The fundamental solution in the theory of eddy currents and forces for conductors in steady motion

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A closed-form solution to the central problem of the steady linear motion of an arbitrary current distribution past materials of constant permeability is presented. The application of the Green's function technique to the field equations yields integral representations of the induction, eddy currents, and electromagnetic forces. Due to interface coupling of the boundary conditions along the surface of the conductor, Green's functions are shown to satisfy integral equations. In the case of a conducting slab, explicit solutions for the Green's functions are derived. Application to magnetic levitation and the calculations of forces on moving coils are developed. Results are compared with experimental drag measurements.

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I. INTRODUCTION

In recent years, the constraints imposed on the development of high-speed ground transportation systems have generated strong interest in magnetic levitation and linear induction motor technologies. The basic principles being well understood, important contributions demonstrating the feasibility of those techniques were made by various research groups,[1–9] the aim being to identify the most promising configurations.

Along those lines, a considerable theoretical effort has been developed in order to obtain expressions for the electromagnetic forces exerted on current loops moving past specific geometries. Pioneering studies[5–7] pertained to coils in uniform motion parallel to conducting plates, whereas some recent calculations were performed for an unsteady situation encountered in a dynamical scaling experiment.[10] On the other hand, numerical methods have been applied to eddy-current distributions,[11] and schemes have been proposed for the modelling of electromagnetic phenomena in general, but with limited performance.[12] The difficulties inherent to those problems suggest the necessity of developing more powerful tools of investigation. This paper is an attempt to present a global analytical approach amenable to most applications. Moreover, the Green's functions formalism proposed here has the great advantage of providing an explicit physical description of the nature of the interaction of fields with moving conductors. Although restricted here for simplicity to uniform motion, it can easily be generalized to time-dependent fields. Finally, the linear properties of Green's functions are fully exploited since we consider conducting materials of constant magnetic permeability, but we believe that the results can serve as a base line for the study of ferromagnetic media.

In Sec. II, the general formulation of the problem of a constant permeability conductor in steady linear motion past an arbitrary current distribution is developed in terms of a vector induction potential starting from Maxwell's field equations. The solution for this potential is then required to be uniquely defined by a convolution over the current source density. It is subsequently shown to be sufficient to take a scalar kernel (Green's function) for this integral representation: upon applying Green's theorem and using the coupled boundary conditions, the problem boils down to the search for appropriate Green's functions solutions of certain integral equations along the conductor's boundary. The magnetic induction, the eddy-current distributions, and the electromagnetic body forces are then each given in terms of specific surface or volume integrals.

In Sec. III, we focus on the infinite slab conductor and solve for the Green's functions which turn out to be elliptic integrals with modified Bessel functions kernels. Symmetry properties of the latter qualitatively explain the lift and drag plots versus velocity. We then show how the essential electromagnetic variables relate to previously published results by simple application of the "faltung theorem", thereby connecting the aforementioned integral representations to Fourier transforms. In Sec. IV, we discuss some calculations performed on moving thick coils and compare with experimental drag measurements recently obtained.

II. FIELD SOLUTION FOR A STEADY MOVING CONDUCTOR

A. Formulation of the problem, notation

Consider a conductor of constant conductivity σ and constant magnetic permeability μ, occupying the domain Ω^- bounded by the surface Σ and moving in steady rectilinear motion with velocity ϑ with respect to a fixed rectangular coordinate system O_{xyz} (Fig. 1). The position vector of a field point P is denoted $P(x,y,z)$. Let Ω^+ be the remainder of the space with permeability μ_0. Inside Ω^+, consider a fixed volume τ carrying a steady current distribution with density $\mathbf{J}_m(\mathbf{r})$.

In the fixed reference frame, Maxwell's field equations for the magnetic induction $\mathbf{B}$ together with Ohm's law for the currents $\mathbf{J}$ can be written

$$\nabla \cdot \mathbf{B} = 0, \tag{1a}$$

$$\nabla \times \mathbf{B} = \mu \mathbf{J}, \tag{1b}$$

$$\nabla \times \mathbf{J} = -\frac{\sigma d\mathbf{B}}{dt}, \tag{1c}$$

Introduction of a vector potential $\mathbf{A}$ for the induction such that

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{2}$$

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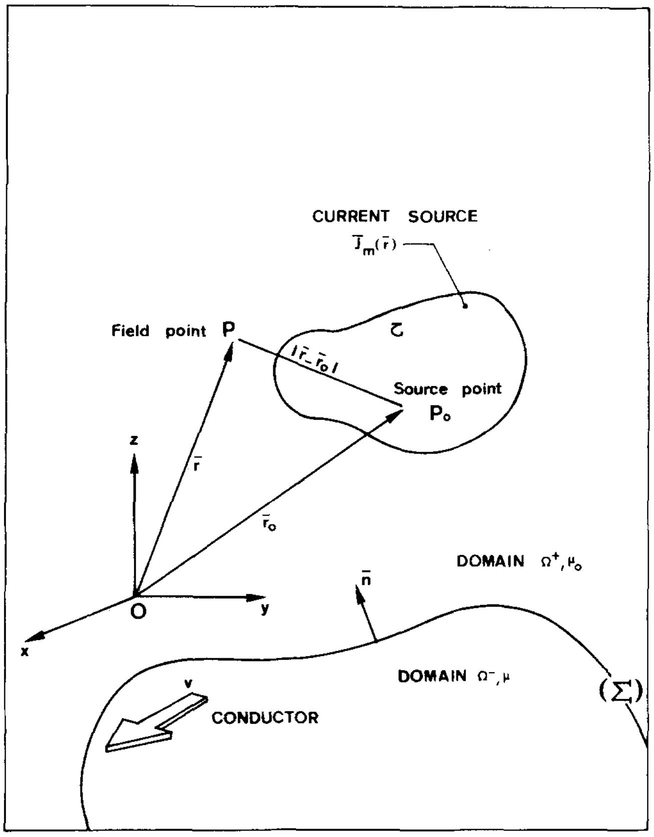

FIG. 1. General configuration.

can greatly simplify Eqs. (1). In general, this vector potential has a gradientlike longitudinal part and a transverse part: by virtue of Eq. (2), we shall consider transverse vector potentials only. This allows us to choose a zero-divergence potential **A**. This choice, however, corresponds to a Coulomb gauge[13] which is not relativistically invariant; it is unimportant here because we only consider a steady-state situation where there are no displacement currents and where **J**, consequently, has zero divergence. Under these conditions, and taking the velocity ϑ of the moving conductor along the O_x direction, the equations for the vector potential $\mathbf{A}(\mathbf{r})$ become

$$\nabla \cdot \mathbf{A} = 0, \tag{3a}$$

$$\nabla^2 \mathbf{A} - \mu\sigma\vartheta \frac{\partial \mathbf{A}}{\partial x} = 0. \tag{3b}$$

The boundary conditions for Eqs. (1a), (1b), and (1c), respectively, express the vanishing of all fields at infinity, the continuity across the interface Σ of the normal component of **B**, and the continuity of the tangential component of the magnetic field $\mathbf{B}/\mu$. Let **n** be the unit normal on the surface Σ pointing outward from the conductor into the domain Ω^+, and let (+) and (−), respectively, denote conditions inside Ω^+ and Ω^-. The boundary conditions determining **A** can then be written

$$\lim_{\mathbf{r}\to\infty} |\mathbf{A}| = 0, \tag{4a}$$

$$(\mathbf{n} \times \mathbf{A})^+ = (\mathbf{n} \times \mathbf{A})^-, \tag{4b}$$

$$(1/\mu_0)[\mathbf{n} \times (\nabla \times \mathbf{A})]^+ = (1/\mu)[\mathbf{n} \times (\nabla \times \mathbf{A})]^-. \tag{4c}$$

In summary, the problem consists in solving the vector equations (3) within Ω^+ and Ω^-, subject to compatibility relations across the free surface Σ as expressed by the boundary conditions (4). The solution of this problem is carried out using Green's functions. However, since Green's functions are the kernel of an integral operator which serves to transform the source density and the boundary conditions into the solution and since the latter have a vectorial nature, the Green's function must be a vector operator, namely, a dyadic function **G**. Finally, it remains to be shown that the magnetic induction, eddy currents, and body forces can be given simple representations in terms of the Green's function itself.

