Thus, only by clearly specifying the geometry used can the experimental results (as for instance numbers such as those given by Young[5]) be correctly interpreted.[11]

For the case of $n_2 < n_1$, $i > i_c$, since the reflected beam is elliptically polarized, a quarter wave plate may be interposed between the reflecting surface and the analyzer. Its "fast" axis should be at the azimuth of the incident plane polarized beam so that the ratio of the axes of the ellipse can be determined from the position for minimum intensity through the analyzer.[12] This result can be checked against Eq. (5).

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The Energy and Momentum of a Classical Electron in Special Relativity

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An equation for the energy of a "classical" electron is derived from Maxwell's equations. The reason for carefully specifying the time at which certain integrations are performed is established. The momentum of the electron is treated in a similar manner. The results are in complete agreement with the special theory of relativity.

1. INTRODUCTION

A "classical" electron is a charged spherical shell in empty space. As such, it constitutes a very simple system. In spite of its simplicity, however, the classical electron has received much attention. An inconsistency seems to arise if the energy and momentum are first calculated in a primed coordinate system in which the electron is at rest and then calculated *ab initio* in an unprimed coordinate system relative to which the electron is in motion with constant velocity. Since momentum and energy form a four-vector, we know how the energy and momentum should transform when we use the Lorentz transformation to transfer from a primed coordinate system to an unprimed system. The paradox is that the two sets of results, when apparently calculated correctly, are not related properly according to the special theory of relativity.

A number of different explanations have been given for the inconsistency. The simplest is that the mass of a classical electron cannot be entirely electromagnetic. This explanation is not very convincing, however, because there appears no good reason why electromagnetic momentum and energy, if correctly calculated, should not transform properly. A more involved approach is to assume that the spherical shell is under stress because of the charge. These (Poincare[1]) stresses or pressures necessary to provide stability happen to transform in such a way as to make the total energy and momentum transform properly. The question still remains, though, as to why electromagnetic momentum and energy do not transform as expected. A third way of resolving the paradox is to bring in a new effect. This was done, for example, by Zink,[2] who talks about induction effects. Here, again, the explanation seems to be artificial. A fourth approach is to redefine[3] electromagnetic momentum and energy in such a way that no inconsistency appears when the calculations are performed in two different reference frames in motion relative to one another. An immediate reaction is to question whether it is really necessary to abandon the usual definitions. A fifth approach is to examine the calculations very carefully to correct any possible errors. This method has been followed by Rohrlich[4] among others.[5] Clearly, this last way of removing any inconsistencies is very attractive. The only problem is that the mathematics used is not sufficiently simple to be within the reach of an undergraduate student. The underlying physics thus becomes somewhat obscured.

Accordingly, the purpose of this paper is to start with Maxwell's equations and derive an expression for the energy of a classical electron. The reason for clearly specifying the time at which certain integrations are performed will be brought out. The momentum of the electron will then be treated in a similar manner. Hopefully, the presentation will have pedagogical advantages.

2. ENERGY OF A CLASSICAL ELECTRON

Poynting's Theorem

By starting with Maxwell's equations and performing the requisite manipulations, we can derive Poynting's theorem[6] in integral form for any system of charges in free space, viz.,

$$-\iiint \mathbf{E}\cdot\mathbf{J}dv=(d/dt)\iiint(\tfrac{1}{2}\epsilon_0\mathbf{E}^2+\tfrac{1}{2}\mu_0\mathbf{H}^2)dv+\iiint\nabla\cdot(\mathbf{E}\times\mathbf{H})dv, \quad (1)$$

where the integrations extend over the volume of the system. The left-hand side term, if positive, represents the work or the heat *supplied* per second by an external agency;

$$(d/dt)\iiint(\tfrac{1}{2}\epsilon_0\mathbf{E}^2+\tfrac{1}{2}\mu_0\mathbf{H}^2)dv$$

is usually interpreted as the rate of increase of the total electromagnetic field energy, and since

$$\iiint\nabla\cdot(\mathbf{E}\times\mathbf{H})dv \quad \text{equals} \quad \oiint(\mathbf{E}\times\mathbf{H})\cdot d\mathbf{A}$$

according to the divergence theorem, we can think of the last term on the right-hand side of Eq. (1) as the total energy that leaves the system as a flux across the bounding surfaces of the system. This view of Poynting's theorem is very adequate for a bounded system that interacts with its surroundings.

