ON THE DARWIN LAGRANGIAN

E.G. BESSONOV
Lebedev Physical Institute AS, 117924, Leninsky prospect 53, Moscow, Russia
E-mail: bessonov@sgi.lpi.msk.su

In this paper we explore some consequences of the retardation effects of Maxwell's electrodynamics to a system of charged particles. The specific cases of three interacting particles are considered in the framework of classical electrodynamics. We show that the solutions of the equations of motion defined by the Darwin Lagrangian in some cases contradict to the energy conservation law.

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1 Introduction

Darwin Lagrangian for interacting particles is an approximate one first derived by Darwin in 1920 [1]. This Lagrangian is considered to be correct to the order of $1/c^2$ inclusive [1−4]. To this order, we can eliminate the radiation modes from the theory and describe the interaction of charged particles in pure action-at-a-distance terms. Although the Darwin Lagrangian has had its most celebrated application in the quantum-mechanical context of the Breit interaction, it has uses in the purely classical domain [3−6]. In this paper we explore some consequences of the retardation effects of Maxwell's electrodynamics to a system of charged particles. The specific cases of three interacting particles are considered in the framework of classical electrodynamics.

Below we will present the detailed and typical derivation of the Darwin Lagrangian and Hamiltonian for a system of charged particles to correct some misprints made by some authors. Then we will show that the solutions of the equations of motion defined by the Darwin Lagrangian in some cases contradict to the energy conservation law.

2 Derivation of the Darwin Lagrangian and consequences

The Lagrangian for a particle of a charge e_a in the external field of an another particle of a charge e_b is

$$L_a = -m_a c^2/\gamma_a - e_a \varphi_b + \frac{e_a}{c}\vec{A}_b \cdot \vec{v}_a, \tag{1}$$

where m_a is the mass of the particle a, c the light velocity, $\gamma_a = 1/\sqrt{1-\beta_a^2}$ relativistic factor of the particle a, $\beta_a = |\vec{v}_a/c|$, $\vec{v}_a$ the vector of a velocity of the particle a, φ_b and $\vec{A}_b$ the scalar and vector retarded potentials produced by the particle b.

The scalar and vector potentials of the field produced by the charge b at the position of the charge a can be expressed in terms of the coordinates and velocities of the particle b (for φ_b to the terms of order v_b^2/c^2, and for $\vec{A}_b$, to terms v_b/c)

$$\varphi_b = \frac{e_b}{R_{ab}}, \qquad \vec{A}_b = \frac{e_b[\vec{v}_b + (\vec{v}_b \cdot \vec{R}_{ab})\vec{R}_{ab}/R_{ab}^2]}{2cR_{ab}}, \tag{2}$$

where $R_{ab} = |\vec{R}_{ab}|$, $\vec{R}_{ab} = \vec{R}_a - \vec{R}_b$, $\vec{R}_a$ and $\vec{R}_b$ are the radius-vectors of the particles a, b respectively, $v_b = |\vec{v}_b|$, $\vec{v}_b$ the velocity-vector of the particle b [2].

Substituting these expressions in (1), we obtain the Lagrangian L_a for the particle a (for a fixed motion of the other particles b). The Lagrangian of the total system of particles is

$$L = L^p + L^{int}, \tag{3}$$

where the Lagrangian of the system of free particles L^p and the Lagrangian of the interaction of particles L^{int} are

$$L^p = -\sum_a m_a c^2/\gamma_a \simeq -\sum_a m_a c^2 + \sum_a \frac{m_a c^2 \beta^2}{2} + \sum_a \frac{m_a c^2 \beta^4}{8},$$

$$L^{int} = -\sum_{a>b} \frac{e_a e_b}{R_{ab}} + \sum_{a>b} \frac{e_a e_b}{2R_{ab}} \vec{\beta}_a \vec{\beta}_b + \sum_{a>b} \frac{e_a e_b}{2R_{ab}^3} (\vec{\beta}_a \vec{R}_{ab})(\vec{\beta}_b \vec{R}_{ab}).$$

