Let us now return to the original case of the stationary solenoid with a current varying in time. We can find a solution for $\mathbf{B}$ that is the curl of a time-varying vector potential $\mathbf{A}$. Then our electric field $\mathbf{E}$ is simply given by the time derivative of this vector potential, and we can calculate the velocity of a given field line directly by using Eq. (4). Thus as desired, we have found a (nonunique) way to calculate field line motion in this case.

## B. Thought experiments in electro-quasi-statics

Now we turn to electro-quasi-statics. Consider the following thought experiment. We have a charged capacitor consisting of two circular, coaxial conducting metal plates. The common axis of the plates is vertical. We place the capacitor on a cart, and move the cart horizontally at a constant velocity $\mathbf{V}$ as seen in the laboratory frame ($V \ll c$). Our intuition is that the electric field lines associated with the charges on the

capacitor plates should move with their source, for example, with the capacitor on the cart. How do we make this intuition quantitative? We realize that in the laboratory frame there will be a "motional" magnetic field $\mathbf{B} = \mathbf{V} \times \mathbf{E}/c^2$. We then imagine placing a hypothetical low energy test magnetic monopole in the electric field of the capacitor. The monopole will gyrate about the electric field and the center of gyration will move in the laboratory frame because it drifts in the $\mathbf{V} \times \mathbf{E}/c^2$ magnetic field. The $\mathbf{E} \times \mathbf{B}$ magnetic monopole drift velocity is given by Eq. (8). An $\mathbf{E} \times \mathbf{B}$ magnetic drift velocity in this motional magnetic field is just $\mathbf{V}$. That is, the test magnetic monopole "hugs" the "moving" electric field line, moving at the velocity our intuition expects. The drift motion of this low energy test magnetic monopole constitutes one way to define the motion of the electric field line. This motion is not unique, but is physically motivated. As above, it also has the advantage that this motion is in the direction of the local Poynting vector.

In the more general case (for example, two sources of electric field moving at two different velocities, or sources which vary in time), we choose the motion of an electric field line to have the same physical basis as above. That is, the electric field line motion is the motion we would observe for hypothetical low energy test magnetic monopoles initially spread along the given electric field line, drifting in the magnetic field that is associated with the electric field and the moving charges.

Finally, consider the animation of the electric field lines in electric dipole radiation that we presented in Sec. VI. In this situation our definition of the motion of field lines using Eq. (8) no longer permits a physical interpretation in terms of monopole drifts, as our "drift" speeds so defined can exceed the speed of light in regions that are not vanishingly small. Even though this velocity is no longer physical in such regions (that is, corresponding to monopole drift motions), it is still in the direction of the local Poynting vector. Thus the animation of the motion using Eq. (8) shows both the field line configuration at any instant of time, and indicates the local direction of electromagnetic energy flow in the system as time progresses. For this reason, even in non-quasi-static situations, animation of the field is useful, even if no longer physically interpretable in terms of the drift motions of test monopoles.

## C. Why are monopole drift velocities so special in electromagnetism?

We now return to the question of why the method of construction of moving field lines described in Sec. IV yields a valid set of field lines at every instant of time. For any general vector field $\mathbf{W}(\mathbf{X}, t)$, the time rate of change of the flux of that field through an open surface $S$ bounded by a contour $C$ that moves with velocity $\mathbf{v}(\mathbf{X}, t)$ is given by

$$\frac{d}{dt} \int_S \mathbf{W} \cdot d\mathbf{A} = \int_S \frac{\partial \mathbf{W}}{\partial t} \cdot d\mathbf{A} + \int_S (\nabla \cdot \mathbf{W}) \mathbf{v} \cdot d\mathbf{A}$$

$$- \oint_C (\mathbf{v} \times \mathbf{W}) \cdot d\mathbf{l}. \tag{13}$$

If we apply Eq. (13) to $\mathbf{E}(\mathbf{X}, t)$ in regions where there are no sources and use $\nabla \cdot \mathbf{E} = 0$ and $\partial \mathbf{E}/\partial t = c^2 \nabla \times \mathbf{B}$, we have

$$\frac{d}{dt}\int_S \mathbf{E}\cdot \mathbf{dA} = c^2 \oint_C (\mathbf{B} - \mathbf{v}\times\mathbf{E}/c^2)\cdot \mathbf{dl}. \qquad (14)$$

