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A new equation of motion for a radiating charged particle

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A new equation of motion for a classical radiating point-charge is proposed. The radiated energy is supplied by a reduction in proper-mass of the particle. Unlike the Lorentz–Dirac equation, the equation proposed is second order: it gives physically reasonable predictions, and in particular has no runaway solutions and no pre-acceleration.

1. INTRODUCTION

Recently there has been considerable interest in the motion of radiating charges (Shen 1970; Sen Gupta 1970, 1972; Jaffe 1972; Herrera 1970, 1973; Cloetens *et al.* 1969; Grandy 1970), partly stimulated by questions about pulsars. This has recalled attention to the difficulties arising from the Lorentz–Dirac (hereafter L–D) equation of motion (Dirac 1938). Among the strange effects allowed by this equation are, first the runaway motion whereby a radiating charge can accelerate indefinitely without apparent cause; secondly, pre-acceleration effects in which charges start to accelerate in preparation for an impulse before it arrives; and thirdly, the extraction of energy from a mysterious acceleration field (Fulton & Rohrlich 1960; Rohrlich 1965; Balazs 1968). These effects arise from a term containing the time derivative of the particle's acceleration, whose origin lies in the damping force believed to act on radiating charges. This is usually called the Schott term.

An alternative approach is possible though it has hardly been contemplated. This is to abandon the assumption that the proper-mass of the charge remains constant, and to allow radiation energy to come from this proper-mass. This proposal was put forward during a lecture in 1912 by Larmor (1929) though he did not go into details. The idea was not taken up, and the only mention I have found of it is in a review by Erber (1961), where it is briefly mentioned with a remark to the effect that it seems to overcome some of the difficulties of the conventional theory.

It is easy to see why Larmor's suggestion has not been followed. When physicists speak of the equation of motion of a radiating charge they usually have in mind the electron, and it is not acceptable for the proper-mass of such a fundamental particle to vary. Against this one can argue that the classical mechanics and electromagnetism used to formulate the equation of motion are not theories of fundamental particles, and the best one can expect of them is to give a sensible equation for macroscopic particles.

I adopt this point of view here. The energy which the charge loses as its proper mass diminishes is supposed to come, in Larmor's words, from a 'change in minute internal configuration'. How radiation can be reconciled with the constant bare-mass of fundamental particles is left to a deeper theory.

It should be emphasized that there is no direct evidence for the Schott term in the L–D equation. In laboratory experiments the effect of the term on radiation emission is always negligible (Shen 1970).

Granted that the proper-mass can vary, the equation of motion follows naturally from momentum balance and Newton's second law in special relativistic form (§ 2). It contains no derivatives of the acceleration, and allows no runaway motions. No pre-acceleration appears in the famous problem, considered by Dirac (1938), of a charged particle subject to a sharp pulse of radiation (§ 3). Sensible predictions also appear for the rectilinear motion of a charge in a uniform electric field (§ 4), and for motion in a uniform magnetic field (§ 5), though in both cases the equation of motion predicts a fairly rapid decrease in proper-mass. In § 6 I discuss the equation in relation to the conservation law derived by Dirac, and there is a brief conclusion (§ 7).

Gaussian units are used in the paper and Latin indices run from 1 to 4.

2. The equation of motion

Let S be an inertial frame with origin O, and (x, y, z, t) the cartesian coordinates and time of an event in S. Introduce the usual notation

$$x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^4 = ct, \tag{2.1}$$

so that the metric of flat space-time is

$$\mathrm{d}s^2 = \eta_{ik}\,\mathrm{d}x^i\,\mathrm{d}x^k, \tag{2.2}$$

η_{ik} being the Minkowski tensor with components

$$\eta_{11} = \eta_{22} = \eta_{33} = -1, \quad \eta_{44} = 1, \quad \eta_{ik} = 0\ (i \neq k). \tag{2.3}$$

Define the proper-time of a moving point P by

$$s = c\tau \tag{2.4}$$

so that if the coordinates of P are $x^i(\tau)$ the velocity of P is

$$v^i = \mathrm{d}x^i/\mathrm{d}\tau \tag{2.5}$$

and

$$\eta_{ik}\,v^i v^k = c^2. \tag{2.6}$$

The four-acceleration of P is defined by

$$\dot{v}^i \stackrel{\text{def}}{=} \frac{\mathrm{d}v^i}{\mathrm{d}\tau} = \frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2}, \tag{2.7}$$

and a dot will always mean differentiation with respect to τ. From (2.6) we have

$$\dot{v}^i v_i = v^i \dot{v}_i = 0, \tag{2.8}$$

indices being lowered by the η_{ik}. Since v^i is time-like, it follows that $\dot{v}^i$ (unless it is zero) must be space-like,