On the other hand, if the system consists of an isolated configuration of free charge that is static in empty space relative to a *primed* reference frame, we must use a somewhat different approach. In such a case, since the system is not being supplied with energy from an external source, we must conclude that

$$-\iiint\mathbf{E}\cdot\mathbf{J}dv=0.$$

Hence, Eq. (1) for an isolated system of unaccelerated charges becomes

$$0=(d/dt)\iiint(\tfrac{1}{2}\epsilon_0\mathbf{E}^2+\tfrac{1}{2}\mu_0\mathbf{H}^2)dv+\iiint\nabla\cdot(\mathbf{E}\times\mathbf{H})dv. \quad (2)$$

Radiation and Velocity Fields

We now consider Eq. (2). The term

$$\iiint\nabla\cdot(\mathbf{E}\times\mathbf{H})dv$$

cannot represent an outflow of radiant energy because **E** and **H** are essentially Coulomb (inverse-square) fields, which extend to infinity. The system is bounded only by the surface of the electron. We shall call the electromagnetic field of a charge configuration moving with constant velocity a velocity field. An electromagnetic velocity field is distinguished from a radiation field by the fact that the velocity field is "attached" to the charge configuration. In fact, it is physically meaningless to consider a charge without its velocity field. Neither can be observed without the other. A Coulomb field must have a source, and a charge configuration must have a Coulomb field. A true radiation field, on the other hand, detaches itself from its source and assumes an independent existence. In this paper, we shall be concerned only with the velocity field of a charge configuration.

Conservation of Energy

To exploit the idea that $\iiint \nabla\cdot(\mathbf{E}\times\mathbf{H})dv$ represents a rate of change of part of the electromagnetic field energy, we write Eq. (2) as follows:

$$0=\frac{d}{dt}\left[\iiint (\tfrac{1}{2}\epsilon_0\mathbf{E}^2+\tfrac{1}{2}\mu_0\mathbf{H}^2)dv + \int_{t_0}^{t}\iiint \nabla\cdot(\mathbf{E}\times\mathbf{H})dvdt\right]. \quad (3)$$

Equation (3) implies that the total energy of the isolated system is conserved and that the electromagnetic field energy U may be defined as follows:

$$U=\iiint (\tfrac{1}{2}\epsilon_0\mathbf{E}^2+\tfrac{1}{2}\mu_0\mathbf{H}^2)dv + \int_{t_0}^{t}\iiint \nabla\cdot(\mathbf{E}\times\mathbf{H})dvdt. \quad (4)$$

The first term on the right-hand side of Eq. (4) represents an increase in energy due to the presence of the charge configuration in previously empty space. To be consistent, the lower limit t_0 in the second term on the right-hand side of Eq. (4) must be a time prior to the existence of the charge configuration and the upper limit t must be an arbitrary time subsequent to the introduction of the charge configuration somewhere in space–time. The quantity U will be called the total "attached" electromagnetic field energy of the charge configuration.

Simplification of the Quadruple Integral

To simplify the quadruple integral in Eq. (4), we shall first write it in four-dimensional notation as follows:

$$(ic)^{-1}\int_{x_{40}}^{x_4}\iiint \nabla\cdot(\mathbf{E}\times\mathbf{H})dxdydzdx_4,$$

where $x_4=ict$. We now note that

$$dxdydzdx_4=J(x,y,z,x_4/x',y',z',x_4')\,dx'dy'dz'dx_4',$$

where the primed reference frame is the rest system of the charge configuration. The Lorentz transformation equations are:

