Waves in a Plasma in a Magnetic Field

IRA B. BERNSTEIN
Project Matterhorn, Princeton University, Princeton, New Jersey
(Received September 20, 1956; revised manuscript received May 27, 1957)

The small oscillations of a fully ionized plasma, in which collisions are negligible, in a constant external magnetic field, is treated by the Laplace transform method. The full set of Maxwell equations is employed and the ion dynamics are included. Various limiting cases are considered. It is shown that self-excitation of waves around thermal equilibrium is impossible. It is also demonstrated that for longitudinal electron oscillations propagating perpendicular to the constant magnetic field, there are gaps in the spectrum of allowed frequencies at multiples of the electron gyration frequency, but zero Landau damping. These particular waves are also associated with a nonuniformity of convergence in the limit of vanishing magnetic field which phenomenon, however, is of no physical consequence. When the ion dynamics are included, two classes of low frequency oscillations are found, the existence of both of which has been predicted by the simple hydrodynamic theory, namely longitudinal ion waves, and transverse hydromagnetic waves. The well known results for the propagation of electromagnetic waves in an ionized atmosphere are also recovered, as well as the effects on such waves in various limiting cases of the magnetic field and particle motion. These calculations indicate that in many cases the transport equations are capable of yielding correct results, apart from such things as Landau damping, for a wide class of waves in a collision-free plasma.

1. INTRODUCTION

THIS work treats the small amplitude oscillations of a fully ionized, quasi-neutral plasma in a uniform, externally produced magnetic field. The ion and electron distribution functions are assumed to depart only slightly from the appropriate zeroth order distribution, which here is taken to be the Maxwell distribution, though the general method is not contingent on this latter assumption. It is further assumed that collisions are negligible. The technique employed is the Laplace transformation first introduced in this context by Landau.[1] This avoids the mathematical difficulties inherent in a simple substitution analysis which assumes exponential time dependence.

The problem of the longitudinal electron oscillations has been treated by Gordeyev.[2] He purported to find self-excitation of waves around thermal equilibrium for a certain regime of plasma parameters. It is shown in Sec. 5a, directly from his dispersion relation, that this is not possible. In Appendix I, it is demonstrated in a proof due to William Newcomb, that within the confines of a collision-free theory, the absence of a mechanism for the degradation of energy implies that the entropy of the system is a constant of the motion, which in turn implies that, in general, there can be no solutions of the linearized equations of motion around thermal equilibrium which increase exponentially in time. Gordeyev's errors are pointed out.

The problem of electron oscillations propagating perpendicular to the uniform external magnetic field has been treated by Gross,[3] and by Sen.[4] The former of these authors purported to find Landau damping of waves whose wavelength is less than the Debye length. It is shown that this conclusion is in error; such waves are undamped. The error, furthermore, vitiates the various limiting results Gross reports. This case is properly treated, and Gross' error indicated, in Sec. 5b.

These perpendicular waves exhibit a curious nonuniform behavior in the limit of vanishing magnetic field. That is, if one first lets the magnetic field go to zero, and then passes to the case of perpendicular propagation, one recovers the results of Landau, which indicate damping. If, however, one first considers perpendicular propagation, and then passes to the limit of vanishing magnetic field, the spectrum of frequencies is undamped, and composed of all the harmonics of the electron gyration frequency. This apparent difficulty is immediately resolved when one observes that the anomalous waves, as one lets the field pass to zero, are confined to an angular region around the direction of perpendicular propagation which itself goes to zero with the magnetic field. That is, mathematically, they form in the limit a set of measure zero. Physically they are effectively nonexistent in the limit.

The development of this work is as follows: Section 2 presents the general formulation of the linearized problem, including arbitrary electromagnetic fields and the ion dynamics. Section 3 discusses the inversion of the Laplace transform. In Sec. 4 the limits of vanishing external magnetic field, and infinite light velocity are treated. Section 5 considers the problem of longitudinal electron oscillations. Section 6 treats the problem of longitudinal oscillations including the ion dynamics, and derives the ion waves. Section 7 considers the general problem, and derives results in the limit of weak magnetic field, and low temperatures. Hydromagnetic waves are also obtained in the appropriate limit. Appendix I presents the general argument of Newcomb which disproves the existence around thermal equilibrium of time increasing solutions. Appendix II

[1] L. D. Landau, J. Phys. U.S.S.R. **10**, 25 (1946).
[2] G. V. Gordeyev, J. Exptl. Theoret. Phys. U.S.S.R. **6**, 660 (1952).
[3] E. P. Gross, Phys. Rev. **82**, 232 (1951).
[4] H. K. Sen, Phys. Rev. **88**, 816 (1952).

evaluates an integral which is characteristic of the theory.

2. LINEARIZED PROBLEM

Let $f(\mathbf{r},\mathbf{v},t)$ be the distribution function for a given kind of particle, ion or electron. The number of particles whose positions lie between $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$, and whose velocities are in the range between $\mathbf{v}$ and $\mathbf{v}+d\mathbf{v}$ is then $fd^3\mathbf{r}\,d^3\mathbf{v}$. The distribution function satisfies the Boltzmann equation[5]

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}\pm\frac{Ze}{m}\left(\mathfrak{E}+\frac{1}{c}\mathbf{v}\times\mathfrak{B}\right)\cdot\frac{\partial f}{\partial \mathbf{v}}=\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}. \quad (1)$$

In Eq. (1) $e=|e|$ represents the electronic charge, Z the charge number, m the particle mass, $\mathfrak{E}$ the electric field intensity, and $\mathfrak{B}$ the magnetic field intensity. The upper sign is to be taken for positively charged particles, the lower for electrons. The term $(\partial f/\partial t)_{\text{coll}}$ represents the rate of change of the distribution function arising from collisions, that is, interparticle interactions whose range is short compared with the distance over which f changes. In what follows it will be assumed negligible, though its effect via a simple relaxation term could be included readily.

Consider systems which depart only slightly from thermal equilibrium in a uniform external magnetic field $\mathfrak{B}_0$. Then one can write

$$f(\mathbf{r},\mathbf{v},t)=f_0(v)+f_1(\mathbf{r},\mathbf{v},t),$$
$$\mathfrak{B}(\mathbf{r},t)=\mathfrak{B}_0+\mathfrak{B}_1(\mathbf{r},t),$$
$$\mathfrak{E}(\mathbf{r},t)=0+\mathfrak{E}_1(\mathbf{r},t),$$

where f_0 is the Maxwell distribution

$$f_0(v)=N\left(\frac{m}{2\pi KT}\right)^{\frac{3}{2}}\exp\left[-\frac{mv^2}{2KT}\right], \quad (2)$$

N the particle density, K Boltzmann's constant, and the subscript 1 refers to a small perturbation of the associated quantity. Then to first order Eq. (1) reads

$$\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{r}}\pm\frac{Ze}{m}\mathfrak{E}_1\cdot\frac{\partial f_0}{\partial \mathbf{v}}\mp\frac{Ze}{mc}\mathfrak{B}_0\cdot\mathbf{v}\times\frac{\partial f_1}{\partial \mathbf{v}}=0. \quad (3)$$

It is convenient to make a Fourier analysis in space,[6] and take Laplace transforms in time,[7] that is, write

$$f_1(\mathbf{r},\mathbf{v},t)=\int d^3\mathbf{v}\, e^{i\mathbf{k}\cdot\mathbf{r}}F^\star(\mathbf{k},\mathbf{v},t), \quad (4)$$

$$F(\mathbf{k},\mathbf{v},s)=\int_0^\infty dt\, e^{-st}F^\star(\mathbf{k},\mathbf{v},t), \quad (5)$$

with similar expressions for the perturbation electric and magnetic fields. In terms of the Fourier-Laplace transforms, Eq. (3) reads

$$(s+i\mathbf{k}\cdot\mathbf{v})F\mp\frac{Ze}{mc}\mathfrak{B}_0\cdot\mathbf{v}\times\frac{\partial F}{\partial \mathbf{v}}=\mp\frac{Ze}{m}\mathbf{E}\cdot\frac{\partial f_0}{\partial \mathbf{v}}+F^\star(\mathbf{k},\mathbf{v},0). \quad (6)$$

Choose in velocity space a rectangular coordinate system such that the 3-direction is parallel to $\mathfrak{B}_0$ and the wave vector $\mathbf{k}$ lies in the 1–3 plane, making an angle θ with the 1-axis. Introducing cylindrical coordinates w, ϕ, u, one can write

$$\mathbf{v}=\mathbf{e}_1 w\cos\phi+\mathbf{e}_2 w\sin\phi+\mathbf{e}_3 u,$$
$$\mathbf{k}=\mathbf{e}_1 k\sin\theta+\mathbf{e}_3 k\cos\theta. \quad (7)$$

Define the gyration frequency Ω by

$$\Omega=Ze\mathfrak{B}_0/mc\geq 0. \quad (8)$$

Then Eq. (6) can be written

$$\frac{\partial F}{\partial \phi}\mp\frac{s+ik(w\sin\theta\cos\phi+u\cos\theta)}{\Omega}F=\mp\frac{1}{\Omega}\left[F^\star(\mathbf{k},\mathbf{v},0)\mp\frac{Ze}{m}\mathbf{E}\cdot\frac{\partial f_0}{\partial \mathbf{v}}\right]. \quad (9)$$