$$\dot{v}^i \dot{v}_i < 0. \tag{2.9}$$

Let P be a charge e with proper-mass m_0 subject to external four-force F^i. The rate at which it radiates four-momentum is

$$-\frac{2}{3}\frac{e^2}{c^5}(\dot{v}^k\dot{v}_k)\,v^i. \tag{2.10}$$

(See for instance Rohrlich (1965) but note the difference in sign which is due to the opposite choice of signs from that in (2.3) and hence in (2.9); see also Schild (1960).) We now consider P from the standpoint of Newtonian mechanics in its special relativistic form. The total rate of change of momentum is equal to the *external* force – that is, the 'self-force' is to be ignored completely. The momentum radiated away must, of course, be included when calculating the total rate of change. The problem is similar to that of rocket motion. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(m_0 v^i) = \frac{2}{3}\frac{e^2}{c^5}(\dot{v}^k\dot{v}_k)\,v^i + F^i. \tag{2.11}$$

Equation (2.11) *will be adopted as the equation of motion of a classical charged particle, m_0 being a function of τ.*

The L–D equation is

$$m_0\,\dot{v}^i = \frac{2}{3}\frac{e^2}{c^5}(\dot{v}^k\dot{v}_k)\,v^i + F^i + \frac{2}{3}\frac{e^2}{c^3}\ddot{v}^i. \tag{2.12}$$

If we write (2.11) in the form

$$m_0\,\dot{v}^i = \frac{2}{3}\frac{e^2}{c^5}(\dot{v}^k\dot{v}_k)\,v^i + F^i - v^i\frac{\mathrm{d}m_0}{\mathrm{d}\tau} \tag{2.13}$$

we see that the proposed equation differs from the usual one in two respects:

(i) the variability of m_0,

(ii) the replacement of the Schott term $\frac{2}{3}(e^2/c^3)\,\ddot{v}^i$ by a term involving the change of proper-mass.

Contracting (2.11) with v_i and supposing

$$v_i F^i = 0 \tag{2.14}$$

(F^i non-dissipative) we find, using (2.6) and (2.8)

$$\frac{\mathrm{d}m_0}{\mathrm{d}\tau} = \frac{2}{3}\frac{e^2}{c^5}(\dot{v}^k\dot{v}_k) \tag{2.15}$$

and substituting this into (2.13) we find

$$m_0\dot{v}^i = F^i. \tag{2.16}$$

Equations (2.15) *and* (2.16) *are an alternative form of the proposed equation of motion if* (2.14) *is satisfied.* (2.15) shows clearly how the radiated energy is supplied by the reduction in proper-mass.

Putting $F^i = 0$ we obtain from (2.15) and (2.16)

$$v^i = \text{const}, \quad m_0 = \text{const}. \tag{2.17}$$

so in the absence of external forces the particle obeys Newton's first law. *There are no runaway solutions of the equation of motion* (2.11). Moreover, if $F^i \neq 0$ the more bizarre effects of the L–D equation are not to be expected. For example, Eliezer (1947) using (2.12) found that in certain circumstances unlike charges repel, and like charges attract; now in (2.16) m_0 is positive (while the particle lasts) so $\dot{v}^i$ and F^i are parallel, have the same sign, and vanish together. This ensures that the motion will have some resemblance to what ordinary physics leads us to expect. This will be illustrated in the next three sections.

3. Charged particle disturbed by a pulse

One of the most puzzling effects of the L–D equation is the pre-acceleration of a charge by a pulse of radiation. Dirac (1938) wrote: 'The electron seems to know about the pulse before it arrives and to get up an acceleration (as the equations of motion allow it to do), just sufficient to balance the effect of the pulse when it does arrive.'

Let us see how this situation is met by the equation of motion (2.11). Suppose the charge is at rest at the origin of a Lorentz frame, and that it encounters a sharp pulse of polarized electromagnetic radiation in which the fields $\boldsymbol{E}$, $\boldsymbol{H}$ are

$$E_x = k\delta(t-c^{-1}y), \quad E_y = E_z = 0, \tag{3.1}$$

$$H_x = H_y = 0, \quad H_z = -k\delta(t-c^{-1}y). \tag{3.2}$$

Using

$$F^i = (e/c)\, F_k^{\ i} v^k, \tag{3.3}$$

$F_k^{\ i}$ being the electromagnetic field tensor, we see that (2.14) is satisfied so that the equation of motion (2.11) reduces to (2.15) together with (2.16). Next, suppose that k is small (more precisely, that the dimensionless quantity $ek/m_0c \ll 1$) so that only motion in the line $y = 0$, $z = 0$ need be considered; we then obtain from (2.16) to order k:

$$m_0 \dot{v}^1 = (e/c)\, E_x v^4, \tag{3.4}$$

$$m_0 \dot{v}^3 = 0, \tag{3.5}$$

v^2 being of order k^2 and $v^4 = 1 + O(k^2)$. (3.5) simply shows that v^3 remains zero, and (3.4) gives, if we use (2.5) and (3.1) and integrate with respect to τ:

$$\int_{0-}^{0+} m_0 \frac{\mathrm{d}v^1}{\mathrm{d}\tau}\, \mathrm{d}\tau = ek, \tag{3.6}$$

