Electrostatic potential energy leading to an inertial mass change for a system of two point charges

Timothy H. Boyer
Department of Physics, City College of the City University of New York, New York, New York 10031
(Received 11 June 1976; revised 12 July 1976)

The system of two charged point particles traveling side by side at arbitrary velocity less than the speed of light and accelerating slowly is analyzed within classical electromagnetism as to the connection between the electrostatic potential energy of the system and the inertial mass. The total external force necessary to accelerate the system is calculated, and it is found that the change of system mass associated with the electrostatic potential energy of the two particles is accounted for exactly by the electromagnetic force of one charged particle on the other during the acceleration. The example provides a striking illustration and detailed mechanism for understanding the mass-energy idea associated with special relativity.

INTRODUCTION

The mass of a system is a measure of its energy content. This fundamental idea of Einstein's which relates the binding energy of a system to its inertial or gravitational mass appears repeatedly in elementary as well as advanced textbooks of physics. The result is well illustrated by experimental work in nuclear physics. However, any illustration of the principle in terms of elementary physical theory seems rare indeed in the textbook literature. The present short paper furnishes a detailed electromagnetic example of the connection between potential energy of a system and its inertia. The example deepens one's understanding of the mass-energy relation by showing the mechanism in a simple case.

We consider two point charges separated by a distance l perpendicular to the direction of motion. The two point charges are subject to a small acceleration in the direction of motion so that the separation between the charges remains unchanged; the acceleration is so small that radiation emitted by the charges is negligible. On the basis of classical electrodynamics we can calculate to lowest order in the acceleration and to all orders in v/c the additional force needed to accelerate each particle in the presence of the other particle, and hence using relativistic dynamics may determine the apparent inertial mass due to the presence of the other particle. This additional inertial mass can be compared with the system electrostatic potential energy divided by c^2. Indeed the relationship between inertial mass and potential energy content appears in detail.

BASIC ANALYSIS OF THE SYSTEM

We consider two point charges q_1 and q_2 of masses m_{10} and m_{20}, located at the points (x_1, y_1, z_1) and (x_2, y_2, z_2) with $x_1 = x_2$, $y_1 = l/2$, $y_2 = -l/2$, $z_1 = z_2 = 0$, thus separated by a distance l in the y direction. Until time $t = 0$ the particles are held near the origin, $x_1 = x_2 = 0$, by external forces $\mathbf{F}^{(0)}_{\text{ext }1}$ and $\mathbf{F}^{(0)}_{\text{ext }2}$ on particles q_1 and q_2, respectively. It is clear that for the particles at rest these external forces exactly cancel the electrostatec force q_1q_2/l^2 between the two charges. Hence the forces $\mathbf{F}^{(0)}_{\text{ext }1}$ and $\mathbf{F}^{(0)}_{\text{ext }2}$ are equal and opposite, and there is a vanishing sum for the external forces acting on the system consisting of the two point charges,

$$\mathbf{F}^{(0)}_{\text{ext }1} + \mathbf{F}^{(0)}_{\text{ext }2} = 0. \tag{1}$$

Now at time $t = 0$ new external forces are applied to the two particles so as to achieve a small, equal acceleration $\mathbf{a} = \hat{i}a$ in the x direction for each particle. Clearly the x coordinates of the particles at time t become $x_1 = x_2 = \frac{1}{2}at^2$, and the velocity is $\mathbf{v} = \hat{i}at$. We imagine the small acceleration continuing for some long time T after which the particles continue to move parallel to each other with some constant large velocity $\mathbf{V} = \hat{i}aT$ less than the speed of light. We will discuss the system at a time t between 0 and T.

The power P radiated by an accelerating charge e traveling at relativistic speed is given by the formula[1]

$$P = (2/3)(e^2/c^3)\, a^2\gamma^6, \tag{2}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$. The total energy $\mathscr{E} = \int P\, dt < P_{v=V}T$ radiated by our particles q_1 and q_2 during the acceleration time T may be made negligible, even while the end velocity $V = aT$ is fixed, by choosing the acceleration a sufficiently small,

$$\mathscr{E} < (2/3)(e^2/c^3)\, aV\gamma^6. \tag{3}$$

We will assume the acceleration a is indeed small and hence will ignore any question of the radiated energy.