$$x=\gamma x'+(\beta\gamma/i)x_4', \quad (5)$$

$$y=y', \qquad z=z', \quad (6)$$

$$x_4=-(\beta\gamma/i)x'+\gamma x_4', \quad (7)$$

and hence,

$$J(x,y,z,x_4/x',y',z',x_4')=1,$$

$$dxdydzdx_4=dx'dy'dz'dx_4'. \quad (8)$$

Thus, since

$$\nabla\cdot(\mathbf{E}\times\mathbf{H})=\nabla\cdot\mathbf{S} = (\partial S_x/\partial x)+(\partial S_y/\partial y)+(\partial S_z/\partial z),$$

the quadruple integral may be written as follows:

$$(ic)^{-1}\int_{x_{40}}^{x_4}\iiint\nabla\cdot(\mathbf{E}\times\mathbf{H})dxdydzdx_4 = (ic)^{-1}\int_{x_{40}'}^{x_4'}\iiint\left(\frac{\partial S_x}{\partial x}+\frac{\partial S_y}{\partial y}+\frac{\partial S_z}{\partial z}\right)\times dx'dy'dz'dx_4'. \quad (9)$$

We continue by replacing each of the derivatives in the integrand on the right-hand side of Eq. (9) by appropriate derivatives with respect to primed variables. For example,

$$\frac{\partial S_x}{\partial x}=\frac{\partial S_x}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial S_x}{\partial y'}\frac{\partial y'}{\partial x}+\frac{\partial S_x}{\partial z'}\frac{\partial z'}{\partial x}+\frac{\partial S_x}{\partial x_4'}\frac{\partial x_4'}{\partial x}.$$

The right-hand side of the last equation may be simplified by using the inverse of Eqs. (5)–(7), i.e.,

$$x'=\gamma x-(\beta\gamma/i)x_4, \quad (10)$$

$$y'=y, \qquad z'=z, \quad (11)$$

$$x_4'=(\beta\gamma/i)x+\gamma x_4. \quad (12)$$

Substituting for $\partial x'/\partial x$, $\partial y'/\partial x$, $\partial z'/\partial x$, and $\partial x_4'/\partial x$, we obtain

$$\partial S_x/\partial x = \gamma(\partial S_x/\partial x') + (\beta\gamma/i)(\partial S_x/\partial x_4'). \quad (13)$$

Using a similar procedure for S_y and S_z, we get

$$\partial S_y/\partial y = \partial S_y/\partial y',$$
$$\partial S_z/\partial z = \partial S_z/\partial z'. \quad (14)$$

Hence, the quadruple integral in Eq. (9) may be written

$$(ic)^{-1}\int_{x_{40}}^{x_4}\iiint \nabla\cdot(\mathbf{E}\times\mathbf{H})\,dxdydzdx_4$$
$$=(ic)^{-1}\int_{x_{40}'}^{x_4'}\iiint\left(\gamma\frac{\partial S_x}{\partial x}+\frac{\beta\gamma}{i}\frac{\partial S_x}{\partial x_4'}+\frac{\partial S_y}{\partial y'}+\frac{\partial S_z}{\partial z'}\right)\times dx'dy'dz'dx_4'. \quad (15)$$

Further Simplification for the Case of the Classical Electron

A classical electron consists of a charge e uniformly distributed over a spherical surface of radius R when observed in the rest frame (primed coordinate system) of the electron. We wish to evaluate the energy U for this special charge configuration. In the primed coordinate system, if the center of the electron is at the origin, the electromagnetic field of the electron is given by

$$E_r' = e/4\pi\epsilon_0 r^2, \qquad 0 = E_\theta' = E_\phi',$$
$$0 = H_r' = H_\theta' = H_\phi', \quad (16)$$

in spherical coordinates, where r is the distance from the origin, θ is the colatitude angle, and ϕ is the longitude angle. In an unprimed reference frame, the electromagnetic field may be expressed[7] in terms of primed values as follows:

$$E_x = E_x' = E_r'\sin\theta\cos\phi, \quad (17)$$

$$E_y = \gamma E_y' = \gamma E_r'\sin\theta\sin\phi, \quad (18)$$

$$E_z = \gamma E_z' = \gamma E_r'\cos\theta, \quad (19)$$

$$H_x = 0, \quad (20)$$

$$H_y = -\gamma\epsilon_0 V E_z' = -\gamma\epsilon_0 V E_r'\cos\theta, \quad (21)$$

$$H_z = \gamma\epsilon_0 V E_y' = \gamma\epsilon_0 V E_r'\sin\theta\sin\phi. \quad (22)$$