Equation (9) must be solved subject to the requirement that F be single valued, that is, periodic in ϕ, with period 2π. In order to effect this it is convenient to introduce a vector

$$\mathbf{v}'=\mathbf{e}_1 w\cos\phi'+\mathbf{e}_2 w\sin\phi'+\mathbf{e}_3 u, \quad (10)$$

and a function, essentially an integrating factor,

$$G=\exp\left[\pm\int_{\phi'}^{\phi}d\phi\frac{s+i\mathbf{k}\cdot\mathbf{v}}{\Omega}\right],$$
$$=\exp\left[\pm\frac{s+iku\cos\theta}{\Omega}(\phi-\phi')\pm\frac{ikw\sin\theta}{\Omega}(\sin\phi-\sin\phi')\right]. \quad (11)$$

Then, noting that as defined in Eq. (5) Re $s>0$, one can write the solution of Eq. (9) as

$$F(\mathbf{k},\mathbf{v},s)=\mp\frac{1}{\Omega}\int_{\pm\infty}^{\phi}d\phi'\times G\left[F^\star(\mathbf{k},\mathbf{v}',0)\mp\frac{Ze}{m}\mathbf{E}(\mathbf{k},s)\cdot\frac{\partial f_0(\mathbf{v}')}{\partial \mathbf{v}'}\right] \quad (12)$$

which can be verified by direct substitution. The periodicity is apparent if one introduces in the integrand of Eq. (12) the new variable $\phi''=\phi-\phi'$. The limits of integration are then independent of ϕ, which occurs in

[5] S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, Cambridge, 1939).

[6] S. Bochner, *Fourierishe Integrale* (Chelsea Publishing Company, New York, 1948).

[7] G. Doetsch, *Laplace Transformation* (Dover Publications, New York, 1943).

the integrand only in the arguments of periodic functions.

In order to proceed it is necessary to write relations between the electric and magnetic fields and the charge and current densities, which latter are given in terms of the particle distribution functions. These are the Maxwell equations

$$\begin{aligned}\nabla\cdot\mathfrak{E}=4\pi\rho,\quad c\nabla\times\mathfrak{E}=-\partial\mathfrak{B}/\partial t,\\ \nabla\cdot\mathfrak{B}=0,\quad c\nabla\times\mathfrak{B}=4\pi\mathbf{j}+\partial\mathfrak{E}/\partial t,\end{aligned}\tag{13}$$

where the charge density ρ and current density $\mathbf{j}$ are given by

$$\rho=\sum(\pm Ze)\int d^3v\, f,\quad \mathbf{j}=\sum(\pm Ze)\int d^3v\,\mathbf{v}f.\tag{14}$$

The summations are extended over all the species of charged particles present.

The Fourier-Laplace transforms of the Maxwell equations read

$$i\mathbf{k}\cdot\mathbf{B}(\mathbf{k},s)=0,\tag{15}$$

$$ic\mathbf{k}\times\mathbf{E}(\mathbf{k},s)=-s\mathbf{B}(\mathbf{k},s)+\mathbf{B}^\star(\mathbf{k},0)\tag{16}$$

$$ic\mathbf{k}\times\mathbf{B}(\mathbf{k},s)=\sum(\pm4\pi Ze)\int d^3v\mathbf{v}F(\mathbf{k},\mathbf{v},s)+s\mathbf{E}(\mathbf{k},s)-\mathbf{E}^\star(\mathbf{k},0),\tag{17}$$

$$i\mathbf{k}\cdot\mathbf{E}(\mathbf{k},s)=\sum(\pm4\pi Ze)\int d^3vF(\mathbf{k},\mathbf{v},s).\tag{18}$$

Equation (15) states that $\mathbf{B}$ is transverse to $\mathbf{k}$. Thus if one forms the cross product of equation (17) with $ic\mathbf{k}$, there results

$$c^2k^2\mathbf{B}=ic\mathbf{k}\times\left[\sum(\pm4\pi Ze)\int d^3v\mathbf{v}F(\mathbf{k},\mathbf{v},s)+s\mathbf{E}(\mathbf{k},s)-\mathbf{E}^\star(\mathbf{k},0)\right].\tag{19}$$

Equation (19) serves to determine $\mathbf{B}$ once $\mathbf{E}$ and F are known. If one forms the cross product of Eq. (16) with $ic\mathbf{k}$ and employs Eq. (12) for F, there results

$$(s^2+c^2k^2)\mathbf{E}-c^2\mathbf{kk}\cdot\mathbf{E}+\mathbf{E}\cdot\mathbf{Q}=\mathbf{a},\tag{20}$$

where the dyadic $\mathbf{Q}$ is given by

$$\mathbf{Q}=s\sum\pm\frac{4\pi Z^2e^2}{m}\int d^3\mathbf{v}\int_{\pm\infty}^{\phi}d\phi'\frac{G}{\Omega}\frac{\partial f_0(\mathbf{v}')}{\partial\mathbf{v}'}\mathbf{v},\tag{21}$$

and the vector $\mathbf{a}$ by

$$\mathbf{a}=s\mathbf{E}^\star(\mathbf{k},0)+ic\mathbf{k}\times\mathbf{B}^\star(\mathbf{k},0)+\sum 4\pi Zes\times\int d^3v\,\mathbf{v}\int_{\pm\infty}^{\phi}d\phi'(G/\Omega)F^\star(\mathbf{k},\mathbf{v}'0).\tag{22}$$

It can readily be shown by taking the scalar product of Eq. (20) with $\mathbf{k}$, that Eq. (20) implies Eq. (18). Thus the problem has been reduced to solving Eq. (20). In order to do so, we introduce a matrix $\mathbf{R}$ such that in terms of the elements of $\mathbf{Q}$

$$R_{ij}=(s^2+c^2k^2)\delta_{ij}-c^2k_ik_j+Q_{ij}.\tag{23}$$

The solution of Eq. (20), which can be written[8]

$$\mathbf{RE}=\mathbf{a},\tag{24}$$

is

$$\mathbf{E}=\mathbf{R}^A\mathbf{a}/|\mathbf{R}|.\tag{25}$$

The elements of the matrix $\mathbf{R}^A$ are the cofactors of their counterparts in the matrix $\mathbf{R}$, and $|\mathbf{R}|$ represents the determinant of $\mathbf{R}$.

3. INVERSION OF TRANSFORMS

In order to go from Eq. (25) to the frequencies of the electric field in the plasma, one employs the inversion theorem for Laplace transforms,[7] which reads

$$\mathbf{E}^\star(\mathbf{k},t)=\frac{1}{2\pi i}\int_{s_0-i\infty}^{s_0+i\infty}ds\,\mathbf{E}(\mathbf{k},s)e^{st}.\tag{26}$$

The contour of integration above is a straight line parallel to the imaginary s-axis, and to the right of all singularities of $\mathbf{E}(\mathbf{k},s)$. The integral is most conveniently evaluated in terms of the poles of the integrand, assuming there are no complications such as branch points. It will be shown that the elements of $\mathbf{R}$ and hence of $\mathbf{R}^A$ are analytic over the whole finite s-plane. Thus any poles which arise from the numerator of Eq. (25) must come from the vector $\mathbf{a}$ which represents the initial conditions, and thus require for their determination a knowledge of the initial distribution function $f_1(\mathbf{r},\mathbf{v},0)$. For a large class of initial distributions, however, for instance all those Maxwellian in the velocity, $\mathbf{a}$ is an analytic function over the whole finite s-plane. Let us consider only such initial distributions.[9] The method can be readily extended to more complicated situations.

The denominator $|\mathbf{R}(s)|$ of Eq. (25) contributes a pole whenever it vanishes. Denote by s_n a root of the equation

$$|\mathbf{R}(s)|=0.\tag{27}$$

Deform the contour of Eq. (26) to the left in the complex s-plane. The integral can then be written as a sum of terms $\mathbf{E}_n^\star(\mathbf{k},t)$ each of which is an integral extended along a circle enclosing the associated pole. These loop integrals can be evaluated by the residue theorem. For example, if the pole is simple, one can

[8] C. C. Macduffee, *Vectors and Matrices* (The Mathematical Association of America, Ithaca, 1943), p. 58.

[9] The inclusion in the theory of singular initial distributions is discussed in a paper by Bernstein, Greene, and Kruskal [Phys. Rev. (to be published)].

write

$$\mathbf{E}_n^\star(\mathbf{k},t)=\frac{1}{2\pi i}\oint ds e^{st}\frac{\mathbf{R}^A(s_n)\mathbf{a}(s_n)+\cdots}{(s-s_n)|\mathbf{R}(s)|'+\cdots}=\frac{\mathbf{R}^A(s_n)\mathbf{a}(s_n)}{|\mathbf{R}(s_n)|'}e^{s_n t}, \tag{28}$$

where a prime indicates the derivative with respect to s. The extension to higher order poles is immediate. In any case the time dependence will be essentially exponential in nature.

The contribution of a term such as that of Eq. (28) to the electric field is obtained by inverting the Fourier transform, namely

$$\begin{aligned}\mathfrak{E}_n(\mathbf{r},t)&=(2\pi)^{-3}\int d^3k\mathbf{E}_n^\star(\mathbf{k},t)e^{-i\mathbf{k}\cdot\mathbf{r}}\\&=(2\pi)^{-3}\int d^3k\frac{\mathbf{R}^A(s_n)\mathbf{a}(s_n)}{|\mathbf{R}(s_n)|'}e^{s_n t-i\mathbf{k}\cdot\mathbf{r}}.\end{aligned} \tag{29}$$

Clearly is_n is to be interpreted as an oscillation frequency. Thus Eq. (27) whose roots relate is_n to $\mathbf{k}$ is termed the dispersion relation. If any of the s_n has a positive real part, the electric field will grow in time, and hence, from Eq. (12), so will the first order distribution function. It is shown in Appendix I that this is not possible around thermal equilibrium.