The motion of a particle a is described by the equation $d\vec{P}_a/dt = \partial L/\partial \vec{R}_a$, where $\vec{P}_\alpha = \partial L/\partial \vec{v}_a$ is the canonical momentum of the particle. This equation according to (3) can be presented in the form (see Appendix)

$$\frac{d\vec{p}_a}{dt} = \sum_{a>b} \frac{e_a e_b}{R_{ab}^3} (1 - \vec{\beta}_a \vec{\beta}_b) \vec{R}_{ab} + \sum_{a>b} \frac{e_a e_b}{R_{ab}^3} (\vec{R}_{ab} \vec{\beta}_a) \vec{\beta}_b + \sum_{a>b} \frac{e_a e_b}{2R_{ab}^3} \beta_b^2 \vec{R}_{ab}$$

$$- \sum_{a>b} \frac{3 e_a e_b}{2R_{ab}^5} (\vec{R}_{ab} \vec{\beta}_b)^2 \vec{R}_{ab} - \sum_{a>b} \frac{e_a e_b}{2c} [\frac{\dot{\vec{\beta}} b}{R_{ab}} + \frac{(\vec{R}_{ab} \dot{\vec{\beta}}_b) \vec{R}_{ab}}{R_{ab}^3}]. \tag{4}$$

where $\vec{p}_a = m_a\gamma_a\vec{v}_a$ is the kinetic momentum of the particle a.

The Hamiltonian of a system of charges in the same approximation must be done by the general rule for calculating H from L ($H = \vec{v}_a\vec{P}_a - L$). According to (3) the value (see Appendix)

$$H = H^p + H^{int}, \tag{5}$$

where

$$H^p = \sum_a m_a c^2\gamma_a = \sum_a \sqrt{m_a^2c^4 + p_a^2c^2} \simeq \sum_a m_a c^2 + \sum_a \frac{p_a^2}{2m_a} - \sum_a \frac{p_a^4}{8c^2m_a^3},$$

$$H^{int} = \sum_{a>b}\frac{e_ae_b}{R_{ab}} + \sum_{a>b}\frac{e_ae_b}{2c^2m_am_bR_{ab}}\vec{p}_a\vec{p}_b + \sum_{a>b}\frac{e_ae_b}{2c^2m_am_bR_{ab}^3}(\vec{p}_a\vec{R}_{ab})(\vec{p}_b\vec{R}_{ab}).$$

The constant value $\sum_a m_a c^2$ in (5) can be omitted. Here we would like to note that contrary to expressions presented in [2] the last two items in the term H^{int} of the equation (5) has the positive sign and the momentum $\vec{p}_a = m_a\gamma_a\vec{v}_a$ includes γ-factor of the particle ($\gamma \simeq 1 + \beta^2/2 + 3\beta^4/8$). The Hamiltonian expressed through the canonical momentum has the form (5), where the ordinary momentum $\vec{p}_a$ is replaced by the canonical one $\vec{P}_a$ and the signs of the last two terms are changed [1] [a]. When the particles are moving in the external electromagnetic field then the term $\sum_a e_a\varphi - e_a(\vec{P}_a\vec{A})/m_ac + e_a^2|\vec{A}|^2/2m_ac^2$ is included in the Hamiltonian, where φ and $\vec{A}$ are the external scalar and vector potentials. In [1] the term $e_a^2|\vec{A}|^2/2m_ac^2$ is omitted.

The Lagrangian (3) does not depend on time. That is why the Hamiltonian (5) is the energy of the system [1].

When particles are moving along the axis x then the Hamiltonian of the system of particles is described by the expression

$$H = \sum_a \sqrt{m_a^2c^4 + p_a^2c^2} + \sum_{a>b}\frac{e_ae_b}{R_{ab}}(1 + \frac{p_ap_b}{c^2m_am_b}), \tag{6}$$

