Lorentz-transformation properties for energy and momentum in electromagnetic systems

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The Lorentz-transformation properties of the energy and momentum in electromagnetic systems are illustrated in simple examples involving systems of two point charges. For systems free of external forces the total energy and momentum consisting of mechanical and electromagnetic contributions are seen to transform as a Lorentz four-vector. The examples reveal the roles played by time nonsimultaneity between inertial frames and by electromagnetic forces of one part of the system upon the other in maintaining the four-vector behavior in special relativity. For systems experiencing external forces the mechanical plus electromagnetic energy and momentum do not transform as a Lorentz four-vector; the actual transformation properties can be understood from descriptions for the assembly of the system as viewed in two different inertial frames.

I. INTRODUCTION

Classical electromagnetism conforms to the theory of special relativity, and textbooks now regularly present Maxwell's equations in manifestly Lorentz-covariant form[1,2] and often give some examples of the interesting interchanges between electric and magnetic fields when seen in different inertial frames.[1-3] However, the literature of electromagnetism contains few elementary examples for the transformation of electromagnetic energy and momentum between two inertial frames. Indeed, the naively expected Lorentz-transformation properties for electromagnetic energy and momentum do not hold for so simple a system as a parallel plate capacitor with small separation between the plates. Thus, for such a capacitor, the energy in the rest frame is $U=(1/8\pi)E^2\mathscr{V}$, where E is the electric field and $\mathscr{V}$ the volume between the plates, but the electromagnetic field momentum seen in a primed frame moving with velocity V parallel to the plates has magnitude

$$P'=\frac{c}{4\pi}E'B'\mathscr{V}'=\frac{c}{4\pi}(\gamma E)(\beta\gamma E)\frac{\mathscr{V}}{\gamma}=2V\gamma U$$

with $\gamma=(1-V^2/c^2)^{-1/2}$, which is twice as large as appropriate for a four-vector transformation rule. Graduate students and even well-trained physicists are sometimes confused by this transformation behavior. It is with the hope of clarifying this confusion that we present here some elementary examples where the energy and momentum in electromagnetic systems exhibit transparent connections when analyzed from two different inertial frames.

The organization of the paper is as follows. First we remind the reader and illustrate that any open system which has outside sources for some quantity will in general not have covariant Lorentz transformations for the quantity unless there is local conservation of the quantity at the sources. This idea, as applied to the Lorentz-transformation properties of energy and momentum in electromagnetic systems, is developed in a qualitative discussion of a simple point-charge system. Next we give detailed calculations for systems consisting of two point charges, showing the transformation properties of energy and momentum between different inertial frames.

A. Failure of Lorentz covariance for nonlocal sources

The first step in understanding the Lorentz-transformation properties of energy and momentum in electromagnetic systems consists of dispelling the erroneous but immediate expectation of some students that physical quantities transform convariantly despite the presence of nonlocal sources. This error is most easily demonstrated in an example of a hypothetical scalar quantity which we will here call "charge."

Suppose that in a certain inertial frame S having total charge Q we have the creation at the instant t_0 of a pair of charges, one a charge $+e$ created at the point $\mathbf{r}_+$ and the other a charge $-e$ created at a different point $\mathbf{r}_-$. In the frame S the total charge of the system is conserved, remaining unchanged at Q despite the charge creation since the balancing plus and minus charges were created at the same time. Next we examine the total charge of the system in a primed inertial frame S' moving with velocity $-\mathbf{V}=-\hat{i}V$ relative to the frame S. Now our hypothetical charge is to be a Lorentz scalar. Thus one may expect that the total charge Q in the frame S' is the same as the total charge Q in S, $Q'=Q$. However, the simultaneity of the charge creation which held in S does not hold in S'. If for our example the x coordinates satisfy $x_+>x_-$, then in S' first the charge $-e'=-e$ is created at time $t'_-=\gamma(t_0+Vx_-/c^2)$ and later the charge $+e'=+e$ is created at time $t'_+=\gamma(t_0+Vx_+/c^2)$. Thus although the total charge is conserved in the frame S, it is not conserved in S' but changes from $Q'=Q$, to $Q-e$, and then back to Q. Only in the special cases when the coordinates are such that $x_+=x_-$, corresponding to a velocity of the inertial frame S' perpendicular to the line joining the two charges, do we again have $t'_+=t'_-$ and hence charge conservation in S'. Indeed we can imagine a continuous creation of equal numbers of positive and negative charges at points $\mathbf{r}_+$ and $\mathbf{r}_-$ beginning at $t=0$ in the S frame. Then after the sources start, the various inertial frames in general will forever disagree as to the total charge in the frame. The source of one sign of charge will have started first and so will have created charges which are never compensated by the opposite source which began later.