4. LIMITS OF VANISHING MAGNETIC FIELD AND $c\to\infty$

In order to show that in the limit of vanishing magnetic field one obtains the results of Landau, it is convenient to break up Eq. (20) into longitudinal and transverse parts, namely

$$\begin{aligned}(s^2+c^2k^2)\mathbf{k}\times\mathbf{E}+\mathbf{E}\cdot\mathbf{Q}\times\mathbf{k}&=\mathbf{k}\times\mathbf{a},\\ s^2\mathbf{k}\cdot\mathbf{E}+\mathbf{E}\cdot\mathbf{Q}\cdot\mathbf{k}&=\mathbf{k}\cdot\mathbf{a}.\end{aligned} \tag{30}$$

If then one applies the asymptotic relation, which follows from Eq. (11),

$$\begin{aligned}&\int_{\pm\infty}^{\phi} d\phi'\frac{G}{\Omega}g(\phi')\\&=\mp\int_{\pm\infty}^{\phi} d\phi'\frac{g(\phi')}{s+i\mathbf{k}\cdot\mathbf{v}'}\frac{\partial G}{\partial\phi'}\\&=\mp\frac{g(\phi')}{s+i\mathbf{k}\cdot\mathbf{v}'}G\bigg|_{\pm\infty}^{\phi}\pm\int_{\pm\infty}^{\phi} d\phi' G\frac{\partial}{\partial\phi'}\frac{g(\phi')}{s+i\mathbf{k}\cdot\mathbf{v}'}\\&=\mp\frac{g(\phi)}{s+i\mathbf{k}\cdot\mathbf{v}}-\frac{\Omega}{s+i\mathbf{k}\cdot\mathbf{v}}\frac{\partial}{\partial\phi}\left[\frac{g(\phi)}{s+i\mathbf{k}\cdot\mathbf{v}}\right]\mp\frac{\Omega^2}{s+i\mathbf{k}\cdot\mathbf{v}}\\&\quad\times\frac{\partial}{\partial\phi}\left\{\frac{1}{s+i\mathbf{k}\cdot\mathbf{v}}\left[\frac{\partial}{\partial\phi}\frac{g(\phi)}{s+i\mathbf{k}\cdot\mathbf{v}}\right]\right\}+\cdots,\end{aligned} \tag{31}$$

and passes to the limit $\Omega\to 0$, Eqs. (30) reduce, after some integration with respect to $\mathbf{v}$ and employment of Poisson's equation, to

$$\mathbf{k}\times\mathbf{E}=\frac{s\mathbf{k}\times\mathbf{E}^\star(\mathbf{k},0)-ick^2\mathbf{B}^\star(\mathbf{k},0)+\sum\pm4\pi Zes\int d^3\mathbf{v}\mathbf{k}\times\mathbf{v}F^\star(\mathbf{k},\mathbf{v},0)/(s+i\mathbf{k}\cdot\mathbf{v})}{s^2+c^2k^2+s\sum(4\pi NZ^2e^2/m)\int d^3\mathbf{v}f_0(\mathbf{v})/(s+i\mathbf{k}\cdot\mathbf{v})}, \tag{32}$$

$$\mathbf{k}\cdot\mathbf{E}=\frac{\sum\pm(4\pi Ze/ik)\int d^3vF^\star(k,\mathbf{v},0)/(s+i\mathbf{k}\cdot\mathbf{v})}{1+\sum(4\pi Z^2e^2/ik^2m)\int d^3v[\mathbf{k}\cdot\partial f_0(\mathbf{v})/\partial\mathbf{v}]/(s+i\mathbf{k}\cdot\mathbf{v})}. \tag{33}$$

In the limit of zero temperature Eq. (32) yields the well known result for the propagation of electromagnetic waves in an ionized atmosphere[10]; namely on setting the denominator of Eq. (32) equal to zero, $-s^2=c^2k^2+\sum(4\pi NZ^2e^2/m)$. Equation (33), apart from notation, is just the result of Landau, if the ion motion is neglected.

Proceeding to the next limit note that at moderate densities and magnetic fields the frequencies of the plasma divide into two classes, one set of order of magnitude ck, and the other set of order $[\omega^2+\Omega^2]^{\frac{1}{2}}\ll ck$, where the electron plasma frequency $\omega=[4\pi Ne^2/m]^{\frac{1}{2}}$. The coupling between longitudinal and transverse oscillations is once more small and Eqs. (30) reduce to

$$\begin{aligned}(s^2+c^2k^2)\mathbf{k}\times\mathbf{E}&=\mathbf{k}\times\mathbf{a},\\ s^2\mathbf{k}\cdot\mathbf{E}+\mathbf{k}\cdot\mathbf{E}\mathbf{k}\cdot\mathbf{Q}\cdot\mathbf{k}/k^2&=\mathbf{k}\cdot\mathbf{a}.\end{aligned} \tag{34}$$

This can be viewed mathematically as the result of passing to the limit $c\to\infty$.

5. LONGITUDINAL ELECTRON OSCILLATIONS

One expects that even in the limit $c\to\infty$ the possible longitudinal oscillations separate into two classes, one

[10] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1940), pp. 327 et. seq.

of relatively high frequency electron oscillations, and another of relatively low frequency ion oscillations. In order to investigate the former class we consider the positive ions to be effectively at rest, serving only to provide a uniform background of positive charge. The dispersion relation then reads, from the second of Eqs. (34),

$$k^2s^2+\mathbf{k}\cdot\mathbf{Q}\cdot\mathbf{k}=0, \tag{35}$$

or

$$k^2s^2-\frac{s\omega^2}{N}\int d^3v\int_{-\infty}^{\phi}d\phi'\mathbf{k}\cdot\mathbf{v}\mathbf{k}\cdot\frac{\partial f_0}{\partial \mathbf{v}'}\frac{G}{\Omega}=0. \tag{36}$$

When f_0 is the Maxwell distribution, Eq. (36) can be shown, by performing in succession three of the indicated integrations, as outlined in Appendix 2, to reduce to

$$1+k^2a^2=\frac{s}{\Omega}\int_0^{\infty}dy\exp\left[-\frac{sy}{\Omega}-\lambda(1-\cos y)-\tfrac{1}{2}\mu y^2\right], \tag{37}$$

where as before $\Omega=e\mathfrak{B}_0/mc$, the electron gyration frequency. The Debye length a is given by $a^2=KT/4\pi Ne^2$, the electron gyration radius ρ by $\rho^2=KT/m\Omega^2$; θ is the angle between $\mathbf{k}$ and $\mathfrak{B}_0$, $\lambda=k^2\rho^2\sin^2\theta$, and $\mu=k^2\rho^2\cos^2\theta$.

Equation (37), apart from notation, is the result of Gordeyev.[2] The integral in Eq. (37) defines an analytic function of s over the whole s-plane as long as $\mu>0$. For the case $\mu=0$ ($\mathbf{k}\cdot\mathfrak{B}_0=0$), that treated by Gross, the integral in Eq. (37) converges only for Re $s>0$. In order to effect its analytic continuation, one writes Eq. (37) in the form

$$\begin{aligned}1+k^2a^2&=\frac{s}{\Omega}\sum_{n=0}^{\infty}\int_{2\pi n}^{2\pi(n+1)}dy\exp\left[-\left(\frac{sy}{\Omega}\right)-\lambda(1-\cos y)\right]\\&=\frac{s}{\Omega}\sum_{n=0}^{\infty}\exp\left(\frac{-2\pi ns}{\Omega}\right)\int_0^{2\pi}dy'\\&\qquad\times\exp\left[-\left(\frac{sy'}{\Omega}\right)-\lambda(1-\cos y')\right]\\&=\frac{\int_0^{2\pi}dy\exp\left[-\left(\frac{sy}{\Omega}\right)-\lambda(1-\cos y)\right]}{1-\exp\left(\frac{-2\pi s}{\Omega}\right)}.\end{aligned} \tag{38}$$

The numerator of Eq. (38) is defined for all values of s. The denominator vanishes when $s/\Omega=0, \pm1, \pm2, \cdots$, where the function has simple poles. Thus Eq. (38) represents the desired continuation. Apart from notation, it is the result of Gross.[3]

(a) Absence of Solutions Which Increase in Time

It is possible to show directly from Eq. (37) that there are no roots of the dispersion relation for which Re $s>0$, and hence no waves which grow in time. To demonstrate this we note that

$$e^{\lambda\cos y}=\sum_{n=-\infty}^{\infty}I_n(\lambda)e^{iny} \tag{39}$$

where $I_n(y)=I_{-n}(y)$ is the Bessel function of the first kind of imaginary argument. The terms in n and $-n$ are to be considered as taken together in any sum. Insertion of Eq. (39) in Eq. (37) yields, one writing $y=\Omega t$,

$$1+k^2a^2=se^{-\lambda}\sum_{n=-\infty}^{\infty}I_n(\lambda)\int_0^{\infty}dt\times\exp[-\tfrac{1}{2}\Omega^2\mu t^2-(s+in\Omega)t]. \tag{40}$$