$0-$ and $0+$ meaning proper times arbitrarily earlier and later than zero. The change in proper mass is of order k^2 by virtue of (2.15), so to order k, m_0 may be taken as constant in (3.6); therefore

$$[m_0 v^1]_{0-}^{0+} = ek. \tag{3.7}$$

Thus to first order in k the effect of the pulse on the charge is simply that of an impulse at time $t = 0$ as in classical mechanics. Before and after the arrival of the pulse the velocity and mass are constant from (2.17), because no force acts. No difficulties arise with pre- or post-acceleration.

4. Rectilinear motion of a charged particle in a uniform electric field

Let E be the field strength and suppose that motion takes place along Ox. The equations of motion (2.16) give

$$m_0 \dot{v}^1 = ec^{-1}Ev^4, \tag{4.1}$$

$$m_0 \dot{v}^4 = ec^{-1}Ev^1, \tag{4.2}$$

where the dot means $\mathrm{d}/\mathrm{d}\tau$. Let $v^1 = 0$ and $m_0 = M$ at $\tau = 0$; then for $\tau > 0$ these equations and (2.15) can be solved in the form

$$\left.\begin{aligned} v^1 = c\sinh\phi(\tau), \quad v^4 = c\cosh\phi(\tau),\\ \phi = (3p/2\alpha)\{1-(1-\alpha\tau)^{\frac{2}{3}}\},\\ m_0 = M(1-\alpha\tau)^{\frac{1}{3}}, \quad -\dot{v}^k\dot{v}_k = c^2p^2(1-\alpha\tau)^{-\frac{2}{3}},\end{aligned}\right\} \tag{4.3}$$

where $\alpha = 2e^4E^2/c^5M^3$, $p = eE/cM$. The speed $|\mathrm{d}x/\mathrm{d}t|$ measured in the Lorentz frame is $c|\tanh\phi|$, which increases monotonically for $\tau > 0$ until the mass becomes zero and the acceleration infinite at proper time

$$\tau_0 = \alpha^{-1}.$$

This is the lifetime of the charged particle for rectilinear motion in the given electric field E. For macroscopic charged particles it is very long indeed: for instance, a particle of 1 g carrying a charge of 100 e.s.u. would have a lifetime of more than 10^{29} years in a field of 10^6 V cm^{-1}. If one extrapolates to an electron in the same field one finds a lifetime of only 15 s; however, this theory is not intended to describe electrons. I suppose that it applies to macroscopic particles up to a certain stage, as yet unknown, in the process of mass-decay, and after that some other theory should replace it.

5. Motion of a charge in a uniform magnetic field

Let the magnetic field be H parallel to Ox. The equations of motion (2.16) give

$$m_0 \dot{v}^2 = eHc^{-1}v^3, \tag{5.1}$$

$$m_0 \dot{v}^3 = -eHc^{-1}v^2, \tag{5.2}$$

$$\dot{v}^1 = \dot{v}^4 = 0. \tag{5.3}$$

For simplicity assume $v^1 = 0$ so that the charge moves in the plane $x = 0$. As initial conditions let us take $v^2 = V$, $v^3 = 0$, $m_0 = M$ at $\tau = 0$, where V and M are constants. Then a solution to (5.1)–(5.3) and (2.15) is

$$\left.\begin{aligned} v^2 = V\cos\psi(\tau), \quad v^3 = V\sin\psi(\tau), \quad v^4 = (c^2+V^2)^{\frac{1}{2}},\\ \psi = \frac{3q}{2\beta}\{1-(1-\beta\tau)^{\frac{2}{3}}\},\\ m_0 = M(1-\beta\tau)^{\frac{1}{3}}, \quad -\dot{v}^k\dot{v}_k = V^2q^2(1-\beta\tau)^{-\frac{2}{3}},\end{aligned}\right\} \tag{5.4}$$

where $\beta = 2e^4H^2V^2/c^7M^3$ and $q = -eH/cM$. As in the motion in a uniform electric field, the mass becomes zero after a certain time, namely β^{-1}. For a particle of 1 g carrying 100 e.s.u. in a field of 10^6 G, with $V = 10^{-2}c$, this time is 4×10^{30} years. Extrapolating the theory to an electron projected with the same velocity in the same field we find β^{-1} about 2 s; but, as stated in § 4, I regard such extrapolation as unjustified.

