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Radiation from a Uniformly Accelerated Charge*

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The electromagnetic field associated with a uniformly accelerated charge is studied in some detail. The equivalence principle paradox that the co-accelerating observer measures no radiation while a freely falling observer measures the standard radiation of an accelerated charge is resolved by noting that all the radiation goes into the region of space time inaccessible to the co-accelerating observer.

I. INTRODUCTION

The question of whether a uniformly accelerated charge radiates has been the subject of a long series of papers with some distinguished authors reaching the conclusion that it does while others, equally distinguished, reach the conclusion that it does not. The most careful treatments are those of Fulton and Rohrlich [1] and Coleman [2] both of whom concluded that the charge does in fact radiate. They both use Minkowski coordinates to describe the motion of the particle with Fulton and Rohrlich giving a careful discussion of the definition of radiation showing that the energy radiated is precisely that which one expects from an accelerated particle (see, e.g., Jackson [3])

$$\frac{dE}{d\tau} = \frac{2}{3}\left(\frac{e^2}{4\pi}\right)\frac{\dot{u}^2}{c^3}\,u^0 \tag{I.1}$$

where $dE/d\tau$ is energy radiated per unit proper time of the particle, $\dot{u}^2$ is the square of the 4-acceleration of the particle and Heaviside–Lorenz units are employed ($\nabla \cdot \mathbf{E} = \rho$).

There are two arguments which have been used to suggest that this immediate (and correct) conclusion is wrong. First, the radiation reaction on the charge is

$$m\dot{u}^{\mu} = -\frac{2}{3}\frac{e^2}{4\pi}\frac{1}{c^3}\{u^{\mu}\dot{u}^2 - \ddot{u}^{\mu}\} \tag{I.2}$$

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and, for a particle undergoing uniform acceleration,

$$\ddot{u}^\mu = \frac{d^2}{d\tau^2} u^\mu = g^2 u^\mu$$
$$\dot{u}^2 = g^2 = \text{constant} \tag{I.3}$$

hence the radiation reaction vanishes and the motion of the charge is unaffected by the radiation which it emits. The paradox is discussed by Fulton and Rohlich [1] and more fully by Coleman [2] who points out that, in the point particle limit in which the radiation reaction (I.2) is valid, the sum of the radiation field energy and the energy of interference between the radiation field and the (Lorentz transformed) Coulomb field do in fact remain constant. Hence there should be no radiation reaction.

The other argument involves the equivalence principle and runs as follows: By the equivalence principle, a uniformly accelerated frame must be indistinguishable from a gravitational field. However, a charged particle at rest in a static gravitational field cannot radiate, hence a uniformly accelerated particle cannot radiate.

Fulton and Rohrlich argued that a uniform gravitational field is unphysical and, since one must determine whether radiation is present by measurements made at large distances from the charge, the global static gravitational field idealization is not appropriate for discussing the question. Otherwise put, the equivalence principle is a local principle, not one which can be applied globally.

Coleman, on the other hand, argues that the principle of equivalence asserts not only that there should be no radiation, but that there should be only a Coulomb field for a uniformly accelerated charge measured by a uniformly accelerated observer. He then notes that Fulton and Rohrlich's field satisfies this condition, hence the principle of equivalence is valid. The way out of the paradox is, then, to deny that the concept of radiation is the same in the accelerated and unaccelerated frames.

This observation too is incomplete. A brief review of the static gravitational field is given in Section II. The most important point, the significance of which has not always been fully appreciated, is that a uniformly accelerated observer has an event horizon. As may be seen by looking at Fig. 1, no matter how long he waits, the observer moving with the charge will never receive any information about half of the space-time (regions II and III). Because he is asymptotically approaching the speed of light, one quarter of the space-time (region III) is everywhere space-like with respect to the observer's world line and another quarter of the space-time (region II) can receive light signals from the observer but cannot send light signals to him.

As a result of the existence of the event horizon and the fact that the observer cannot send signals to region IV, if one transforms to coordinates with respect to which the uniformly accelerated observer is at rest, those coordinates can only cover region I. There are coordinate singularities at the boundaries of region I with regions II and III. The metric in region I is static in that the uniformly accelerated observer sees no change with respect to his time τ. However, the time coordinate measures the observer's position along the hyperbola and the observer has a different velocity for each time. Thus, changes in time (τ) are changes in velocity and the time translation

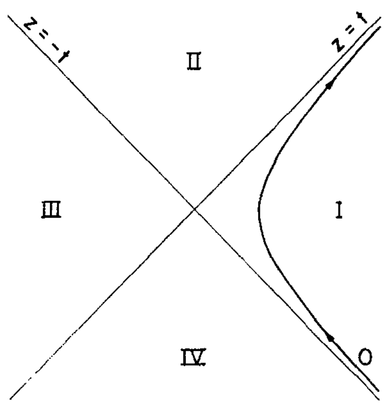

FIG. 1. The straight lines indicate the null planes $z = \pm t$. The uniformly accelerated observer whose path is indicated (O) can receive signals from regions I and IV and can send signals to regions I and II. Region III is everywhere spacelike with respect to the observer's world line.

invariance is the invariance of the Minkowski coordinates under Lorentz transformations. One may naturally extend these coordinates to regions II, III, and IV by requiring that changes in the coordinate τ be the changes under Lorentz transformations just as in region I; the resultant metric in regions II and IV is independent of τ (invariant under Lorentz transformations) but the invariant hyperbolic cylinders are spacelike surfaces and τ is a space coordinate. The other coordinate, Z, which is the space coordinate in region I is the time coordinate in region II and the metric does depend on Z. There is no time independent coordinate system for region II in which a uniformly accelerated charge in region I is at rest.

The electromagnetic field produced by the charge is discussed in Section III where it is shown that the retarded field is restricted to regions I and II, plus a delta function field restricted to the null surface separating regions I and II from regions III and IV. The delta function field which is necessary for the field to satisfy Maxwell's equations may be understood by considering a charge which initially has a constant velocity of approach and begins its uniform acceleration at some finite time. The limit in which the velocity of approach goes to the velocity of light and the time at which the acceleration began goes to minus infinity produces the field of the uniformly accelerated charge in regions I and II. Further, the initial Lorentz transformed Coulomb field goes over into the delta function field in that limit.