Clearly the total charge does not transform as a Lorentz scalar in this hypothetical example involving nonlocal sources of charge, but rather changes its value depending upon the inertial frame in which it is evaluated. In the physical world electric charge is a Lorentz scalar and the creation of particle–antiparticle pairs actually occurs at a single space-time point, corresponding to $x_+=x_-$ and hence always $t'_+ = t'_-$ in the example above. In this physical case, where the sources of charge observe local conservation, the charge is conserved in every inertial frame and indeed is a Lorentz scalar. However, the moral of our hypothetical example is clear, and it is crucial for understanding the Lorentz transformation of energy and momentum in the electromagnetic field. If a quantity has sources which do not obey local conservation, then the values assumed by this quantity in various inertial frames do not correspond to covariant behavior under Lorentz transformation.

B. Poynting's theorem and sources of electromagnetic energy and momentum

Now in this article we are interested in the Lorentz transformation properties of not charge but rather energy and momentum in the electromagnetic fields. Hence in order to apply the example given above, we must consider the sources of energy and momentum for the electromagnetic fields.

The connections between the sources and densities of energy and momentum in the electromagnetic field are provided by Poynting's theorem[4] and its momentum analog in the absence of material media,

$$-\int_V d^3x\,\mathbf{J}\cdot\mathbf{E} = \frac{d}{dt}\left(\frac{1}{8\pi}\int_V d^3x(E^2+B^2)\right) + \frac{c}{4\pi}\int_S (\mathbf{E}\times\mathbf{B})\cdot\hat{n}\,da, \tag{1}$$

$$-\int_V d^3x\left(\rho\mathbf{E}+\frac{1}{c}\mathbf{J}\times\mathbf{B}\right) = \frac{d}{dt}\left(\frac{1}{4\pi c}\int_V d^3x\,\mathbf{E}\times\mathbf{B}\right) + \frac{1}{4\pi}\int_S\left[\mathbf{E}\,\mathbf{E}+\mathbf{B}\,\mathbf{B} - \tfrac{1}{2}\mathscr{I}(E^2+B^2)\right]\cdot\hat{n}\,da, \tag{2}$$

where the electric and magnetic fields are denoted by **E** and **B**, and the charge and current densities by ρ and **J**. These equations follow directly from Maxwell's equations and hold in any inertial frame. The left-hand sides of the equations give the sources, the rates of production of electromagnetic energy, and momentum, respectively, in the volume $\mathscr{V}$, and these are equated to the rates of change of electromagnetic energy and momentum, respectively, in the volume $\mathscr{V}$ plus the rates at which electromagnetic energy and momentum, respectively, are flowing outward across the surface S which forms the boundary of the volume $\mathscr{V}$.

Following the example of Sec. I A involving "charge" we expect here to find convariant transformation properties for the energy and momentum in electromagnetic fields only in the absences of sources for energy and momentum in electromagnetic fields. From Eqs. (1) and (2) we see that these sources vanish in general only when ρ and **J** vanish. Thus only for electromagnetic fields in the absence of charges ρ and currents **J** do we find vanishing sources of electromagnetic energy and momentum, and consequently only in this case do we have four-vector Lorentz-transformation properties for the total electromagnetic energy and momentum of a system. In all other cases we expect noncovariant behavior for the energy and momentum in the electromagnetic fields.

It should be emphasized that Eqs. (1) and (2) do not specify the nature of the reservoirs of energy and momentum which are fed through the electromagnetic sources given on the left-hand side. However, in elementary electromagnetism we are familiar with the conversion of mechanical energy and momentum into electromagnetic form on the collision of charged particles, and we are also familiar with external forces of unspecified origin which balance the electromagnetic forces and hence produce electromagnetic energy and momentum. Specifically we can rewrite the sources of electromagnetic energy and momentum in terms of the changes in the reservoirs which feed them:

$$-\int_V d^3x\,\mathbf{J}\cdot\mathbf{E} = -\frac{d}{dt}U_{\text{mech}} - \frac{d}{dt}U_{\text{ext}}, \tag{3}$$