In order to transform Eq. (40) into a more useful form, one defines

$$\phi(u)=(m/2\pi KT)^{\frac{1}{2}}\exp(-mu^2/2KT\cos^2\theta). \tag{41}$$

Note that

$$\int_{-\infty}^{\infty}du\phi(u)=1,$$

and that as $\theta\to\pi/2$, $\phi(u)\to\delta(u)$. If Re $s>0$

$$\begin{aligned}\int_{-\infty}^{\infty}&du\phi(u)/(s+iku)\\&=\int_{-\infty}^{\infty}du\phi(u)\int_0^{\infty}dt\exp[-(s+iku)t]\\&=\int_0^{\infty}dt\exp[-st-\tfrac{1}{2}(KT/m)k^2t^2\cos^2\theta].\end{aligned} \tag{42}$$

Thus, defining $s=i\beta+\gamma$, $\gamma>0$ and employing Eq. (42) to transform Eq. (40), one can write on splitting into

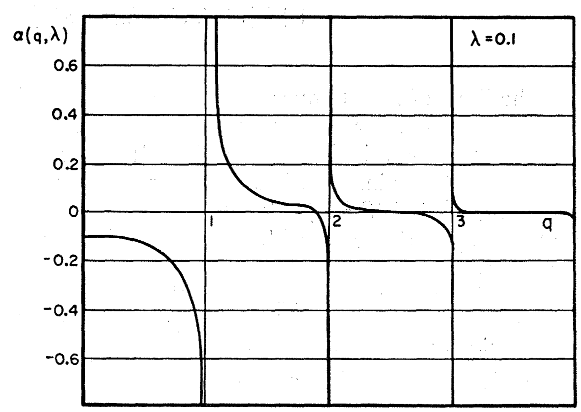

FIG. 1. The function $\alpha(q,\lambda)$ vs q for $\lambda=0.1$.

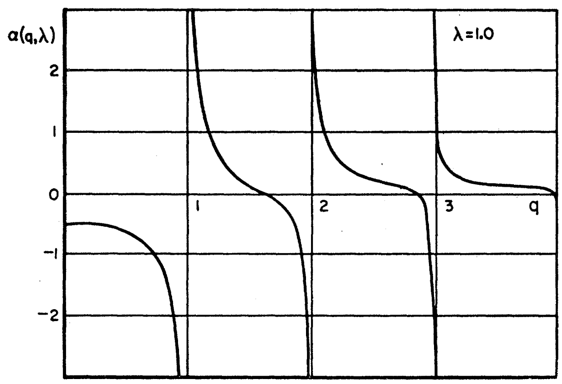

FIG. 2. The function $\alpha(q,\lambda)$ *vs* q for $\lambda=1.0$.

real and imaginary parts

$$1+k^2a^2=\sum_{n=-\infty}^{\infty} e^{-\lambda}I_n(\lambda)\int_{-\infty}^{\infty} du\phi(u)\times\left[1-\frac{(n\Omega+ku)(\beta+n\Omega+ku)}{\gamma^2+(\beta+n\Omega+ku)^2}\right], \quad (43)$$

$$0=\gamma\sum_{n=-\infty}^{\infty} e^{-\lambda}I_n(\lambda)\int_{-\infty}^{\infty} du\phi(u)\frac{n\Omega+ku}{\gamma^2+(\beta+n\Omega+ku)^2}. \quad (44)$$

Multiply Eq. (44) by β/γ and add it to Eq. (43). There results

$$1+k^2a^2=\sum_{n=-\infty}^{\infty} e^{-\lambda}I_n(\lambda)\int_{-\infty}^{\infty} du\phi(u)\times\left[1-\frac{(n\Omega+ku)^2}{\gamma^2+(n\Omega+ku+\beta)^2}\right]$$

$$=1-\sum_{n=-\infty}^{\infty} e^{-\lambda}I_n(\lambda)\int_{-\infty}^{\infty} du\phi(u)\times\frac{(n\Omega+ku)^2}{\gamma^2+(n\Omega+ku+\beta)^2}, \quad (45)$$

where we have employed Eq. (39) with $y=0$. Clearly Eq. (45) has no solution since the left-hand side is always greater than the right. Thus there can be no roots with Re $s>0$, and hence no time increasing solutions. This is in contradiction with the result of Gordeyev,[2] who purported to show that self-excitation of oscillations is possible. Gordeyev's error lies in making approximations on the equivalent of Eq. (42) the results of which are incompatible with the assumptions motivating them. A separate demonstration of the absence of time increasing solutions, essentially on the basis of an entropy argument, due to William Newcomb, is given in Appendix I.

(b) Case $k\cdot\mathfrak{B}_0=0$

For the case of waves propagating perpendicular to the zeroth order magnetic field the dispersion relation can be solved completely. Since in this case $\theta=\pi/2$, Eq. (40) yields for Re $s>0$

$$1+k^2a^2=se^{-\lambda}\sum_{n=-\infty}^{\infty} I_n(\lambda)/(s+in\Omega)$$

$$=e^{-\lambda}I_0(\lambda)+2\sum_{n=1}^{\infty}\frac{(s/\Omega)^2e^{-\lambda}I_n(\lambda)}{(s/\Omega)^2+n^2}. \quad (46)$$

The series in Eq. (46), however, apart from simple poles at $s/\Omega=0, \pm1, \cdots$ converges for all values of s. The right-hand side of Eq. (46) is an even function of s. Thus the general impossibility of a root with Re $s>0$ implies in this case that there can be no root with Re $s<0$. Hence the roots are pure imaginary. This is in contradiction with the conclusion of Gross, who purported to find damping of waves for which $k^2a^2\gg1$. Gross's error lies in his saddle-point integral, in particular, in his statement in Appendix II of his paper that the integral along I_3 and I_4 gives $-e^{2\pi i\omega/\omega_c}(I_1+I_2)$, when in fact it gives $e^{2\pi i\omega/\omega_c}(I_1^*+I_2^*)$. This error also vitiates the low-temperature results he reports. Upon making the correction, the same results are obtained as in this work.

In order to solve Eq. (46), let $s/\Omega=iq$, and define

$$\alpha(q,\lambda)=e^{-\lambda}I_0(\lambda)+q\sum_{n=1}^{\infty} I_n(\lambda)\left[\frac{1}{q-n}+\frac{1}{q+n}\right]-1. \quad (47)$$

Then the dispersion relation can be written

$$\alpha=k^2a^2. \quad (48)$$

The function $\alpha(q,\lambda)$ is plotted *versus* q for various values of λ in Figs. 1, 2, and 3. The roots of the dispersion relation are the intersections of an α *vs* q curves with the horizontal line $\alpha=k^2a^2$. As $|q|\to\infty$ the curve α *vs* q approaches a function which is zero in the interval $n<q<n+1$ and lies very close to the vertical line $q=n$ when $q\sim n$. Note that the roots depend in con-

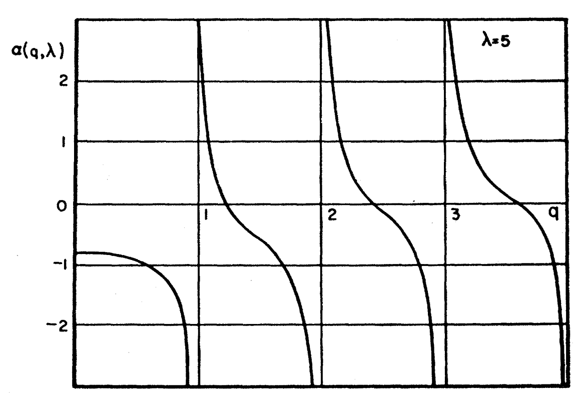

FIG. 3. The function $\alpha(q,\lambda)$ *vs* q for $\lambda=5$.

tinuous fashion on the parameters $k^2\rho^2$, k^2a^2, etc. If one fixes Ω, and $\lambda=k^2\rho^2\sin^2\theta=k^2\rho^2$, but permits k^2a^2 to vary, then there are gaps in the spectrum of possible frequencies,[11] namely those values of $s=-i\Omega q$ which make $\alpha<0$. The width of these gaps, which correspond to frequencies which cannot propagate, can readily be computed in the limit $\lambda=k^2\rho^2\ll 1$, the case of gyration radius small compared with wavelength. Namely one can employ in Eq. (47) the leading significant terms in the series expansion for the Bessel functions,[12]

$$I_n(\lambda)=\sum_{m=0}^{\infty}\left(\frac{\lambda}{2}\right)^{n+2m}\frac{1}{m!(n+m)!}. \tag{49}$$

This yields

$$\frac{\Omega^2}{\omega^2}=\frac{1-\lambda}{q^2-1}+\frac{\lambda}{q^2-4}+\sum_{n=3}^{\infty}\frac{n^2}{n!}\left(\frac{\lambda}{2}\right)^{n-1}\frac{1}{q^2-n^2}, \tag{50}$$

where as before the electron plasma frequency ω is given by $\omega^2=4\pi Ne^2/m$. For the root in the interval $1<q<2$ it is sufficient to keep the first two terms on the right. The result is

$$-\frac{s^2}{\Omega^2}=\frac{1}{2}\left\{5+\frac{\omega^2}{\Omega^2}-\left[\left(3-\frac{\omega^2}{\Omega^2}\right)^2+3\lambda\frac{\omega^2}{\Omega^2}\right]^{\frac{1}{2}}\right\}. \tag{51}$$

For $\Omega^2\gg\omega^2$, Eq. (51) yields

$$-s^2=\Omega^2+\omega^2-(\omega^2/\Omega^2)(KT/m)k^2. \tag{52}$$

For $\Omega^2\ll\omega^2$, there results

$$-s^2=4\Omega^2-3(KT/m)k^2. \tag{53}$$

The gap for the interval $\Omega^2<|s^2|<4\Omega^2$ corresponds to the range in q for which the sum of the first two terms in Eq. (50) are negative. Its width in frequency is $\frac{3}{8}k^2KT/m\Omega$. Clearly similar results can be obtained for the other branches of the curve α *vs* q. The gap width is always proportional to KT in the limit $\lambda\ll 1$.