The resultant field has a number of unusual properties. In region I the field is precisely the retarded field of the accelerated charge, including the appropriate radiation field with the $1/r$ behavior where r is the radius of the future light cone of the charge. In region I the field is *also* precisely the *advanced* field of the accelerated charge including the inflowing radiation with its $1/r'$ behavior where r' is radius of the backward light cone of the charge. That is, at every point in region I, the field may be viewed as *either* the retarded (Coulomb plus outgoing radiation) field *or* the advanced (Coulomb plus incoming radiation to be absorbed) field of the charge. No measurements within region I can distinguish the alternative interpretations. Furthermore, within region I the radii of the light cones are restricted so that the observer cannot consider a limit of large r so as to readily distinguish the Coulomb field from the radiation field.

There is a paradox here. The field was calculated as the retarded field, *complete with incoming radiation.* However, the backward light cone, sufficiently extended, reaches into region IV where there is no field. Where did the radiation come from? The answer is the delta function field along the null surface separating regions I and IV. The field in region I cannot be maintained with a vanishing field in region IV without a delta function field on the surface to feed the field in region I. It is the familiar phenomenon of ordinary accelerated charges: after the acceleration, the charge is not in the right place to support the original Coulomb field, hence the difference between the original and final Coulomb fields is converted to radiation. What is unusual is that the resultant radiation is focused back onto the world line of the particle.

The situation in region I may be summarized as follows: The field at a given point may be regarded *either* as the Coulomb field plus outgoing radiation field of the charge at the intersection of its world line with the backward light cone *or* as the Coulomb field plus incoming radiation field of the charge at the intersection of its world line with the forward light cone of the field point. If the radiation field is defined to be one-half the difference between the retarded field and the advanced field, there is no radiation. If one identifies the radiation by the $1/r$ dependence of the field along the light cone one cannot decide whether the radiation is retarded or advanced and, furthermore, one cannot remain within region I and let r become large enough for the radiation field to dominate. Thus, the observer whose measurements are restricted to region I will not be able to decide whether there is any radiation and may conclude that all the radiation is absorbed and reemitted by the charge. Thus, there is consistency with the conclusions of the accelerated observer whose measurements are restricted to region I and who only detects a Coulomb field, with no radiation at all.

The situation is region II is quite different. There is no coordinate frame covering the region in which the accelerated charge is at rest and the metric static. As a result, one cannot argue that an accelerated observer finds no radiation. Further, if one calculates the field, one again finds that it is a Coulomb field going as $1/r^2$ plus a radiation field going as $1/r$. There is outgoing radiation which cannot be interpreted as incoming radiation because region II is outside the backward light cone of the charge. The radius of the light cone can now be made arbitrarily large and the radiation field can be made very large compared to the Coulomb field. The radiation is certainly present and may be identified by any of the standard methods. There is, however, a subtlety. The field in region II is invariant under reflection in the plane through the point where all four regions meet and the replacement of the charge e by $-e$. Thus, the field in region II may be regarded as *either* the field due to a uniformly accelerated charge, e, in region I *or* as the field due to a uniformly accelerated charge $-e$ in region III; no measurements restricted to region II can ever distinguish the two situations.

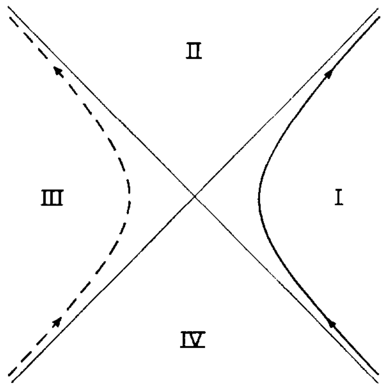

FIG. 2. Under reflection in z and $e \to -e$, the fields in Region II are invariant and the charge in Region I becomes a charge $-e$ in Region III.

The last section is devoted to a discussion of the energy flows which exhibit the properties which one immediately infers from the preceeding discussion including the outgoing energy flow in region II. There is however, one additional point which is not so obvious. The full radiation reaction vanishes, hence there is no net flow of energy into the electromagnetic field. This is verified; however, there is a net flow of energy into the radiation field which is exactly compensated by a decrease in the energy of interference between the continuous retarded field and the delta function field along the null surface separating regions I and II from regions II and IV.

II. STATIC GRAVITATIONAL FIELD

The metric associated with a static gravitational field has been discussed in several places (see, e.g., Rohrlich [4] and Misner, Thorne, and Wheleer [5]); the arguments are summarized here for completeness. The space is assumed to be invariant under time translations and under the Euclidean group, E_2, of translations and rotations in the plane. As a result, the most general metric is

$$ds^2 = -\phi^2(z)\, d\tau^2 + A^2(z)\, dz^2 + B^2(z)(dx^2 + dy^2) \tag{II.1}$$

however, the function A may be set equal to one by a change of variable,

$$z \to \zeta = \int^z d\tilde{z}\, A(\tilde{z})$$

and the non-zero curvature components are

$$\begin{aligned} R^{\tau}_{\zeta,\tau\zeta} &= -(\phi''/\phi), \\ R^{\tau}_{x\ \tau x} &= R^{\tau}_{y,\tau y} = \frac{-\phi' B' B}{\phi} \\ R^{x}_{\zeta,x\zeta} &= R^{y}_{\zeta,y\zeta} = -(B''/B) \\ R^{x}_{y,xy} &= -(B')^2. \end{aligned} \tag{II.2}$$

In order for the space to be equivalent to a flat space, the curvature must vanish, hence

$$B' = 0 = \phi'', \qquad \phi = (1 + g\zeta), \qquad B = 1$$

and

$$ds^2 = -(1 + g\zeta)^2 \, d\tau^2 + d\zeta^2 + dx^2 + dy^2 \tag{II.3}$$

is the most general coordinate system. The coordinates x, y and τ have been scaled so that $\phi = B = 1$ at $\zeta = 0$ and g is an arbitrary constant which turns out to be the proper acceleration of a body sitting at $\zeta = 0$. For the purposes of this work, it is convenient to translate the coordinates

$$Z = \frac{1}{g} + \zeta$$

so that the metric becomes,

$$ds^2 = -g^2 Z^2 \, d\tau^2 + dZ^2 + dx^2 + dy^2 \tag{II.3}$$

which is related to the Minkowski coordinates by the transformation

$$z = Z \cosh g\tau, \qquad t = Z \sinh g\tau. \tag{II.4}$$

This transformation does not cover the entire space. The $z - t$ plane is shown in Fig. 1 and the coordinate transformation in each of the regions is

$$\begin{array}{lll} z = Z \cosh g\tau, & t = Z \sinh g\tau & \text{I} \\ z = Z \sinh g\tau, & t = Z \cosh g\tau & \text{II} \\ z = -Z \cosh g\tau, & t = -Z \sinh g\tau & \text{III} \\ z = -Z \sinh g\tau, & t = -Z \cosh g\tau & \text{IV} \end{array} \tag{II.5}$$

with the metric

$$ds^2 = \epsilon(-g^2 Z^2 \, d\tau^2 + dZ^2) + dx^2 + dy^2 \tag{II.6}$$

where

$$\epsilon = \begin{cases} 1 & x \text{ in I, III} \\ -1 & x \text{ in II, IV.} \end{cases}$$

These coordinates are known as Rindler coordinates [6].