Consequently,

$$S_x = E_y H_z - E_z H_y$$
$$= \gamma^2\epsilon_0 V E_r'^2(\sin^2\theta\sin^2\phi + \cos^2\theta), \quad (23)$$

$$S_y = E_z H_x - E_x H_z$$
$$= -\gamma\epsilon_0 V E_r'^2\sin^2\theta\sin\phi\cos\phi, \quad (24)$$

$$S_z = E_x H_y - E_y H_x$$
$$= -\gamma\epsilon_0 V E_r'^2\sin\theta\cos\theta\cos\phi. \quad (25)$$

We now find the value of the quadruple integral in Eq. (15) in a straightforward manner.

$$\frac{\partial S_x}{\partial x'} = \frac{\partial S_x}{\partial r}\frac{\partial r}{\partial x'} + \frac{\partial S_x}{\partial \theta}\frac{\partial \theta}{\partial x'} + \frac{\partial S_x}{\partial \phi}\frac{\partial \phi}{\partial x'};$$

since

$$r = (x'^2 + y'^2 + z'^2)^{1/2},$$
$$\partial r/\partial x' = x'/r = \sin\theta\cos\phi,$$
$$\cos\theta = z'/r,$$
$$\partial\theta/\partial x' = \cos\theta\cos\phi/r,$$
$$\tan\phi = y'/x',$$
$$\partial\phi/\partial x' = -\sin\phi/r\sin\theta,$$

we have

$$\frac{\partial S_x}{\partial x'} = \frac{\partial S_x}{\partial r}\sin\theta\cos\phi + \frac{\partial S_x}{\partial\theta}\frac{\cos\theta\cos\phi}{r} + \frac{\partial S_x}{\partial\phi}\left(-\frac{\sin\phi}{r\sin\theta}\right). \quad (26)$$

Remembering that E_r' is a function of r only, we can obtain the values of $\partial S_x/\partial r$, $\partial S_x/\partial\theta$, and $\partial S_x/\partial\phi$ by using Eq. (23). To perform the integrations required in Eq. (15), we recall that

$$dx'dy'dz' = r^2\sin\theta d\theta d\phi dr. \quad (27)$$

Since the limits for the ϕ integrations are 0 and 2π, we get

$$\int_R^\infty\int_0^{2\pi}\int_0^\pi \gamma\frac{\partial S_x}{\partial x'}r^2\sin\theta d\theta d\phi dr = 0.$$

In a similar manner, we find that

$$\int_R^\infty \int_0^{2\pi} \int_0^\pi \left(\frac{\partial S_y}{\partial y'} + \frac{\partial S_z}{\partial z'}\right) r^2 \sin\theta d\theta d\phi dr = 0.$$

Thus, the only integral in Eq. (15) that we need to consider is

$$(ic)^{-1} \int_{x_{40}'}^{x_4'} \iiint \frac{\beta\gamma}{i} \frac{\partial S_x}{\partial x_4'} dx'dy'dz'dx_4'.$$

We integrate first with respect to x_4'. At the lower limit x_{40}', the electron has not yet been introduced into the otherwise completely empty space. Hence, S_x is zero at x_{40}'. At the upper limit, S_x has the value given by Eq. (23). Consequently, the quadruple integral has the value

$$\iiint (-\beta\gamma/c) S_x dx'dy'dz' = \iiint (-\mathbf{V}\cdot\mathbf{S}/c^2) dx'dy'dz'.$$

It is very important to note that S_x in the latter integral may assume only those values that are consistent with the condition that $t' = x_4'/ic$ is constant. In an unprimed coordinate system, S_x is a function of x, y, z, and x_4, but, according to Eqs. (5) and (7), if x_4' is constant, x and x_4 are not independent. Of course, when expressed in primed coordinates, S_x is a function of x', y', and z' only, and there is no difficulty in imposing the condition that x_4' be held constant.