Now we specialize to those configurations of electric, magnetic, and velocity fields that preserve the electric flux, that is for which the left-hand side of Eq. (14) is zero, and thus for which the right-hand side of Eq. (14) is also zero. Because the surface $S$ and its contour are arbitrary (other than the requirement that they not sweep over sources of electric field), we must have the integrand on the right-hand side be zero as well in those regions. Note that because the integrand is zero, we are implicitly assuming that the electric field is perpendicular to the magnetic field at all points in space. This requirement is satisfied for all of the examples we considered above. We further assume that the velocity field is everywhere perpendicular to $\mathbf{E}$. This restriction is reasonable because there is no meaning to the motion of a field line parallel to itself. Then the requirement that the integrand be zero leads to our equation for magnetic monopole drift, Eq. (8).

Thus the velocity field defined by our magnetic monopole drifts preserves electric flux in source free regions. In particular, if we define the open surface $S$ at time $t$ to be such that the electric field $\mathbf{E}$ is everywhere perpendicular to its surface normal, then the electric flux through that surface is zero at time $t$. If we let that surface evolve according to the velocity field in Eq. (8), then Eq. (14) guarantees that the electric flux continues to remain zero. Note that the surface does not have to be infinitesimally small for this argument to hold, because the velocity $\mathbf{v}(\mathbf{x},t)$ varies at every point around the contour according the local values of $\mathbf{E}$ and $\mathbf{B}$ on the contour, in such a way that the integrand on the right-hand side of Eq. (14) is zero. Thus the $\mathbf{E}$ field continues to be locally tangent to the surface as it evolves.

We can now argue that our method of constructing electric field lines in Sec. IV using Eq. (8) yields a valid set of field lines at every instant of time, as follows. For our surface at time $t$, take the field line joining the points $\mathbf{x}_0$ and $\mathbf{x}_1$ discussed in Sec. IV, extended an infinitesimal amount perpendicular to itself in the local direction of $\mathbf{B}$. The electric flux through this surface is zero by construction. Let this surface evolve to time $t+\Delta t$ using the velocity field described by Eq. (8). Equation (14) guarantees that the flux through this evolved surface at time $t+\Delta t$ remains zero, and thus the electric field is still everywhere tangent to that surface. This means that the points $\mathbf{x}_0 + \Delta t\, \mathbf{V}_{d,B}(\mathbf{x}_0,t)$ and $\mathbf{x}_1 + \Delta t\, \mathbf{V}_{d,B}(\mathbf{x}_1,t)$ discussed in Sec. IV still lie on the same field line at time $t+\Delta t$, as was to be shown. Similar considerations apply to our construction in magneto-quasi-statics, except that we are not limited here to point sources, because the divergence of $\mathbf{B}$ is zero everywhere.

## VIII. SUMMARY AND DISCUSSION

It is worth noting that the idea of moving field lines has long been considered suspect because of a perceived lack of physical meaning.[10,11] This is in part due to the fact that there is no unique way to define the motion of field lines. Nevertheless, the concept has become an accepted and useful one in space plasma physics. In this paper we have focused on a particular definition for the velocity of electric and magnetic field lines that is useful in quasi-static situations in which the

$\mathbf{E}$ and $\mathbf{B}$ fields are mutually perpendicular, and that can be identified with physically meaningful quantities in appropriate limits.

In situations that are not quasi-static, for example, dipole radiation, the velocities of field lines calculated according to the above prescriptions are not physical. That is, they do not correspond to the drift motions of low-energy test monopoles. For example, in some regions of nontrivial extent, these velocities exceed the speed of light. Even so, the animation of field line motion using these definitions is informative. Such animations provide both the field line configuration at any instant of time, and the local direction of electromagnetic energy flow in the system as time progresses. Such a visual representation is useful in understanding the energy flows during electric dipole radiation, or in the creation and destruction of electric fields.