Region I is the region of space time such that the uniformly accelerated observer at $Z = g^{-1}$ can both receive signals from any point and send signals to any point. The set of points from which the observer can receive signals is

$$t < \max\{ g^{-1} \sinh g\tau - [(g^{-1} \cosh g\tau - z)^2 + \rho^2]^{1/2} \} = z$$

or $z - t > 0$, and the set of points to which the observer can send signals is

$$t > \min\{g^{-1} \sinh g\tau + [(g^{-1} \cosh g\tau - z)^2 + \rho^2]^{1/2}\} = -z$$

or $z + t > 0$.

As seen by the uniformly accelerated observer, a world line which crosses the boundaries of region I, $z = \pm t$, does so at the time (for the observer) $\tau = \pm\infty$; also, it takes an infinite time, τ, for a signal emitted from the world line as it crosses the future boundary, $z = t$, to catch up with the observer, and a signal must be emitted in the infinite past to meet the world line as it enters region I. The observer attributes the strange behavior of light and other freely falling bodies to the extremely strong gravitational field which produces a future event horizon at $Z = 0$ ($z = t$) so that there is an infinite red shift for signals emitted from there.

The coordinates associated with the general static metric, Eq. (II.3), cover the entire space only if one translates $Z \to g^{-1} + Z$ and takes the limit $g \to 0$ in which case it becomes the Minkowski metric which is globally static. If $g \neq 0$, the metric only applies to a portion of the space, I, However, as a consequence of having required that the curvature vanish, the metric is simply a coordinate transformation of the Minkowski metric. The metric is invariant under τ translation; in terms of the Minkowski coordinates (z, t), these are simply the Lorentz transformations: under $\tau \to \tau + \alpha$,

$$z \pm t \to (z \pm t)\, e^{\pm g\alpha}. \tag{II.7}$$

A particle at rest at $Z = Z_0$ in sector I of the (Z, τ) coordinate system, has the Minkowski trajectory

$$z = Z_0 \cosh g\tau, \qquad t = Z_0 \sinh g\tau$$

or

$$u^\nu = \frac{dx^\nu}{d\lambda} = gZ_0(\cosh g\tau, \sinh g\tau)\, \frac{d\tau}{d\lambda}$$

where λ is the proper time of the particle. But,

$$u^2 = -1 = \left(\frac{d\tau}{d\lambda}\right)^2 (-g^2 Z_0^{\,2})$$

thus,

$$\frac{d\tau}{d\lambda} = \frac{1}{gZ_0} \tag{II.8}$$

and

$$\frac{du^\mu}{d\lambda} = \frac{1}{Z_0} (\sinh g\tau, \cosh g\tau) = a^\mu$$

and the square of the proper acceleration, a^μ, is $1/Z_0^{\,2}$. The (essentially unique) coordinate frame describing a static gravitational field indeed has bodies at rest in it undergoing constant acceleration in an inertial frame. However, bodies located at different points undergo different accelerations: it is not possible to find a single static coordinate system in which bodies at rest at different points undergo the same proper acceleration because two bodies experiencing the same proper acceleration do not maintain the same proper distance.

To understand this, note that the rest frame for an accelerated observer is simply $\tau =$ constant, and the distance to a point at Z', τ is just $| Z - Z' |$. If two bodies start out at rest but initially separated by l_0, their trajectories are

$$\begin{aligned} z_1 &= g^{-1}\cosh g\tau, & z_2 &= \pm l_0 + g^{-1}\cosh g\tau_2 \\ t_1 &= g^{-1}\sinh g\tau, & t_2 &= g^{-1}\sinh g\tau_2 \\ Z_1 &= (z_1{}^2 - t_1{}^2)^{1/2} = g^{-1} & Z_2 &= (z_2{}^2 - t_2{}^2)^{1/2} \\ & & &= (g^{-2} \times l_0^2 \pm 2g^{-1}l_0\cosh g\tau_2)^{1/2} \end{aligned} \tag{II.9}$$

Thus, the distance between the particles is

$$L(\tau) = \pm[(g^{-2} + l_0^2 \pm 2g^{-1}l_0\cosh g\tau_2)^{1/2} - g^{-1}].$$

If particle 1 is chasing particle 2, $(+)$, it never even gains; the lead as measured by its co-accelerating observer is always increasing. On the other hand, if particle 1 is leading particle 2, $(-)$, its lead as measured by its co-accelerating observer is increasing but only until the square root vanishes at which point $L_{\max} = 1/g > l_0$. This limit occurs at infinite proper time for particle 1 as the second particle's world line crosses the $Z = 0$ surface between regions I and II. As seen by the leading particle the following particle lags further behind, asymptoting to a distance g^{-1}.