[a] In [2] the Hamiltonian includes small letters for momentum $\vec{p}_a = m_a\vec{v}_a$ that is $\vec{p}_a$ in [2] is the kinetic momentum. It differ from (5) because of its derivation is based on erroneous connection of small corrections to Lagrangian and Hamiltonian. If $L = L_0 + L_1$ then without any approximation $H = H_0 + H_1$, where $H_0 = \sum_{a>b}\vec{v}_a\vec{P}_{a0} - L_0$, $H_1 = \sum_{a>b}\vec{v}_a\vec{P}_{a1} - L_1$, $\vec{P}_a = \vec{P}_{a0} + \vec{P}_{a1}$, $\vec{P}_{a0} = \partial L_0/\partial\vec{v}_a$, $\vec{P}_{a1} = \partial L_1/\partial\vec{v}_a$ is the extra term to the canonical (conjugate) momentum. In [2] this connection was used but the term $\sum_{a>b}\vec{v}_a\vec{P}_{a1}$ was omitted. In our case this term differ from zero as L_1 depends on velocity. At the same time if we will start from the definition $H = \vec{v}_a\partial L/\partial\vec{v}_a - L$ then we will receive (5) [1].

where β_i, p_i are the x-components of the particle relative velocity and kinetic momentum respectively.

The x-component of the force applied to the particle a from the particle b according to (4) in this case is $dp_a/dt = e_a e_b / R_{ab}^2 \gamma_b^2 - e_a e_b \dot{\beta}_b / c R_{ab}$. This force corresponds to the electric field strength $\vec{E}_b = -\nabla \varphi_b - (1/c)(\partial \vec{A}_b / \partial t)$ produced by the particle b and determined by the equations (2). As was to be expected in the case of the uniform movement of the particle b ($\dot{\beta}_b = 0$; the case $m_b \gg m_a$) the electric field strength produced by the particle b in the direction of its movement is γ_b^2 times less then in the state of rest[b].

3 The dynamics of three particles

Next we consider the dynamics of three particles a, b, d according to the Darwin Lagrangian and Hamiltonian (see Fig.1). Let particles a, b have charges $e_a = e_b = e > 0$, masses $m_a = m_b = m$ and velocities $v_a = -v_b = v = c\beta$. The particle d is located at the position $x = 0$ at rest ($v_d = 0$), its charge and mass are q, M.

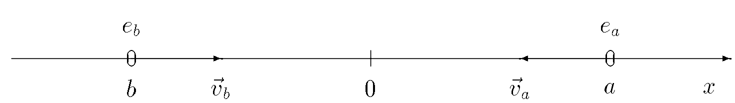

Fig.1. A scheme of two particle interaction.

In this case the Hamiltonian is the energy of the system which according to (6) can be presented in the form

$$H = Mc^2 + 2mc^2\gamma_0 = Mc^2 + 2mc^2\gamma + \frac{e^2}{2R\gamma^2} + \frac{2eq}{R}, \tag{7}$$

where γ_0 is the initial relativistic factor of the particles a, b corresponding to the limit $R \to \infty$, $R = |\vec{R}_a|$ the distance between the particle a and the origin of the coordinate system, $\gamma - 1 \simeq \beta^2/2 \ll 1$.

[b]In accordance with the fact that electric and magnetic fields defined by potentials (2) and expressions (3) - (5) take into account the retardation.

From the equation (7) we can receive the dependence between the distance R and the γ-factor of the particles a, b. In this case the next solutions exist. When $q \neq -e/4\gamma_0^2$, then

$$R = \frac{e^2/2\gamma^2 + 2eq}{2mc^2(\gamma_0 - \gamma)}. \tag{8}$$

When $q = -e/4\gamma_0^2$, then according to (4), (7) the particle is moving with the constant momentum, velocity and energy.

From these solutions it follows the next conclusions.

1. When $q > -e/4\gamma_0^2$ then the turning point exist at which $p = v = 0$ and $\gamma = 1$. According to (8) the minimal distance between particle a and the origin of the coordinate system

$$R_{min} = \frac{e^2 + 4eq}{4mc^2(\gamma_0 - 1)} = \frac{e^2 + 4eq}{4T_0}, \tag{9}$$

where T_0 is the initial kinetic energy of the particle a.

According to (9) the value $R_{min} > r_a/2$, where $r_a = e^2/m_a c^2$ is the classical radius of the particle a. In conformity with the energy conservation law the potential energy $eU = (e^2 + 4eq)/2R_{min}$ of two particles at the turning point is equal to the initial kinetic energy of the particles $2T_0$. Retardation does not lead to any results which contradict to common sense. The term in the electric field strength and in the force (4) which is determined by the acceleration of the opposite particle will compensate the decrease of the repulsive forces corresponding to the uniformly moving particles.