If only one charge q_1 were present in our system and the other absent, then in order to provide a particle acceleration $\mathbf{a} = \hat{i}a = \hat{i}dv/dt$, we would need a force

$$\mathbf{F}^{(a)}_{\text{ext }1} = \frac{d}{dt}(m_{10}\gamma v\hat{i}) = \hat{i}m_{10}\gamma^3 a, \tag{4}$$

where m_{10} is the (renormalized) rest mass of the first particle, and $\gamma = (1 - v^2/c^2)^{-1/2}$. Similarly if particle q_2 were present while q_1 were absent, we would need

$$\mathbf{F}^{(a)}_{\text{ext }2} = \hat{i}m_{20}\gamma^3 a. \tag{5}$$

The vector sum of the forces $\mathbf{F}^{(a)}_{\text{ext }1}$ and $\mathbf{F}^{(a)}_{\text{ext }2}$ does not vanish but gives

$$\mathbf{F}^{(a)}_{\text{ext }1} + \mathbf{F}^{(a)}_{\text{ext }2} = \hat{i}(m_{10} + m_{20})\gamma^3 a. \tag{6}$$

For the individual point masses, the renormalization program of classical electromagnetism[2] insures that we indeed have the usual relativistic particle dynamics. However, a new element is introduced if both particles are present. Now the rest energy of the system is not simply

 0002-9505/78/4604-0383$00.50 © 1978 American Association of Physics Teachers

$$E_0 = m_{10}c^2 + m_{20}c^2, \quad (7)$$

but rather includes the electrostatic potential energy so that

$$E_0 = m_{10}c^2 + m_{20}c^2 + q_1q_2/l \quad (8)$$

and the rest mass should be E_0/c^2,

$$M_0 = m_{10} + m_{20} + q_1q_2/lc^2. \quad (9)$$

Accordingly the total force needed to accelerate the system in the direction of motion should be

$$\mathbf{F}^{(a)}_{\text{ext tot}} = \hat{i}(m_{10} + m_{20} + q_1q_2/lc^2)\gamma^3 a. \quad (10)$$

By comparing Eqs. (10) and (6), we see that the force necessary to accelerate the system is not just the sum of the forces $\mathbf{F}^{(a)}_{\text{ext }1}$ and $\mathbf{F}^{(a)}_{\text{ext }2}$ which were required when each particle was alone, but includes an additional contribution to the force $\Delta\mathbf{F}^{(a)}_{\text{ext tot}}$ associated with the rest energy from the electrostatic potential energy of the system,

$$\Delta\mathbf{F}^{(a)}_{\text{ext tot}} = \hat{i}(q_1q_2/lc^2)\gamma^3 a. \quad (11)$$

Now the system of two point charges which we are considering is a classical electromagnetic system and hence it is possible to derive in detail all the forces on various particles of the system. We will simply calculate from classical electromagnetic theory what force is necessary to accelerate the system and we will find that it provides exactly the addition term (11) required by the mass-energy ideas of special relativity. Our calculation is carried out to lowest order in the acceleration and to all orders in v/c.

When both particles q_1 and q_2 are accelerated together, each produces electromagnetic fields which cause forces on the other particle. Considering, for the moment, each charge as an individual system in the presence of the other, we must have the sum of all the forces acting on each charge equal to the rate of change of the momenum of the particle as in Eqs. (4), (5),

$$\mathbf{F}^{(a)}_{\text{ext }1} + \Delta\mathbf{F}^{(a)}_{\text{ext }1} + \mathbf{F}^{(a)}_{\text{em }1} = \hat{i}m_{10}\gamma^3 a, \quad (12)$$

$$\mathbf{F}^{(a)}_{\text{ext }2} + \Delta\mathbf{F}^{(a)}_{\text{ext }2} + \mathbf{F}^{(a)}_{\text{em }2} = \hat{i}m_{20}\gamma^3 a. \quad (13)$$

Comparing (4), (5), 12), and (13), we see that the additional external forces must just balance the interparticle electromagnetic forces

$$\Delta\mathbf{F}^{(a)}_{\text{ext }1} = -\mathbf{F}^{(a)}_{\text{em }1}, \quad (14)$$

$$\Delta\mathbf{F}^{(a)}_{\text{ext }2} = -\mathbf{F}^{(a)}_{\text{em }2}. \quad (15)$$

If we now go back and consider the system of interest as the two point charges q_1 and q_2 together with their electromagnetic fields, then the electromagnetic forces are internal forces and the external forces are just $\mathbf{F}^{(a)}_{\text{ext }1} + \Delta\mathbf{F}^{(a)}_{\text{ext }1}$ and $\mathbf{F}^{(a)}_{\text{ext }2} + \Delta\mathbf{F}^{(a)}_{\text{ext }2}$, so that the sum of the external forces on the two-particle system is

$$\mathbf{F}^{(a)}_{\text{ext tot}} = \mathbf{F}^{(a)}_{\text{ext }1} + \Delta\mathbf{F}^{(a)}_{\text{ext }1} + \mathbf{F}^{(a)}_{\text{ext }2} + \Delta\mathbf{F}^{(a)}_{\text{ext }2} \quad (16)$$

and

$$\Delta\mathbf{F}^{(a)}_{\text{ext tot}} = \Delta\mathbf{F}^{(a)}_{\text{ext }1} + \Delta\mathbf{F}^{(a)}_{\text{ext }2}. \quad (17)$$