To be consistent, when the energy U is calculated from Eq. (4), the triple integral,

$$\iiint (\tfrac{1}{2}\epsilon_0 \mathbf{E}^2 + \tfrac{1}{2}\mu_0 \mathbf{H}^2) dv,$$

should also be evaluated at constant x_4'. Otherwise, the meaning of U would be obscure. Recalling that at constant x_4'

$$\begin{aligned} dxdydz &= J(x, y, z/x', y', z') dx'dy'dz' \\ &= \gamma dx'dy'dz', \end{aligned} \tag{28}$$

we may write Eq. (4) as follows:

$$U = \gamma \iiint [\tfrac{1}{2}\epsilon_0 \mathbf{E}^2 + \tfrac{1}{2}\mu_0 \mathbf{H}^2 - (\mathbf{V}\cdot\mathbf{S}/c^2)] dx'dy'dz'. \tag{29}$$

Basically, U is the total electromagnetic energy that is "attached" to a moving electron. Rohrlich[8] derives Eq. (29) in a different way.

Calculation of the Energy U

We now wish to complete the calculation of the energy U of a classical electron. First, we wish to note that in the primed reference frame, because $\mathbf{H}' = 0$ and $\mathbf{V} = 0$, Eq. (29) reduces to

$$U' = \iiint \tfrac{1}{2}\epsilon_0 E_r'^2 dv'. \tag{30}$$

Using Eq. (16), we find that

$$\begin{aligned} U' &= \int_R^\infty \int_0^{2\pi} \int_0^\pi \tfrac{1}{2}\epsilon_0 \left(\frac{e^2}{16\pi^2\epsilon_0^2 r^4}\right) r^2 \sin\theta d\theta d\phi dr \\ &= e^2/8\pi\epsilon_0 R. \end{aligned} \tag{31}$$

If we wish, we may write

$$U' = m_0 c^2, \tag{32}$$

where m_0 is the rest mass corresponding to U', and hence,

$$m_0 = e^2/8\pi\epsilon_0 c^2 R. \tag{33}$$

To find the energy U in an unprimed reference frame, we use Eq. (29). For convenience, we express all quantities in terms of primed coordinates. Equations (17)–(22) give the equivalents of the components of $\mathbf{E}$ and $\mathbf{H}$. By making the proper substitutions, we obtain

$$\begin{aligned} \tfrac{1}{2}\epsilon_0 \mathbf{E}^2 + \tfrac{1}{2}\mu_0 \mathbf{H}^2 &= \tfrac{1}{2}\epsilon_0 E_x'^2 + \tfrac{1}{2}\gamma^2(1+\beta^2)\epsilon_0(E_y'^2 + E_z'^2), \\ -\mathbf{V}\cdot\mathbf{S}/c^2 &= -\gamma^2\beta^2\epsilon_0(E_y'^2 + E_z'^2), \end{aligned}$$

and Eq. (29) becomes

$$\begin{aligned} U &= \gamma \iiint [\tfrac{1}{2}\epsilon_0 E_x'^2 + \tfrac{1}{2}\gamma^2(1-\beta^2) \\ &\qquad \times \epsilon_0(E_y'^2 + E_z'^2)] dx'dy'dz' \\ &= \gamma \iiint \tfrac{1}{2}\epsilon_0(E_x'^2 + E_y'^2 + E_z'^2) dx'dy'dz' \\ &= \gamma \iiint \tfrac{1}{2}\epsilon_0 E_r'^2 r^2 \sin\theta d\theta d\phi dr \\ &= \gamma m_0 c^2 \\ &= \frac{m_0 c^2}{[1-(V^2/c^2)]^{1/2}}. \end{aligned} \tag{34}$$

Comparing U with U', we can say that the result expressed in Eq. (34) is precisely what one would expect on the basis of the special theory of relativity.