B. Vector potential and fundamental solution

Consider a dyadic Green's function $\mathbf{G}(\mathbf{r}|\mathbf{r}_0)$ of field points $\mathbf{r}(x,y,z)$ and source points $\mathbf{r}_0(x_0,y_0,z_0)$ associated with the vector potential equation (3). Such a dyadic must have the same general properties as the usual scalar Green's functions, namely, it must satisfy a reciprocity relationship, it will serve to generate the solution from both boundary conditions and source functions, and the resulting solutions will have discontinuities just outside the boundaries. Consequently, it must satisfy the following equations: (i) inside the domain Ω^+,

$$\nabla^2 \mathbf{A}^+ = -\mu_0 \mathbf{J}_m(\mathbf{r}), \tag{5a}$$

$$\nabla^2 \mathbf{G}^+ = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)\mathbf{I}; \tag{5b}$$

(ii) inside the conductor's domain Ω^-

$$\nabla^2 \mathbf{A}^- - 2\omega \frac{\partial \mathbf{A}^-}{\partial x} = 0, \tag{5c}$$

$$\nabla^2 \mathbf{G}^- - 2\omega \frac{\partial \mathbf{G}^-}{\partial x} = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)\mathbf{I}. \tag{5d}$$

Here **I** denotes the idemfactor dyadic, whereas $\delta(\mathbf{r} - \mathbf{r}_0)$ represents the three-dimensional generalized δ function of Dirac. The parameter ω has the unit of a wave number and is defined as

$$2\omega = \mu\sigma\vartheta. \tag{6}$$

When this wave number is multiplied by some length h characteristic of the spatial variation of **B**, the resulting dimensionless number $R_m = \mu\sigma\vartheta h$ is the magnetic Reynolds number associated with the problem. Finally, the vector potential equation (3) being of the diffusion type, the Green's function must satisfy the following reciprocity relationship where a velocity reversal is involved by virtue of the time sequence demanded by causality:

$$\mathbf{G}(\mathbf{r}|\mathbf{r}_0;\omega) = \mathbf{G}(\mathbf{r}_0|\mathbf{r};-\omega). \tag{7}$$

In order to apply the vector Green's theorem to Eqs. (5), it is useful to imagine the Ω domains to be bounded at large distance by regular surfaces labelled Σ^+ and Σ^- and sharing Σ as a common surface boundary. Upon applying Green's theorem,[13] namely, writing Eqs. (5) in source space taking Eq. (7) into account, contracting the vector Laplacians with **G** and the dyadic Laplacians with **A**, subtracting, integrating over each Ω domain, and using Gauss's theorem, one deduces

$$\frac{\mu_0}{4\pi}\int_\tau \mathbf{G}^+(\mathbf{r}|\mathbf{r}_0)\cdot\mathbf{J}_m(\mathbf{r}_0)\,d\tau_0$$
$$-\frac{1}{4\pi}\int_{\Sigma+\Sigma^+}\{(\nabla_0\cdot\mathbf{A}^+)[\mathbf{G}^+(\mathbf{r}|\mathbf{r}_0^s)\cdot\mathbf{n}_0]$$
$$-(\nabla_0\cdot\mathbf{G}^+)(\mathbf{A}^+\cdot\mathbf{n}_0)\}\,d\sigma_0$$
$$+\frac{1}{4\pi}\int_{\Sigma+\Sigma^+}\mathbf{G}^+(\mathbf{r}|\mathbf{r}_0^s)\cdot[\mathbf{n}_0\times(\nabla_0\times\mathbf{A}^+)]$$
$$+(\nabla_0\times\mathbf{G}^+)\cdot(\mathbf{n}_0\times\mathbf{A}^+)\}\,d\sigma_0$$
$$=\mathbf{A}^+(\mathbf{r})\text{ if }\mathbf{r}\text{ is inside }\Omega^+$$
$$=0\text{ if }\mathbf{r}\text{ is outside }\Omega^+; \tag{8a}$$

$$\frac{1}{4\pi}\int_{\Sigma+\Sigma^-}\{(\nabla_0\cdot\mathbf{A}^-)(\mathbf{G}^-\cdot\mathbf{n}_0)-(\nabla_0\cdot\mathbf{G}^-)(\mathbf{A}^-\cdot\mathbf{n}_0)\}\,d\sigma_0$$
$$-\frac{1}{4\pi}\int_{\Sigma+\Sigma^-}\{\mathbf{G}^-\cdot[\mathbf{n}_0\times(\nabla_0\times\mathbf{A}^-)]$$
$$+(\nabla_0\times\mathbf{G}^-)\cdot(\mathbf{n}_0\times\mathbf{A}^-)\}\,d\sigma_0$$
$$-2\omega\int_{\Omega^-}\frac{\partial}{\partial x_0}(\mathbf{A}^-\cdot\mathbf{G}^-)\,d\tau_0$$
$$=\mathbf{A}^-(\mathbf{r})\text{ if }\mathbf{r}\text{ is inside }\Omega^-$$
$$=0\text{ if }\mathbf{r}\text{ is outside }\Omega^-. \tag{8b}$$

Here the subscript 0 refers to source variables and the superscript s refers to points S belonging to the surface; thus, for the surface integrals, all the functions are evaluated at $\mathbf{r}_0^s$.

Because of the vanishing of the field $\mathbf{A}$ at infinity [Eq. (4a)], a similar boundary condition is imposed on the dyadic $\mathbf{G}$, thereby discarding the surface integrals over Σ^+ and Σ^-. Furthermore, the surface integrals along Σ which involve $\nabla_0\cdot\mathbf{A}(\mathbf{r}_0^s)$ identically vanish by our gauge choice [Eq. (3a)]. The Green's theorem as written above naturally separates the normal components of $\mathbf{A}$ from the tangential ones. Since no information is supplied on the normal projection $\mathbf{A}\cdot\mathbf{n}$ of the potential, it would seem natural to eliminate the remainder of the second surface integral by choosing a transverse dyadic. However, it is not necessary to do so because the contribution of this integral corresponds to a longitudinal component of $\mathbf{A}$. As a matter of fact, if we take the curl of Eq. (8) in order to obtain the magnetic induction $\mathbf{B}$, this troublesome longitudinal field disappears and, therefore, has no effect on the physical solution: in the following, it will consequently be ignored.

