

On the Darwin Lagrangian

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In this paper we explore some surprising consequences of the retardation effects of Maxwell's electrodynamics to a system of charged particles. The specific cases of three interacting particles are considered in the framework of classical electrodynamics. We show that the solutions of the equations of motion defined by the Darwin Lagrangian in some cases contradict to common sense.

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Darwin Lagrangian for interacting particles is an approximate one first derived by Darwin in 1920 [1]. This Lagrangian is considered to be correct to the order of $1/c^2$ inclusive [2], [3], [4]. To this order, we can eliminate the radiation modes from the theory and describe the interaction of charged particles in pure action-at-a-distance terms. Although the Darwin Lagrangian has had its most celebrated application in the quantum-mechanical context of the Breit interaction, it has uses in the purely classical domain [3] - [5]. In this paper we explore some surprising consequences of the retardation effects of Maxwell's electrodynamics to a system of charged particles. The specific cases of three interacting particles are considered in the framework of classical electrodynamics.

Below we will present the detailed and typical derivation of the Darwin Lagrangian and Hamiltonian for a system of charged particles to correct some misprints made by authors. Then we will show that the solutions of the equations of motion defined by the Darwin Lagrangian in some cases contradict to common sense.

The Lagrangian for a particle of a charge e_a in the external field of another particle of a charge e_b is

$$L_a = -m_a c^2 / \gamma_a - e_a \phi_b + \frac{e_a}{c} \vec{A}_b \cdot \vec{v}_a, \quad (1)$$

where m_a is the mass of the particle a , c the light velocity, $\gamma_a = 1/\sqrt{1-\beta_a^2}$ relativistic factor of the particle a , $\beta_a = |\vec{v}_a/c|$, \vec{v}_a the vector of a velocity of the particle a , ϕ_b and \vec{A}_b the scalar and vector retarded potentials produced by the particle b .

The scalar and vector potentials of the field produced by the charge b at the position of the charge a can be expressed in terms of the coordinates and velocities of the particle b (for ϕ_b to the terms of order $(v_b/c)^2$, and for \vec{A}_b , to terms (v_b/c))

$$\phi_b = \frac{e_b}{R_{ab}}, \quad \vec{A}_b = \frac{e_b [\vec{v}_b + (\vec{v}_b \cdot \vec{R}_{ab}) \vec{R}_{ab} / R_{ab}^2]}{2c R_{ab}}, \quad (2)$$

where $R_{ab} = |\vec{R}_{ab}|$, $\vec{R}_{ab} = \vec{R}_a - \vec{R}_b$, \vec{R}_a and \vec{R}_b are the radius-vectors of the particles a, b respectively, $v_b = |\vec{v}_b|$, \vec{v}_b the vector of a velocity of the particle b [2].

Substituting these expressions in (1), we obtain the Lagrangian L_a for the particle a (for a fixed motion of the other particles b). The Lagrangian of the total system of particles is

$$L = L^p + L^{int}, \quad (3)$$

where the Lagrangian of the system of free particles L^p and the Lagrangian of the interaction of particles L^{int} are

$$L^p = - \sum_a m_a c^2 / \gamma_a \simeq - \sum_a m_a c^2 + \sum_a \frac{m_a c^2 \beta^2}{2} + \sum_a \frac{m_a c^2 \beta^4}{8},$$

$$L^{int} = - \sum_{a>b} \frac{e_a e_b}{R_{ab}} + \sum_{a>b} \frac{e_a e_b}{2R_{ab}} \vec{\beta}_a \vec{\beta}_b + \sum_{a>b} \frac{e_a e_b}{2R_{ab}^3} (\vec{\beta}_a \vec{R}_{ab}) (\vec{\beta}_b \vec{R}_{ab}).$$

The equation of motion of a particle a is described by the equation $d\vec{P}_a/dt = \partial L / \partial \vec{R}_a$, where $\vec{P}_a = \partial L / \partial \vec{v}_a$ is the canonical momentum of the particle. This equation according to (3) can be presented in the form (see Appendix A)

$$\frac{d\vec{p}_a}{dt} = \sum_{a>b} \frac{e_a e_b}{R_{ab}^3} (1 - \vec{\beta}_a \vec{\beta}_b) \vec{R}_{ab} + \sum_{a>b} \frac{e_a e_b}{R_{ab}^3} (\vec{R}_{ab} \vec{\beta}_a) \vec{\beta}_b + \sum_{a>b} \frac{e_a e_b}{2R_{ab}^3} \beta_b^2 \vec{R}_{ab}$$

$$- \sum_{a>b} \frac{3e_a e_b}{2R_{ab}^5} (\vec{R}_{ab} \vec{\beta}_b)^2 \vec{R}_{ab} - \sum_{a>b} \frac{e_a e_b}{2c} \left[\frac{\dot{\vec{\beta}}_b}{R_{ab}} + \frac{(\vec{R}_{ab} \dot{\vec{\beta}}_b) \vec{R}_{ab}}{R_{ab}^3} \right]. \quad (4)$$

where $\vec{p}_a = m_a \gamma_a \vec{v}_a$ is the kinetic (non-canonical) momentum of the particle a .