$$-\int_V d^3x\left(\rho\,\mathbf{E}+\frac{1}{c}\mathbf{J}\times\mathbf{B}\right) = -\frac{d}{dt}\mathbf{P}_{\text{mech}} - \frac{d}{dt}\mathbf{P}_{\text{ext}}, \tag{4}$$

where U_{mech}, $\mathbf{P}_{\text{mech}}$ refer to mechanical energy and momentum and U_{ext},$\mathbf{P}_{\text{ext}}$ refer to the energy and momentum arising from external forces of unspecified origin in the volume $\mathscr{V}$. In general we are interested in the Lorentz-transformation properties of energy and momentum given by the sum of the mechanical and electromagnetic contributions for a system. Even with this extension, however, external forces still provide nonlocal sources which change the system energy and momentum.

C. Qualitative discussion of the examples of energy and momentum

We now turn to a qualitative discussion of some illustrative examples for the energy and momentum in electromagnetic systems. For the sake of simplicity our detailed calculations are carried out for systems of two charged particles.

The qualitative aspects of the point-charge systems are most quickly understood by considering two point charges approaching each other along a straight line, although our subsequent calculations are not limited to this case. When the charges are far apart, the forces of each charge upon the other are negligible and all the system energy and momentum can be regarded as that associated with the system mechanical mass. As the particles approach each other, the electromagnetic forces of each charge upon the other increase from their previously zero values, and parts of the mechanical energy and momentum are converted into energy and momentum in the electromagnetic field. In this situation where no external forces are present, the total energy and momentum of the system transform as a Lorentz four-vector. This is immediate when the particles are far apart so that all energy and momentum are in mechanical form. Furthermore, since the total system energy and momentum are conserved in each inertial frame, the four-vector behavior for the sum of mechanical and electromagnetic energy and momentum holds at any time. One interesting aspect arising in the explicit verification of this four-vector behavior is the necessity for evaluating the contributions to the energy and momentum at a single time in the inertial frame of interest; aspects of nonsimultaneity for space-time events in different inertial frames appear crucially in the calculations.

Most examples used in courses in electromagnetism involve not moving charges but rather static charge configurations. We can obtain a static configuration of two point charges by applying external forces to the system of two point charges discussed above. Specifically, if we go to the center-of-momentum frame for the system, then two point charges of like sign which are approaching a head-on collision will come to rest simultaneously. At this instant of closest approach we can imagine external stabilizing forces applied to the system. The stabilizing forces balance the forces of electrostatic repulsion so that the charges are held at the fixed separation rather than again separating to infinite separation. In the center-of-momentum frame the stabilizing forces are applied to the particles at zero velocity and at the same instant so that they introduce neither energy nor momentum into the point charge system. Thus the system energy in this frame is the same as when the particles were infinitely far away. However, in a general inertial frame the application of the stabilizing forces will not be simultaneous, and hence these forces introduce both energy and momentum into the system. It follows that the energy and momentum of the two-particle system, including mechanical and electromagnetic energy, which had behaved as a Lorentz four-vector before the application of the stabilizing forces, will not have such behavior afterwards. The energy and momentum of a system stabilized by external forces is not a Lorentz four-vector. In each inertial frame the departure from the covariant behavior can be calculated explicitly and is found equal to the net work and impulse imparted by the stabilizing forces during their nonsimultaneous application.

II. SYSTEMS OF TWO CHARGED PARTICLES

A. Systems without external forces

Our detailed calculations involve a relativistic system of two charged particles of mass m_1,m_2 and charge q_1,q_2. For simplicity in the analysis and in order to avoid the enormous complications of radiation emission we will limit our calculations to second order v_i^2/c^2, $i=1,2$ in the particle velocities and to first order V/c in the relative velocities between the inertial frames. The total relativistic energy and momentum of the system consisting of the particles and their fields involve mechanical and electromagnetic field contributions

$$U_{\rm tot}=m_1(1+\tfrac{1}{2}v_1^2/c^2)c^2+m_2(1+\tfrac{1}{2}v_2^2/c^2)c^2 + \frac{1}{8\pi}\int d^3x(2\mathbf{E}_1\cdot\mathbf{E}_2+2\mathbf{B}_1\cdot\mathbf{B}_2), \tag{5}$$

