Classical electrodynamics with vacuum polarization: electron self-energy and radiation reaction

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Abstract

The region very close to an electron ($r \lesssim r_0 = e^2/mc^2 \approx 2.8 \times 10^{-13}$ cm) is, according to quantum electrodynamics, a seething maelstrom of virtual electron-positron pairs flashing in and out of existence. To take account of this well-established physical reality, a phenomenological representation for vacuum polarization is introduced into the framework of classical electrodynamics. Such a model enables a consistent picture of classical point charges with finite electromagnetic self-energy. It is further conjectured that the reaction of a point charge to its own electromagnetic field is tantamount to interaction with its vacuum polarization charge or "aura." This leads to a modification of the Lorentz-Dirac equation for the force on an accelerating electron, a new differential-difference equation which avoids the pathologies of preacceleration and runaway solutions.

1 Introduction

The singularities in fields and energies associated with point charges in classical electrodynamics has been a pervasive flaw in what has been an otherwise beautifully complete and consistent theory. An immense number of attempts to address this problem have been based, roughly speaking, on one of the following lines of argument: (1) Actual point charges do *not* exist—*real* particles have a finite size—hence the problem is artificial; (2) By a clever limiting procedure in the formalism, the radius of a charge can be reduced to zero *without* introducing infinities; (3) Point charges are quantum objects and classical electrodynamics has no business dealing with them. The last point of view, espoused by Frenkel[1] and others, asserts that any classical model is futile because the electron is a quantum-mechanical object with no substructure. It is nonetheless of at least academic interest to have a consistent classical relativistic model which connects to macroscopic electrodynamics, while remaining cognizant of its limitations. The purpose of the present paper is a modified theory able to handle the singularities produced by point charges while reducing to standard electrodynamics for $r \gg r_0$.

We propose to provide possible finishing touches to Maxwell's electromagnetism *without* making any *ad hoc* modifications of the fundamental equations of the theory. The key to our approach is the physical reality of vacuum polarization in the submicroscopic vicinity of charged elementary particles. We begin therefore with a review of the problem in the context of particle physics.

2 Structure of the Electron

Since the discovery of the electron by J. J. Thomson a century ago, the structure of "the first elementary particle" has been the subject of extensive theoretical contemplation by some of the leading figures of 20th Century physics[2]. The earliest models (Thomson, Poincaré, Lorentz, Abraham, Schott)[3] pictured the electron as a finite charged sphere, on the scale of the classical electron radius $r_0 = e^2/mc^2 \approx 2.818 \times 10^{-13}$ cm. The electromagnetic self energy of such a finite structure would be of the order of $W \approx e^2/r_0 \approx mc^2$ and thus implies an electron rest mass predominantly electromagnetic in origin.

Yet all experimental evidence implies an electron radius much smaller than r_0, consistent, in fact, with a particle of point mass and point charge[4]. Recent results of high-energy electron-positron scattering[5] imply an upper limit of 2×10^{-16} cm on the electron size.

A number of ingeneous schemes to avoid a divergent electromagnetic self-energy for a point electron have been proposed over the years by Dirac[6], Wheeler and Feynman[7], Rohrlich[8], Teitelboim[9] and many others. The more recent approaches invoke such arcana as *advanced* solutions of Maxwell's equations (superposed on the conventional retarded solutions) and/or renormalization of mass and charge infinities. This enables the divergent part of the self-interaction to be avoided while leaving intact the radiation reaction, an effect long known and thoroughly tested.

We will proceed on the premise that the electron rest mass (0.511 MeV/c^2) is *totally* electromagnetic, which was the original idea of Lorentz and Abraham (see, however, Section 5). This is consistent with the (nearly!) zero rest mass of the electron's uncharged weak isodoublet partner—the neutrino—and with order of magnitude of the neutron-proton mass difference (1.29 MeV/c^2). There is no need to invoke any non-electromagnetic forces within the electron—collectively known as Poincaré stresses. It should be noted that theories have been proposed with counterbalancing *gravitational* fields[10] but these have been regarded with disfavor by Einstein[11] among others.

3 Stationary Point Charge

The energy of an electromagnetic field in a rest frame is given by

$$W = \frac{1}{8\pi} \int (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})\, d^3\mathbf{r} \tag{1}$$

The field produced by a point charge e in vacuum has $D = E = e/r^2$ and

$$W = \frac{1}{8\pi} \int \frac{e^2}{r^4}\, 4\pi r^2\, dr = \infty \tag{2}$$

unless a lower cutoff is introduced.