On the other hand, if instead of the vector potential equation (3b), we were to consider three scalar equations for each rectangular component of $\mathbf{A}$, we would then use, in Green's theorem, three identical scalar equations for the same scalar Green's function G. Thus, inasmuch as we do not have to separate out longitudinal and transverse parts of the dyadic $\mathbf{G}$, a spherical dyadic such as

$$\mathbf{G}=G\mathbf{I} \tag{9}$$

is a perfectly acceptable Green's dyadic in which G is to satisfy

$$\nabla^2G^+=-4\pi\delta(\mathbf{r}-\mathbf{r}_0)\quad\text{inside }\Omega^+, \tag{10a}$$

$$\nabla^2G^- - 2\omega\frac{\partial G^-}{\partial x}=-4\pi\delta(\mathbf{r}-\mathbf{r}_0)\quad\text{inside }\Omega^-. \tag{10b}$$

It remains to establish the link between G^+ and G^-. This is accomplished by imposing $\mathbf{A}^+$ to be uniquely determined by the current sources $\mathbf{J}_m(\mathbf{r})$; thus, the last surface integral over Σ in Eq. (8a) is required to vanish identically. Upon denoting the ratio μ_0/μ by m, introducing the projection n_x of the normal $\mathbf{n}$ along the O_x direction, and substituting the remaining boundary conditions (4b) and (4c) into Eq. (8b), the solution for the vector potential takes the form

$$\mathbf{A}^+(\mathbf{r})=(\mu_0/4\pi)\int_\tau G^+(\mathbf{r}|\mathbf{r}_0)\mathbf{J}_m(\mathbf{r}_0)\,d\tau_0, \tag{11a}$$

$$\mathbf{A}^-(\mathbf{r})=(1/4\pi)\int_\Sigma\{(1/m)G^-(\mathbf{r}|\mathbf{r}_0^s)[\mathbf{n}_0\times(\nabla_0\times\mathbf{A}^+)]$$
$$-\nabla G^-(\mathbf{r}|\mathbf{r}_0^s)\times(\mathbf{n}_0\times\mathbf{A}^+)$$
$$-2\omega n_{0_x}G^-(\mathbf{r}|\mathbf{r}_0^s)\mathbf{A}^+\}\,d\sigma_0. \tag{11b}$$

The constraint on G^+ can then be easily derived by taking the curl of $\mathbf{A}^+$ from Eq. (11a), substituting into Eq. (11b), and using condition (4a). The relationship takes the form of an integral equation which serves as the boundary condition imposed along Σ on the solution G^+ of Eq. (10a) according to

$$G^+(\mathbf{r}^s|\mathbf{r}_0)=\frac{1}{2\pi}\int_\Sigma\left[\frac{1}{m}G^-(\mathbf{r}^s|\mathbf{r}')\frac{\partial G^+}{\partial n'}(\mathbf{r}'|\mathbf{r}_0)\right.$$
$$\left.-G^+(\mathbf{r}'|\mathbf{r}_0)\left(\frac{\partial G^-}{\partial n'}(\mathbf{r}^s|\mathbf{r}')+2\omega n_xG^-(\mathbf{r}^s|\mathbf{r}')\right)\right]d\sigma'. \tag{12}$$

It should be understood here that the prime superscripts refer to dummy integration variables. Finally, G^- is determined by noting that a particular solution of Eq. (10b) is the fundamental solution denoted G_f^- which corresponds to an infinite domain Ω^- with no infinities, namely,

$$G_f^-(\mathbf{r}|\mathbf{r}_0)=\frac{\exp[\omega(x-x_0)-\omega|\mathbf{r}-\mathbf{r}_0|]}{|\mathbf{r}-\mathbf{r}_0|}. \tag{13}$$

If now Ω^- is bounded by a free surface Σ, and if G^- is interpreted as a potential solution of Eq. (10b) in the domain Ω^-, then across Σ the boundary condition analogous to Eq. (4c) is, for such a potential,

$$\left.\frac{\partial G^-}{\partial n}\right|^+=m\left.\frac{\partial G^-}{\partial n}\right|^-. \tag{14}$$

Upon solving Eq. (10b) for G^-, making use of the fundamental solution G_f^- and the jump condition (14), it is straightforward to derive the relationship imposed on G^- along Σ as the solution of the Fredholm equation of the second kind:

$$G^-(\mathbf{r}^s|\mathbf{r}_0^s)+\frac{1}{2\pi}\int_\Sigma G^-(\mathbf{r}^s|\mathbf{r}')\left(\frac{\partial G_f^-}{\partial n'}+2\omega n'_xG_f^-\right)d\sigma'$$
$$=\frac{2}{1+m}G_f^-(\mathbf{r}^s|\mathbf{r}_0^s)+\frac{\omega}{\pi}\int_\Sigma n'_x(G_f^-)^2\,d\sigma'. \tag{15}$$

Thus, we have demonstrated that Eqs. (11a) and (11b) do indeed constitute a solution of our problem with its boundary conditions. The fact that there can be no other solution is implicitly deduced from the uniqueness property of physical boundary value problems.

C. Induction, forces, and eddy currents

From the vector potential $\mathbf{A}(\mathbf{r})$, one can deduce straightforwardly the magnetic induction. For example, taking the curl of Eq. (11a), one derives in the domain Ω^+ the generalization of the Biot and Savart Law:

$$\mathbf{B}^+(\mathbf{r}) = (\mu_0/4\pi) \int_\tau \nabla G^+ \times \mathbf{J}_m(\mathbf{r}_0)\, d\tau_0. \quad (16)$$

The electromagnetic body forces exerted on the current source distribution $\mathbf{J}_m(\mathbf{r})$ enclosed in τ can then be obtained from the usual definition as the volume integral of the cross product of $\mathbf{J}_m(\mathbf{r})$ with $\mathbf{B}^+(\mathbf{r})$. If F_i is the component of the force in the x_i direction, one has the general formula for the body force,

$$F_i = \frac{\mu_0}{4\pi} \int_\tau \int_\tau [\mathbf{J}_m(\mathbf{r}) \cdot \mathbf{J}_m(\mathbf{r}_0)] \frac{\partial G^+}{\partial x_i}\, d\tau\, d\tau_0. \quad (17)$$

A similar relationship can be derived for the torque which is then given by replacing the gradient ∇G^+ in Eq. (17) by the vector product $\mathbf{r} \times \nabla G^+$.