$$P_{\rm tot}=m_1(1+\tfrac{1}{2}v_1^2/c^2)\mathbf{v}_1+m_2(1+\tfrac{1}{2}v_2^2/c^2)\mathbf{v}_2 + \frac{1}{4\pi c}\int d^3x(\mathbf{E}_1\times\mathbf{B}_2+\mathbf{E}_2\times\mathbf{B}_1), \tag{6}$$

where the factors of $(1+\frac{1}{2}v_i^2/c^2)$ arise from the expansion of $\gamma_i=(1-v_i^2/c^2)^{-1/2}$ to second order in v_i/c. The integrals in (5) and (6) over the electromagnetic field densities involve only the interference terms for the fields; the divergent field self-energies $\int d^3x(E_i^2+B_i^2)/8\pi$ and momenta $\int d^3x\mathbf{E}_i\times\mathbf{B}_i/4\pi c$, $i=1,2$ are incorporated into the mass renormalizations producing m_1 and m_2.

The electromagnetic fields of a point charge accurate to order $1/c^2$ are given by[5]

$$\mathbf{E}=\frac{q\mathbf{r}}{r^3}-\frac{q}{2c^2}\left[\frac{\mathbf{a}}{r}+\left(\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}-\frac{v^2}{r^3}+\frac{3(\mathbf{v}\cdot\mathbf{r})^2}{r^5}\right)\mathbf{r}\right], \tag{7}$$

$$\mathbf{B}=(q/c)(\mathbf{v}\times\mathbf{r}/r^3), \tag{8}$$

where $\mathbf{r}$ is the displacement vector from the charge to the field point, $\mathbf{v}$ is the velocity of the charge, and $\mathbf{a}$ is the acceleration. All quantities are evaluated at the same time t in the inertial frame of interest. The lowest-order contributions to the electromagnetic energy and momentum needed in (5) and (6) can be found by evaluating the integrals[5]

$$U_{\rm em}=\frac{1}{8\pi}\int d^3x(2\mathbf{E}_1\cdot\mathbf{E}_2+2\mathbf{B}_1\cdot\mathbf{B}_2)\cong\frac{q_1q_2}{r_{12}}, \tag{9}$$

$$\mathbf{P}_{\rm em}=\frac{1}{4\pi c}\int d^3x(\mathbf{E}_1\times\mathbf{B}_2+\mathbf{E}_2\times\mathbf{B}_1) =\frac{q_1q_2}{2c^2}\left[\left(\frac{\mathbf{v}_1}{r_{12}}+\frac{\mathbf{v}_1\cdot\mathbf{r}_{12}\mathbf{r}_{12}}{r_{12}^3}\right)+\left(\frac{\mathbf{v}_2}{r_{12}}+\frac{\mathbf{v}_2\cdot\mathbf{r}_{12}\mathbf{r}_{12}}{r_{12}^3}\right)\right], \tag{10}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the particle velocities and $\mathbf{r}_{12}=\mathbf{r}_1-\mathbf{r}_2$ is the displacement of q_1 relative to q_2. The total momentum $\mathbf{P}_{\rm tot}$ of the system in the center-of-momentum frame S vanishes by definition, $\mathbf{P}_{\rm tot}=0$, giving the condition for the sum of the mechanical and electromagnetic momentum to order $1/c^2$ from (6) and (10):

$$\mathbf{P}_{\rm tot}=0=m_1(1+\tfrac{1}{2}v_1^2/c^2)\mathbf{v}_1+m_2(1+\tfrac{1}{2}v_2^2/c^2)\mathbf{v}_2 +(q_1q_2/2c^2)[(\mathbf{v}_1+\mathbf{v}_2)/r_{12} +(\mathbf{v}_1+\mathbf{v}_2)\cdot\mathbf{r}_{12}\mathbf{r}_{12}/r_{12}^3]. \tag{11}$$

We wish to consider the momentum of this system as seen in a primed inertial frame S' moving with a small velocity $-\mathbf{V}=-\hat{i}V$ relative to the center-of-momentum frame S. Since there are no external forces applied to this system, we expect the total system energy and momentum to transform as a Lorentz four-vector

$$U'_{\rm tot}=\gamma U_{\rm tot}, \tag{12}$$

$$\mathbf{P}'_{\rm tot}=\mathbf{V}\gamma U_{\rm tot}/c^2, \tag{13}$$

where $\gamma=(1-V^2/c^2)^{-1/2}$. Through first order in the small velocity $\mathbf{V}$, these reduce to

$$U'_{\rm tot}\cong U_{\rm tot}, \tag{14}$$

$$\mathbf{P}'_{\rm tot}\cong\mathbf{V}U_{\rm tot}/c^2, \tag{15}$$

so that only the momentum shows a change in this approximation. We will now apply the traditional definitions of mechanical and electromagnetic momentum in S' so as to evaluate $\mathbf{P}_{\rm tot}$ independently. In so doing we will find those aspects of the analysis which are relevant to provide the four-vector character.