It was suggested a long time ago by Furry and Oppenheimer[12] that quantum-electrodynamic effects could give the vacuum some characteristics of a polarizable medium, which Weisskopf[13] represented phenomenologically by an inhomogeneous dielectric constant, viz

$$\mathbf{D}(r) = \epsilon(r)\mathbf{E}(r) \tag{3}$$

Accordingly,

$$W = \frac{1}{8\pi}\int_0^\infty \frac{1}{\epsilon(r)}\frac{e^2}{r^4}\,4\pi r^2\,dr \tag{4}$$

and equating this to the self-energy of the electron

$$W = \frac{e^2}{2}\int_0^\infty \frac{dr}{r^2\epsilon(r)} = mc^2 \tag{5}$$

Remarkably, the functional form of $\epsilon(r)$ need not be further specified, provided only that it satisfies the limiting conditions

$$\epsilon(\infty) = 1 \qquad \text{and} \qquad \epsilon(0) = \infty \tag{6}$$

Maxwell's first equation $\nabla\cdot\mathbf{E} = 4\pi\varrho$ applied to the electric field

$$\mathbf{E} = \frac{e\mathbf{r}}{\epsilon(r)r^3} \tag{7}$$

determines the charge density

$$\rho(r) = -\frac{e\,\epsilon'(r)}{4\pi r^2[\epsilon(r)]^2} \tag{8}$$

Note that this represents the *net* or *total* charge density, the sum of the free and polarization densities. This function is appropriately normalized since

$$\int_0^\infty \rho(r)\,4\pi r^2\,dr = -e\int_0^\infty \frac{\epsilon'(r)\,dr}{[\epsilon(r)]^2} = e\left[\frac{1}{\epsilon(\infty)} - \frac{1}{\epsilon(0)}\right] = e \tag{9}$$

An explicit functional form for $\epsilon(r)$ does follow if it is conjectured that the net charge density (8) is proportional to the field energy density from (5). For then,

$$\frac{\epsilon'(r)}{\epsilon(r)} = -\frac{e^2}{2mc^2r^2} \tag{10}$$

with the solution

$$\epsilon(r) = \exp\left(\frac{e^2}{2mc^2r}\right) = \exp\left(\frac{r_0}{2r}\right) \tag{11}$$

It should be emphasized for the benefit of QED theorists who might be reading this that our use of the term "vacuum polarization" is intended only in a classical phenomenological context. The leading contribution to vacuum polarization in real life comes from the interaction of the electron with the transverse radiation field, which does not enter in our model. We are thereby overlooking additional self-energy contributions arising from fluctuations in the vacuum radiation field. Accordingly, our representation of vacuum polarization is *not* to be compared with QED computations.

Somewhat of a rationalization for the functional form of $\epsilon(r)$ is suggested by Debye-Hückel theory for ionic solutions and plasmas. The dielectric constant depends on a Boltzmann factor $e^{-\mathcal{E}/kT}$. If in place of the average thermal energy kT, we substitute the relativistic energy of pair formation $2mc^2$, regarding the vacuum as an effective thermal reservior, then Eq (11) follows with $\mathcal{E} = e^2/r$.

An explicit expression for the charge density follows by substituting (11) into (8):

$$\rho(r) = \frac{er_0}{8\pi r^4} e^{-r_0/2r} \tag{12}$$

Since $\rho_{\text{free}}(r) = e\delta(\mathbf{r})$, the density from vacuum polarization must equal

$$\rho_{\text{VP}}(r) = \frac{er_0}{8\pi r^4} e^{-r_0/2r} - e\delta(\mathbf{r}) \tag{13}$$

According to this model, the free point charge is exactly cancelled by the delta-function term of the polarization charge. The corresponding electrostatic potential is given by

$$\Phi(\mathbf{r}) = \frac{2e}{r_0}\left(1 - e^{-r_0/2r}\right) \approx \frac{e}{r} \qquad \text{when} \begin{cases} r_0 \to 0 \\ \quad \text{or} \\ r \to \infty \end{cases} \tag{14}$$