Induced currents are generated within the conductor and have two causes. On one hand, the **B** field generated by the current source induces eddy currents within the core of the conductor in motion, and on the other hand, surface currents or "polarization currents" are established along the conductor's boundary Σ in order to account for the jump in the tangential component of the magnetic field. Considerable care must be taken in handling such a singular current distribution which mathematically is characterized by a δ function $\delta(\mathbf{r}^s)$ of the surface Σ. Upon recalling the field equation (1b) and upon using Eq. (3), one obtains the eddy currents in terms of the vector potential **A** as

$$\mathbf{J} = -\sigma\vartheta \frac{\partial \mathbf{A}^-}{\partial x} + \left(\frac{1}{\mu_0}[\mathbf{n} \times (\nabla \times \mathbf{A}^+)] - \frac{1}{\mu}[\mathbf{n} \times (\nabla \times \mathbf{A}^-)]\right) \delta(\mathbf{r}^s). \quad (18)$$

It is interesting to note that when the velocity is infinite, G^+ has a limit denoted G^*, whereas G^- vanishes identically. So does $\mathbf{A}^-$, which physically expresses the fact that for such a velocity the **B** field cannot diffuse inside the conductor while the current reduces to a sheet of current density $\mathbf{J}^*_\Sigma$ given by

$$\mathbf{J}^*_\Sigma(\mathbf{r}^s) = -\frac{\delta(\mathbf{r}^s)}{4\pi} \int_\tau \frac{\partial G^*}{\partial n}(\mathbf{r}^s|\mathbf{r}_0)\mathbf{J}_m(\mathbf{r}_0)\, d\tau_0. \quad (19)$$

Consequently, there results a body force F^* exerted on τ which takes the form

$$\mathbf{F}^* = -\mu_0 \int_\tau [\mathbf{J}_m(\mathbf{r}) \cdot \mathbf{J}^*_\Sigma(\mathbf{r}^s)]\, \mathbf{n}\, d\tau. \quad (20)$$

In summary, the main results obtained in this section concern the integral representations (11a) and (11b) of the vector potential $\mathbf{A}(\mathbf{r})$. The kernels of these representations are the Green's functions G^+ and G^- solutions of Eqs. (10a) and (10b) and determined from the boundary conditions stated in integral equations (12) and (15), respectively; the **B** field is represented by Eq. (16) so that the body forces are simply expressed by Eq. (17). Finally, the essential feature of these results lies in their clear physical content.

III. INFINITE SLAB CONDUCTOR

A. Fundamental solutions

We now consider the central problem of an infinite slab conductor in motion past an arbitrary current source enclosed in a volume τ and proceed to determine explicitly the Green's functions. Let Σ be the plane $z = 0$. By virtue of the vanishing of the normal projection n_x, the integral equation (15) becomes

$$G^-(\mathbf{r}^s|\mathbf{r}^s_0) + \frac{1}{2\pi}\int_\Sigma G^-(\mathbf{r}^s|\mathbf{r}')\frac{\partial G^-_f}{\partial n'}\, d\sigma' = \frac{2}{1+m} G^-_f(\mathbf{r}^s|\mathbf{r}^s_0). \quad (21)$$

Since G^- is to be a joint solution of Eq. (10b) and (21), it is natural to introduce the mirror images $\mathbf{r}^-_0(x_0, y_0, -z_0)$ of a source point $\mathbf{r}_0$ with respect to the plane $z_0 = 0$ and to seek G^- in terms of a linear superposition of fundamental solutions $G^-_f(\mathbf{r}|\mathbf{r}_0)$ [as given by Eq. (13)] with arguments $\mathbf{r}_0$ and $\mathbf{r}^-_0$. The integration thus yields

$$G^-(\mathbf{r}|\mathbf{r}_0) = G^-_f(\mathbf{r}|\mathbf{r}_0) + [(1-m)/(1+m)]G^-_f(\mathbf{r}|\mathbf{r}^-_0). \quad (22)$$

Physically, the regular part of this Green's function can be interpreted as being weighted by a factor expressing the "effective polarization" of the medium when the relative permeabilities differ. Moreover, at zero velocity, one recovers the classical result of magnetostatics[14] since the Green's function then exhibits the usual $1/r$ dependency. Under these conditions, the integral equation (12) determining the condition satisfied by the upper Green's function G^+ along the plane $z = 0$ takes the remarkable form

$$G^+(\rho|\mathbf{r}_0) = \frac{1}{2\pi m}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\exp[\omega(x-x') - \omega|\rho - \rho'|]}{|\rho - \rho'|} \times \left.\frac{\partial G^+(\rho'|\mathbf{r}_0)}{\partial z'}\right|_{z=0} d\sigma'. \quad (23)$$

Here the polar radius ρ denotes the distance $(x^2+y^2)^{1/2}$ taken in the $z = 0$ plane. The general solution of the harmonic equation (10a) is sought by superposing the well-known particular singular solution corresponding to the infinite domain

$$G^+_f(\mathbf{r}|\mathbf{r}_0) = 1/|\mathbf{r} - \mathbf{r}_0| \quad (24)$$

with a harmonic function, regular in the upper half-space $z \geqslant 0$ and adjusted so as to satisfy the free-surface condition (23). Upon employing the shortened notation $R = |\mathbf{r} - \mathbf{r}_0|$ and $R' = |\mathbf{r} - \mathbf{r}^-_0|$, it is convenient to split the solution according to

$$G^+(\mathbf{r}|\mathbf{r}_0) = 1/R - 1/R' + V(\mathbf{r}|\mathbf{r}^-_0), \quad (25)$$

where, for the case of identical relative permeabilities ($m = 1$), the solution V is given in rectangular coordinates by

$$V(\mathbf{r}|\mathbf{r}^-_0) = -\frac{1}{2\pi\omega}\frac{\partial}{\partial z}\int_0^\omega dt \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \times \frac{\exp[t\zeta - t(\zeta^2+\eta^2)^{1/2}]\, d\zeta\, d\eta}{(\zeta^2+\eta^2)^{1/2}[(x-x_0-\zeta)^2 + (y-y_0-\eta)^2 + (z+z_0)^2]^{1/2}}. \quad (26)$$

For completeness, it should be stated that the trivial derivative with respect to z and the integration with respect to t have not been carried out here in order to keep V in a compact form.

Equations (25) and (26) show that the fundamental solution $G^+(\mathbf{r}|\mathbf{r}_0)$, that is, the potential at point $P(\mathbf{r})$ of a unit source located at point $M(\mathbf{r}_0)$, may be written as the sum of three contributions:

(i) the familiar potential $1/R$ of the source in the infinite domain as if there were no free surface;

(ii) an image sink potential $-1/R'$ which is associated with a slab conductor propagating infinitely fast (these two terms represent the Green's function of the so-called "infinite Reynolds number approximation" in which the lines of induction are rigidly transported);

(iii) a "trailing" potential $V(\mathbf{r}|\mathbf{r}_0^-)$ which characterizes the field disturbance caused by the conductor's motion at finite velocity. As a matter of fact, for a unit relative permeability conductor at rest, one recovers the infinite domain solution as expected: the verification that in Eq. (26) the limit of V when ω approaches zero is $1/R'$ and cancels out the second contribution is left to the reader. It may be worth emphasizing that since Green's functions in three-dimensional space are homogeneous to the inverse of a length, the function V, after appropriately introducing a characteristic length such as h mentioned in Sec. II B, up to a factor $1/h$, becomes a universal function of the dimensionless vector $(\mathbf{r} - \mathbf{r}_0^-)/h$ and the magnetic Reynolds number $R_m = 2\omega h$, which can conceivably be tabulated.

Finally, it is interesting to note that the nature of the above solution is typical of problems involving a free-surface condition whether in differential or integral form. A first example can be taken from hydrodynamics in the theory of steady motion of a ship where the linearized disturbance velocity potential of the steady inviscid free-surface gravity flow, the so-called Havelock source potential, is given by a solution[15] completely analogous to Eq. (25). Another example is provided by the Sommerfeld solution[16] for the propagation of radio signals in a homogeneous medium over a finite conductivity earth: it only difers from the above in the basic equation which is a Helmholtz equation.

