proved by cutting the first plate along the dotted line, in which case the values between brackets are valid. These four waves interfere, two by two, to give two sine-wave interference patterns of different wavelength on the screen; Fig. 2(a) is the interference of waves 1 and 3; and Fig. 2(b) is created by waves 2 and 4. In Fig. 2(c) we see the interference of the four waves; to a first approximation it can be considered as a superposition of two perfect sine waves, and consequently it shows a typical beat note.

In order to simulate dispersion, one has to move those waves; this is done in the setup of Fig. 3. The two glass plates are held at a distance d_2 by two studs; one of them is slightly compressible with a screw. The light beam from the laser is focused on the glass plates, and the reflected beams are directed towards a distant screen. Because the two plates are not exactly parallel, the reflected beams overlap at the screen. Fig. 2(c) shows a picture of the interferogram, with the following values for the parameters: $d_1 = 0.70$ mm; $d_2 = d_3 = 1.05$ mm; $f = 10$ cm; $D_1 = 1$ m; $D_2 = 10$ m; and we used (of course) a low-power He–Ne laser. When turning the screw in Fig. 3, one changes not only the angle between beams (1,2) and (3,4), but also the thickness of the air gap d_2; so the sine waves on the screen move, and the beat note changes consequently. One can clearly see that the speed of the wave (phase velocity) differs from the speed of the envelope (group velocity). In a typical experiment the wave moves two or three times faster than the envelope; but we have even found a situation in which both moved in opposite direction!

More on the Feynman's Disk Paradox

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Reference Gabriel Lombardi's excellent article about the Feynman Disk Paradox.[1,2] A simple example that illustrates the principles and is easy to analyze is as follows:

An infinite solenoid of radius R, current i, and n turns per meter is concentric with two cylindrical tubes of charge Q and $-Q$, and radii a and b, respectively. The charge is distributed uniformly over the cylindrical surfaces and both tubes have length l, where $l \gg b > R > a$. The coil and the cylindrical tubes are all stationary initially but free to rotate without friction about their common axis.[3] Thus the initial mechanical angular momentum is zero.

The apparent paradox arises when the current in the solenoid is interrupted, say by raising the temperature above the super conducting point. The changing magnetic flux causes a tangential electric field that acts on charged tubes, giving them a mechanical angular momentum as follows:

$$L_{ma} = \int (\text{torque}) dt = \int aQE_a \, dt,$$
$$L_{mb} = \int (\text{torque}) dt = \int bQE_b \, dt, \qquad (1)$$

where E_a and E_b are the tangential electric fields induced at the inner and outer tubes, respectively. According to Faraday's law of induction and Ampere's circuital rule we get[4]

$$E_a = \frac{d\Phi_a/dt}{2\pi a} = \pi a^2 \frac{dB/dt}{2\pi a} = a\frac{dB/dt}{2},$$
$$E_b = \frac{d\Phi_b/dt}{2\pi b} = \pi R^2 \frac{dB/dt}{2\pi b} = R^2 \frac{dB/dt}{2b}.$$

(In each case the field is tangential in the direction of the original current.)

Substituting these values into Eqs. (1) we get

$$L_{ma} = Qa^2 \int dB/2 = Qa^2 B/2$$

(in the direction of the solenoid axis. The rotation is in the same direction as the original current.).

$$L_{mb} = QR^2 \int dB/2 = QR^2 B/2$$

(direction opposite to that of L_{ma}),

where B is the initial magnetic field within the solenoid. The total final mechanical angular momentum is

$$L_m = L_{mb} - L_{ma} = QB(R^2 - a^2)/2$$

(in the direction of L_{mb}). (2)

Where does this mechanical angular momentum come from? The answer according to Lombardi's proof is that the initial electromagnetic field possesses angular momentum and that this is transferred to the cylindrical tubes as the current in the solenoid drops to zero in such a way that angular momentum is conserved. The initial field angular momentum is thus equal to the final mechanical angular momentum.

To check this for the present example we note first that initially B is equal to zero everywhere except within the solenoid where it is uniform. The electric field is nearly zero everywhere except in the region between the cylindrical shells where it is radially outward and of magnitude $E = Q/2\pi\epsilon_0 rl$, where r is the cylindrical radius.[5] The field angular momentum is, following the procedure given by Lombardi,

$$\mathbf{L}_F = \epsilon_0 \int \mathbf{r}\times(\mathbf{E}\times\mathbf{B}) d^3r,$$
$$L_F = \epsilon_0 \int_{r=a}^{R} r(Q/2\pi\epsilon_0 rl)(B)(2\pi rl \, dr)$$
$$= QB \int_{r=a}^{R} r\, dr = QB(R^2 - a^2)/2$$

(in the direction of L_m). (3)

Hence the initial field angular momentum (3) is equal to the final mechanical angular momentum (2).