This implies a deviation from Coulomb's law of the same magnitude as the fine structure in atoms, but totally negligible on a macroscopic scale. An alternative evaluation of the electromagnetic self-energy follows from transformation of Eq (1) as follows:

$$W = \frac{1}{8\pi} \int \mathbf{E} \cdot \mathbf{D}\, d^3\mathbf{r} = \frac{1}{2} \int \Phi_{\text{free}}\, \rho\; d^3\mathbf{r} \tag{15}$$

using

$$\mathbf{D} = -\nabla \Phi_{\text{free}} = \frac{e\,\mathbf{r}}{r^3} \tag{16}$$

and assuming the requisite vanishing of integrands at infinity. Thus

$$W = \frac{1}{2} \int_0^\infty \Phi_{\text{free}}(r)\rho(r)\, 4\pi r^2\, dr = \frac{e^2 r_0}{4} \int_0^\infty \frac{e^{-r_0/2r}}{r^3}\, dr = mc^2 \tag{17}$$

in agreement with the previous result, and further justification for the conjectured functional form of $\epsilon(r)$.

The preceding result suggests that the self-interaction of an electron is in some sense equivalent to the interaction between a point charge and its *net* polarization density—which we will denote as its "aura" (in New Age jargon, an energy field which emanates from a body). In the following section, we will utilize this picture to derive the radiation reaction for an accelerated charge.

4 Accelerating Point Charge

The Lorentz-Dirac equation for the force on an accelerating electron is given by[6]

$$F^\lambda_{\rm ext} = m a^\lambda - \frac{2\,e^2}{3\,c^3}\left(\dot a^\lambda + \frac{1}{c^2} a^2 v^\lambda\right) \tag{18}$$

However, this equation has fallen into disfavor in recent years because it admits pathological solutions, including preacceleration and runaway behavior[14]. Such unphysical behavior is the result of taking the limit of the electron radius to zero. It can be avoided by treating the electron as a finite charged sphere, leading to a differential-difference equation without such pathology[15]. Our picture of the electron as a point charge interacting with its aura provides such an extended structure in a physically natural way.

The configuration of the aura surrounding an accelerating electron is most likely quite complicated. At the very least, the aura is distorted from its original spherical symmetry by Lorentz contraction. In addition, complicated processes involving creation and relaxation of vacuum polarization in the vicinity of the accelerating electron are certain to be occurring. We shall assume a highly idealized model for the aura, treating it as a point charge trailing the electron at a distance R^* with a proper-time delay τ^*. Analogously, the simplest model for an ionic crystal idealizes a lattice consisting of point charges. We will work in covariant notation throughout, thus avoiding the "4/3 problem" and other relativistic pitfalls. The Liénard-Wiechert 4-potential for a moving point charge is given by

$$A^\lambda(x) = e\,\left[\frac{v^\lambda}{v\cdot R}\right]_{\rm ret} \tag{19}$$

and the corresponding field tensor is

$$F^{\lambda\mu} = \partial^\lambda A^\mu - \partial^\mu A^\lambda = \Bigg[\frac{e}{(v\cdot R)^2}\left(R^\lambda a^\mu - a^\lambda R^\mu\right) + \frac{e}{(v\cdot R)^3}\left(c^2 - a\cdot R\right)\left(R^\lambda v^\mu - v^\lambda R^\mu\right)\Bigg]_{\rm ret} \tag{20}$$

Four-dimensional scalar products are expressed $a\cdot b = a^\mu b_\mu = a_0 b_0 - \mathbf{a}\cdot\mathbf{b}$ ("West Coast metric"). The relevant variables are

$$R^\lambda = (R, \mathbf{R}), \qquad v^\lambda = (\gamma c, \gamma \mathbf{v}),$$

$$a^\lambda = \frac{dv^\lambda}{d\tau} = \left(\gamma^4\,\mathbf{a}\cdot\mathbf{v}/c, \gamma^2\,\mathbf{a} + \gamma^4\,(\mathbf{a}\cdot\mathbf{v})\,\mathbf{v}/c^2\right) \tag{21}$$

$\mathbf{R}$ is the displacement from the charge at the retarded time to the field point at the present time. Thus R^λ, lying on the light cone, is a null 4-vector with $R^\mu R_\mu = 0$. We will also require the relations

$$\dot a^\lambda = \frac{da^\lambda}{d\tau}, \qquad v^\mu v_\mu = c^2, \qquad v^\mu a_\mu = 0 \tag{22}$$

We picture the point charge representing the aura to be chasing the electron along the same trajectory, with an effective time delay τ^* relative to the proper time τ. Additionally, let the displacement R^λ produced during the time τ^* be parametrized as

$$R^\lambda = \frac{[v^\lambda]_{\rm ret}}{c}\, R^* \tag{23}$$

in terms of an effective separation R^* between the electron and its aura (not necessarily to the center of the aura). The parameters τ^* and R^* are independent and R^λ is no longer restricted to the light cone since $R^\lambda R_\lambda = R^{*2} \not\equiv 0$.

