different positions of Q on the ellipse, when the theoretical eight shape figure corresponding to the ellipse is obtained. Compare this with the experimental curve.

The data processing and curve plotting for the above exercise and the main construction can be done very conveniently using a personal computer, especially when ellipses for several ϕ values are examined. This was done to obtain Fig. 4.

DISCUSSION

The simple construction discussed above helps in realizing the full potential of the standard undergraduate experiment on the basic types of polarization: linear, circular, and elliptical. The standard experiments are confined to observing visually and noting that as the analyzer is rotated, the light intensity goes through a maximum and minimum for elliptically polarized light, the minimum is zero for linearly polarized light. In experiments employing a photocell the photocurrent (i) is measured and i vs $\cos^2 \Theta$ is plotted for linear polarized light. For elliptical polarization the ratio of maximum and minimum currents is measured and compared with the square of the ratio of the major and minor axes of the theoretical polarization ellipse. A student does not observe the shape of the ellipse, as we do in the experiment of this paper.

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Time-dependent, generalized Coulomb and Biot–Savart laws: A derivation based on Fourier transforms

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On the basis of the solution of the Fourier transformed inhomogeneous wave equation expressed in terms of both the charge and current density source, two basic field equations are derived, from which the time-dependent generalized Coulomb and Biot–Savart laws are readily obtained. For the specific case of an arbitrarily moving point charge, the use of the Fourier-transform approach makes the derivation of two intermediate field equations, required to get the fields in both the Heaviside–Feynman and Liénard–Wiechert form, straightforward. © *1996 American Association of Physics Teachers.*

I. INTRODUCTION

The electric and magnetic fields of a point charge in arbitrary motion *in vacuo* are traditionally obtained by making use of the Liénard–Wiechert potentials,[1] whereas for arbitrary charge and current distributions the fields can be advantageously obtained by means of the time-dependent generalized Coulomb and Biot–Savart laws as given by Jefimenko.[2] Recently, two alternative derivations of these time-dependent laws have been given, one based on the standard retarded scalar and vector potentials of electrodynamics[3] and the other one based on a light cone transformation.[4] A generalization of Jefimenko's formulas to include magnetic monopoles has also been considered.[5]

This paper gives another alternative derivation of these formulas, using the Fourier-space description, that has notable advantages when compared with the traditional approach to electromagnetic theory.[6] Instead of the coordinate-space description of the more conventional treatment of electromagnetic theory, in the Fourier-space description the electromagnetic fields are described in terms of their Fourier transforms in both space and time, so that they are functions of frequency ω and wave vector $\mathbf{k}$, rather than of time t and position $\mathbf{r}$. As a result, the inhomogeneous wave equation obtained from Maxwell's equations is an algebraic equation, rather than a differential equation, whose solution, which is straightforward *in vacuo*, has a singular part, the causal treatment of which requires the use of a proper contour integration, as shown in Sec. II. In the more conventional approach this corresponds to choosing the retarded solution of d'Alembert's equation for the potentials.

On making use of two basic field equations derived in Sec. II, the time-dependent generalized Coulomb and Biot–Savart laws are readily obtained, as shown in Sec. III.

Another example where the Fourier-space description reveals major advantages with respect to the traditional

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Liénard–Wiechert potentials approach is the derivation, given in Sec. IV, of two field equations needed to arrive at the fields in both the Heaviside–Feynman and Liénard–Wiechert form.

II. BASIC FIELD EQUATIONS

Describing the electromagnetic field in terms of its Fourier transform, i.e., assuming that the space-time variation of both the electric field $\mathbf{E}(\mathbf{r},t)$ and the magnetic field $\mathbf{B}(\mathbf{r},t)$ is of the form $\exp\{i(\mathbf{k}\cdot\mathbf{r}-\omega t)\}$, Fourier transformed Faraday's and Ampère–Maxwell's laws (Gaussian units) yield, on eliminating the magnetic field, the inhomogeneous wave equation which relates the (Fourier transform of the) electric field $\mathbf{E}(\mathbf{k},\omega)$ to the (Fourier transform of the) current density source $\mathbf{j}(\mathbf{k},\omega)$.[6] For free space, such a wave equation has the form[7]

$$\omega^2\mathbf{E}(\mathbf{k},\omega)+c^2\mathbf{k}\times(\mathbf{k}\times\mathbf{E}(\mathbf{k},\omega))=-4\pi i\omega\mathbf{j}(\mathbf{k},\omega). \quad (1)$$