There is an alternate method for finding the field angular momentum transferred to mechanical angular momentum that uses vector potential. A vector potential for the infinite solenoid that works is[6]

$$\mathbf{A} = \mathbf{B}\times\mathbf{r}/2, \quad r<R, \quad \nabla\times\mathbf{A} = \mathbf{B},$$
$$\mathbf{A} = R^2(\mathbf{B}\times\mathbf{r})/2r^2, \quad r>R, \nabla\times\mathbf{A} = 0.$$

The angular momentum associated with **A** is[7]

$$L_{Aa} = \left|\int \mathbf{r}\times\mathbf{A}\, dQ\right| = QBa^2/2$$

(in the direction of L_{ma}),

$$L_{Ab} = \left|\int \mathbf{r}\times\mathbf{A}\, dQ\right| = QBR^2/2$$

(in the direction of L_{mb}),

for the inner and outer shells, respectively. The total field angular momentum transferred to the tubes as **A** drops to zero is the sum

$$L_A = L_{Ab} - L_{Aa} = B^2Q(R^2 - a^2)/2$$

(in the direction of L_m),

which is the same result given by Eq. (3).

This second method for showing that angular momentum is conserved as the current in the solenoid drops to zero is sometimes much easier to carry out, particularly if **E** and **B** are not so neatly confined as they are in the setup described here. An example where this is the case is the setup described above with the inner tube omitted.

[1]R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures* (Addison-Wesley, Reading, MA, 1984), Vol. II, p. 17-5.
[2]G. Lombardi, Am. J. Phys. **51**, 213 (1983).
[3]A similar setup with two rotating charged cylinders but no solenoid is analyzed by E. Corinaldesi, Am. J. Phys. **48**, 83 (1980).
[4]D. Halliday and R. Resnick, *Physics* (Wiley, New York, 1978), 3rd ed., Part 2, Sec. 35-5.
[5]Reference 4, Sec. 28-8.
[6]Reference 1, p. 14-3.
[7]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), 2nd ed., p. 574.

Note on the Coulomb gauge condition in magnetostatics

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The magnetic induction vector **B** for a magnetostatic situation is determined by Maxwell's equations

$$\nabla\cdot\mathbf{B} = 0, \tag{1}$$
$$\nabla\times\mathbf{B} = 4\pi\mathbf{j}/c, \tag{2}$$

and some boundary conditions. Here we use Gaussian units; **j** is the current density and c is the speed of light. The boundary conditions will be abbreviated "*BC*." It is often convenient to express

$$\mathbf{B} = \nabla\times\mathbf{A}, \tag{3}$$

introducing a vector potential **A** [whose existence is guaranteed by condition (1), Ref. 1, Sec. 4.5]. Equation (2) then becomes

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} = 4\pi\mathbf{j}/c. \tag{4}$$

If a particular solution $\mathbf{A}_c$ of (3) can be found which satisfies the Coulomb gauge condition

$$\nabla\cdot\mathbf{A}_c = 0, \tag{5}$$

then the Cartesian components of $\mathbf{A}_c$ satisfy Poisson's equation (compare Ref. 2, p. 140):

$$\nabla^2\mathbf{A}_c = -4\pi\mathbf{j}/c. \tag{6}$$

The Coulomb gauge is particularly useful since there is a wealth of knowledge concerning the Poisson equation (its Green's function, homogeneous solutions in many coordinate systems, etc.). For instance, a particular solution of (6) is given by

$$\mathbf{A}_p = \frac{1}{c}\iiint \frac{\mathbf{j}}{r}\, dV. \tag{7}$$

Here r is the distance between field point and source point and dV denotes a volume integral (Ref. 2, p. 141). Not so obvious, but nonetheless true, is the fact that $\mathbf{A}_p$ in (7) satisfies the Coulomb gauge (5).

However, $\mathbf{A}_p$ will not, in general, satisfy *BC* for a particular situation ($\mathbf{A}_p$ is the free-space solution), so solutions of (5) and the homogeneous form of (6)

$$\nabla^2\mathbf{A} = 0 \tag{6'}$$

must be superposed with $\mathbf{A}_p$ to solve the problem.

Now, the fact of the matter is that the vector potential **A** is only determined up to a gradient:

$$\mathbf{B} = \nabla\times\mathbf{A} = \nabla\times(\mathbf{A} + \nabla\psi), \tag{8}$$

and one may get the impression from textbooks that the extra freedom afforded by this arbitrary ψ permits one to satisfy the Coulomb gauge (5) as an afterthought, after solving (6) or (6′) and *BC*.

We shall show shortly that this impression is false! Lest this cast doubt on the feasibility of the Coulomb gauge scheme, however, we shall also prove:

Proposition. Every magnetostatic problem can be solved with a vector potential satisfying the Coulomb gauge.

This, of course, is a well-known corollary of the general gauge theory for Maxwell fields, but our new proof is perhaps more illuminating inasmuch as it confines itself strictly to the consideration of magnetostatic fields.

To demonstrate the futility of using ψ to satisfy (5) *a posteriori*, suppose we are given $\mathbf{A}_1$ satisfying (6) and *BC*. For $\mathbf{A}_1 + \nabla\psi$ to satisfy (5) clearly we must have