It follows from (14) and (15) that $\Delta\mathbf{F}^{(a)}_{\text{ext tot}}$ in (17) is

$$\Delta\mathbf{F}^{(a)}_{\text{ext tot}} = -\mathbf{F}^{(a)}_{\text{em }1} - \mathbf{F}^{(a)}_{\text{em }2}. \quad (18)$$

If all is consistent, the value obtained here in (18) should be exactly that from special relativity above in (11).

EVALUATION OF THE ELECTROMAGNETIC FORCES

The electromagnetic forces on each charge due to the presence of the other charge may be evaluated from standard formulas of classical electromagnetism,[3]

$$\mathbf{E} = e\left[\frac{(\hat{n} - \beta)(1 - \beta^2)}{K^3R^2}\right]_{\text{ret}} + \frac{e}{c}\left[\frac{\hat{n}}{K^3R} \times \{(\hat{n} - \beta) \times \dot{\beta}\}\right]_{\text{ret}}, \quad (19)$$

$$\mathbf{B} = [\hat{n}]_{\text{ret}} \times \mathbf{E}, \quad (20)$$

$$\mathbf{F} = q(\mathbf{E} + \beta \times \mathbf{B}), \quad (21)$$

where e is the charge of the source particle, q that of the particle experiencing the force, $\hat{n}$ is the unit vector pointing from source to field point, R is the separation between the source and field points, $K = 1 - \hat{n}\cdot\beta$, $\beta = \mathbf{v}/c$, and all quantities are evaluated at the retarded time. From the symmetry of our two-particle system under reflection through the xy plane (or xz plane), it can be seen that any y components (and z components) of the forces on the particles are in opposite directions for the two particles and hence vanish in the sum (18) for $\Delta\mathbf{F}^{(a)}_{\text{ext tot}}$. Hence only the x components of $\mathbf{F}^{(a)}_{\text{em }1}$ and $\mathbf{F}^{(a)}_{\text{em }2}$ can contribute in the sum, and only the electric field (19) can give an x component of force.

In order to evaluate the electric field $\mathbf{E}$ at particle q_1 due to particle q_2, we must first find the retarded time t_{ret} corresponding to time t, $0 < t < T$, when the particles are moving with velocity $\mathbf{v} = \hat{i}at$ and are located at positions $x_1 = x_2 = \frac{1}{2}at^2$, $y_1 = l/2$, $y_2 = -l/2$, $z_1 = z_2 = 0$. The retarded time is such that a signal emitted by q_2 and traveling in a straight line with speed c will arrive at q_1 at time t. During this time, particle q_2 has moved a distance d given by $d = \frac{1}{2}at^2 - \frac{1}{2}at^2_{\text{ret}}$. Since q_1 and q_2 differ only in their y coordinate by a separation l, the condition for the retarded time follows from the Pythagorean formula

$$[c(t - t_{\text{ret}})]^2 = l^2 + [\tfrac{1}{2}at^2 - \tfrac{1}{2}at^2_{\text{ret}}]^2. \quad (22)$$

This equation can be rewritten in terms of the particle velocity at t, $v = at$, and the small time interval $\Delta t = t - t_{\text{ret}}$,

$$c^2(\Delta t)^2 = l^2 + [v(\Delta t) - \tfrac{1}{2}a(\Delta t)^2]^2. \quad (23)$$

Now we are assuming that the particle acceleration is very slow so that the change in speed during Δt is very small compared with v,

$$a(\Delta t) \ll v. \quad (24)$$

In this approximation, we may solve for Δt as a power series in the small parameter a, $\Delta t = (\Delta t)_0 + (\Delta t)_1 + \cdots$. To zero order in a, we find the result $(\Delta t)_0$ appropriate for particles moving with constant velocity,

$$(\Delta t)_0 = l(c^2 - v^2)^{-1/2}. \quad (25)$$

Substituting this result back into (23) and comparing terms first order in the acceleration a, we find

$$(\Delta t)_1 = -\tfrac{1}{2}avl^2(c^2 - v^2)^{-2}. \quad (26)$$

It is not necessary to go beyond this order of accuracy. Thus the retarded time is found to be

$$t_{ret} = t - l(c^2 - v^2)^{-1/2} + \tfrac{1}{2} a v l^2 (c^2 - v^2)^{-2} + O(a^2). \quad (27)$$