The particle velocities in the primed frame S' are related to those in the center-of-momentum frame by Lorentz transformation:

$$\mathbf{v}'_i=\frac{\hat{i}(v_{ix}+V)}{1+v_{ix}V/c^2}+\frac{\hat{j}v_{iy}(1-V^2/c^2)^{1/2}}{1+v_{ix}V/c^2} \cong\mathbf{v}_i+\mathbf{V}-\mathbf{V}\cdot\mathbf{v}_i\mathbf{v}_i/c^2,\quad i=1,2, \tag{16}$$

where we retain terms only through first order in the small velocity $\mathbf{V}=\hat{i}V$ and through second order in v_i/c. The evaluation of the momentum in the primed inertial frame S' requires that velocities v'_1 and v_2 be given at the same time t'. However, Lorentz transformation of the time involves $t'=\gamma(t+Vx/c^2)$ which leaves to first order in V,

$$t'=t_i+Vx_i/c^2,\quad i=1,2, \tag{17}$$

corresponding to different times t_1 and t_2 for the two point charges as seen in the center-of-momentum frame S. Then expanding the velocity $\mathbf{v}_i(t_i)$ about the common time t', we find to first order in $\mathbf{V}$

$$\mathbf{v}_i(t_i) = \mathbf{v}_i(t' - Vx_i/c^2) \cong \mathbf{v}_i(t') - Vx_i\mathbf{a}_i/c^2, \quad i = 1,2, \tag{18}$$

where $\mathbf{a}_i = d\mathbf{v}_i/dt$. Substituting this expression back into Eq. (16), we have

$$\mathbf{v}_i'(t') \cong \mathbf{v}_i(t') - \mathbf{V}\cdot\mathbf{r}_i\mathbf{a}_i/c^2 + \mathbf{V} - \mathbf{V}\cdot\mathbf{v}_i\mathbf{v}_i/c^2, \quad i = 1,2. \tag{19}$$

The total momentum of the system as seen in the primed frame at time t' is

$$\mathbf{P}'_{\text{tot}} = m_1(1 - v_1'^2/c^2)^{-1/2}\,\mathbf{v}_1' + m_2(1 - v_2'^2/c^2)^{-1/2}\,\mathbf{v}_2' + \mathbf{P}'_{\text{em}}. \tag{20}$$

In evaluating the electromagnetic field momentum $\mathbf{P}'_{\text{em}}$ given in (10) we may ignore all terms in the velocity which are of order $1/c^2$ since P_{em} is itself of order $1/c^2$. Substituting the velocity expression (19) into the momentum (20), and expanding to first order in the small velocity $\mathbf{V}$ and to second order in $1/c$, we have

$$\mathbf{P}'_{\text{tot}} = \sum_{i=1}^{2}\left[m_i(1 + \tfrac{1}{2}v_i^2/c^2)\mathbf{v}_i + m_i(1 + \tfrac{1}{2}v_i^2/c^2)\mathbf{V} - \mathbf{V}\cdot\mathbf{r}_i m_i\mathbf{a}_i/c^2\right] + (q_1q_2/2c^2)[(\mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{V})/r_{12} + (\mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{V})\cdot\mathbf{r}_{12}\mathbf{r}_{12}/r_{12}^3]. \tag{21}$$

Now the center-of-momentum frame S is defined by the condition (11), corresponding to exactly the expression (21) with $\mathbf{V} = 0$. Substituting the condition (11) into (21), we find the total momentum in the primed frame S',

$$\mathbf{P}'_{\text{tot}} = \left[m_1 + m_2 + \tfrac{1}{2}(m_1v_1^2 + m_2v_2^2)/c^2\right]\mathbf{V} - (\mathbf{V}\cdot\mathbf{r}_1m_1\mathbf{a}_1 + \mathbf{V}\cdot\mathbf{r}_2m_2\mathbf{a}_2)/c^2 + (q_1q_2/c^2)(\mathbf{V}/r_{12} + \mathbf{V}\cdot\mathbf{r}_{12}\mathbf{r}_{12}/r_{12}^3). \tag{22}$$