On noting that $\mathbf{k}\times(\mathbf{k}\times\mathbf{E})=(\mathbf{k}\cdot\mathbf{E})\mathbf{k}-k^2\mathbf{E}$ and making use of the Fourier transformed form of Poisson's equation, i.e., $\mathbf{k}\cdot\mathbf{E}(\mathbf{k},\omega)=-4\pi i\rho(\mathbf{k},\omega)$ with $\rho(\mathbf{k},\omega)$ the Fourier transform of the charge density source $\rho(\mathbf{r},t)$, Eq. (1) yields at once

$$\mathbf{E}(\mathbf{k},\omega)=\frac{4\pi i}{k^2c^2-\omega^2}\{\omega\mathbf{j}(\mathbf{k},\omega)-c^2\rho(\mathbf{k},\omega)\mathbf{k}\}. \quad (2)$$

The main feature of the electric field given by Eq. (2) is the singularity for $k^2c^2=\omega^2$, that is, just the dispersion relation for the electromagnetic field *in vacuo*.[6]

The electric field in space and time is obtained from solution (2) by means of the inverse Fourier transform

$$\mathbf{E}(\mathbf{r},t)=\int\frac{d\omega}{2\pi}\int\frac{d\mathbf{k}}{(2\pi)^3}\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{E}(\mathbf{k},\omega), \quad (3)$$

where the integrals are over all ω (from $-\infty$ to ∞) and the whole $\mathbf{k}$ space. Let us consider the $\mathbf{k}$ integration first. The steps in carrying out such an integration involve: (i) expressing $\mathbf{j}(\mathbf{k},\omega)$ and $\rho(\mathbf{k},\omega)$, occurring in Eq. (2), by means of their spatial Fourier transforms; i.e.,

$$\{\mathbf{j}(\mathbf{k},\omega),\rho(\mathbf{k},\omega)\}=\int d\mathbf{r}'\;e^{-i\mathbf{k}\cdot\mathbf{r}'}\{\mathbf{j}(\mathbf{r}',\omega),\rho(\mathbf{r}',\omega)\}, \quad (4)$$

the exponential factor is combined with the corresponding one in Eq. (3) to yield $\exp\{i(\mathbf{k}\cdot\mathbf{R})\}$, where $\mathbf{R}\equiv\mathbf{r}-\mathbf{r}'$ denotes the vector from the source point $\mathbf{r}'$, at which the sources $\mathbf{j}$ and ρ are evaluated, to the field point $\mathbf{r}$, at which the field is evaluated; (ii) using spherical coordinates (k,ϑ,φ), such that $\mathbf{k}=k(\cos\varphi\sin\vartheta,\sin\varphi\sin\vartheta,\cos\vartheta)$ and $\mathbf{k}\cdot\mathbf{R}=kR\cos\vartheta$, the φ integration is trivial, and the $\cos\vartheta$ integration is carried out by parts; (iii) an integration over k, from $-\infty$ to ∞, with singularities connected with (simple) poles at $k=\pm(\omega/c)$ is carried out in the complex k plane along the closed path shown in Fig. 1.[8] The relevant two singular integrals are thus evaluated by means of the Cauchy integral theorem, with the result

$$\int_{-\infty}^{\infty}dk(k,k^2)\,\frac{e^{ikR}}{k^2c^2-\omega^2}=\frac{\pi i}{c^2}\left(1,\frac{\omega}{c}\right)e^{i(\omega/c)R}. \quad (5)$$

It is to be noted that the choice of the path of integration in Fig. 1 amounts to choosing the retarded solution of

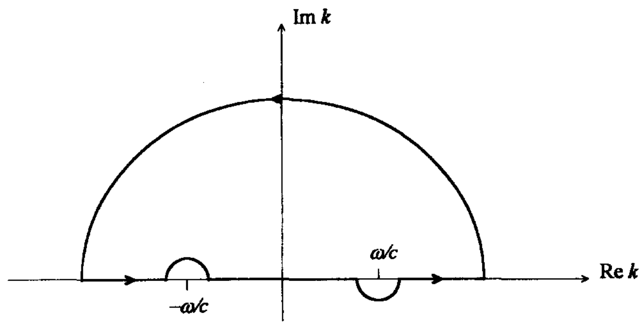

Fig. 1. Schematic of the closed contour of integration in the complex k-plane: the semicircle extends over the whole upper semiplane, and only the pole $k=\omega/c$ contributes to the contour integral.