Having obtained the retarded time, we may now evaluate the quantities $\hat{n}$, β, $\dot{\beta}$, $K = 1 - \hat{n}\cdot\beta$, $R = c\Delta t$, which appear in Eq. (19),

$$\begin{aligned}\hat{n} &= (\hat{i}d + \hat{j}l)/R \\ &= \hat{i}(v/c - \tfrac{1}{2} a\Delta t/c) + \hat{j}l/(c\Delta t) \\ &= \hat{i}[v/c - \tfrac{1}{2} alc^{-1}(c^2 - v^2)^{-1/2}] \\ &+ \hat{j}[(c^2 - v^2)^{1/2}c^{-1} + \tfrac{1}{2} avlc^{-1}(c^2 - v^2)^{-1/2}] + O(a^2),\end{aligned} \quad (28)$$

$$\begin{aligned}\beta &= \hat{i}(v/c - a\Delta t/c) \\ &= \hat{i}[v/c - alc^{-1}(c^2 - v^2)^{-1/2}] + O(a^2),\end{aligned} \quad (29)$$

$$\dot{\beta} = \hat{i}a/c, \quad (30)$$

$$K = 1 - v^2/c^2 + (\tfrac{3}{2})avlc^{-2}(c^2 - v^2)^{-1/2} + O(a^2), \quad (31)$$

$$R = cl(c^2 - v^2)^{-1/2} - (\tfrac{1}{2})\, avcl^2(c^2 - v^2)^{-2} + O(a^2). \quad (32)$$

When substituting into Eq. (19), we notice that $\dot{\beta}$ and also the x component of $(\hat{n} - \beta)$ are both linear in the acceleration a. Since we are evaluating the x component of the electric field only through first order in the acceleration, we need only the zero-order terms in all the remaining expressions. Thus Eq. (19) becomes

$$E_{x1} = q_2 \left[\frac{\{[v/c - \tfrac{1}{2} al/c\,(c^2 - v^2)^{1/2}] - [v/c - al/c(c^2 - v^2)^{1/2}]\}(1 - v^2/c^2)}{(1 - v^2/c^2)^3[cl/(c^2 - v^2)^{1/2}]}\right] + q_2 \left[\frac{-[(c^2 - v^2)^{1/2}/c]\{[(c^2 - v^2)^{1/2}/c]a/c\}}{(1 - v^2/c^2)^3[cl/(c^2 - v^2)^{1/2}]}\right] = -q_2 a/2c^2 l(1 - v^2/c^2)^{3/2}. \quad (33)$$

The x component of the electromagnetic force on q_1, $F^{(a)}_{\text{em}\,1\,x} = q_1 E_{x\,1}$, is thus

$$F^{(a)}_{\text{em}\,1\,x} = \frac{-q_1 q_2 a}{2c^2 l(1 - v^2/c^2)^{3/2}}. \quad (34)$$

By symmetry the x component of the electromagnetic force on q_2 is exactly equal. Hence from Eq. (18), we have

$$\begin{aligned}\Delta \mathbf{F}^{(a)}_{\text{ext tot}} &= -\mathbf{F}^{(a)}_{\text{em}1} - \mathbf{F}^{(a)}_{\text{em}\,2} \\ &= -\hat{i}(F^{(a)}_{\text{em}\,1\,x} + F^{(a)}_{\text{em}\,2\,x}) \\ &= \hat{i}(q_1 q_2/lc^2)\gamma^3 a\end{aligned} \quad (35)$$

with $\gamma = (1 - v^2/c^2)^{-1/2}$. This is exactly the result required in (11) from the mass-energy connection of special relativity.

SUMMARY

Special relativity arose at the beginning of the twentieth century as an expression of the kinematics and dynamics implicit within classical electromagnetism. Today special relativity is often taught without particular reference to electromagnetism, and the mass-energy relationship appears as a result of the general analysis. Within the present paper we examine a special case of inertial mass associated with the electrostatic potential energy of two point charges. The system is a purely electromagnetic one and hence contains within the classical theory all the relationships of special relativity. In this case we can see explicitly the appearance of the mass-energy connection and see the dynamical mechanism within electromagnetism which provides the mass-energy connection. The potential energy which changes the system rest mass is associated with electromagnetic forces. During the acceleration of the system, the electromagnetic fields change in just such a way that each charge causes a new electromagnetic force on the other. It is precisely this electromagnetic force which must be balanced by the external force accelerating the system and which is interpreted as a change in the inertial mass of the system.

[1]See, for example, J. D. Jackson, Classical Electrodynamics (Wiley, New York, 1962), p. 474, Eq. (14.43).

[2]S. Coleman, Rand Corp. Report, 1961 (unpublished).

[3]See Ref. 1, p. 467, Eqs. (14.12) and (14.14).