III. The Uniformly Accelerated Charge

The electromagnetic field associated with a uniformly accelerated charge has been discussed in many places: Fulton and Rohrlich [1], Born [7], Pauli [8], and Bondi and Gold [9]. A brief review will be given here. The Liénard–Wiechert potential for a particle of charge e moving on the world line $x(\lambda)$ is given by [3]

$$A^\mu(x) = \frac{e}{2\pi}\int_{-\infty}^{\infty} d\lambda' \,\frac{dx^\mu}{d\lambda}(\lambda')\,\theta(t - x^0(\lambda'))\,\delta((x - x(\lambda'))^2) \tag{III.1}$$

where, for a uniformly accelerated charge,

$$x(\lambda) = 0 = y(\lambda), \qquad t = g^{-1}\sinh g\lambda, \qquad z(\lambda) = g^{-1}\cosh g\lambda$$

$$\frac{dx^\mu}{d\lambda} = (\cosh g\lambda, 0, 0, \text{singh}\, g\lambda).$$

The vector potential is most simply calculated in Rindler coordinates, (II.5),

$$\begin{aligned}
A_\tau &= \frac{\partial t}{\partial \tau} A_t + \frac{\partial z}{\partial \tau} A_z \\
&= -\frac{egZ}{2\pi} \int_{-\infty}^{\infty} d\lambda \cosh g(\tau-\lambda)\, \theta(\tau-\lambda)\, \delta(g^{-2}+\rho^2+Z^2-2Zg^{-1}\cosh g(\tau-\lambda)) \\
&\qquad\qquad z > |t| \\
&= -\frac{eg\tilde{Z}}{2\pi} \int_{-\infty}^{\infty} d\lambda \sinh g(\tau-\lambda)\, \delta(g^{-2}+\rho^2-\tilde{Z}^2-2\tilde{Z}g^{-1}\sinh g(\tau-\lambda)) \\
&\qquad\qquad t > |z| \\
&= 0 \qquad t+z<0
\end{aligned} \tag{III.2}$$

where $\rho^2 = x^2 + y^2$. Thus,

$$A_\tau = \begin{cases} -(eg/4\pi)(g^{-2}+\rho^2+Z^2)/[(g^{-2}+\rho^2+Z^2)^2-4g^{-2}Z^2]^{1/2}, & z>|t| \\ -(eg/4\pi)(g^{-2}+\rho^2-\tilde{Z}^2)/[(g^{-2}+\rho^2-\tilde{Z}^2)^2+4g^{-2}\tilde{Z}^2]^{1/2}, & t>|z| \\ 0, \quad t+z<0 \end{cases} \tag{III.3}$$

and, similarly,

$$\begin{aligned}
A_Z &= \frac{\partial t}{\partial Z} A_t + \frac{\partial z}{\partial Z} A_z \\
&= \begin{cases} -e/4\pi Z, & z>|t| \\ -e/4\pi\tilde{Z}, & t>|z| \\ 0, & t+z<0 \end{cases}
\end{aligned} \tag{III.4}$$

hence,

$$\begin{aligned}
A^t &= \frac{\partial t}{\partial \tau} A^\tau + \frac{\partial t}{\partial Z} A^Z = -(z/gZ^2)\, A_\tau + (t/Z)\, A_Z \\
&= (e/4\pi Z^2)\{[zg(g^{-2}+\rho^2+Z^2)/2R] - t\}\, \theta(t+z)
\end{aligned} \tag{III.5}$$

where

$$Z^2 = z^2 - t^2$$

and

$$R \equiv (g/2)[(g^{-2}+\rho^2+Z^2)^2 - 4g^{-2}Z^2]^{1/2}.$$

Further

$$A^z = (e/4\pi Z^2)\{[tg(g^{-2}+\rho^2+Z^2)/2R] - z\}\, \theta(t+z). \tag{III.6}$$

The electromagnetic fields may be obtained from the potentials,

$$\begin{aligned}
\tilde{F}^{tz} &= E^z = -\partial A^z/\partial t - \partial A^t/\partial z \\
&= (e/4\pi)[g(Z^2-g^{-2}-\rho^2)/2R^3]\, \theta(t+z) \\
\tilde{F}^{t\boldsymbol{\rho}} &= E^{\boldsymbol{\rho}} = -\nabla_\rho A^t \\
&= (e/4\pi)(\boldsymbol{\rho} gz/R^3)\, \theta(t+z) \\
\tilde{F}^{z\boldsymbol{\rho}} &= -\hat{\mathbf{e}}_k \times \mathbf{B} = -\nabla_\rho A^z \\
&= (e/4\pi)(\boldsymbol{\rho} gt/R^3)\, \theta(t+z).
\end{aligned} \tag{III.7}$$

These fields satisfy Maxwell's equations for $z + t > 0$ and, trivially, for $z + t < 0$. However, as was pointed out by Bondi and Gold [9], they do not satisfy Maxwell's equations along the null surface $t + z = 0$,

$$\begin{aligned} \partial_\nu \tilde{F}^{t\nu} &= -\partial_\nu \tilde{F}^{z\nu} = -[e4g^2/4\pi(1 + g^2\rho^2)^2]\, \delta(z + t) \\ \partial_\nu \tilde{F}^{\rho\nu} &= 0 \end{aligned} \tag{III.8}$$

The expression for the electromagnetic field is correct except along the null surface where the expression for the vector potential (III.1), is singular. Since the charge world line is invariant under Lorentz transformations along the z axis, the resultant field must also be invariant. The only field, $\Delta F^{\mu\nu}$, which is invariant and restricted to the null surface $z + t = 0$ is of the form

$$\begin{aligned} \Delta F^{tz} &= 0 \\ \Delta F^{t\rho} &= -\Delta F^{z\rho} = \boldsymbol{\rho} A(\rho)\, \delta(z + t) \\ \partial_\nu\, \Delta F^{t\nu} &= -\partial_\nu\, \Delta F^{z\nu} = \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\, \rho^2 A(\rho)\right] \delta(z + t) \\ \partial_\nu\, \Delta F^{\rho\nu} &= 0. \end{aligned} \tag{III.9}$$

Then, $F^{\mu\nu} = \tilde{F}^{\mu\nu} + \Delta F^{\mu\nu}$ will satisfy Maxwell's equations provided

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\, \rho^2 A(\rho) = \frac{e}{4\pi}\frac{4g^2}{(1 + g^2\rho^2)^2} \tag{III.10}$$

or

$$A(\rho) = (2eg^2/4\pi[1 + g^2\rho^2])$$

and the full electromagnetic field is,

$$\begin{aligned} F^{tz} &= (e/4\pi)\, g\{(Z^2 - g^{-2} - \rho^2)/2R^3\}\, \theta(z + t) \\ F^{t\rho} &= (e/4\pi)\, \boldsymbol{\rho}\{(gz/R^3)\, \theta(z + t) + (2g^2/[1 + g^2\rho^2])\, \delta(z + t)\} \\ F^{z\rho} &= (e/4\pi)\, \boldsymbol{\rho}\{(gt/R^3)\, \theta(z + t) - (2g^2/[1 + g^2\rho^2])\, \delta(z + t)\}. \end{aligned} \tag{III.11}$$