Because of the paramount importance of the Green's function, it is extremely useful to develop alternative expressions for $V(\mathbf{r}|\mathbf{r}_0^-)$: eigenfunction expansions in particular can yield good insight into the physical content of the Green's function. Two such expansions are derived in the Appendix and are worth mentioning. With the notations of the Appendix, the first one is

$$V = -\frac{1}{\pi}\sum_{m=0}^{\infty}\epsilon_m \cos m\psi \frac{\partial}{\partial z} \times \int_0^{\infty} S_m(\omega R'u) Q_{m-1/2}\left(\frac{1+u^2}{2u\sin\theta}\right)\frac{du}{(u\sin\theta)^{1/2}} \quad (27)$$

in which the functions $Q_{m-1/2}(t)$ are the toroidal harmonics of the second kind and where the kernel $S_m(t)$ is a recursive combination of modified Bessel functions of the first kind defined as

$$S_m(t) = \exp(-t)[I_0(t) + I_1(t)] + m\,\frac{\exp(-t)I_0(t) - 1}{t} + 2\,\frac{\exp(-t)}{t}\sum_{k=1}^{m-1}(m-k)I_k(t). \quad (28)$$

It is worth emphasizing that $Q_{-1/2}(t)$ and $Q_{1/2}(t)$ are complete elliptic integrals[17] so that higher harmonics can be obtained by recursion. Another useful development in terms of well-behaved functions is the following expansion where

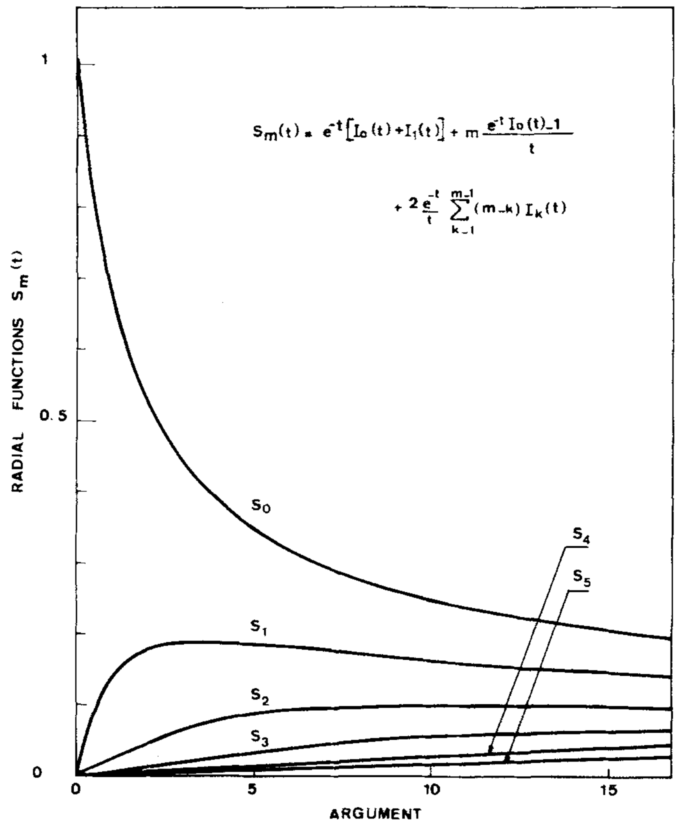

FIG. 2. Radial functions.

the eigenfunctions are defined by twofold integral representations, the angular part of which are precisely related to the foregoing elliptic integrals:

$$V = \frac{\cos\theta}{\pi R'}\sum_{m=0}^{\infty}\epsilon_m\cos m\psi \int_0^{\pi}\int_0^{\infty} \times \frac{S_m(\omega R'u)\cos m\varphi}{(1-2u\sin\theta\cos\varphi + u^2)^{3/2}}\,du\,d\varphi. \quad (29)$$

An interesting piece of information which can be gathered from these expansions relates to the Reynolds-number dependency of $G^+(\mathbf{r}|\mathbf{r}_0)$. As can be observed on the graph of the "radial eigenfunctions $S_m(t)$" (Fig. 2), the contribution of higher-order harmonics is decreasing with the argument, except perhaps in the asymptotic region. For medium-range magnetic Reynolds numbers, the two lowest eigenfunctions provide the essential velocity dependency and, thus, explain qualitatively the shape of the well-known lift and drag curves associated with magnetic levitation systems.[1,7] This fundamental remark is deeply rooted in the parity properties of the regular part of the Green's function in Eq. (25). The starting point of the argument consists in deriving a symmetrized formula for the body force: this is achieved by recalling the result [Eq. (17)] and exchanging field points with source points, thus obtaining

$$\mathbf{F} = \frac{\mu_0}{8\pi}\int_\tau\int_\tau [\mathbf{J}_m(\mathbf{r})\cdot\mathbf{J}_m(\mathbf{r}_0)][\nabla G^+ + \nabla_0 G^+]\,d\tau\,d\tau_0. \quad (30)$$

Now this result is totally invariant in the coordinate exchange if it is simultaneously accompanied by a velocity exchange in the Green's function according to the reciprocity relationship (7). Upon decomposing the Green's function into a velocity-even and a velocity-odd component, it is easy to check that the lift is associated with the even component only because of the symmetry of the argument $(z + z_0)$ used in the computation of the gradient, whereas the drag (or, rather, all horizontal forces) is uniquely associated with the odd component of V because of the antisymmetry of the arguments $(x - x_0)$ and $(y - y_0)$. Thus, insofar as the higher harmonics are unimportant, the lift force exhibits a velocity (Reynolds number) dependency analogous to that of the function $1 - S_0(\omega R')$, while the drag forces display the velocity dependency of the function $S_1(\omega R')$.

For large Reynolds numbers, the eigenfunction expansions become less useful since all radial eigenfunctions have the same asymptotic behavior. However, upon applying the method of steepest descent to integral (26), one can derive the following asymptotic expression for G^+ in terms of an incomplete elliptic integral of the first kind:

$$G^+(\mathbf{r}|\mathbf{r}_0) = \frac{1}{R} - \frac{1}{R'} - \frac{1}{[2\pi\omega(x - x_0)]^{1/2}} \frac{\partial}{\partial z} \times \int_0^1 \frac{dt}{(1-t)^{1/2}[t^2 - 1 + R^{1/2}/(x - x_0)^2]^{1/2}}. \quad (31)$$

This Green's function in the "high Reynolds number approximation" displays a $R_m^{-1/2}$ dependency analogous to the boundary-layer flow behavior encountered in hydrodynamics. This circumstance arises here because large **B**-field gradients due to the conductor's motion completely thwart the field diffusion which normally prevails for small Reynolds numbers: rather a convective-type correction is superposed on the image sink Green's function $-1/R'$. In a similar way, boundary layers are introduced as a perturbation to inviscid potential flows in hydrodynamics.