An interesting feature of this dispersion relation for waves propagating perpendicular to the zero-order magnetic field is that it corresponds to a nonuniformity in the limit $\Omega\to 0$. That is, as one tends to this limit, the dispersion relation [Eq. (48)] obtained from the general dispersion relation [Eq. (37)] by first setting $\theta=\pi/2$ and then letting $\Omega\to 0$ still yields frequencies spaced roughly at multiples of the gyration frequency, while one expects that the frequencies obtained be those derived by Landau, which corresponds in the general dispersion relation to first setting $\Omega=0$, and then choosing $\theta=\pi/2$. The resolution of this apparent difficulty is that the anomalous waves are a set of measure zero. That is to say, as $\Omega\to 0$, those waves which depart in character from those found by Landau are confined to a sector of width $\delta\theta$ about $\theta=\pi/2$, but $\delta\theta\to 0$ with Ω.

[11] Gaps of this kind were first noted by Gross, reference 3, for the case of a delta function equilibrium distribution.

[12] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, New York, 1948).

(c) Arbitrary Direction of Propagation

Let us return to the case of arbitrary direction of propagation. Consider the situation of low temperature and weak magnetic field. If one defines $\nu^2=KTk^2/m$ and $y=\Omega t$, the dispersion relation (37) reads

$$\begin{aligned}1+\frac{\nu^2}{\omega^2}&=s\int_0^\infty dt\,\exp\left\{-st-\tfrac{1}{2}\nu^2t^2\cos^2\theta-\sin^2\theta\left[\tfrac{1}{2}\nu^2t^2-\frac{1}{24}\nu^4t^4+\cdots\right]\right\}\\&=s\int_0^\infty dt e^{-st}\left\{1-\tfrac{1}{2}\nu^2t^2+\left[\frac{1}{24}\nu^2\Omega^2\sin^2\theta+\tfrac{1}{8}\nu^4\right]t^4+\cdots\right\}\\&=1-\frac{\nu^2}{s^2}+\frac{\nu^2\Omega^2\sin^2\theta+3\nu^4}{s^4}-\cdots.\end{aligned} \tag{54}$$

Equation (54) is an asymptotic expansion valid for $(\mathrm{Im}\,s)^2\gg\Omega^2$, ν^2, $|\mathrm{Re}\,s|^2$. It yields, to first order in the small quantities,[13]

$$-s^2=\omega^2+\Omega^2\sin^2\theta+3\nu^2. \tag{55}$$

If one wishes to compute Re s, it is necessary to write $s=i\omega-\gamma$ in the second line of Eq. (54) and to evaluate asymptotically the imaginary part of the integral. The result is

$$\gamma=\left(\frac{\pi}{8}\right)^{\frac{1}{2}}\frac{\omega^4}{\nu^3}\exp\left(-\frac{\omega^2}{2\nu^2}\right)\left\{1+\frac{\sin^2\theta}{24}\frac{\omega^4\Omega^2}{\nu^6}\right\}. \tag{56}$$

Equations (55) and (56) are the same results as have been obtained by Gordeyev. Note that they reduce to Landau's results in the limit $\Omega\to 0$.

Consider now the case where $k^2\rho^2\ll k^2a^2\ll 1$, the case of large magnetic field and low temperature. One anticipates a root of the dispersion relation $s\sim i\omega\cos\theta$. Then, in the dispersion relation in the form

$$1+k^2a^2=s\int_0^\infty dt\,\exp\{-st-\tfrac{1}{2}\nu^2t^2\cos^2\theta-2k^2\rho^2\sin^2\theta\sin^2(2\Omega t)\}, \tag{57}$$

the term $\sin^2(2\Omega t)$ can be replaced by its average value of $\frac{1}{2}$, whence on asymptotic expansion of the resultant integral, as treated in the preceding case, there results[13]

$$-s^2=\omega^2\cos^2\theta[1+3k^2a^2-\rho^2/a^2], \tag{58}$$

$$\gamma=\left(\frac{\pi}{8}\right)^{\frac{1}{2}}\frac{\omega^4\cos\theta}{\nu^3}\exp\left(-\frac{\omega^2}{2\nu^2}\right). \tag{59}$$

[13] The results reported in Eqs. (55) and (58) have been obtained independently by William Newcomb by a different method which will be published in the near future.

Equation (58) indicates that the plasma behaves as though the electrons were free to move only along the lines of force. To lowest order the frequency is just that of a plasma wave whose wave vector is $\mathbf{k}$ projected onto the zeroth order magnetic field $\mathfrak{B}_0$. Note that the exponentially small Landau damping falls off to zero as $\theta\rightarrow\pi/2$.

In order to investigate the case of large Landau damping which occurs in the zero magnetic field case for $k^2a^2=\nu^2/\omega^2\gg 1$, one can consider first the case of small magnetic field, and write Eq. (54) in the form

$$1+k^2a^2=s\exp\left(\frac{s^2}{2\nu^2}\right)\int_0^\infty dt\exp\left\{-\frac{1}{2}\nu^2\left(t+\frac{s}{\nu^2}\right)^2\right\}\times[1+(1/24)\nu^2\Omega^2t^4\sin^2\theta+\cdots]. \quad (60)$$

One anticipates a root for which $-\mathrm{Re}\,s\gg|\mathrm{Im}\,s|$, in which case the lower limit in the integral in Eq. (60) can be taken to be $-\infty$, while in the power series in the integrand it is sufficient to drop all terms past the one in t^4, in which term one can set $t=-s/\nu^2$. The result is

$$1+k^2a^2=(2\pi)^{\frac{1}{2}}(s/\nu)[1+(1/24)\Omega^2s^4\sin^2\theta/\nu^6]\times\exp[\tfrac{1}{2}s^2/\nu^2]. \quad (61)$$

When $\Omega=0$, Eq. (61) is just Landau's result. As long as the term in Ω^2 in the above expression is small compared with unity, which of course is the assumption, the magnetic field has only a logarithmic effect on the determination of s. The damping is then given to good approximation by Landau's result, which can be readily obtained by taking the logarithm of Eq. (61), namely

$$s=-\nu[4\ln ka-\ln 2\pi]^{\frac{1}{2}}-\pi i\nu[4\ln ka-\ln 2\pi]^{-\frac{1}{2}}. \quad (62)$$

If $k^2a^2\gg 1$, but Ω is large, one can as before employ Eq. (57) with $\sin^2(2\Omega t)=\frac{1}{2}$, whence

$$1+k^2a^2=s\exp\left(\frac{s^2}{2\nu^2\cos^2\theta}\right)\int_0^\infty dt\times\exp\left[-\frac{1}{2}\nu^2\cos^2\theta\left(t+\frac{s}{\nu^2\cos^2\theta}\right)\right]. \quad (63)$$

Equation (63) can be treated exactly like Eq. (60). The result is

$$s=-\nu\cos\theta[4\ln ka-\ln 2\pi]^{\frac{1}{2}}-\pi i[4\ln ka-\ln 2\pi]^{\frac{1}{2}}\nu\cos\theta. \quad (64)$$

Note that the damping decreases as θ increases from zero. This is in agreement with the exact result of no damping of waves propagating perpendicular to $\mathfrak{B}_0$.

Consider the waves for which $\theta\sim\pi/2$ ($\mathbf{k}\cdot\mathfrak{B}_0\sim 0$, whence $\mu\ll 1$). If one employs in Eq. (40) the variable $q=-is/\Omega$, there results after appropriate integration by parts

$$1+k^2a^2-e^{-\lambda}I_0(\lambda)\left[1+\frac{\mu}{q^2}+\frac{3\mu^2}{q^4}+\cdots\right]=\sum_{n=1}^{\infty}\frac{2q^2e^{-\lambda}I_n(\lambda)}{q^2-n^2}\left\{1+\frac{\mu(q^2+3n^2)}{q^2-n^2}+\frac{3\mu^2(q^2+10n^2q^2+5n^4)}{(q^2-n^2)^4}+\cdots\right\}. \quad (65)$$

If also $k^2a^2\gg 1$, we expect from our previous results for $\mu=0$ that the roots in q lie near the integers. Thus for the root $q\sim m\gg 1$, one can in Eq. (65) neglect all terms with $n\neq m$, and all powers of μ greater than the first. There results

$$q^2=m^2+\frac{2m^2e^{-\lambda}I_m(\lambda)}{k^2a^2}\left\{1+\frac{\mu}{m^2}\left[\frac{k^2a^2}{e^{-\lambda}I_m(\lambda)}\right]^2\right\}. \quad (66)$$

Equation (66) is valid so long as $2e^{-\lambda}I_m(\lambda)/k^2a^2\ll 1$, and $(\mu/m^2)[k^2a^2/e^{-\lambda}I_m(\lambda)]^2\ll 1$.