3. MOMENTUM OF A CLASSICAL ELECTRON

Conservation of Momentum

We now wish to derive an expression for the total electromagnetic momentum of a non-radiating system. By starting with

$$\iiint(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B})dv=d\mathbf{p}_{\text{mech}}/dt \qquad (35)$$

and using Maxwell's equations, we can show[9] that

$$\oint[T][dA]=(d\mathbf{p}_{\text{mech}}/dt)+(d/dt)\iiint[(\mathbf{E}\times\mathbf{H})/c^2]dv, \qquad (36)$$

where $[T]$ is the matrix of the Maxwell stress tensor. When dealing with four-dimensional space manifolds, it is more convenient to manipulate volume integrals than surface integrals. We therefore convert the surface integral on the left-hand side of Eq. (36) to a volume integral by means of the divergence theorem, namely,

$$\oint[T][dA]=\iiint[\nabla][T]dv,$$

where $[\nabla]$ is the row vector $[\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z]$. Equation (36) can now be written

$$0=\frac{d}{dt}\left(\mathbf{p}_{\text{mech}}+\iiint\frac{\mathbf{E}\times\mathbf{H}}{c^2}dv-\int_{t_0}^{t}\iiint[\nabla][T]dvdt\right). \qquad (37)$$

We note that Eq. (37) suggests that the momentum of an isolated system is conserved provided we interpret

$$\int_{t_0}^{t}\iiint[\nabla][T]dvdt$$

as part of the total electromagnetic momentum. Accordingly, we shall define the total electromagnetic momentum of an isolated system to be

$$\mathbf{G}=\iiint\frac{\mathbf{E}\times\mathbf{H}}{c^2}dv-\int_{t_0}^{t}\iiint[\nabla][T]dvdt. \qquad (38)$$

In four-dimensional notation, Eq. (38) is

$$\mathbf{G}=\iiint\frac{\mathbf{E}\times\mathbf{H}}{c^2}dxdydz-(ic)^{-1}\times\int_{x_{40}}^{x_4}\iiint[\nabla][T]dxdydzdx_4. \qquad (39)$$

Simplification of the Quadruple Integral

As in the case of the energy U of a classical electron, we wish to simplify the quadruple integral in Eq. (39). We note immediately that we may substitute $dx'dy'dz'dx_4'$ for $dxdydzdx_4$. Hence, considering G_x for simplicity, we may write

$$G_x=\iiint\frac{(\mathbf{E}\times\mathbf{H})_x}{c^2}dxdydz-(ic)^{-1}\times\int_{x_{40}'}^{x_4'}\iiint\left(\frac{\partial T^{11}}{\partial x}+\frac{\partial T^{21}}{\partial y}+\frac{\partial T^{31}}{\partial z}\right)dx'dy'dz'dx_4', \qquad (40)$$

where

$$T^{11}=\epsilon_0E_x^2+\mu_0H_x^2-(\tfrac{1}{2}\epsilon_0\mathbf{E}^2+\tfrac{1}{2}\mu_0\mathbf{H}^2),$$
$$T^{21}=T^{12}=\epsilon_0E_xE_y+\mu_0H_xH_y,$$
$$T^{31}=T^{13}=\epsilon_0E_xE_z+\mu_0H_xH_z.$$

Proceeding in a manner similar to the derivations of Eqs. (13) and (14), we find that

$$\frac{\partial T^{11}}{\partial x}+\frac{\partial T^{21}}{\partial y}+\frac{\partial T^{31}}{\partial z}=\gamma\frac{\partial T^{11}}{\partial x'}+\frac{\beta\gamma}{i}\frac{\partial T^{11}}{\partial x_4'}+\frac{\partial T^{21}}{\partial y'}+\frac{\partial T^{31}}{\partial z'}. \qquad (41)$$