The Hamiltonian of a system of charges in the same approximation must be done by the general rule for calculating H from L ($H = \vec{v}_a \vec{P}_a - L$). According to (3) (see Appendix A) the value

$$H = H^p + H^{int}, \quad (5)$$

where

$$H^p = \sum_a m_a c^2 \gamma_a = \sum_a \sqrt{m_a^2 c^4 + p_a^2 c^2} \simeq \sum_a m_a c^2 + \sum_a \frac{p_a^2}{2m_a} - \sum_a \frac{p_a^4}{8c^2 m_a^3},$$

$$H^{int} = \sum_{a>b} \frac{e_a e_b}{R_{ab}} + \sum_{a>b} \frac{e_a e_b}{2c^2 m_a m_b R_{ab}} \vec{p}_a \vec{p}_b + \sum_{a>b} \frac{e_a e_b}{2c^2 m_a m_b R_{ab}^3} (\vec{p}_a \vec{R}_{ab}) (\vec{p}_b \vec{R}_{ab}).$$

The constant value $\sum_a m_a c^2$ in (5) can be omitted. Here we would like to note that contrary to [2] the last two items in the term H^{int} of the equation (5) has the positive sign and the momentum $\vec{p}_a = m_a \gamma_a \vec{v}_a$ includes γ -factor of the particle ($\gamma \simeq 1 + \beta^2/2 + 3\beta^4/8$). The Hamiltonian expressed through the canonical momentum has the form (5), where the ordinary momentum \vec{p}_a is replaced by the canonical one \vec{P}_a and the signs of the last two terms are changed [1]¹. When the particles are moving in the external electromagnetic field then the term $\sum_a e_a \phi - e_a (\vec{P}_a \vec{A})/m_a c + e_a^2 |\vec{A}|^2/2m_a c^2$ is included in the Hamiltonian, where ϕ and \vec{A} are the external scalar and vector potentials. In [1] the term $e_a^2 |\vec{A}|^2/2m_a c^2$ is omitted.

The Lagrangian (3) does not depend on time. That is why the Hamiltonian (5) is the energy of the system [1].

Further we consider a special case when particles are moving along the axis x (see Fig.1). In this case the Lagrangian and Hamiltonian of the system of particles are described by the expressions

$$L = - \sum_a m_a c^2 / \gamma_a - \sum_{a>b} \frac{e_a e_b}{R_{ab}} \left(1 - \frac{\beta_a \beta_b}{2} \right), \quad (6)$$

$$H = \sum_a \sqrt{m_a^2 c^4 + p_a^2 c^2} + \sum_{a>b} \frac{e_a e_b}{R_{ab}} \left(1 + \frac{p_a p_b}{c^2 m_a m_b} \right), \quad (7)$$

where β_i, p_i are the x -components of the particle relative velocity and kinetic momentum respectively.

The x -component of the force applied to the particle a from the particle b according to (4) in this case is $dp_a/dt = e_a e_b / R_{ab}^2 \gamma_b^2 - e_a e_b \dot{\beta}_b / c R_{ab}$. This force corresponds to the electric field strength $\vec{E}_b = -\nabla \phi_b - (1/c)(\partial \vec{A}_b / \partial t)$ produced by the particle b and determined by the equations (2).

As was to be expected in the case of the uniform movement of the particle b ($\dot{\beta}_b = 0$; the case $m_b \gg m_a$) the electric field strength produced by the particle b in the direction of its movement is γ_b^2 times less then in the state of rest.

Next we consider the dynamics of three particles a, b, d according to the Darwin Lagrangian and Hamiltonian. Let particles a, b have charges $e_a = e_b = e > 0$, masses $m_a = m_b = m$ and velocities $v_a = -v_b = v = c\beta$. The particle d is located at the position $x = 0$ at rest ($v_d = 0$) its charge and mass are q, M .