d'Alembert's equation in the more conventional approach to electromagnetic theory.[1] As a result of steps (i)–(iii), the electric field (3) reduces to

$$\mathbf{E}(\mathbf{r},t)=\int d\mathbf{r}'\,\frac{1}{R}\int\frac{d\omega}{2\pi}\,e^{-i\omega(t-R/c)}\times\left\{\frac{i\omega}{c^2}\,\mathbf{j}(\mathbf{r}',\omega)-\left(\frac{i\omega}{c}-\frac{1}{R}\right)\rho(\mathbf{r}',\omega)\hat{\mathbf{R}}\right\}, \quad (6)$$

where $\mathbf{R}\equiv\mathbf{r}-\mathbf{r}'$ and $\hat{\mathbf{R}}\equiv\mathbf{R}/R$.

As for the magnetic field, from the Fourier transformed Faraday's law one has

$$\mathbf{B}(\mathbf{k},\omega)=\frac{c}{\omega}\,\mathbf{k}\times\mathbf{E}(\mathbf{k},\omega) \quad (7a)$$

$$=\frac{4\pi ic}{k^2c^2-\omega^2}\,\mathbf{k}\times\mathbf{j}(\mathbf{k},\omega), \quad (7b)$$

Eq. (7b) following from (7a) on using Eq. (2). The evaluation of the magnetic field as a function of space and time proceeds along the same lines as for the evaluation of the electric field, with the result

$$\mathbf{B}(\mathbf{r},t)=\int d\mathbf{r}'\,\frac{1}{cR}\int\frac{d\omega}{2\pi}\,e^{-i\omega(t-R/c)}\times\left(\frac{i\omega}{c}-\frac{1}{R}\right)\hat{\mathbf{R}}\times\mathbf{j}(\mathbf{r}',\omega). \quad (8)$$

The field equations (6) and (8) are the basic equations of this paper.

III. TIME-DEPENDENT GENERALIZED COULOMB AND BIOT–SAVART LAWS

To carry out the ω-integration in (6) and (8), $\mathbf{j}(\mathbf{r}',\omega)$ and $\rho(\mathbf{r}',\omega)$ are replaced by their temporal Fourier transform; i.e.,

$$\{\mathbf{j}(\mathbf{r}',\omega),\rho(\mathbf{r}',\omega)\}=\int dt'\;e^{i\omega t'}\{\mathbf{j}(\mathbf{r}',t'),\rho(\mathbf{r}',t')\}, \quad (9)$$

where t' is the time related to the source, distinct, in general, from the time t related to the field. Now, for the two terms of Eqs. (6) and (8) containing the factor $i\omega$, a straightforward integration over t' is available; the remaining ω integral is then just the integral representation of the Dirac δ function, namely,

$$\int \frac{d\omega}{2\pi} e^{i\omega(t'-t+R/c)} = \delta(t'-t+R/c),$$

with the consequence that the t' integration amounts to just taking the corresponding integrand at $t'=t-R/c$; i.e.,

$$t'=t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}, \tag{10}$$

referred to as the retarded time. One thus obtains, from Eq. (6),

$$\mathbf{E}(\mathbf{r},t)=\int d\mathbf{r}'\left(\frac{[\rho(\mathbf{r}',t')]}{R^2}\hat{\mathbf{R}}+\frac{1}{cR}\left[\frac{\partial\rho(\mathbf{r}',t')}{\partial t'}\right]\hat{\mathbf{R}} - \frac{1}{c^2R}\left[\frac{\partial\mathbf{j}(\mathbf{r}',t')}{\partial t'}\right]\right) \tag{11}$$

and, from Eq. (8)

$$\mathbf{B}(\mathbf{r},t)=\int d\mathbf{r}'\left(\frac{[\mathbf{j}(\mathbf{r}',t')]}{cR^2}+\frac{1}{c^2R}\left[\frac{\partial\mathbf{j}(\mathbf{r}',t')}{\partial t'}\right]\right)\times\hat{\mathbf{R}}, \tag{12}$$

where the square brackets mean that the quantity within is to be evaluated at the retarded time (10). Equations (11) and (12) express, respectively, the time-dependent generalized Coulomb and Biot-Savart laws.[2–4]