At this point, a final remark concerning the so-called "infinite Reynolds number approximation" can be made: from Eq. (31), it is clear that in this limit all horizontal efforts vanish, and the only remaining force is the maximum allowed lift, usually called the "image force" and given according to Eq. (20) by

$$F_z^* = \frac{\mu_0}{2\pi} \int_\tau \int_\tau \frac{z_0[\mathbf{J}_m(\mathbf{r}) \cdot \mathbf{J}_m(\mathbf{r}_0)]}{[(x - x_0)^2 + (y - y_0)^2 + z_0^2]^{3/2}} \, d\tau \, d\tau_0. \quad (32)$$

B. Green's functions and Fourier transforms

A great deal of the analytical results published about magnetic levitation systems pertains to forces exerted on moving coils: most computations have been performed by employing Fourier transform techniques.[5–7] Such techniques succeed chiefly because they take advantage of the nature of the coil geometry as well as of the simplicity afforded in Fourier space by the boundary conditions along the interface $z = 0$. Green's functions, on the other hand, provide a more general framework encompassing all of the aforementioned results and relying exclusively on the free-surface condition. The crucial step is to establish a connection between the physical space used so far and the Fourier space: this relationship is ensured by means of the "faltung theorem". Indeed, Green's functions are particularly well suited for this process since they constitute diagonal kernels [with respect to the field and source points by virtue of the reciprocity principle (7)] used within the various convolutions to be carried out in all of the integral solutions obtained in Secs. II–III A.

Let $\mathbf{k}(k_1,k_2)$ be a two-dimensional wave vector, and let the two-dimensional Fourier transform $\hat{\phi}(\mathbf{k},z)$ associated with the function $\phi(\mathbf{r})$ be defined as

$$\hat{\phi}(\mathbf{k},z) = (1/2\pi) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \phi(\mathbf{r}) \exp(-i\mathbf{k} \cdot \rho) \, d^2\rho, \quad (33)$$

where we denote

$$k^2 = k_1^2 + k_2^2, \qquad \rho^2 = x^2 + y^2,$$
$$\mathbf{k} \cdot \rho = k_1 x + k_2 y, \qquad d^2\rho = dx\, dy. \quad (34)$$

In the case of Green's functions, the arguments are $(x - x_0)$ and $(y - y_0)$. Consequently, there appears a phase factor $\exp[-i(k_1x_0 + k_2y_0)]$ which, since no ambiguity can arise, will subsequently be discarded in order to keep the results in compact form: it must naturally be restored when appropriate. The physical content of the faltung theorem[18] expresses the reciprocal relationship existing between the convolution product or "faltung" of two functions $f(\rho)$ and $g(\rho)$ and the product of their respective Fourier transforms according to

$$f * g = (1/2\pi) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\rho')g(\rho - \rho') \, d^2\rho'$$
$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{f}(k)\hat{g}(k) \exp(i\mathbf{k} \cdot \rho) \, d^2k. \quad (35)$$

Instead of taking the transform of all the basic equations [Eqs. (5a)–(5c) and (4a)–(4c) for the vector potential $\mathbf{A}(\mathbf{r})$ and Eqs. (10a), (10b), (22), and (23) for the Green's functions] and solving in Fourier space, it is equivalent to apply the faltung theorem and to "transcribe" all of the results of Secs. II B, II C, and III A in Fourier variables: in particular, integral equations such as (21) or (23) break down into products. For example, the vector potential $\mathbf{A}^+(\mathbf{r})$ transforms into

$$\hat{A}^+(k_1z) = \frac{1}{2}\mu_0 \int_0^\infty \hat{G}^+(k,z|z_0)\hat{J}_m(k_1z_0) \, dz_0, \quad (36)$$

where the Green's function $\hat{G}^+(k,z|z_0)$ is the joint solution of the two equations

$$\frac{d^2\hat{G}^+}{dz^2} - k^2\hat{G}^+ = -2\delta(z - z_0), \quad (37a)$$

$$\hat{G}^+(k,0|z_0) = \frac{1}{m\alpha} \frac{d\hat{G}^+}{dz}(k,0|z_0). \quad (37b)$$

These two equations are, respectively, the transform of Eq. (10a) (omitting the cumbersome phase factor) and the solution of the integral equation (23) obtained by the faltung theorem. Here the function α is defined as the principal complex root of

$$\alpha^2 = k^2 + 2i\omega k_1 \quad (37c)$$

It is left to the reader to check that $1/\alpha$ is indeed the following Fourier transform:

$$\frac{1}{\alpha} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\exp[\omega(x - x_0) - \omega|\rho - \rho_0|]}{|\rho - \rho_0|} \times \exp^{[-i(k_1x + k_2y)]} d^2\rho. \quad (38)$$

From Eqs. (37a) and (37b), it is straightforward to deduce the expression of the upper Green's function $\hat{G}^+(k,z|z_0)$:

$$\hat{G}^+(k,z|z_0) = \frac{\exp(-k|z-z_0|)}{k} + \frac{k-m\alpha}{k+m\alpha} \times \frac{\exp[-k(z+z_0)]}{k}, \quad z,z_0 \geqslant 0. \quad (39)$$

Naturally, up to a phase factor, this is the Fourier transform of the result given for $m = 1$ in Eqs. (25) and (26); in particular, the first term $\exp(-k|z-z_0|)/k$ is the familiar transform of the fundamental singular solution (24). Similarly, the transform of the lower Green's function (22) is just a generalization of Eq. (38):

$$\hat{G}^-(k,z|z_0) = \frac{\exp(-\alpha|z-z_0|)}{\alpha} + \frac{1-m}{1+m} \times \frac{\exp(-\alpha|z+z_0|)}{\alpha}, \quad z,z_0 \leqslant 0. \quad (40)$$

Note that at rest ($\alpha = k$) both functions coincide and represent a well-known result of magnetostatics.[14] It is clear now that expressions (39) and (40) are the kernels of many of the particular results already published. Obviously, all of the investigations regarding either the singularities of the Green's functions or their Reynolds-number dependency are recovered, as well as the results concerning the currents [Eq. (18)] and, also, the body forces which, upon using Parseval's theorem,[18] take the following form:

$$F_i = \frac{\mu_0}{2}\int_0^{\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\hat{J}_m|^2 \frac{\partial \hat{G}^+}{\partial x_i} d^2k\, dz. \quad (41)$$

In summary, the main results obtained in Secs. III A and III B on one hand are the analytical expressions (22) and (25), respectively, for $G^-(\mathbf{r}|\mathbf{r}_0)$ and $G^+(\mathbf{r}|\mathbf{r}_0)$ and the representation in terms of elliptic integrals with modified Bessel functions kernels (27). On the other hand, the exceedingly useful faltung theorem enables one to derive their Fourier transforms (40) and (39), thereby considerably enlarging the framework of previously published results.

C. Application to magnetic levitation

One of the simplest tests with which it is possible to compare the foregoing theory against experimental data is the calculation of forces on moving coils. In terms of potential application to future transportation systems, it is also interesting to evaluate the useful range of variation of such parameters as the velocity, the coil flight height, the guideway width, etc., and their possible tradeoffs. The object of this short section is to give preliminary results demonstrating that the method presented in previous sections is well adapted to magnetic levitation engineering calculations.