In conclusion, we return to the question of the pedagogical usefulness of such animations. Emphasizing the shape of field lines and their dynamical effects in instruction underscores the fact that electromagnetic forces between charges and currents arise due to stresses transmitted by the fields in the space surrounding them, and not magically by "action at a distance" with no need for an intervening mechanism. For this reason, our approach is advocated as a useful pedagogical approach to teaching electromagnetism.[4] This approach allows the student to construct conceptual models of how electromagnetic fields mediate the interactions of the charged objects that generate them, by basing that conceptual understanding on simple analogies to strings and ropes.

Our further contention is that animation greatly enhances the student's ability to perceive this connection between shape and dynamical effects. This is because animation allows the student to gain additional insight into the way in which fields transmit stress, by observing how the motions of material objects evolve in time in response to the stresses associated with those fields. Moreover, watching the evolution of field structures (for example, their compression and stretching) provides insight into how fields transmit the forces that they do. Finally, because our field line motions are in the direction of the Poynting vector, animations provide insight into the energy exchanges between material objects and the fields they produce. The actual efficacy of these animations when used in instruction is the subject of an ongoing investigation,[9] which will be reported elsewhere.

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[1] In one of his first papers on the subject, Maxwell remarked that "...In order therefore to appreciate the requirements of the science [of electromagnetism], the student must make himself familiar with a considerable body of most intricate mathematics, the mere retention of which in the memory materially interferes with further progress...," J. C. Maxwell, "On Faraday's lines of force," as quoted by T. K. Simpson, *Maxwell on the Electromagnetic Field* (Rutgers U.P., New Brunswick, NJ, 1997), p. 55.

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[28] Many of the mathematical and physical concepts in this paper are discussed at a much more detailed and sophisticated level in an expanded version of this paper available at the URL given in Ref. 7.

[29] This statement is true on a global scale, even though the local energy density in the electric field just above the moving charge is decreasing, rather than increasing. If we calculate the total electric energy inside a cube fixed in space whose dimensions are large compared to the transverse structure shown in Fig. 3, and compare the value of this field energy when the charge is at the center of the cube to its value when the charge is a distance $\Delta z$ above the center, we find that the total electrostatic energy in the fixed cube increases by an amount $q E_0 \Delta z$ as the particle moves from the center of the cube to a distance $\Delta z$ above the center. This is the amount of decrease in the kinetic energy of the particle as it moves upward over this same distance, as it must be.

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# NOTES AND DISCUSSIONS

## Comments on the motion of magnetic field lines

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Belcher and Olbert recently showed that the concept of the motion of magnetic field lines can be helpful in teaching classical electromagnetism. Although this concept holds in many situations, it has important limitations. It is shown that the most common definition, $\mathbf{v}=\mathbf{E}\times\mathbf{B}/B^2$, which is the one used by Belcher and Olbert, is not appropriate when an electrostatic field is present, unless the field satisfies special conditions. In an infinitely conducting medium where the electric field has no component parallel to the magnetic field, $\mathbf{E}\times\mathbf{B}/B^2$ is still a meaningful definition of the motion of magnetic field lines (which follow the plasma motion as if "frozen-in"). It used to be assumed that space plasmas could be treated as infinitely conducting and therefore the concept of magnetic field line motion was used extensively. But local nonvanishing values of $\mathbf{E}\cdot\mathbf{B}$ can "cut" magnetic field lines and invalidate the frozen-in condition. © *2006 American Association of Physics Teachers.*
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## I. INTRODUCTION

Although the concept of the motion of magnetic field lines is inherently meaningless for reasons discussed by Feynman *et al.*,[1] it can sometimes be usefully defined as $\mathbf{v}=\mathbf{E}\times\mathbf{B}/B^2$, where $\mathbf{E}$ is the electric field, $\mathbf{B}$ the magnetic field, and $\mathbf{v}$ is the local velocity of the magnetic field line. Belcher and Olbert[2] have shown several examples of the use of the concept of magnetic field line motion that apply to vacuum fields. In Sec. II the vacuum case is discussed and it is shown that this definition of field line motion breaks down in the presence of an electrostatic field. It remains valid under certain conditions, such as in the absence of electric field components parallel to the magnetic field. For brevity the term "parallel electric field" will be used for a nonvanishing electric field component along the magnetic field.