Substituting (23) into (20), noting that $v \cdot R = c\, R^*$ and writing $\tau - \tau^*$ for the retarded time, we obtain a major simplification to

$$F^{\lambda\mu}(\tau) = \frac{e}{c^3 R^*}\, [v^\lambda a^\mu - v^\mu a^\lambda]_{\tau-\tau^*} \tag{24}$$

According to Lorentz and Abraham, if the electron is a purely electromagnetic entity, the self force should exactly balance the external force. Thus

$$F^\lambda_{\rm ext}(\tau) = -F^\lambda_{\rm self}(\tau) = -\frac{e}{c}\, F^{\lambda\mu}(\tau) v_\mu(\tau) \tag{25}$$

We obtain thereby a differential-difference equation for the force on an accelerating electron:

$$F^\lambda_{\rm ext}(\tau) = \frac{e^2}{c^4 R^*}\, [a^\lambda v^\mu - a^\mu v^\lambda]_{\tau-\tau^*}\, v_\mu(\tau) \tag{26}$$

The values of R^* and τ^* can be inferred by considering the nonrelativistic limit, as $\mathbf{v}$ and τ^* approach zero. Expanding $[v^\lambda]$ and $[a^\lambda]$ and doing the summations over μ, we obtain

$$F^\lambda_{\rm ext} \approx \frac{e^2}{c^2\, R^*}\, a^\lambda \; - \; \frac{e^2\, \tau^*}{c^4\, R^*}\, \dot{a}^\lambda \tag{27}$$

Since this should reduce to the original Abraham-Lorentz equation[16]

$$\mathbf{F}_{\rm ext} \approx m\, \mathbf{a} - \frac{2\, e^2}{3\, c^3}\, \dot{\mathbf{a}} \tag{28}$$

(as well as Newton's second law when $\tau^* = 0$) we can identify

$$R^* = \frac{e^2}{mc^2} \equiv r_0, \tag{29}$$

the classical electron radius, and

$$\tau^* = \frac{2e^2}{3mc^3} \equiv \tau_0 \tag{30}$$

Remarkably, the parameter $\tau_0 \approx 6.26 \times 10^{-24}$ sec is the same "relaxation time" that occurs in the integration of the Lorentz-Dirac equation—the immeasurably brief time interval during which classical acausal behavior is tolerated.

Finally, writing $\beta^\lambda = v^\lambda/c$, we obtain a compact differential-difference formulation for the force on an accelerating electron:

$$F^\lambda_{\rm ext}(\tau) = m\,[a^\lambda\beta^\mu - a^\mu\beta^\lambda]_{\tau-\tau_0}\,\beta_\mu(\tau) \tag{31}$$

Expansion of the bracketed quantity for small τ_0 reacquires the conventional Lorentz-Dirac equation:

$$F^\lambda_{\rm ext} = ma^\lambda - \frac{2\,e^2}{3\,c^3}\left(\dot{a}^\lambda + \frac{1}{c^2}a^2v^\lambda\right) + \mathcal{O}(\tau_0) \tag{32}$$

noting that $\dot{a}^\mu v_\mu = -a^\mu a_\mu = -a^2$.

The nonoccurrence of runaway solutions to the modified Lorentz-Dirac equation (31) is easy to prove. In the absence of external forces,

$$[a^\lambda v^\mu - a^\mu v^\lambda]_{\tau-\tau_0}\,v_\mu(\tau) = 0 \tag{33}$$

Premultiplying by $v_\lambda(\tau-\tau_0)$ and summing, we obtain

$$a^\mu(\tau-\tau_0)v_\mu(\tau) = 0 \tag{34}$$

Writing out the components explicitly, using Eq (21),

$$[\gamma^4\,\mathbf{a}\cdot\mathbf{v}/c]_{\tau-\tau_0}\,\gamma c - [\gamma^2\,\mathbf{a} + \gamma^4\,(\mathbf{a}\cdot\mathbf{v})\,\mathbf{v}/c^2]_{\tau-\tau_0}\cdot\gamma\mathbf{v}(\tau) = 0 \tag{35}$$