However, in the center-of-momentum frame S, Newton's second law to zero order in $1/c$ gives

$$m_i\mathbf{a}_i = q_1q_2(\mathbf{r}_i - \mathbf{r}_j)/|\mathbf{r}_i - \mathbf{r}_j|^3, \quad i,j = 1,2, \quad i \neq j, \tag{23}$$

so that the terms involving the acceleration in (22) can be eliminated as

$$(\mathbf{V}\cdot\mathbf{r}_1m_1\mathbf{a}_1 + \mathbf{V}\cdot\mathbf{r}_2m_2\mathbf{a}_2)/c^2 = (q_1q_2/c^2)\mathbf{V}\cdot\mathbf{r}_{12}\mathbf{r}_{12}/r_{12}^3, \tag{24}$$

with $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$. The total momentum in (23) then simplifies to give

$$\mathbf{P}'_{\text{tot}} = \left[m_1 + m_2 + \tfrac{1}{2}(m_1v_1^2 + m_2v_2^2)/c^2 + q_1q_2/r_{12}c^2\right]\mathbf{V}, \tag{25}$$

corresponding exactly to the expression (15) obtained from four-vector Lorentz transformation where U_{tot} follows from (5) and (9). Thus we have followed through the explicit evaluation of the momentum in S', have noted the use of time nonsimultaneity and of the force of one part of the system upon the other, and have indeed verified the four-vector character of the system energy momentum.

Our analysis has been given for a general system of two point charges free of external forces. However, one special case seems worthy of mention. If two point charges are bound by mutual electrostatic attraction in orbits about their common center-of-mass, then no external stabilizing forces are required to the order $1/c^2$ which neglects radiation reaction. Thus the system is stable under electrostatic forces. Since there are no external forces on the system, the system is a special case covered by the general calculation above. Indeed, for the model of an atom in classical electron theory we find valid the Lorentz four-vector transformation of the system energy and momentum.

B. Point charge systems stabilized by external forces

Often in electromagnetism one considers systems which are electrostatic in the center-of-momentum frame S. Since such systems cannot be stable under electrostatic forces alone, they must be stabilized by some external forces. And, as was emphasized at the beginning of this article, the introduction of external forces means changing the Lorentz-transformation properties of the system energy and momentum.

The basic aspects involved in the transformation of mechanical and electromagnetic energy U^{MeEl} and momentum $\mathbf{P}^{\text{MeEl}}$ for all electrostatic systems can be found in the simple illustration of two point charges m_1, q_1, m_2, q_2 held at points $\mathbf{r}_{s1}$ and $\mathbf{r}_{s2}$ in an inertial frame S. The mechanical and electromagnetic energy U_S^{MeEl} and momentum $\mathbf{P}_S^{\text{MeEl}}$ in S are simply

$$U_S^{\text{MeEl}} = m_1c^2 + m_2c^2 + q_1q_2/r_{12}, \tag{26}$$

$$\mathbf{P}_S^{\text{MeEl}} = 0. \tag{27}$$

In the primed inertial frame S' moving with a small velocity $-\mathbf{V} = \hat{\imath}V$ relative to S, we find the mechanical and electromagnetic momentum of two particles with velocity $\mathbf{v}_1' = \mathbf{v}_2' = +\mathbf{V}$, which to first order in $\mathbf{V}$ is

$$\mathbf{P}_{S'}^{\text{MeEl}\prime} = (m_1 + m_2)\mathbf{V} + (q_1q_2/c^2)(\mathbf{V}/r_{12} + \mathbf{V}\cdot\mathbf{r}_{12}\mathbf{r}_{12}/r_{12}^3). \tag{28}$$

It is apparent immediately that this momentum (28) does not follow by four-vector Lorentz transformation from the expressions (26) and (27) holding in S. The external forces provide nonlocal sources of energy and momentum, and hence destroy the covariant transformation properties of mechanical and electromagnetic energy and momentum.