The concept of magnetic field line motion has been used extensively in space plasma physics. In the context of Alfvén's discovery of what is now called Alfvén waves, he noted[3] that "the matter of the medium in which the waves travel appear to be 'fastened' to the lines of force;" that is, the magnetic field lines appear to move with the matter. This behavior is now described as a "frozen-in magnetic field."[4] In the late 1950s it was generally believed that the parallel electric field was everywhere zero in space plasmas, because the essentially unimpeded motion of electrons and ions along the magnetic field in nearly collisionless space plasmas would cause any such components to be "shorted out," that is, extinguished by the rearrangement of electrons and ions in the plasma, and a state of frozen-in magnetic field would prevail. Therefore, little attention was given to Alfvén's suggestion[5] that magnetic-field aligned electric fields exist above the ionosphere. As soon as *in situ* measurements were made in the space plasma, the first indications in support of Alfvén's suggestion were found,[6] and the existence and importance of magnetic-field aligned electric fields in space plasmas is now generally accepted. Therefore, the usefulness of the concept of moving field lines is limited, and as emphasized by Alfvén,[5] it can even be misleading.

## II. THE VACUUM CASE

To illustrate the limited applicability of the concept of field line motion, consider the example of a magnetic dipole field in the presence of a homogeneous electric field parallel to the dipole axis. We have in spherical coordinates $r$, $\theta$, $\varphi$:

$$B_r = B_p \cos\theta, \quad B_\theta = 0.5 B_p \sin\theta, \quad B_\varphi = 0, \tag{1}$$

and

$$E_r = E_o \cos\theta, \quad E_\theta = -E_o \sin\theta. \tag{2}$$

For the same definition as used by Belcher and Olbert,[2] the local "field line velocity" $\mathbf{v}=\mathbf{E}\times\mathbf{B}/B^2$ is given by

$$v_r = v_\theta = 0, \quad v_\varphi = (6E_o/B_p)\sin\theta\cos\theta/(1+3\cos^2\theta) \tag{3}$$

which implies an angular velocity $\omega = (6E_o/B_p r)\cos\theta/(1+3\cos^2\theta)$. Equation (3) implies that different parts of a magnetic field line would rotate at an angular velocity that not only varies along the field line but even changes sign. Thus $\mathbf{v}=\mathbf{E}\times\mathbf{B}/B^2$ describes a velocity distribution along the field line that would continually distort it, while in reality the magnetic field remains undistorted, and each of its field lines remains confined to a plane $\varphi=$constant.

Although this example is simple, we can imagine other configurations where the local field line velocity $\mathbf{v}=\mathbf{E}\times\mathbf{B}/B^2$ does not describe the real evolution of the field line. Indeed, for it to do so, it would need to have the property that different segments of a field line will remain segments of a common field line in the real magnetic field. It has been shown by Newcomb[7] that $\mathbf{v}=\mathbf{E}\times\mathbf{B}/B^2$ satisfies this requirement, and thus is a meaningful definition of field line velocity if

$$\mathbf{B}\times\mathrm{curl}[\mathbf{B}(\mathbf{E}\cdot\mathbf{B}/B^2)]=0. \tag{4}$$

This condition is trivially fulfilled if the parallel electric field is zero everywhere.
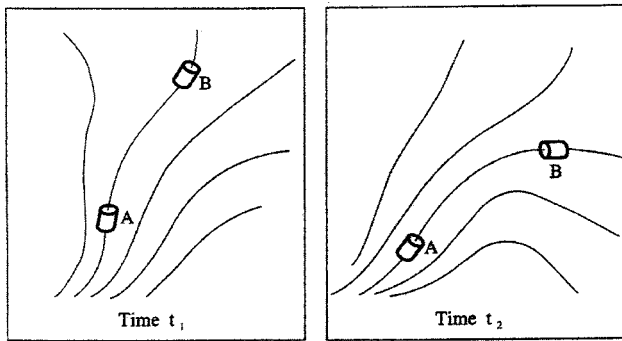
Fig. 1. Frozen-in magnetic field. Two elements of plasma that are on a common magnetic field line at any one time, $t_1$, will be on a common magnetic field line at any other time, $t_2$.