If $\lambda\ll 1$ (low temperature and/or large magnetic field) we can expand the Bessel functions in powers of λ via Eq. (49). For the root in the interval $1<|q|<2$, it is sufficient to write

$$1+k^2a^2=1+\frac{\lambda}{q^2-1}+\frac{\mu}{q^2}+\frac{3\lambda^2}{(q^2-1)(q^2-4)}+\frac{\mu\lambda(-q^6+4q^2-1)}{q^2(q^2-1)}+\frac{3\mu^2}{q^4}. \quad (67)$$

If one drops terms quadratic in λ and μ, there results

$$q^2=-(s^2/\Omega^2)=\tfrac{1}{2}\{1+(\omega^2/\nu^2)-[(1-\omega^2/\nu^2)^2+4\omega^2\sin^2\theta/\nu^2]^{\frac{1}{2}}\}. \quad (68)$$

6. LONGITUDINAL OSCILLATIONS INCLUDING THE ION DYNAMICS

Consider as before the limit $c\rightarrow\infty$, but assume that there is present one kind of positive ion. The second of Eqs. (34) then yields the dispersion relation

$$k^2s^2=\left(\frac{s\omega_+^2}{N}\right)\int d^3v\int_\infty^\phi d\phi'\mathbf{k}\cdot\mathbf{v}\left(\frac{G_+}{\Omega_+}\right)\mathbf{k}\cdot\frac{\partial f_{0+}(\mathbf{v}')}{\partial\mathbf{v}'}+\left(\frac{s\omega_-^2}{N}\right)\int d^3v\int_{-\infty}^{\nu} d\phi'\mathbf{k}\cdot\mathbf{v}\left(\frac{\mathbf{G}}{\Omega_-}\right)\mathbf{k}\cdot\frac{\partial f_{0-}(\mathbf{v}')}{\partial\mathbf{v}'}, \quad (69)$$

where $\omega_\pm^2=4\pi Ne^2/m_\pm$, $\Omega_\pm=|e\mathfrak{B}_0/m_\pm c|$, $f_{0\pm}$ is given by Eq. (2) with the mass and temperature chosen appropriately, and $G_\pm$ by Eq. (11), Equation (69) yields, on performing three of the indicated integrations in the manner of Appendix II,

$$1+(T_+/T_-)+k^2a_-^2$$
$$=s\int_0^\infty dt\exp\{-st-2k^2\rho_+^2\sin^2\theta\sin^2(2\Omega_+t)$$
$$-\tfrac{1}{2}\nu_+^2\cos^2\theta t^2\}+s(T_+/T_-)\int_0^\infty dt$$
$$\times\exp\{-st-2k^2\rho_-^2\sin^2\theta\sin^2(2\Omega_-t)$$
$$-\tfrac{1}{2}\nu_-^2\cos^2\theta t^2\},\quad(70)$$

where $a_-^2=KT_-/4\pi Ne^2$, $\rho_\pm^2=KT_\pm/m_\pm\Omega_\pm^2$, and $\nu_\pm^2=KT_\pm k^2/m_\pm$.

If one considers the case $T_+=T_-$, it can readily be shown that when $\mathfrak{B}_0=0$ the inclusion of the ion dynamics makes corrections to the frequencies calculated in Sec. V of order $(m_-/m_+)^2\ll 1$. If, however, $T_-\gg T_+$ there appear frequencies characteristic of so-called ion oscillations. In order to see this, consider for simplicity the case of large magnetic field, so large that $1\gg k^2\rho_+^2\gg k^2\rho_-^2$. Then $k^2\rho_\pm^2$ in Eq. (70) can be replaced by zero, and if one then expands the first integral in descending and the second in ascending powers of s, there results

$$1+\frac{T_+}{T_-}+k^2a_-^2=1-\frac{\nu_+^2\cos^2\theta}{s^2}+\frac{3\nu_+^4\cos^4\theta}{s^4}-\cdots$$
$$+\frac{T_+}{T_-}\left[\frac{s}{\nu_-\cos\theta}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}-\frac{s^2}{\nu_-^2\cos^2\theta}+\cdots\right].\quad(71)$$

On the right in Eq. (71) it is sufficient to keep only the terms in s^0, s^{-2}, and s^1. If $k^2a_-^2\gg T_+/T_-$, the result is[14]

$$s=\pm i(KT_-k^2/m_+)^{\frac{1}{2}}\cos\theta[\pm i-(m_-/m_+)^{\frac{1}{2}}(\pi/8)^{\frac{1}{2}}].\quad(72)$$

Note that because $m_-/m_+\ll T_+/T_-\ll 1$, it follows that $|\text{Im}\,s|\gg|\text{Re}\,s|$, and there will be many oscillations before damping occurs. The same results for Im s is derived by Spitzer,[15] for the case of zero magnetic field, from the hydrodynamic equations. He, however, derives no damping. This is because the stress tensor is not isotropic, as is conventionally assumed in hydrodynamic treatments, but rather obeys a complicated equation of state. The case of zero magnetic field is effectively realized in Eq. (72) by setting $\theta=0$. Note that the result of Eq. (72) can be interpreted by saying that in a strong magnetic field the charged particles are tied to the lines of force.

7. GENERAL ELECTRON OSCILLATIONS

(a) Weak Magnetic Field

Consider the case of a weak magnetic field and neglect the motion of the positive ions. One can then employ Eq. (31) to expand the dyadic Q of Eq. (21) to the second order in Ω^2. The result is readily shown to be

$$\mathbf{Q}=(\mathbf{n}_1\mathbf{n}_1+\mathbf{n}_2\mathbf{n}_2)sJ+\mathbf{n}_3\mathbf{n}_3[(s^2/\nu^2)-(s^3/\nu^3)J]$$
$$+\Omega s\{(\mathbf{n}_1\mathbf{n}_1-\mathbf{n}_2\mathbf{n}_2)\cos\theta\,\partial J/\partial s+(\mathbf{n}_3\mathbf{n}_2-\mathbf{n}_2\mathbf{n}_3)$$
$$\times\tfrac{1}{2}\sin\theta(\partial/\partial s)[J+(s^2/\nu^2)-(s^3/\nu^3)J]\}$$
$$+\Omega^2 s\{-(\mathbf{n}_1\mathbf{n}_1+3\mathbf{n}_2\mathbf{n}_2+\mathbf{n}_3\mathbf{n}_3)(1/24)\nu^2\sin^2\theta\partial^4J/\partial s^4$$
$$+\mathbf{n}_2\mathbf{n}_2\tfrac{1}{2}(\cos^2\theta/\nu^2)(\partial^2/\partial s^2)(s^2-s^3J)$$
$$+\tfrac{1}{2}\partial^2J/\partial s^2[(\mathbf{n}_1\mathbf{n}_1+\mathbf{n}_2\mathbf{n}_2)\cos^2\theta+\mathbf{n}_3\mathbf{n}_3\sin^2\theta$$
$$+(\mathbf{n}_1\mathbf{n}_3+\mathbf{n}_3\mathbf{n}_1\sin\theta\cos\theta]\},\quad(73)$$

where $\nu^2=KTk^2/m$, $\mathbf{n}_3$ is a unit vector in the direction of $\mathbf{k}$, $\mathbf{n}_2$ is a unit vector in the direction of $\mathbf{k}\times\mathfrak{B}_0$, $\mathbf{n}_1=\mathbf{n}_2\times\mathbf{n}_3$, and

$$J=\int_0^\infty dt\exp[-st-\tfrac{1}{2}\nu^2t^2]$$
$$\doteqdot\frac{1}{s}-\frac{\nu^2}{s^3}+\frac{3\nu^4}{s^5}-\cdots+\frac{1}{\nu}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\exp(s^2/2\nu^2)[1+\cdots].\quad(74)$$

In Eq. (74) the asymptotic representation indicated by $\doteqdot$ is useful when $|\text{Im}\,s|\gg|\text{Re}\,s|$, which corresponds to $\nu^2\ll\omega^2=4\pi Ne^2/m$. In this latter case the dispersion relation (27) reads, if one neglects exponentially small terms which yield the here-negligible Landau damping, $|\mathbf{R}|=0$ where the elements of the matrix $\mathbf{R}$ are

$$\begin{aligned}
R_{11}&=s^2+c^2k^2-(\omega^2/s^2)(\nu^2+\Omega^2\cos^2\theta),\\
R_{21}&=\omega^2\Omega\cos\theta/s,\\
R_{31}&=-\omega^2\Omega^2\sin\theta\cos\theta/s^2,\\
R_{12}&=-\omega^2\Omega\cos\theta/s,\\
R_{22}&=s^2+c^2k^2+\omega^2-(\omega^2/s^2)(\nu^2+\Omega^2),\qquad(75)\\
R_{32}&=-\omega^2\Omega\cos\theta/s,\\
R_{13}&=-\omega^2\Omega^2\sin\theta\cos\theta/s^2,\\
R_{23}&=\omega^2\Omega\sin\theta/s,\\
R_{33}&=s^2+\omega^2(\omega^2/s^2)(3\nu^2+\Omega^2\sin^2\theta).
\end{aligned}$$

In the determinant, which is a function of s^2, Ω^2 and ν are to be considered as of comparable smallness, say of order ϵ. Equation (75) is then of the form $f(s^2,\epsilon)=0$. The roots of $f(s^2,0)=0$ are $s_1^2=s_2^2=-(\omega^2+c^2k^2)$ corresponding to the electromagnetic waves, and $s_3^2=-\omega^2$ corresponding to longitudinal electron oscillations. In order to solve the determinantal equation, it is convenient to expand $f(s^2,\epsilon)$ in a joint Taylor series about $s^2=s_i^2$ $(i=1, 2, 3)$, and about $\epsilon=0$. In computing a root, it is necessary to retain only the lowest order nonvanishing terms separately in ϵ and $s^2-s_i^2$. For the root near s_3^2, it is sufficient to keep only first derivatives. The result is

$$-s^2=\omega^2+3\nu^2+\Omega^2\sin^2\theta+\omega^2\Omega^2\sin^2\theta/c^2k^2.\quad(76)$$

---

[14] The observation that Eq. (72) follows from Eq. (71) is due to John M. Greene.