The origin of the delta function field may appear somewhat obscure. It may be understood by considering the field of a charge which has not always undergone uniform acceleration. Let the charge be at rest at $z = 1/g$ for $t < 0$ and undergo uniform acceleration after that. Then the field is simply the Coulomb-field of the charge outside the forward light cone of the point at which the acceleration started and is the field of the uniformly accelerated charge, $\tilde{F}$, inside the light cone,

$$\begin{aligned} F^{tz} &= \frac{e}{4\pi}\frac{(z - 1/g)}{r^3}\, \theta(r - t) + \frac{e}{4\pi}\frac{g(Z^2 - g^{-2} - \rho^2)}{2R^3}\, \theta(t - r) \\ F^{t\rho} &= \frac{e}{4\pi}\frac{\boldsymbol{\rho}}{r^3}\, \theta(r - t) + \frac{e}{4\pi}\frac{\boldsymbol{\rho} gz}{R^3}\, \theta(t - r) \\ F^{z\rho} &= \frac{e}{4}\frac{\boldsymbol{\rho} gt}{R^3}\, \theta(t - r) \end{aligned} \tag{III.12}$$

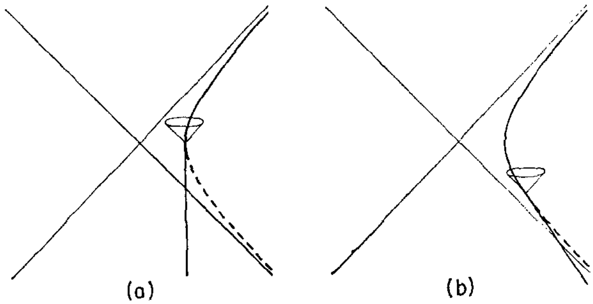

FIG. 3. (a) The world line of a particle initially at rest and subsequently undergoing uniform acceleration. (b) The Lorentz transformed line. In both cases the forward light cone of the spacetime point at which the acceleration starts is shown. The field inside the light cone is that of the uniformly accelerated charge; outside the light cone, the field is the (Lorentz transformed) Coulomb field of the unaccelerated charge.

where

$$r \equiv [\rho^2 + (z - g^{-1})^2]^{1/2}.$$

If the field is now Lorentz transformed so that the charge has a negative initial velocity when the acceleration starts

$$z \to z \cosh \alpha + t \sinh \alpha, \qquad t \to t \cosh \alpha + z \sinh \alpha,$$

the fields become,

$$\begin{aligned}
F^{tz} &= \frac{e}{4\pi} \frac{(z \cosh \alpha + t \sinh \alpha - g^{-1})}{r^3} \theta(r - t \cosh \alpha - z \sinh \alpha) \\
&\quad + \frac{e}{4\pi} \frac{g(Z^2 - g^{-2} - \rho^2)}{2R^3} \theta(t \cosh \alpha + z \sinh \alpha - r) \\
F^{t\rho} &= \frac{e}{4\pi} \frac{\rho \cosh \alpha}{r^3} \theta(r - t \cosh \alpha - z \sinh \alpha) \\
&\quad + \frac{e}{4\pi} \frac{\rho g z}{R^3} \theta(t \cosh \alpha + z \sinh \alpha - r) \qquad \text{(III.13)} \\
F^{z\rho} &= \frac{e}{4\pi} \frac{\rho \sinh \alpha}{r^3} \theta(r - t \cosh \alpha - z \sinh \alpha) \\
&\quad + \frac{e}{4\pi} \frac{\rho g t}{R^3} \theta(t \cosh \alpha + z \sinh \alpha - r)
\end{aligned}$$

where

$$r = [\rho^2 + (z \cosh \alpha + t \sinh \alpha - g^{-1})^2]^{1/2}.$$

In the limit $\alpha \to \infty$, the time at which the uniform acceleration began goes to $-\infty$

and the initial velocity of approach goes to c. In that limit any point with $z + t > 0$ lies inside the forward light cone of the point at which the acceleration started,

$$t \cosh \alpha + z \sinh \alpha - r \sim g^{-1} + O(e^{-\alpha}) > 0$$

while any point with $z + t < 0$ lies outside the light cone,

$$t \cosh \alpha + z \sinh \alpha - r \sim 2(t + z)\, e^{\alpha} - g^{-1} < 0.$$

The value of the field inside the light cone, being invariant under the Lorentz transformations, does not depend upon α, hence, for $z + t > 0$, the field simply goes over into the field of a uniformly accelerated charge. For $z + t < 0$, the field is the limit of the Lorentz transformed Coulomb field but $r \sim -\frac{1}{2}(z + t)\, e^{\alpha} + g^{-1}$, hence the field vanishes as $e^{-2\alpha}$ and there is indeed no field for $z + t < 0$. The null surface $z + t = 0$ is more complicated. There, $r \sim g^{-1}(1 + g^2\rho^2)^{1/2}$ and the field becomes infinite; also the integral of the field over t from $-\infty$ to the light cone is

$$\int_{-\infty}^{\infty} dt\, \theta(r - t \cosh \alpha - z \sinh \alpha)\, F^{tz} = -\frac{e}{4\pi} \frac{1}{\sinh \alpha} \frac{1}{r} \bigg|_{r = t\cosh\alpha + z\sinh\alpha}$$
$$\underset{\alpha\to\infty}{\longrightarrow} -\frac{e}{4\pi} \frac{3}{\sinh \alpha} \frac{2g}{1 + g^2\rho^2} = 0$$

and

$$\int_{-\infty}^{\infty} dt\, \theta(r - t \cosh \alpha - z \sinh \alpha)[F^{t\rho} \pm F^{z\rho}]$$
$$\underset{\alpha\to\infty}{\longrightarrow} \frac{e}{4\pi} \begin{cases} 0 \\ 4g^2\rho/[1 + g^2\rho^2] \end{cases}$$

hence there is a delta function at the null surface and the fields given in (III.11) are reproduced. The delta function field is the original Lorentz transformed Coulomb field of the charge "before" it began its acceleration.

The delta function field can also be calculated directly from the retarded field of the uniformly accelerated charge. If the field is carefully treated as a distribution, the fields in (III.11) are again reproduced.