## III. INFINITELY CONDUCTING FLUID

In an infinitely conducting medium, where infinite electrical conductivity ensures that "parallel electric fields" are everywhere zero, a "frozen in magnetic field" is a simple consequence of ideal magnetohydrodynamics. More rigorously, the state of frozen-in magnetic field lines is one where any two elements of the medium that are at one instant on a common magnetic field line will be on a common magnetic field line at any other instant (see Fig. 1). This definition does not imply any concept of motion of magnetic field lines. However, if this frozen-in condition holds, elements of the medium can be used to "label" the magnetic field lines and thus define their "motion." When ideal magnetohydrodynamics is applicable, the frozen field condition is an attractive tool for simplifying magnetohydrodynamic problems.

## IV. SPACE PLASMA

Almost all of the known matter in our universe is in a state of magnetized plasma. In some regions, such as in the sun and stars, the high electrical conductivity makes magnetohydrodynamics an acceptable approximation and therefore magnetic field line motion is a useful concept. In the Earth's magnetosphere, solar and stellar winds, and in interstellar and intergalactic space the plasma is very nearly collisionless, that is, the density is so low that collisions between particles are very infrequent and can be disregarded. The electrical conductivity is very anisotropic. As electrons and ions move almost unimpeded along the magnetic field, it was commonly assumed that the electric field component along the magnetic field would be zero, because the unimpeded motion of charge carriers along the magnetic field would "short out" the electric field component along the magnetic field. Therefore the concept of moving and "frozen-in" magnetic field lines has been extensively used in space plasma physics.

If there exists regions where the parallel electric field is nonzero, the magnetic field lines can be "cut" so that different segments of a field line move differently, as illustrated by

Alfvén and Fälthammar,[8] because the condition in Eq. (4) is violated.

The assumption of vanishing parallel electric fields was challenged by Alfvén[4] who proposed that such fields exist above the ionosphere and cause downward acceleration of auroral primary electrons, but his suggestion was generally disregarded. The first indication in support of Alfvén's idea was McIlwains[6] observation of auroral primary electrons. Since then, an extensive body of evidence from space measurements has been accumulated, and the existence and importance of magnetic-field aligned electric fields in space plasmas is now generally accepted. For example, in the auroral acceleration region upward directed parallel electric fields accelerate electrons down into the ionosphere where they cause aurora; at the same time ionospheric ions go into the magnetosphere. Beautiful examples of such electron and ion fluxes have been reported in Ref. 9. In the auroral return current region, downward directed parallel electric fields accelerate ions down and electrons up.

The question of how it is possible for an essentially collisionless plasma to sustain a parallel electric field has not been fully answered, but a number of possible mechanisms have been identified. (For a recent review see Ref. 10.) Nevertheless, the existence of these parallel fields is an empirical fact, which means that the concept of moving, frozen-in magnetic field lines must be applied with caution. Alfvén seems to have been the first to draw attention to frozen-in magnetic field lines and he became convinced by his studies of the aurora that this concept could be dangerously misleading. Especially in his later years he vigorously warned against unjustified use of the concept.[4]

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# Erratum to "Comment on 'A simple special relativistic perturbation scheme for yielding the general relativistic behavior of point particles and photons in the gravitational field of stars,' " by A. Alaniz [Am. J. Phys. 70, 498–501 (2002)] [Am. J. Phys. 71, 1321–1322 (2003)]

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The expressions

$$\frac{1}{2}m\dot{r}^2 + r^2\dot{\theta}^2$$

in Eqs. (7) and (8) should be replaced by

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2),$$

and the expressions

$$m\ddot{r} - 2r\dot{\theta}^2$$

in Eqs. (10) and (11) by

$$m(\ddot{r} - r\dot{\theta}^2).$$

The conclusions of the Comment are unchanged.

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## ONLINE COLOR FIGURES AND AUXILIARY MATERIAL

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