We now substitute primed values for T^{11}, T^{21}, and T^{31} by using Eqs. (17) through (22), which apply, of course, to a "classical" electron. We find that

$$T^{11}=\tfrac{1}{2}\epsilon_0[E_x'^2-\gamma^2(1+\beta^2)(E_y'^2+E_z'^2)]=\tfrac{1}{2}\epsilon_0E_r'^2[\sin^2\theta\cos^2\phi-\gamma^2(1+\beta^2)\times(\sin^2\theta\sin^2\phi+\cos^2\theta)], \qquad (42)$$

$$T^{21}=\gamma\epsilon_0E_r'^2\sin^2\theta\sin\phi\cos\phi, \qquad (43)$$

$$T^{31}=\gamma\epsilon_0E_r'^2\sin\theta\cos\theta\cos\phi. \qquad (44)$$

The next step is to express $\partial T^{11}/\partial x'$, $\partial T^{21}/\partial y'$, and $\partial T^{31}/\partial z'$ in spherical coordinates and integrate with respect to $dx'dy'dz' = r^2 \sin\theta d\theta d\phi dr$. The outcome of the integration with respect to ϕ between the limits 0 and 2π is that the integrals of the three space derivatives are zero. Thus, the quadruple integral in Eq. (40) has been reduced to

$$\int_{x_{40}'}^{x_4'} \iiint \frac{\beta\gamma}{i} \frac{\partial T^{11}}{\partial x_4'} dx'dy'dz'dx_4'.$$

We integrate first with respect to x_4' and recall that T^{11} at x_{40}' is zero and at x_4' is given by Eq. (42). Consequently, Eq. (40) becomes

$$G_x = \iiint [(\mathbf{E} \times \mathbf{H})_x / c^2] dxdydz + (\beta\gamma/c) \iiint T^{11} dx'dy'dz'. \quad (45)$$

It is very important to note that T^{11} in Eq. (45) may assume only those values that are in agreement with the condition that $t' = x_4'/ic$ is constant. To be consistent, we now impose the same condition on the first integral on the right-hand side of Eq. (45). Otherwise, the meaning of G_x would be obscure. Thus, since

$$dxdydz = \gamma dx'dy'dz'$$

for constant x_4', we have

$$G_x = \gamma \iiint \{ [(\mathbf{E} \times \mathbf{H})_x / c^2] + (VT^{11}/c^2) \} dx'dy'dz'. \quad (46)$$

Equation (46) agrees with the result given by Rohrlich.[10]

Calculation of G

We now proceed to compute the momentum of the electromagnetic field "attached" to a classical electron. We express the integrand in Eq. (46) in terms of primed values and find that

$$(\mathbf{E} \times \mathbf{H})_x + VT^{11} = \tfrac{1}{2}\epsilon_0 V E_r'^2.$$

Therefore

$$G_x = \gamma \int_R^\infty \int_0^{2\pi} \int_0^\pi \frac{1}{2} \left(\frac{\epsilon_0 V}{c^2} \right) E_r'^2 r^2 \sin\theta d\theta d\phi dr$$
$$= \gamma (V/c^2)(m_0 c^2)$$
$$= \frac{m_0 V}{[1 - (V^2) c^2]^{1/2}}. \quad (47)$$

Clearly, Eq. (47) is precisely what one would expect on the basis of the special theory of relativity.

A similar calculation shows that G_y and G_z are zero. This is exactly what one would anticipate for an electron moving along the x axis.

4. SUMMARY

Evidently, the calculations for U and $\mathbf{G}$ verify the accuracy of our assumption that U and $\mathbf{G}$ be consistently computed at constant time t'. From the standpoint of the special theory of relativity, U and $\mathbf{G}$ form a four-vector given by $(\mathbf{G}, iU/c)$. This same four-vector is given by $(0, iU'/c)$ in the primed reference frame. In fact

$$\mathbf{G}^2 - U^2/c^2 = \gamma^2 m_0^2 V^2 - \gamma^2 m_0^2 c^2$$
$$= -m_0^2 c^2$$
$$= -U'^2/c^2.$$

Certainly, if we wish to compute a four-vector, we must be consistent with regard to the time at which the calculation is performed.

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[8] F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965), p. 132.
[9] Reference 6, p. 182.
[10] Reference 8, p. 132.