To study the trajectory we introduce polar coordinates in the yz plane. Then

$$(v^2)^2+(v^3)^2 = r^2\dot{\theta}^2+\dot{r}^2 = V^2, \tag{5.5}$$

so the speed is constant. Also

$$r\dot{\theta} = \dot{z}\cos\theta-\dot{y}\sin\theta = V\sin(\psi-\theta),$$
$$\dot{r} = \dot{y}\cos\theta+\dot{z}\sin\theta = V\cos(\psi-\theta),$$

and it follows that ψ may be taken as the angle between the tangent to the trajectory and the initial line of polar coordinates. If σ is the arc length we have from (5.5) $\mathrm{d}\sigma = V\,\mathrm{d}\tau$, so the radius of curvature is

$$\rho = |\mathrm{d}\sigma/\mathrm{d}\psi| = |V/q|\,(1-\beta\tau)^{\frac{1}{3}}$$

from (5.4). ρ decreases monotonically from $|V/q|$ to zero as τ runs from zero to β^{-1}, so the particle spirals inwards to a point.

6. Relation to Dirac's conservation law

Investigating the conservation of momentum and energy in a world tube of radius ϵ containing the charge, Dirac (1938) introduced a vector B_i such that

$$\dot{B}_i = \tfrac{1}{2}e^2\epsilon^{-1}c^{-1}\dot{v}_i - ev_k\{F_i{}^k+\tfrac{2}{3}ec^{-4}(\ddot{v}_iv^k-\ddot{v}^kv_i)\}, \tag{6.1}$$

where $F_i{}^k$ is the incident electromagnetic field and no other external forces act; B_i satisfies

$$\dot{B}_iv^i = 0, \tag{6.2}$$

because the right-hand side of (6.1) does so, but is otherwise arbitrary. Dirac chose

$$B_i = kv_i, \tag{6.3}$$

where k is a constant, remarking that other choices for B_i are all much more complicated, and adding: 'one would hardly expect them to apply to a simple thing like an electron', which today seems somewhat ironic. Taking for k

$$k = \tfrac{1}{2}e^2\epsilon^{-1}c^{-1}-m_0c, \tag{6.4}$$

where m_0 is another constant independent of ϵ, he obtained the equation of motion (2.12) with F^i given by (3.3).

An alternative form for B_i is

$$B_i = P(\tau)\,v_i+Q(\tau)\,\dot{v}_i. \tag{6.5}$$

This form has been previously considered and rejected (Bhabba 1939; Eliezer 1946). The reason given was that, because of conservation of momentum, and of angular momentum, $\dot{B}_i$ and $v_i B_k - v_k B_i$ have both to be time-derivatives of some quantities depending only on v_i and its derivatives, and constants. There are two reasons why I do not regard this argument as compelling: first, orbital angular momentum is origin-dependent and so may be expected to depend on the coordinates as well as the velocity; secondly, in this paper another function, $m(\tau)$, is associated with the particle, in addition to v_i.

Equation (6.5) will satisfy (6.2) if

$$\dot{P}c^2 + Q\ddot{v}_i v^i = 0; \tag{6·6}$$

noting that from (2.8)

$$\ddot{v}_i v^i + \dot{v}_i \dot{v}^i = 0$$

we can write (6.6) as

$$\dot{P}c^2 = Q\dot{v}_i \dot{v}^i. \tag{6.7}$$

We now choose

$$P = \tfrac{1}{2}e^2\epsilon^{-1}c^{-1} - m_0(\tau)\,c, \quad Q = -\tfrac{2}{3}e^2c^{-2}, \tag{6.8}$$

in which $m_0(\tau)$ satisfies (2.15). Then (6.7) is satisfied and (6.1) reduces to (2.11) with F^i of form (3.3).

This shows that our equations (2.15) and (2.16) are consistent with Dirac's conservation law provided we adopt a B_i different from that chosen by Dirac.

7. Conclusion

We have seen that the equation of motion (2.11) is physically reasonable in several situations for which the L–D equation gives mysterious results: for instance, it has no runaway solutions, and predicts no pre-acceleration in the circumstances described in § 3. For the motion of macroscopic charges it seems that (2.11) may be correct, but it cannot apply to a fundamental particle with constant proper-mass, like the electron. This is because according to (2.11) the radiated energy is supplied by a reduction in the proper-mass. In strong electric and magnetic fields this can cause a rapid decrease in mass, as was seen in §§ 4 and 5.

Equation (2.11) has the merit of showing clearly where the radiated energy comes from. As several authors have noted, this cannot be said of the L–D equation. It seems that the whole of the radiated energy does not come from the external field and the kinetic energy (Cloetens *et al.* 1969; Grandy 1970), at least part coming from the 'acceleration energy', which derives from the Schott term, and which presumably is part of the internal energy of the particle. If internal energy is being radiated away it is hard to see how the proper-mass can remain constant as required by the L–D equation.

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