2. When $q = -e/4\gamma_0^2$ then according to (4), (6) the particles a, b are moving uniformly ($\dot{\beta} = 0$, $v = v_0, \gamma = \gamma_0$). In this case particles can reach the distance $R = x = 0$, which is not reachable for them under the condition of the same energy expense $2T_0$ and a non-relativistic bringing closer of the particles. This conclusion is valid in the arbitrary relativistic case as in this case there is no emission of the electromagnetic radiation. It contradicts to common sense as the particles can be stopped at any position R_s to give back the kinetic energy $2T_0$ (in the form of heat and so on) and moreover contrary to the energy conservation law they will produce an extra energy $eU(R_s) > 0$ under the process of slow moving aside of these particles by extraneous forces under conditions of repulsive electromagnetic forces.

3. When $-e/4 < q < -e/4\gamma_0^2$ then the particles a, b will be brought closer under the condition of an acceleration by attractive forces and "fall in" toward each other. At the same time under such value of charge q of the particle d

in the non-relativistic case the particles a, b will repel each other such a way that the position $R = x = 0$ will not be reachable for them if the same energy expense $2T_0$ will be used for slow bringing closer of the particles.

4. When $q = -e/4$, $\gamma_0 > 1$ then the particles a, b will be brought closer under the condition of an acceleration by attractive forces. After a stop the particles will not experience any force.

5. When $q < -e/4$ then the particles will be brought closer under the condition of an acceleration by attractive forces and "fall in" toward each other. At that there is no necessity in the initial acceleration of particles to the energy $mc^2\gamma_0$. They can be released at some finite distance with zero velocity.

In the cases (3) - (5) the velocities of particles can reach the values compared with the light velocity when Darwin Lagrangian does not valid. Nevertheless in these cases obviously the process of "fall in" and contradiction with the energy conservation law will take place as well. In the cases (3), (4) the sum of the kinetic energies of particles before a stop by extraneous forces will be higher than the initial ones ($2T > 2T_0$). Moreover the potential energy of the particles after the stop will be positive ($eU(R_s) \geq 0$).

In the case (5) the unphysical situation will appear when particles will be stopped at the distance $R_s \ll r_e$ and the total energy of the system at this position will be negative: $eU(R_s) + Mc^2 + 2m_ac^2 < 0$. This result is the known fact for a system of two particles of the opposite sign which is beyond of the present consideration. In this case the contradiction with the energy conservation law will take place as well as the attractive forces in the process of bringing particles closer will be higher then the attractive forces in the process of slow moving aside of these particles.

4 Conclusion

In the framework of classical electrodynamics there are many "open" or "perpetual" problems such as the problem of the particle stability, the problem of the self-energy and momentum of particles, the nature of the particle's mass, the problem of the runaway solutions. There is a spectrum of opinions concerning the importance and the ways of a finding of the answers on these questions [7]. Unfortunately the efforts of the majority of authors are directed to avoid similar questions but not to solve them (see e.g. [3,8,9,10]). In addition they base themselves on the laws of conservation of energy and momentum ostensibly following from the electrodynamics in the most general case and presenting electrodynamics as the consistent theory. In such stating the arising questions by their opinion do not have a physical subject of principle and the difficulties in their solution are on the whole in the field of the mathematicians [2]. In reality the corresponding equations for the energy and momentum including the system of particles and fields does not describe the laws of conservation of energy and momentum. These equations are not treated correct way in the existing textbooks. The electrodynamics of particles and fields is not consistent theory [11].

The result presented in this paper is the reminiscence of the non-consistency of the classical Maxwell-Lorentz electrodynamics. It can be considered as a new open question of the classical electrodynamics. The existence of the received solution is a genuine effect of electrodynamics of point particles with retardation [11].