1. Experimental drag measurement: thick rectangular coil

We have performed a numerical calculation corresponding to the drag force measured in a recent experiment[19] carried out within our laboratory. The geometrical configuration is presented in Fig. 3 together with the experimental data and the numerical results. The 200 × 100-mm coil has its long dimension in the direction of motion; it carries a 4000-A turn current. During the measurement, the coil is suspended at a height of about 5 mm above the rim of a solid rotating wheel. This wheel is 600 mm in diameter, 400 mm wide, and

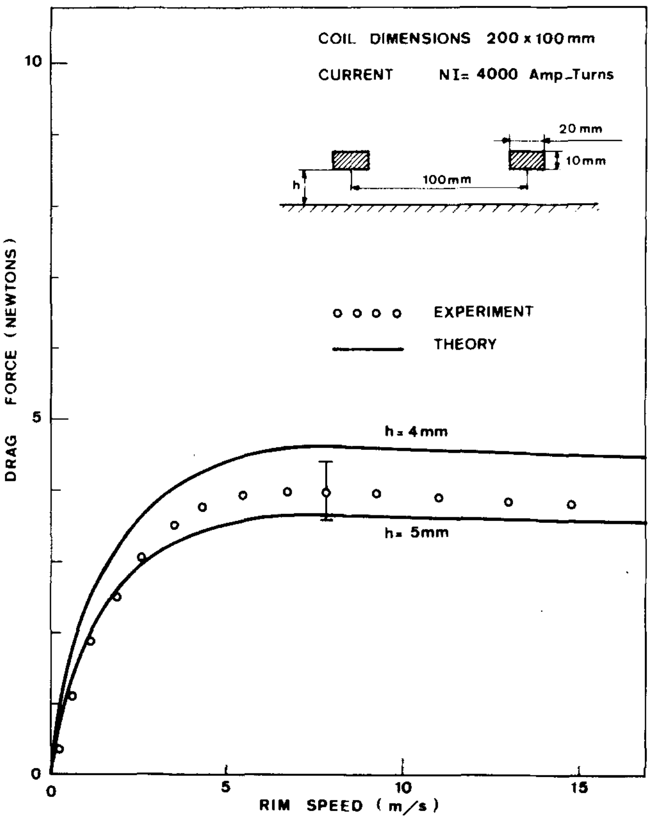

FIG. 3. Drag force and measurement.

can reach a maximum rim speed of about 16 m/s; it is made out of an aluminum alloy such that the product $\mu\sigma$ approaches 20 in mksA units. The drag force is measured with a conventional dynamometric gauge.

Two calculations have been performed with this coil, respectively with a 4-mm air gap and a 5-mm gap. The reason stems from the great difficulty encountered during the experiment in monitoring the flight height with reasonable accuracy. As can be observed the agreement between experimental and theoretical values fluctuates within ±12%. It should also be stated that, owing to the rather small diameter of the wheel relative to the large coil length, the coil is given some curvature for the experiment and this introduces an extra component within the drag force which has proven difficult to estimate. The overall experiment is in the process of being improved and we expect to obtain more refined data. Nevertheless, the agreement obtained so far between theory and experiment can be qualified as reasonable.

2. Forces on a rectangular coil

Although many aspects of magnetic levitation systems can be investigated within the framework of this theory, and since considerable research is currently devoted to the comparison of design tradeoffs, we have chosen to present a nondimensional chart of lift and drag forces as a function of the basic scaling parameter, namely, the magnetic Reynolds number based on the flight height h. We consider a rectangular flat coil for which the dimensionless length and width are $2\mathbf{a} \times 2\mathbf{b}$; the forces acting on the rectangular coil are made dimensionless after dividing by the image force [Eq. (32)]. The parametrizing of the results in terms of the rela-

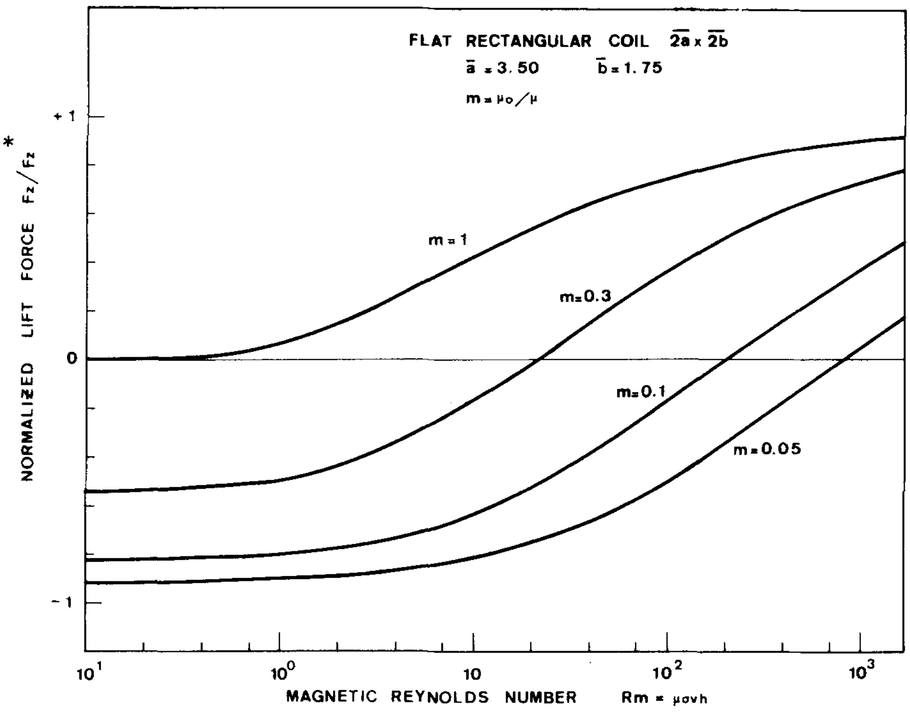

FIG. 4. Lift chart for magnetic levitation.

tive permeability ratio $m = \mu_0/\mu$ as presented on the logarithmic plot (Figs. 4 and 5) is intended to simulate the possible use of ferromagnetic tracks. In fact, since saturation and hysteresis of the material play an important role in such cases, a nonlinear theory accounting for the bulk of these effects should be constructed. However, by arbitrarily increasing the linear permeability of the material, one gets a good qualitative idea of the physical situation. As can be seen, the lift is attractive at low Reynolds numbers and repulsive at sufficiently higher ones: this is explained by the intense surface current [Eq. (18)] caused by the strong magnetization of the conductor which at a low Reynolds numbers overcomes the convective volume currents. For a particular flight condition, the forces in mksA units can be recovered after multiplying by the image force which for a rectangular coil can easily be evaluated directly using Eq. (32) and is given by the well-known formula

$$F_z^* = \frac{\mu_0 I^2}{\pi}\left[(1+\mathbf{a}^2) + (1+\mathbf{b}^2)^{1/2} + \frac{1}{(1+\mathbf{a}^2)^{1/2}} + \frac{1}{(1+\mathbf{b}^2)^{1/2}} - \left(\frac{1}{1+\mathbf{a}^2} + \frac{1}{1+\mathbf{b}^2}\right)(1+\mathbf{a}^2+\mathbf{b}^2)^{1/2} - 2\right]. \quad (42)$$

IV. CONCLUSION

The scope of this paper has been the development of a comprehensive physical theory of eddy currents and forces