In region I, $z > |t| \geqslant 0$, the fields are just the retarded fields, $\tilde{F}$, given in (III.7). Note that these fields are invariant under Lorentz transformations along the z axis, if,

$$\begin{aligned} z' &= z \cosh \alpha + t \sinh \alpha \\ t' &= z \sinh \alpha + t \cosh \alpha \end{aligned} \tag{III.14}$$

$Z' = Z$, then

$$\begin{aligned} F^{t'z'}(z', t') &= F^{tz}(z, t) = \frac{e}{4\pi} g(Z'^2 - g^{-2} - \rho^2)/2R^3 \\ F^{t'\rho}(z', t') &= F^{t\rho}(z, t) \cosh \alpha + F^{z\rho}(z, t) \sinh \alpha \\ &= \left(\frac{e}{4\pi}\right) \rho(gz'/R^3) = F^{t\rho}(z', t') \end{aligned} \tag{III.15}$$

and

$$F^{z'\rho}(z', t') = F^{z\rho}(z, t) \cosh \alpha + F^{t\rho}(z, t) \sinh \alpha$$
$$= \left(\frac{e}{4\pi}\right) \rho(gt'/R^3) = F^{z\rho}(z', t).$$

Also, the fields are invariant under time reversal, $t \to -t$, $F^{t\nu} \to F^{t\nu}$, $F^{z\rho} \to -F^{z\rho}$. However, under time reversal, retarded fields are transformed into advanced fields, hence the retarded field is equal to the advanced field.

The field at the point (ρ, z, t) was calculated as the retarded field of some point along the world line of the charge. Lorentz transform so that the charge is at rest at that point; then

$$z = \frac{1}{g} + r \cos \theta \qquad \rho = \hat{\rho}\, r \sin \theta$$

and

$$t = r$$
$$Z^2 = g^{-2} + 2g^{-1} r \cos \theta - r^2 \sin^2 \theta \tag{III.16}$$
$$R = \frac{g}{2} [(g^{-2} + \rho^2 + Z^2)^2 - 4g^{-2}Z^2]^{1/2} = r$$

where r is the radius of the light cone centered on the world line of the charge at the instant at which the charge is at rest. The angle θ is the angle between the acceleration of the charge and the vector to the field point in the instantaneous rest frame of the charge.

The field along the forward light cone of the point at which the charge is at rest is, then,

$$F^{tz} = \left(\frac{e}{4\pi}\right) \left\{\frac{\cos \theta}{r^2} - \frac{g}{r} \sin^2 \theta\right\}$$
$$F^{t\rho} = \left(\frac{e}{4\pi}\right) \hat{\rho} \left\{\frac{\sin \theta}{r^2} + \frac{g}{r} \sin \theta \cos \theta\right\} \tag{III.17}$$
$$F^{z\rho} = \left(\frac{e}{4\pi}\right) \hat{\rho}\, \frac{g \sin \theta}{r}$$

The terms proportional to $1/r^2$ are the Coulomb field of the charge and the terms proportional to $(g \sin \theta/r)$ are the familiar radiation field of a charge undergoing acceleration g. The field along the *backward* light cone is exactly the same, except that the magnetic field, $F^{z\rho}$, is opposite in direction. However, the point along the backward light cone, A', is also along the forward light cone of O' in Fig. 4 and the field there is *also* the retarded field of O' as may be readily verified by Lorentz transforming the points (A, O) into the points (A', O').

As a result the advanced field is equal to the retarded field. The field at any point in region I may be interpreted *either* as the Coulomb field plus *outgoing* radiation field

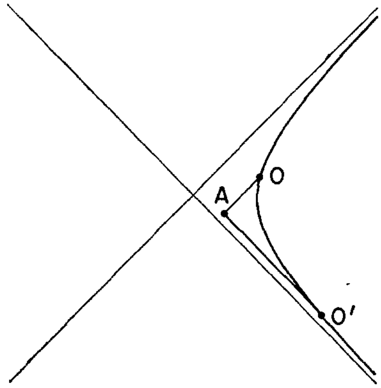

FIG. 4. The field point A is on the backward light cone of O and on the forward light cone of O'. Points in region II are on the forward light cone of some point on the world line of the charge but not on the backward light cone of any point on the world line.

of the charge at the retarded time, *or* as the Coulomb plus *incoming* radiation field at the advanced time.

Although the field in region I is the Coulomb-plus outgoing radiation field, the experimental situation is complicated. First, detailed measurements of the field must be made to determine that fact; second, the measurements will be consistent with the field's being the Coulomb plus incoming radiation field; third, except very near the direction of the acceleration, no limit can be taken such that the radiation field is large compared with the Coulomb field and; fourth, the field, as measured by the distance to the world line of the charge, does not drop off as $1/l$ but rather as $1/l^2$ in region I. To understand the latter two points, note that the forward light cone of the point $z = 1/g$, $t = 0$ at which the charge is at rest is given by (III.16) and

$$(|\mathbf{E}_{\text{rad}}|/|\mathbf{E}_{\text{coulomb}}|) = rg \sin\theta \tag{III.18}$$

but, in region I,

$$0 < z - t = g^{-1} + r(\cos\theta - 1)$$

hence

$$gr < \frac{1}{(1 - \cos\theta)}$$

or

$$(|E_{\text{rad}}|/|E_{\text{coulomb}}|) < \sin\theta/(1 - \cos\theta) \tag{III.19}$$

which is of order 1 except for θ very near zero. For θ very near zero, there is little radiation since that is the direction of the acceleration, and, furthermore, the charge is accelerated so that the distance to the world line of the charge does not increase linearly with r, the radius of the light cone. The distance to the world line is the maximum invariant distance, i.e., the distance in the Lorentz frame in which the charge is instantaneously at rest. Thus, to find the distance, Lorentz transform so that $t = 0$; in region I this is always possible and the distance squared is

$$l^2 = \rho^2 + (Z - g^{-1})^2$$

where Z is the Rindler coordinate

$$Z = [g^{-2} + 2g^{-1}r\cos\theta - r^2\sin^2\theta]^{1/2}$$

and

$$\tfrac{1}{2}(gl)^2 = 1 + gr\cos\theta - [(1 + gr\cos\theta)^2 - (gr)^2]^{1/2} \tag{III.20}$$

however, in region I,

$$y = 1 + gr(\cos\theta - 1)$$

must lie between zero and one, hence,

$$\tfrac{1}{2}(gl)^2 = gr + y - [2gry + y^2]^{1/2}$$

and for gr large

$$gr \sim \tfrac{1}{2}(gl)^2 \tag{III.21}$$

which, of course, entails $\theta \sim 0$. Thus, either the radiation field is comparable in size to the Coulomb field or it drops off with distance to the charge as

$$E\cdot\mathrm{re} \sim \frac{e}{4\pi}\frac{g\sin\theta}{r} \sim \frac{e}{4\pi}\frac{2\sin\theta}{l^2} \tag{III.22}$$

and displays the characteristic $1/l^2$ behavior of a Coulomb field.