[15] L. Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., London, 1956), Chap. 4.

Note that in the limit $c\to\infty$ Eq. (76) goes over into Eq. (55). For the double root $s_1^2=s_2^2$, it is necessary to go to second order in both parameters of smallness. The result is

$$-s^2=c^2k^2+\omega^2\pm\omega\Omega\cos\theta/(\omega^2+c^2k^2)^{\frac{1}{2}}. \tag{77}$$

(b) Arbitrary Parameters

Consider the case of a plasma composed of an equal number of electrons and protons. It is convenient to revert to the representation of Eq. (7) and introduce the unitary matrix

$$P=\frac{-i}{\sqrt{2}}\begin{Vmatrix}1 & -1 & 0\\ 1 & i & 0\\ 0 & 0 & \sqrt{2}\end{Vmatrix}, \tag{78}$$

where the scalar factor $i/\sqrt{2}$ is understood to multiply each element of the matrix. If one defines $\mathbf{R}'=\mathbf{PRP}^{-1}$, since $|\mathbf{R}|=|\mathbf{P}^{-1}\mathbf{R}'\mathbf{P}|=|\mathbf{R}'|$, the dispersion relation (27) can be written $|\mathbf{R}'|=0$. The transformation essentially represents $\mathbf{E}$ in terms of circularly polarized waves.

The matrix $\mathbf{Q}'=\mathbf{PQP}^{-1}$ which enters R', where Q is given by Eq. (21), can be expressed in terms of variables $\alpha=\frac{1}{2}(\phi'+\phi)$ and $\beta=\mp\frac{1}{2}(\phi'-\phi)$, where the upper sign is to be taken for positive particles, and the lower for negative. Namely,

$$\mathbf{Q}'=-\sum\left(\frac{s}{\Omega}\right)\left(\frac{\omega^2}{N}\right)\int_0^\infty dw\,w\int_{-\infty}^\infty du\times\int_0^\infty d\beta\int_\beta^{\beta+2\pi}d\alpha\,G\mathbf{T}, \tag{79}$$

where the summation is extended over all the particles present, and where the elements of the matrix $\mathbf{T}$ are

$$\begin{aligned}
T_{11}&=e^{\mp 2i\beta}w\partial f_0/\partial w,\\
T_{21}&=e^{2i\alpha}w\partial f_0/\partial w,\\
T_{31}&=\sqrt{2}e^{i(\alpha\mp\beta)}u\partial f_0/\partial w,\\
T_{12}&=e^{-2i\alpha}w\partial f_0/\partial w,\\
T_{22}&=e^{\pm 2i\beta}w\partial f_0/\partial w,\\
T_{32}&=\sqrt{2}e^{-i(\alpha\mp\beta)}u\partial f_0/\partial w,\\
T_{13}&=\sqrt{2}e^{-i(\alpha\pm\beta)}u\partial f_0/\partial u,\\
T_{23}&=\sqrt{2}e^{i(\alpha\pm\beta)}u\partial f_0/\partial u,\\
T_{33}&=2u\partial f_0/\partial u.
\end{aligned} \tag{80}$$

The first index on T_{ij} indicates the row, the second the column.

Three of the four integrations indicated in Eq. (79), effectively those over w, u, and α, can be performed (see Appendix II). The result is, in terms of the elements of $\mathbf{R}'$,

$$\begin{aligned}
R_{11}'&=s^2+\tfrac{1}{2}c^2k^2(1+\cos^2\theta)+\textstyle\sum\omega^2M_\mp,\\
R_{21}'&=-\tfrac{1}{2}c^2k^2\sin^2\theta+\textstyle\sum\omega^2L,\\
R_{31}'&=-2^{-\frac{1}{2}}c^2k^2\sin\theta\cos\theta+\textstyle\sum\omega^2N_\mp,\\
R_{12}'&=-\tfrac{1}{2}c^2k^2\sin^2\theta+\textstyle\sum\omega^2L,\\
R_{22}'&=s^2+\tfrac{1}{2}c^2k^2(1+\cos^2\theta)+\textstyle\sum\omega^2M_\pm,\\
R_{32}'&=-2^{-\frac{1}{2}}c^2k^2\sin\theta\cos\theta+\textstyle\sum\omega^2N_\pm,\\
R_{13}'&=-2^{-\frac{1}{2}}c^2k^2\sin\theta\cos\theta+\textstyle\sum\omega^2N_\mp,\\
R_{23}'&=-2^{-\frac{1}{2}}c^2k^2\sin\theta\cos\theta+\textstyle\sum\omega^2N_\pm,\\
R_{33}'&=s^2+c^2k^2\sin^2\theta+\textstyle\sum\omega^2W,
\end{aligned} \tag{81}$$

where again the upper sign refers to positive particles, and where

$$\begin{aligned}
L&=\lambda(\partial/\partial\lambda)\Big\{(s/\Omega)\int_0^\infty dy\times\exp[-(s/\Omega)y-\lambda(1-\cos y)-\tfrac{1}{2}\mu y^2]\Big\},\\
M_\pm&=[1+\lambda(\partial/\partial\lambda)]\Big\{(s/\Omega)\int_0^\infty dy\times\exp[-(s/\Omega\mp i)y-\lambda(1-\cos y)-\tfrac{1}{2}\mu y^2]\Big\},\\
N_\pm&=-(\lambda\mu)^{\frac{1}{2}}(s/\Omega)\int_0^\infty dy\sin\tfrac{1}{2}y\times\exp[-(s/\Omega\mp\tfrac{1}{2})y-\lambda(1-\cos y)-\tfrac{1}{2}\mu y^2],\\
W&=[1+2\mu(\partial/\partial\mu)]\Big\{(s/\Omega)\int_0^\infty dy\times\exp[-(s/\Omega)y-\lambda(1-\cos y)-\tfrac{1}{2}\mu y^2]\Big\}.
\end{aligned} \tag{82}$$

As usual $\lambda=KTk^2\sin^2\theta/m$, $\mu=KTk^2\cos^2\theta/m$, $\Omega=Ze\mathfrak{B}_0/mc$, and $\omega^2=4\pi Nze^2/m$.

Consider first the situation $T=0$. This corresponds to the case where the organized velocity is very much greater than the random velocity. Then $L=0$, $M_\pm=s/(s\mp i\Omega)$, $N_\pm=0$, and $W=1$. If $\theta=0$, the dispersion relation $|\mathbf{R}'|=0$ reads, for a plasma composed of an equal number of electrons and protons,

$$[s^2+c^2k^2+\textstyle\sum\omega^2s/(s\pm i\Omega)]\times[s^2+c^2k^2+\textstyle\sum\omega^2/(s\mp i\Omega)][s^2+\textstyle\sum\omega^2]=0. \tag{83}$$

One solution of Eq. (83) is $s^2=-\sum\omega^2$, which corresponds to an electron plasma oscillation along $\mathfrak{B}_0$, since $\omega_-^2\gg\omega_+^2$. The other solutions of (83) are of two kinds. First there are high-frequency solutions $s\sim ick$ for which the proton motion can be neglected. These can be obtained by writing $s^2+c^2k^2+\omega_-^2 s/s(s\pm i\Omega_-)=0$, a result which corresponds to Eq. (4–13) of Spitzer.[15]

The second class corresponds to hydromagnetic waves for which $|s^2|\ll c^2k^2,\ \Omega_+^2$. In this case one can write for the first factor in Eq. (83) (similar results are obtained from the second),

$$\begin{aligned}0&=s^2+c^2k^2+\sum \omega^2 s/(s\mp i\Omega)\\&=s^2+c^2k^2+(s\omega_+^2/i\Omega_+)[1+(s/i\Omega_+)+\cdots]\\&\qquad-(s\omega_-^2/i\Omega_-)[1-(s/i\Omega_-)+\cdots].\end{aligned}\tag{84}$$

But $\omega_+^2/\Omega_+=\omega_-^2/\Omega_-=4\pi Nec/\mathfrak{B}_0$. Thus, if we neglect small terms, Eq. (84) reads

$$0=c^2k^2+s^2\omega_+^2/\Omega_+^2,\tag{85}$$

whence

$$-s^2/k^2=\mathfrak{B}_0^2/4\pi Nm_+.\tag{86}$$

The expression on the right above is the well-known expression for the square of the phase velocity of a hydromagnetic wave.