Clearly, if $\mathbf{v}$ is not identically zero for all τ, then $\mathbf{a} = 0$. But if $\mathbf{v} = 0$ for all τ, then $\mathbf{a} = 0$ again. Thus the only solutions for zero external force have zero acceleration. Further, the absence of preacceleration is strongly implied by the dependence on no time variables other than τ and $\tau - \tau_0$. By contrast, the Lorentz-Dirac equation (18) contains the derivative of acceleration, which can be approximated by the finite difference

$$\dot{a}^\lambda(\tau) \approx \frac{1}{2\tau_0}\left[a^\lambda\right]^{\tau+\tau_0}_{\tau-\tau_0} \tag{36}$$

with the possibility of preacceleration attributed to the occurrence of the time variable $\tau + \tau_0$. Note that the occurrence or absence of preacceleration can *not* be readily discerned from the expanded form (32) of the L-D equation.

5 Non-Electromagnetic Mass

Although we have emphasized the case of a charged particle with purely electromagnetic self energy, the treatment can easily be generalized to include non-electromagnetic contributions to mass. In place of m in all preceding formulas, substitute $m_{\rm EM}$. For example, $r_0 = e^2/m_{\rm EM}c^2$. The total self-energy can now be written

$$W = m_{\rm total}c^2 = (m_{\rm bare} + m_{\rm EM})c^2 \tag{37}$$

This might pertain to elementary charged particles such as the muon, tauon, quarks and W bosons—and possibly even to the electron if one accepts, for example, the QED computation[17] giving

$$W_{\mathrm{QED}} \approx \frac{3\alpha}{2\pi}\, mc^2 \, \log\left(\frac{M}{m}\right) \tag{38}$$

where $M \gg m$, defines a relevant mass scale.

References

[1] J. Frenkel J *Zeit. Phys.* **32** 518 (1925).

[2] For a retrospective on the electron centennial, see S. Weinberg "The first elementary particle" *Nature* **386** 213 (1997).

[3] A definitive review of classical electron theories is given by F. Rohrlich *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1990).

[4] See, for example D. H. Perkins *Introduction to High Energy Physics* (Addison-Wesley, Reading, MA, 1987).

[5] I. Levine *et al Phys Rev. Lett.* **78** 424 (1997); see the artist's conception of the electron's structure on the AIP Physics News website: **http://www.aip.org/physnews/graphics/html/bare.htm**.

[6] P. A. M. Dirac *Proc. Roy. Soc. (London)* **A167** 148 (1938).

[7] J. A. Wheeler and R. P. Feynman R P *Revs. Mod. Phys.* **17** 157; 1949 **21** 425 (1945).

[8] F. Rohrlich *Phys. Rev. Lett.* **12** 375 (1964).

[9] C. Teitelboim *Phys. Rev. D* **1** 1572 (1970); **3** 297 (1970); **4** 345 (1971).

[10] R. Arnowitt, S. Deser and C. W. Misner *Phys. Rev. Lett.* **4** 375 (1960); *Phys. Rev.* **120** 313 (1960).

[11] A. Einstein *Sitzungsberichte Preussiche Akademie der Wissenschaften Physikalische Mathematische Klasse* (1919); translation in A. Einstein, *et al The Principle of Relativity* (Dover, New York, 1952)

[12] W. Furry and J. R. Oppenheimer *Phys. Rev.* **45** 245 (1934).

[13] V. F. Weisskopf *Det. Kgl. Danske Videnskab. Selskab. Mat.-Fys. Medd.* **14** 1 (1936); Reprinted in J. Schwinger *Quantum Electrodynamics* (Dover, New York, 1958).

[14] J. L. Jiménez and I. Campos *Am. J. Phys.* **55** (1987).

[15] A. D. Yaghian *Relativistic Dynamics of a Charged Sphere* (Springer, Berlin, 1992); P. Caldirola *Nuovo Cimento* **3**, Suppl. 2, 297 (1956); E. J. Moniz and D. H. Sharp *Phys. Rev.* **15** 2850 (1977); F. Rohrlich *Am. J. Phys.* **65** 1051 (1997).

[16] For details of the derivation, see J. D. Jackson *Classical Electrodynamics, 3rd Ed.* (Wiley, New York, 1999) Chap. 16.

[17] V. F. Weisskopf *Phys. Rev.* **56** 72 (1939).