The situation in region II is quite different. There the field is the retarded field of the charge; the field is unambiguously the retarded field since the advanced field vanishes. Further the radius of the light cone may be arbitrarily large for any nonzero angle θ. Thus, the radiation field may be made arbitrarily large compared to the Coulomb field. Also, there is no Lorentz frame which passes through a field point in region II and in which the charge is instantaneously at rest, thus there is no invariant distance to the world line of the charge and, since the motion is not bounded, one cannot establish a distance to the charge and misidentify the fields as $1/l^2$ fields. Further, there is no coordinate frame in which the charge is at rest, which is static and which covers region II; the analytic continuation of the accelerated frame from region I to region II yields the coordinates of (II.5) in which the metric is time (Z) dependent. In summary, there is no equivalence principle argument against radiation in region II nor is there any difficulty in identifying the radiation. It is there and is precisely the radiation of the accelerated charge.

Unfortunately that is not the whole story. The fields in region II given by (III.7) are invariant under $z \to -z$ and $e \to -e$. However, under that transformation the charge e undergoing uniform acceleration in region I becomes a charge $-e$ undergoing uniform acceleration in region III: no measurements restricted to region II can distinguish between the Coulomb plus outgoing radiation field of a uniformly accelerated charge e in region I and the Coulomb plus outgoing radiation field of a uniformly accelerated charge $-e$ in region III! Observers in region II will proclaim the presence of the radiation but be totally unable to determine whence it came! The location of the charge in region I is signaled either by 1) the presence of the field in region I, 2) the absence of the field in region III, or 3) the delta function field along the surface $z + t = 0$; any one of the three is sufficient to locate the charge in region I.

IV. The Stress-Energy Tensor

Given the electromagnetic fields, the stress energy tensor,

$$T^{\mu\nu} = F^{\mu\lambda}F_{\lambda}{}^{\nu} - \tfrac{1}{4}g^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}\,, \tag{IV.1}$$

may be easily obtained. The result is

$$\begin{aligned}
T^{tt} &= \left(\frac{e^2}{4\pi}\right)\frac{1}{4\pi}\left\{\frac{g^2}{8R^6}\left[(\rho^2 + g^{-2} - Z^2)^2 + 4\rho^2(t^2 + z^2)\right]\theta(z+t)\right.\\
&\quad \left. + \frac{16g^2 z\rho^2}{(1+g^2\rho^2)^4}\,\delta(z+t) + \frac{4g^4\rho^2}{(1+g^2\rho^2)^2}\,\delta(0)\,\delta(z+t)\right\}\\
T^{tz} &= \left(\frac{e^2}{4\pi}\right)\frac{1}{4\pi}\left\{\frac{g^2 tz\rho^2}{R^6}\,\theta(z+t) - \frac{16g^6\rho^2 z}{(1+g^2\rho^2)^4}\,\delta(z+t)\right.\\
&\quad \left. - \frac{4g^4\rho^2}{(1+g^2\rho^2)^2}\,\delta(0)\,\delta(z+t)\right\}\\
T^{t\boldsymbol{\rho}} &= \left(\frac{e^2}{4\pi}\right)\frac{\boldsymbol{\rho}}{4\pi}\left\{\frac{g^2(\rho^2 + g^{-2} - Z^2)\,t}{2R^6}\,\theta(z+t) - \frac{4g^4}{(1+g^2\rho^2)^3}\,\delta(z+t)\right\}\\
T^{zz} &= \left(\frac{e^2}{4\pi}\right)\frac{1}{4\pi}\left\{\frac{g^2}{8R^6}\left[4(t^2+z^2)\,\rho^2 - (\rho^2 + g^{-2} - Z^2)^2\right]\theta(z+t)\right.\\
&\quad \left. - \frac{g^6 16\rho^2 t}{(1+g^2\rho^2)^4}\,\delta(z+t) + \frac{4g^4\rho^2}{(1+g^2\rho^2)^2}\,\delta(0)\,\delta(z+t)\right\}\\
T^{z\boldsymbol{\rho}} &= \left(\frac{e^2}{4\pi}\right)\frac{\boldsymbol{\rho}}{4\pi}\left\{\frac{g^2}{2R^6}(\rho^2 + g^{-2} - Z^2)\,z\theta(z+t) + \frac{4g^4}{(1+g^2\rho^2)^3}\,\delta(z+t)\right\}\\
T^{\boldsymbol{\rho\rho}} &= \left(\frac{e^2}{4\pi}\right)\frac{1}{4\pi}\left\{\frac{-g^2Z^2\boldsymbol{\rho\rho}}{R^6} + \frac{\tilde{I}}{2R^4}\right\}\theta(z+t),
\end{aligned} \tag{IV.2}$$

where $\tilde{I}$ is the two dimensional unit dyadic in the $x - y$ plane.

It is straightforward to verify that the stress-energy tensor is conserved, except along the world line of the charged particle. The only subtlety occurs in the calculations of the product of the delta function terms of F with the discontinuous terms. There one must use

$$\delta(z+t)\,\theta(z+t)[\cdots] = \tfrac{1}{2}\delta(z+t)[\cdots]$$

in order to obtain a stress-energy which is conserved along the null surface $z + t = 0$. The term arising from the square of the delta function field is formally conserved by itself.

The stress-energy tensor measured relative to the coordinate frame of the uniformly accelerated observer does not contain any delta functions (the null surfaces $z = \pm t$ are outside his coordinate patch) and are easily obtained by coordinate transforming the continuous parts

$$\begin{aligned}
T^{\tau\tau} &= \left(\frac{e^2}{4\pi}\right)\frac{1}{8\pi}\frac{1}{Z^2 g^2 R^4}\\
T^{\tau Z} &= 0 = T^{\tau\boldsymbol{\rho}}\\
T^{ZZ} &= -\left(\frac{e^2}{4\pi}\right)\frac{1}{8\pi}\frac{1}{R^4}\\
T^{Z\hat{\rho}} &= \left(\frac{e^2}{4\pi}\right)\frac{1}{4\pi}\frac{\boldsymbol{\rho}}{R^6}\{g^2 Z(\rho^2 + g^{-2} - Z^2)\}
\end{aligned} \tag{IV.3}$$

and

$$T^{\hat{\rho}\hat{\rho}} = \left(\frac{e^2}{4\pi}\right)\frac{1}{4\pi}\left\{\frac{\tilde{I}}{2R^4} - \frac{g^2 Z^2 \boldsymbol{\rho}\boldsymbol{\rho}}{R^6}\right\}.$$

In the accelerated frame, there is no energy flux, $T^{\tau Z} = 0 = T^{\tau\hat{\rho}}$, and no radiation. As measured in the unaccelerated frame, the situation is more complicated.