IV. THE FIELD OF AN ARBITRARILY MOVING POINT CHARGE

Here, we consider the specific case in which the source of the electromagnetic field is a point charge, but, rather than using directly the time-dependent, generalized Coulomb and Biot–Savart laws, i.e., Eqs. (11) and (12), respectively, to evaluate the corresponding fields,[3,4] we make use of Eqs. (6) and (8), for which the inverse Fourier transform with respect to ω is yet to be carried out, $\mathbf{j}(\mathbf{r}',\omega)$ and $\rho(\mathbf{r}',\omega)$ being given by (9) with

$$\mathbf{j}(\mathbf{r}',t')=\dot{\mathbf{r}}_q(t')\rho(\mathbf{r}',t')=q\dot{\mathbf{r}}_q(t')\delta(\mathbf{r}'-\mathbf{r}_q(t')), \tag{13}$$

where $\dot{\mathbf{r}}_q(t')\equiv d\mathbf{r}_q(t')/dt'$ is the velocity of the charge q, the instantaneous position of which is given by $\mathbf{r}_q(t')$. The presence of the δ function in Eq. (13) makes the $\mathbf{r}'$-integration in both Eqs. (6) and (8) straightforward. Furthermore, an integration by parts over t' of the terms proportional to $i\omega$ is readily carried out and the remaining ω integration produces the Dirac δ function:

$$\delta\left(t'-t+\frac{R(t')}{c}\right)=\frac{\delta(t'-t'(t))}{g(t')}, \tag{14a}$$

$$R(t')\equiv|\mathbf{r}-\mathbf{r}_q(t')|, \tag{14b}$$

$$g(t')\equiv 1+\dot{R}(t')/c=1-\hat{\mathbf{R}}(t')\cdot\boldsymbol{\beta}(t'), \tag{14c}$$

where $\boldsymbol{\beta}(t')\equiv\dot{\mathbf{r}}_q(t')/c$ and the retarded time $t'(t)$ is the solution of the equation

$$t'-t+\frac{|\mathbf{r}-\mathbf{r}_q(t')|}{c}=0. \tag{15}$$

Finally, making use of δ function (14a) to carry out the t' integration yields

$$\mathbf{E}(\mathbf{r},t)=q\left(\left[\frac{\hat{\mathbf{R}}(t')}{g(t')R^2(t')}\right]+\frac{1}{c}\left[\frac{1}{g(t')}\frac{d}{dt'}\frac{\hat{\mathbf{R}}(t')-\boldsymbol{\beta}(t')}{g(t')R(t')}\right]\right), \tag{16}$$

$$\mathbf{B}(\mathbf{r},t)=-q\left(\left[\frac{\hat{\mathbf{R}}(t')\times\boldsymbol{\beta}(t')}{g(t')R^2(t')}\right]+\frac{1}{c}\left[\frac{1}{g(t')}\frac{d}{dt'}\frac{\hat{\mathbf{R}}(t')\times\boldsymbol{\beta}(t')}{g(t')R(t')}\right]\right), \tag{17}$$

with square brackets denoting retarded values. The novel derivation of the fields (16) and (17) given here appears somewhat more straightforward than the standard one based on the use of the Liénard–Wiechert potentials.[1,9] From Eqs. (16) and (17) one can obtain the fields both in the Heaviside–Feynman and Liénard–Wiechert form.[9]

V. SUMMARY

In summary, on the basis of the solution of the Fourier transformed inhomogeneous wave equation, two basic field equations, namely Eqs. (6) and (8), are obtained, which allow a straightforward derivation of the time-dependent generalized Coulomb and Biot–Savart laws. Also, in the context of the derivation of the fields in both the Heaviside–Feynman and Liénard–Wiechert form, the intermediate field equations (16) and (17) are obtained in a more direct manner than the standard one based on the use of the Liénard–Wiechert potentials.

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