The case $\theta=\pi/2$ yields the dispersion relation

$$\begin{aligned}0=\{&[s^2+\tfrac{1}{2}c^2k^2+\sum \omega^2 s/(s\pm i\Omega)]\\&\times[s^2+\tfrac{1}{2}c^2k^2+\sum \omega^2 s/(s\mp i\Omega)]\\&-\tfrac{1}{4}c^4k^4\}[s^2+c^2k^2+\sum \omega^2],\end{aligned}\tag{87}$$

where again the upper sign refers to the ions. Consider first the high-frequency solutions, for which the ion motion can be neglected. These are

$$\begin{aligned}-s^2=c^2k^2+\omega_-^2,\ \omega_-^2\pm\tfrac{1}{2}(\Omega_-^2+c^2k^2)\\\pm[(\Omega_-^2-c^2k^2)^2+4\Omega_-^2\omega_-^2]^{\frac{1}{2}},\end{aligned}$$

a result already found by Gross. The low-frequency solutions which correspond to hydromagnetic waves are, as before, given by $-s^2/k^2=B_0^2/4\pi Nm_+$, after neglecting small quantities. Thus for these limiting cases the results are just those which hydromagnetic theory predicts.*

ACKNOWLEDGMENTS

The author would like to indicate his indebtedness to his colleagues in the Theoretical Group at Project Matterhorn, in particular to Dr. John M. Greene, and Dr. William Newcomb, for many stimulating discussions. He is also beholden to Professor Lyman Spitzer, Jr., for criticism and advice.

APPENDIX I. DIRECT THERMODYNAMIC PROOF OF THE ABSENCE OF TIME INCREASING SOLUTIONS

It is possible to show directly, without having to explicitly solve the linearized equations, that the most general small motions about thermal equilibrium of a fully ionized plasma in the absence of collisions cannot exhibit a monotonic increase in time. The proof is due to William Newcomb, and relies on the fact that in the absence of mechanisms for the degradation of energy (e.g., collisions) the entropy of the system must be a constant of the motion.

In order to effect the proof, note that if f_0 denotes the space- and time-independent Maxwell distribution of Eq. (2), which characterizes each kind of particle in thermal equilibrium, then the entropy S can be written

$$S=-K\sum\int d^3r\, d^3v[f\ln f-f_0\ln f_0].\tag{A-1}$$

The summation is extended over all the classes of particle present. The expression above is so normalized that $S=0$ for thermal equilibrium. If the departure from thermal equilibrium is of bounded extent at any finite time, Eq. (A-1) is well defined when one extends the integration in d^3r over all space. If one wishes to consider disturbances which vary spatially like $e^{i\mathbf{k}\cdot\mathbf{r}}$, the space integration can be considered as extending over a cube each edge of which is one wavelength $2\pi/k$ in length, and one face of which is perpendicular to $\mathbf{k}$.

The time rate of change of S is given by

$$\frac{dS}{dt}=-K\sum\int d^3r\, d^3v\left[\frac{\partial f}{\partial t}+\frac{\partial f}{\partial t}\ln f\right].\tag{A-2}$$

The term in $\partial f/\partial t$ alone vanishes in virtue of conservation of the total number of particles. The term in $(\partial f/\partial t)\ln f$ can be transformed by employing the Boltzmann equation (1) with $(\partial f/\partial t)_{\text{coll}}=0$ to yield, if q is the algebraic charge, and we revert to Latin letters for the electric and magnetic fields,

$$\frac{dS}{dt}=K\sum\int d^3r\, d^3v\ln f\left\{\frac{\partial}{\partial\mathbf{r}}\cdot[\mathbf{v}f]+\frac{\partial}{\partial\mathbf{v}}\cdot\left[\frac{q}{m}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right)f\right]\right\}.\tag{A-3}$$

If one integrates by parts so as to shift the differential operators on to $\ln f$, since all surface terms vanish, Eq. (A-3) results in

$$\frac{dS}{dt}=-K\sum\int d^3r\, d^3v\left[\mathbf{v}\cdot\frac{\partial f}{\partial\mathbf{r}}+\frac{q}{m}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial f}{\partial v}\right]=0.\tag{A-4}$$

Thus S is a constant of the motion.

It is also easily demonstrated from Eq. (1), and Maxwell's equations (13), and (14), that the total energy W, if $\mathbf{B}_0$ denotes the constant external magnetic

---

* *Note added in proof.*—The results of T. Pradhan [Phys. Rev. **107**, 1222 (1957)] for the propagation of electromagnetic waves along $\mathfrak{B}_0$ can be obtained by setting $\theta=0$ in Eq. (81) and writing $|R'|=0$.

field, is given by

$$W=\frac{1}{8\pi}\int d^3r[\mathbf{E}^2+\mathbf{B}^2-\mathbf{B}_0{}^2]+\sum\int d^3r\, d^3v\frac{m}{2}v^2[f-f_0], \quad \text{(A-5)}$$

is also a constant of the motion. Note that W has been normalized to zero for equilibrium.

If one expands each distribution function f, essentially in powers of the small initial departure from thermal equilibrium

$$f=f_0+f_1+f_2+\cdots, \quad \text{(A-6)}$$

and writes correspondingly

$$\mathbf{E}=0+\mathbf{E}_1+\mathbf{E}_2+\cdots,\quad \mathbf{B}=\mathbf{B}_0+\mathbf{B}_1+\mathbf{B}_2+\cdots, \quad \text{(A-7)}$$

then correct to second order in the parameter of smallness

$$S=-K\sum\int d^3r\, d^3v\left\{f_1\left(1+A-\frac{mv^2}{2KT}\right)+f_2\left(1+A-\frac{mv^2}{2KT}\right)+\frac{f_1{}^2}{2f_0}\right\}. \quad \text{(A-8)}$$

The constant $A=\frac{3}{2}\ln[N^{\frac{2}{3}}m/2\pi KT]$. The terms above in $(1+A)$ vanish in virtue of particle conservation. The terms in $mv^2/2KT$ can be transformed by employing Eq. (A-5) expressed correct to second order.

The result is

$$S=-K\left\{\left(\frac{1}{8\pi KT}\right)\int d^3r[\mathbf{E}_1{}^2+\mathbf{B}_1{}^2]+\tfrac{1}{2}\int d^3r\, d^3v\frac{f_1{}^2}{f_0}\right\}. \quad \text{(A-9)}$$

Note that the quantity in brackets above is the sum of essentially positive terms, while $S=\text{const}$ by Eq. (A-4). Thus there can be no solutions for which any of the quantities $\mathbf{E}_1$, $\mathbf{B}_1$, or f_1 increase monotonically and hence in particular exponentially in time, since the others could not compensate so as to keep S constant. Clearly, however, an exponential decrease in time is admissible, since then compensation can occur.

APPENDIX II. EVALUATION OF CHARACTERISTIC INTEGRALS

Consider a typical integral occurring in $\mathbf{Q}'$, namely [see Eqs. (2), (7), (10), (11), and (76)]

$$\begin{aligned}Q_{33}'=&\sum(\omega^2s/\Omega)\int_0^\infty dww\int_{-\infty}^\infty du\int_0^\infty d\beta\\&\times\int_\beta^{\beta+2\pi}d\alpha(2mu^2/KT)(m/2\pi KT)^{\frac{3}{2}}\\&\times\exp\{-m(u^2+w^2)/2KT-2i\beta ku\cos\theta/\Omega\\&\qquad-2i\cos\alpha\sin\beta kw\sin\theta/\Omega-2\beta s/\Omega\}\\=&\sum(\omega^2s/\Omega)\int_0^\infty dww\int_{-\infty}^\infty du\int_0^\infty d\beta\\&\times\int_0^{2\pi}d\alpha(2mu^2/KT)(m/2\pi KT)^{\frac{3}{2}}\\&\times\exp\{-(m/2KT)[u+2i\beta k\cos\theta KT/m\Omega]^2\\&-2\beta^2k^2KT\cos^2\theta/m\Omega^2-2\beta s/\Omega\\&\qquad-2i\cos\alpha\sin\beta kw\sin\theta/\Omega\}. \quad \text{(A-10)}\end{aligned}$$

The integral over u is elementary. The integral over α can be effected by noting that[12]

$$2\pi J_0(z)=\int_0^{2\pi}d\alpha e^{iz\cos\alpha}. \quad \text{(A-11)}$$

The result is

$$\begin{aligned}Q_{33}'=&\sum(\omega^2s/\Omega)\int_0^\infty d\beta\int_0^\infty dww(2m/KT)(1-4\mu\beta^2)\\&\times J_0[2(\lambda m/KT)^{\frac{1}{2}}w\sin\beta]\\&\times\exp\{-(mw^2/2KT)-2\mu\beta^2-2\beta s/\Omega\}, \quad \text{(A-12)}\end{aligned}$$

where $\lambda=KTk^2\sin^2\theta/m$, and $\mu=KTk^2\cos^2\theta/m$. One can next carry out the integration over w by employing the formula[12]

$$\int_0^\infty dww^{\nu+1}J_\nu(aw)\exp(-wp^2)=a^\nu(2p^2)^{-\nu-1}\exp(-a^2/4p^2). \quad \text{(A-13)}$$

The result is, if one sets $2\beta=y$,

$$\begin{aligned}Q_{33}'=&\sum(\omega^2s/\Omega)\int_0^\infty dy[1-\mu y^2]\\&\times\exp\{-2\lambda\sin^22y-\tfrac{1}{2}\mu y^2-(s/\Omega)y\}\\=&\sum(\omega^2s/\Omega)\left[1+2\mu\frac{\partial}{\partial\mu}\right]\int_0^\infty dy\\&\times\exp\{-(s/\Omega)y-\lambda(1-\cos y)-\tfrac{1}{2}\mu y^2\}. \quad \text{(A-14)}\end{aligned}$$