First consider the forward light cone of the charge at the space time point $z = 1/g$, $t = 0$ at which it is at rest (Fig. 5). (See Rohrlich [9].) The light cone is described by, (III.16),

$$\begin{aligned}
T^{tz} &= \frac{e^2}{4\pi}\frac{1}{4\pi}\left\{\frac{g\sin^2\theta}{r^3} + \frac{g^2\sin^2\theta\cos\theta}{r^2}\right\}\\
T^{t\rho} &= \frac{e^2}{4\pi}\frac{\hat{\rho}}{4\pi}\left\{\frac{-g\sin\theta\cos\theta}{r^3} + \frac{g^2\sin^3\theta}{r^2}\right\},
\end{aligned} \tag{IV.4}$$

and the flux out of a sphere of radius r is

$$\int d\mathbf{S}\cdot\mathbf{T}^t = \frac{e^2}{4\pi}g^2\frac{1}{4\pi}\int d\Omega\sin^2\theta = \frac{e^2}{4\pi}\frac{2}{3}g^2, \tag{IV.5}$$

exactly the usual result. This determination of the energy flow emanating from the charge is non-local in that one must carefully construct a sphere on the light cone centered on the line of the particle.

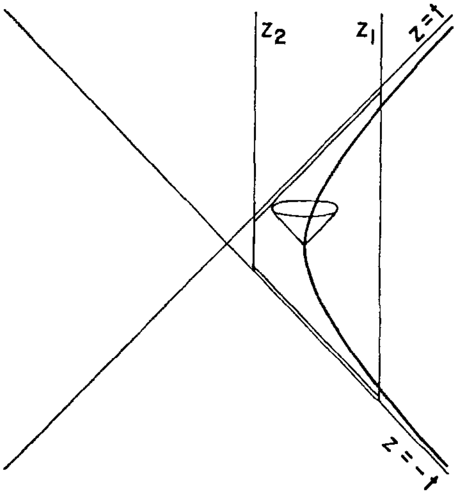

FIG. 5. The energy flux through the sphere centered on the world line of the charge is $(e^2/4\pi)/(2/3)(g^2/c^3)$. The net energy flow through the surfaces z_1 and z_2, $|t| < z$ is zero. There is a net flow through the surfaces $z_{1,2}$, $t > |z|$; this energy arises from the decreasing energy concentrated on the null surface, $z + t = 0$.

Let us now calculate, the radiation another way: Calculate the total energy flow through a fixed z surface (Fig. 5). We integrate over all ρ to obtain the rate at which energy passes through the surface, then integrate over t to find the total energy which has passed through the surface. If $z > 1/g$, then the particle passes through the surface and the energy flow includes the particle's field self-energy, however, if we integrate so that we include both the particle's entering and it's leaving, it will enter and leave with the same kinetic energy and the two will cancel with the net integral representing the negative of the total energy which moved from right to left, plus the energy which crossed from left to right, i.e., the particle energy plus radiation energy *after* the particle has been accelerated. It is immediate from Eq. (IV.2) that T^{tz} is odd in t, hence if we integrate from $t = -z + 0$ to $t = z - 0$, we obtain zero and there is *no* net flow of energy through the surface in the region of space time accessible to the accelerated observer.

Similarly the net flow of energy through the z_2 surface which is not intersected by the particle's world line is zero. Thus, the total energy passing through the surfaces accessible to the accelerated observer is zero.

The remaining two surfaces do have energy passing through them; the energy passing through the null surface $z = t$ is

$$\int d^2\rho \int_{z_2}^{z_1} dz\, [T^{tt} - T^{tz}](\rho, z, z)$$

$$= (z_1 - z_2) \frac{e^2}{4\pi} g^2 2 \int_0^\infty \frac{d\rho^2\, g^2}{(1 + g^2\rho^2)^4} = (z_1 - z_2) \frac{e^2}{4\pi} \frac{2}{3} g^2\,;$$

however, exactly the same amount of energy enters through the $t = -z + 0$ surface. Thus, the net energy produced in the region is zero. This is a reflection of the fact that the radiation reaction is zero. As much energy is being absorbed by the charge as is being emitted. To see this consider a sphere on the past light cone of the charge: the Poynting vector is exactly the negative of its value on the future light cone, hence the energy flow is into the charge and exactly equal to the subsequent outward flow.

From where does the inward flow of energy come? It flows into the region $t > -z$ from the $t = -z$ surface. For $t < -z$, there is no stress energy so it does not come from there. The stress energy confined to the $t = -z$ surface consists of two terms. First, the infinite delta function term which is the square of the delta function field does not vary along the surface; the associated energy simply passes through the region $z_1 > z > z_2$. The second, finite, delta function arises from the interference between the continuous field in the $t > -z$ region and the delta function field. It varies with $t(z)$, becoming smaller as t increases (z decreases). The energy released propagates from the null surface $z + t = 0$ to the world line of the charge to be absorbed and reemitted.

In any radiation process, one may propose two equivalent pictures: One may say that the charge is accelerated and that this acceleration produces the radiation. Equivalently, one may say that the charge before its acceleration is supporting an electromagnetic field. After the acceleration its state of motion is different, hence it will support a different field. However, the field does not change instantaneously into the field which the charge in its new state of motion will support. The difference of the two fields is propagated away as radiation.

The release of energy along the delta function is precisely the phenomenon of the original Coulomb field being converted to radiation. The energy flow is just sufficient, for $z > 0$, to produce the advanced field of the charge including the radiation focussed on the world line of the charge. For $z < 0$, the energy flow from the $z = -t$ surface is the same as that which would be associated with the field produced by the charge moving in region III which, as was discussed in the previous section, is exactly the same as the field produced by the charge in region I.

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