In this case the Hamiltonian is the energy of the system which according to (7) can be presented in the form

¹In [2] the Hamiltonian includes small letters for momentum $\vec{p}_a = m_a \vec{v}_a$ that is \vec{p}_a in [2] is the kinetic momentum. It differ from (5) because of its derivation is based on erroneous connection of small corrections to Lagrangian and Hamiltonian. If the Lagrangian have the form $L = L_0 + L_1$ then without any approximation $H = H_0 + H_1$, where $H_0 = \sum_{a>b} \vec{v}_a \vec{P}_{a0} - L_0$, $H_1 = \sum_{a>b} \vec{v}_a \vec{P}_{a1} - L_1$, $\vec{P}_a = \vec{P}_{a0} + \vec{P}_{a1}$, $\vec{P}_{a0} = \partial L_0 / \partial \vec{v}_a$, $\vec{P}_{a1} = \partial L_1 / \partial \vec{v}_a$ is the extra term to the canonical (conjugate) momentum. In [2] this connection was used but the term $\sum_{a>b} \vec{v}_a \vec{P}_{a1}$ was omitted. In our case this term differ from zero as L_1 depends on velocity. At the same time if we will start from the definition $H = \vec{v}_a \partial L / \partial \vec{v}_a - L$ then we will receive (5) [1].

$$H = Mc^2 + 2mc^2\gamma_0 = Mc^2 + 2mc^2\gamma + \frac{e^2}{2R\gamma^2} + \frac{2eq}{R}, \quad (8)$$

where γ_0 is the initial relativistic factor of the particles a, b corresponding to the limit $R \rightarrow \infty$, $R = |\vec{R}_a|$ the distance between the particle a and the origin of the coordinate system.

It follows from the equation (8) the dependence between the distance R and the γ -factor of the particles a, b

$$R = \frac{e^2/2\gamma^2 + 2eq}{2mc^2(\gamma_0 - \gamma)}. \quad (9)$$

1. We can see that when $q > -e/4\gamma_0^2$ then the turning point exist at which $p = v = 0$ and $\gamma = 1$. According to (8) the minimal distance between particle a and the origin of the coordinate system

$$R_{min} = \frac{e^2 + 4eq}{4mc^2(\gamma_0 - 1)} = \frac{e^2 + 4eq}{4T_0}, \quad (10)$$

where T_0 is the initial kinetic energy of the particle a . The value $eU = (e^2 + 4eq)/2R_{min}$ is the potential energy of two particles at rest at the position of the turning point. According to (10) the value $R_{min} > r_a/2$, where $r_a = e^2/m_a c^2$ is the classical radius of the particle a .

In that case according to (10) and in conformity with the energy conservation law the potential energy of two particles at the turning point is equal to the initial kinetic energy of the particles $2T_0$. Retardation does not lead to any results which are contradict to common sense. The term in the electric field strength and in the force (4) which is determined by the acceleration will compensate the decrease of the repulsive forces corresponding to the uniformly moving particles.

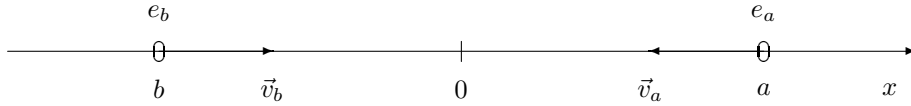


Fig.1. A scheme of two particle interaction.

2. When $q = -e/4\gamma_0^2$ then according to (4), (7) the particles a, b are moving uniformly ($\dot{\beta} = 0$, $v = v_0$, $\gamma = \gamma_0$). In that case particles can reach the distance $R = x = 0$, which is not reachable for them under the condition of the same energy expense $2T_0$ and a non-relativistic bringing closer of the particles. This conclusion is valid in the arbitrary relativistic case as in this case there is no emission of the electromagnetic radiation. It contradicts to common sense as the particles can be stopped at any position R to give back the kinetic energy $2T_0$ (in the form of heat and so on) and moreover contrary to the energy conservation law they will produce an extra energy $eU(R)$ under the process of slow moving aside of these particles under conditions of repulsive forces.

3. When $-e/4 < q < -e/4\gamma_0^2$ then the particles a, b will be brought closer under the condition of an acceleration by attractive forces and "fall in" toward each other. At the same time under such value of charge q of the particle d in the non-relativistic case the particles a, b will repel each other such a way that the position $R = x = 0$ will not be reachable for them if the same energy expense $2T_0$ will be used for slow bringing closer of the particles. In that case we have the same result which contradicts to common sense as well.

4. When $q = -e/4$, $\gamma_0 > 1$ then the particles will acquire an additional energy when bringing closer. After stop by extraneous forces at any position R to give back the kinetic energy $2T > 2T_0$ the particles will not experience any force.

5. When $q < -e/4$ then the particles will acquire the higher value of the energy then necessary for non-relativistic separation of the particles. The particles can be stopped by extraneous forces at some distance between them and then separated. Some gain of energy will take place as well.

In the cases (3), (5) the velocities of particles may be compared with the light velocity when Darwin Lagrangian does not valid because of in that case the radiation can not be neglected. But the process of "fall in" will be kept. In the case (5) the unphysical solution can appear when particles will be stopped at the distance $R \ll r_e$ and the total energy of the system at this position (new Hamiltonian) will be negative ($eU(R) + Mc^2 + 2m_a c^2 < 0$). This result is the known fact for a system of two particles of the opposite sign which is beyond of the present consideration.