5 Appendix

The canonical momentum of the particle a is

$$\vec{P}_a = \frac{\partial L}{\partial \vec{v}_a} = \vec{p}_a + \Delta\vec{p}_a, \tag{10}$$

where

$$\Delta\vec{p}_a = \sum_{b\neq a} \frac{e_a e_b}{2c}[\frac{\vec{\beta}_b}{R_{ab}} + \frac{\vec{R}_{ab}(\vec{R}_{ab}\vec{\beta}_b)}{R^3_{ab}}].$$

The time derivative of the canonical momentum is

$$\frac{d\vec{P}_a}{dt} = \frac{d}{dt}\frac{\partial L}{\partial \vec{v}_a} = \dot{\vec{p}}_a + \Delta\vec{F}_a, \tag{11}$$

where $\dot{\vec{p}}_a = d\vec{p}_a/dt$, $\Delta\vec{F}_a = d(\Delta\vec{p}_a)/dt$ or

$$\Delta\vec{F}_a = \sum_{b\neq a}\frac{e_a e_b}{2R^3_{ab}}[\vec{R}_{ab}(\vec{\beta}_a - \vec{\beta}_b, \vec{\beta}_b) + (\vec{\beta}_a - \vec{\beta}_b)(\vec{R}_{ab}\vec{\beta}_b) - \vec{\beta}_b(\vec{R}_{ab}, \vec{\beta}_a - \vec{\beta}_b)]$$

$$-\sum_{b\neq a}\frac{3e_a e_b}{2R^5_{ab}}(\vec{R}_{ab}\vec{\beta}_b)(\vec{R}_{ab}, \vec{\beta}_a - \vec{\beta}_b)\vec{R}_{ab} + \sum_{b\neq a}\frac{e_a e_b}{2c}[\frac{\dot{\vec{\beta}}_b}{R_{ab}} + \frac{\vec{R}_{ab}(\vec{R}_{ab}\dot{\vec{\beta}}_b)}{R^3_{ab}}].$$

The directional derivative of the Lagrangian is

$$\frac{\partial L}{\partial \vec{R}_a} = \sum_{a>b}\frac{e_a e_b \vec{R}_{ab}}{R^3_{ab}}(1 - \frac{\vec{\beta}_a\vec{\beta}_b}{2}) + \sum_{a>b}\frac{e_a e_b}{2R^3_{ab}}[\vec{\beta}_a(\vec{R}_{ab}\vec{\beta}_b) + \vec{\beta}_b(\vec{R}_{ab}\vec{\beta}_a)] -$$

$$\sum_{a>b} \frac{3e_a e_b}{2R_{ab}^5} \vec{R}_{ab}(\vec{R}_{ab}\vec{\beta}_a)(\vec{R}_{ab}\vec{\beta}_b). \tag{12}$$

From the equation of motion $d\vec{P}_a/dt = \partial L/\partial \vec{R}_a$ and equations (11),(12) it follows the equation (4).

The value $\vec{v}_k \vec{P}_k$ and the Hamiltonian $H = \vec{v}_a \partial L/\partial \vec{v}_a - L$ are equal

$$\vec{v}_a \vec{P}_a = \sum_{a \neq b} \frac{e_a e_b}{2} [\frac{\vec{\beta}_a \vec{\beta}_b}{\vec{R}_{ab}} + \frac{(\vec{\beta}_a \vec{R}_{ab})(\vec{\beta}_b \vec{R}_{ab})}{P_{ab}^3}] + m_a c^2 \gamma_a \beta_a^2, \tag{13}$$

$$H = \sum_a \vec{v}_a \vec{P}_a - L = \sum_a \sqrt{m_a^2 c^4 + p_a^2 c^2} + \sum_{a>b} \frac{e_a e_b}{R_{ab}} [1+$$

$$\frac{c^2(\vec{p}_a \vec{p}_b)}{2\sqrt{m_a^2 c^4 + c^2 p_a^2}\sqrt{m_b^2 c^4 + c^2 p_b^2}} + \frac{c^2(\vec{R}_{ab}\vec{p}_a)(\vec{R}_{ab}\vec{p}_b)}{2R_{ab}^2 \sqrt{m_a^2 c^4 + c^2 p_a^2}\sqrt{m_b c^4 + c^2 p_b^2}}]. \tag{14}$$

In the approximation $(1/c^2)$ the Hamiltonian (14) leads to (5).

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