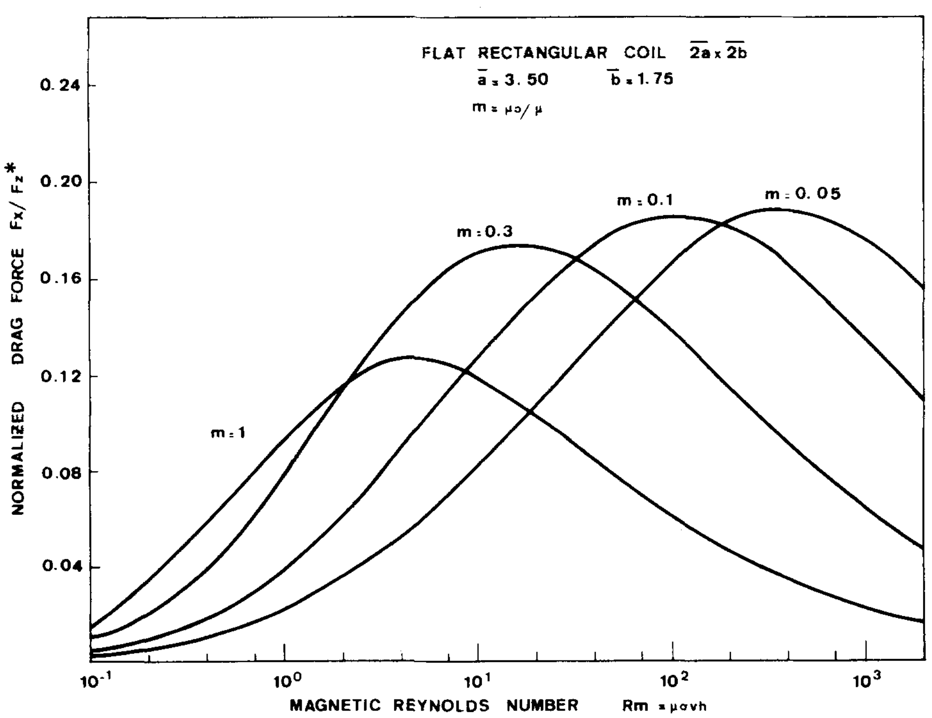

FIG. 5. Drag chart for magnetic levitation.

for conductors in steady motion past arbitrary current sources. The overall formulation is carried out in terms of a vector potential from which the induction, eddy currents, and electromagnetic forces are simply derived. By requiring this potential to be exclusively determined outside the conductor in terms of the current source, it follows that the kernel or Green's function of this integral representation must satisfy a precise integral equation along the outer surface of the conductor because of the boundary conditions imposed on the potential. Closed-form solutions are then derived when the conductor is an infinite slab. The symmetries of the foregoing Green's function, its magnetic Reynolds-number dependency, and its asymptotic form are then fully exploited in order to analyze the coupling of the field with the moving conductor and to predict the properties of the body force, namely, the lift and drag such as are commonly measured on magnetic levitation systems. The connection between previously published results in Fourier space is subsequently established by showing that the relevant integral representations are nothing but "faltungs" owing to the reciprocity relationship satisfied by Green's functions in general. Also, some numerical calculations of drag forces performed for the purpose of comparison are found to be in reasonable agreement with recent experimental results obtained within our laboratories. Because of the versatility of the theory, we believe that many interesting engineering calculations can now be envisioned for future applications.

APPENDIX

We seek an eigenfunction expansion of solution (26). Let us introduce cylindrical coordinates

$$\begin{aligned} x - x_0 = X = r\cos\psi, &\quad \zeta = \rho\cos\varphi, \\ y - y_0 = Y = r\sin\psi, &\quad \eta = \rho\sin\varphi, \\ z + z_0 = Z. & \end{aligned} \tag{A1}$$

Then proceeding with the following expansion in terms of the toroidal harmonics of the second kind,

$$[Z^2 + r^2 - 2\rho r\cos(\varphi - \psi)]^{-1/2} = \sum_{m=0}^{\infty} \epsilon_m \cos m(\varphi - \psi)\frac{1}{\pi(\rho r)^{1/2}} Q_{m-1/2}\left(\frac{Z^2 + r^2 + \rho^2}{2\rho r}\right) \tag{A2}$$

(ϵ_m is the Neumann symbol which is unity when m is zero and takes the value 2 otherwise), we can write

$$V = -\frac{1}{2\pi\omega}\int_0^{\omega} dt \frac{\partial}{\partial z}\sum_{m=0}^{\infty}\epsilon_m \times \int_{-\pi}^{+\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp(t\rho\cos\varphi - t\rho) \times \frac{\cos m(\varphi-\psi)}{\pi(\rho r)^{1/2}} Q_{m-1/2}\left(\frac{Z^2 + r^2 + \rho^2}{2\rho r}\right) d\rho\, d\varphi. \tag{A3}$$

We first evaluate the angular integral in terms of modified Bessel functions of the first kind,

$$(1/2\pi)\int_{-\pi}^{+\pi}\exp(t\rho\omega\varphi)\cos m(\varphi - \psi)\, d\varphi = \cos m\psi I_m(\rho t), \tag{A4}$$

and then perform the ω integration

$$(1/\omega)\int_0^{\omega}\exp(-\rho t)\, I_m(\rho t)\, dt = S_m(\omega\rho), \tag{A5}$$

where the "radial function" $S_m(t)$ is obtained recursively according to

$$S_m(t) = \exp(-t)[I_0(t) + I_1(t)] + m\frac{\exp(-t)I_0(t) - 1}{t} + 2\frac{\exp(-t)}{t}\sum_{k=1}^{m-1}(m-k)I_k(t), \tag{A6}$$

$$S_m = 2S_{m-1} - S_{m-2} + \frac{2}{t}\exp(-t)I_{m-1}. \tag{A7}$$

We now turn to a spherical coordinate system such that

$$Z = R\cos\theta, \quad r = R\sin\theta, \quad R^2 = X^2 + Y^2 + Z^2. \tag{A8}$$

Under these conditions, the expansion takes the form

$$V = -\frac{1}{\pi}\sum_{m=0}^{\infty}\epsilon_m \cos m\,\psi\frac{\partial}{\partial z} \times \int_0^{\infty} S_m(\omega R u) Q_{m-1/2}\left(\frac{1+u^2}{2u\sin\theta}\right)\frac{du}{(u\sin\theta)^{1/2}}. \tag{A9}$$

For $m = 0$ and $m = 1$, the toroidal functions are related to complete elliptic integrals[17]; thus, higher-order harmonics can be obtained by recursion from the general recurrence relations of Legendre's functions. The other expansion of interest is deduced from the development[13]

$$(ch\mu - \cos\varphi)^{-1/2} = (\sqrt{2}/\pi)\sum_{n=0}^{\infty} Q_{n-1/2}(ch\mu)\cos n\varphi. \tag{A10}$$

Upon using the orthogonality properties of circular functions, and upon carrying out the derivative with respect to Z, it may be shown that

$$V = \frac{\cos\theta}{\pi R}\sum_{m=0}^{\infty}\epsilon_m \cos m\psi \times \int_0^{\pi}\int_0^{\infty}\frac{S_m(\omega R u)\cos m\varphi}{(1 - 2u\sin\theta\cos\varphi + u^2)^{3/2}}\, du\, d\varphi. \tag{A11}$$

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