This curious results are the reminiscent of the non-consistency of the classical Maxwell-Lorentz electrodynamics. The existence of these solutions is a genuine effect of electrodynamics with retardation.

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APPENDIX A:

The canonical momentum of the particle a is

$$\vec{P}_a = \frac{\partial L}{\partial \vec{v}_a} = \vec{p}_a + \Delta \vec{p}_a, \quad (\text{A1})$$

where

$$\Delta \vec{p}_a = \sum_{b \neq a} \frac{e_a e_b}{2c} \left[\frac{\vec{\beta}_b}{R_{ab}} + \frac{\vec{R}_{ab}(\vec{R}_{ab}\vec{\beta}_b)}{R_{ab}^3} \right].$$

The time derivative of the canonical momentum is

$$\frac{d\vec{P}_a}{dt} = \frac{d}{dt} \frac{\partial L}{\partial \vec{v}_a} = \dot{\vec{p}}_a + \Delta \vec{F}_a, \quad (\text{A2})$$

where $\dot{\vec{p}}_a = d\vec{p}_a/dt$, $\Delta \vec{F}_a = d(\Delta \vec{p}_a)/dt$ or

$$\begin{aligned} \Delta \vec{F}_a = & \sum_{b \neq a} \frac{e_a e_b}{2R_{ab}^3} [\vec{R}_{ab}(\vec{\beta}_a - \vec{\beta}_b, \vec{\beta}_b) + (\vec{\beta}_a - \vec{\beta}_b)(\vec{R}_{ab}\vec{\beta}_b)] - \vec{\beta}_b(\vec{R}_{ab}, \vec{\beta}_a - \vec{\beta}_b) \\ & - \sum_{b \neq a} \frac{3e_a e_b}{2R_{ab}^5} (\vec{R}_{ab}\vec{\beta}_b)(\vec{R}_{ab}, \vec{\beta}_a - \vec{\beta}_b) \vec{R}_{ab} - \sum_{b \neq a} \frac{e_a e_b}{2c} \left[\frac{\dot{\vec{\beta}}_b}{R_{ab}} + \frac{\vec{R}_{ab}(\dot{\vec{R}}_{ab}\vec{\beta}_b)}{R_{ab}^3} \right]. \end{aligned}$$

The directional derivative of the Lagrangian is

$$\frac{\partial L}{\partial \vec{R}_a} = \sum_{a>b} \frac{e_a e_b \vec{R}_{ab}}{R_{ab}^3} \left(1 - \frac{\vec{\beta}_a \vec{\beta}_b}{2}\right) + \sum_{a>b} \frac{e_a e_b}{2R_{ab}^3} [\vec{\beta}_a(\vec{R}_{ab}\vec{\beta}_b) + \vec{\beta}_b(\vec{R}_{ab}\vec{\beta}_a)] - \sum_{a>b} \frac{3e_a e_b}{2R_{ab}^5} \vec{R}_{ab}(\vec{R}_{ab}\vec{\beta}_a)(\vec{R}_{ab}\vec{\beta}_b). \quad (\text{A3})$$

From the equation of motion and equations (A2),(A3) it follows the equation (4).

The value $\vec{v}_k \vec{P}_k$ and the Hamiltonian are equal respectively

$$\vec{v}_a \vec{P}_a = \sum_{a \neq b} \frac{e_a e_b}{2} \left[\frac{\vec{\beta}_a \vec{\beta}_b}{R_{ab}} + \frac{(\vec{\beta}_a \vec{R}_{ab})(\vec{\beta}_b \vec{R}_{ab})}{P_{ab}^3} \right] + m_a c^2 \gamma_a \beta_a^2, \quad (\text{A4})$$

$$\begin{aligned} H = \sum_a \vec{v}_a \vec{P}_a - L = & \sum_a \sqrt{m_a^2 c^4 + p_a^2 c^2} + \sum_{a>b} \frac{e_a e_b}{R_{ab}} \left[1 + \frac{c^2 (\vec{p}_a \vec{p}_b)}{2\sqrt{m_a^2 c^4 + c^2 p_a^2} \sqrt{m_b^2 c^4 + c^2 p_b^2}} \right. \\ & \left. + \frac{c^2 (\vec{R}_{ab} \vec{p}_a)(\vec{R}_{ab} \vec{p}_b)}{2R_{ab}^2 \sqrt{m_a^2 c^4 + c^2 p_a^2} \sqrt{m_b^2 c^4 + c^2 p_b^2}} \right]. \end{aligned} \quad (\text{A5})$$

In the approximation $(1/c^2)$ the Hamiltonian (A5) leads to (5).

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