C. Assembly of point charge systems stabilized by external forces

A deeper understanding of the transformation properties of energy and momentum for electromagnetic systems can be obtained by following the assembly of an electrostatic system and determining the momentum introduced upon the application of the external stabilizing forces. Specifically let us consider the assembly of our electrostatic system of two point charges by projecting two charges of like sign toward each other from spatial infinity with just sufficient initial kinetic energy that the particles come to rest at time t_0 at the desired electrostatic separation $\mathbf{r}_{s12} = \mathbf{r}_{s1} - \mathbf{r}_{s2}$. At this instant t_0 the stabilizing forces are applied so that the particles are held in the desired electrostatic configuration. Thus during the assembly, the system changes from the situation of no external forces, where covariant transformation properties hold, over to the electrostatic configuration where the covariant properties fail.

In the approximation considered here that the electromagnetic fields are computed to order $1/c^2$, the energy lost to radiation is neglected and the kinetic energy of the particles at spatial infinity is fully converted into electrostatic potential energy as the particles arrive at $\mathbf{r}_{s1} = \mathbf{r}_1(t_0)$ and $\mathbf{r}_{s2} = \mathbf{r}_2(t_0)$. Thus the system mechanical and electromagnetic energy at any time t

$$U_S^{\text{MeEl}} = m_1(1 + \tfrac{1}{2} v_1^2/c^2)c^2 + m_2(1 + \tfrac{1}{2} v_2^2/c^2)c^2 + q_1q_2/|\mathbf{r}_1(t) - \mathbf{r}_2(t)| \quad (29)$$

is equal to the system energy U_S^{MeEl} in (26) as the particles come to rest, $\mathbf{v}_1 = 0, \mathbf{v}_2 = 0$. Since the stabilizing forces are applied at zero velocity and are equal in magnitude and opposite in direction, their sum is zero and they introduce neither energy nor momentum in the S frame.

In the primed frame S' moving with velocity $-\mathbf{V} = -\hat{i}V$ relative to S, there are no external forces for times t' less than some t'_S, and hence we can apply the analysis given earlier where four-vector transformation properties held. The mechanical and electromagnetic momentum $U_{S'}^{\text{MeEl}\prime}$ to lowest order in $\mathbf{V}$ is

$$P_{S'}^{\text{MeEl}\prime} = \mathbf{V}\, U_S^{\text{MeEl}}/c^2 = (m_1 + m_2 + q_1q_2/|\mathbf{r}_{s1} - \mathbf{r}_{s2}|c^2)\mathbf{V}. \quad (30)$$

However, the application of the external forces, which was instantaneous in the S frame, is not in general instantaneous in the S' frame because of the relativity of simultaneity. In S' the external forces are applied at times t' following from Lorentz transformation as

$$t'_i = \gamma(t_0 + Vx_{si}/c^2) \cong t_0 + \mathbf{V}\cdot\mathbf{r}_{si}/c^2, \quad i = 1,2. \quad (31)$$

Thus if $x_{s1} > x_{s2}$, the external force $\mathbf{F}'_{\text{ext}\,2}$ is applied to the charge q_2 a time

$$\Delta t' \cong \mathbf{V}\cdot(\mathbf{r}_{s1} - \mathbf{r}_{s2})/c^2 \quad (32)$$

before the external force $\mathbf{F}'_{ext\,1} = -\mathbf{F}'_{ext\,2}$ is applied to q_1. Hence the unbalanced force $\mathbf{F}'_{\text{ext}\,2}$ delivers to the system a net impulse

$$\mathbf{I}' = \mathbf{F}'_{\text{ext}\,2}\, \Delta t'. \quad (33)$$

But through first order in the small velocity $\mathbf{V}$, the external forces $\mathbf{F}'_{\text{ext}\,1}, \mathbf{F}'_{\text{ext}\,2}$ in the S' frame are the same as those in S. These forces just balance the electrostatic forces so that

$$\mathbf{F}'_{\text{ext}\,i} \cong -q_1q_2(\mathbf{r}_{si} - \mathbf{r}_{sj})/|\mathbf{r}_{si} - \mathbf{r}_{sj}|^3, \quad i,j = 1,2, \quad i \neq j. \quad (34)$$

Thus the net impulse introduced by the external forces in S' follows from (32), (33), and (34) as

$$\mathbf{I}' = q_1q_2\mathbf{V}\cdot(\mathbf{r}_{s1} - \mathbf{r}_{s2})(\mathbf{r}_{s1} - \mathbf{r}_{s2})/|\mathbf{r}_{s1} - \mathbf{r}_{s2}|^3c^2. \quad (35)$$

But this is exactly the discrepancy $\Delta\,\mathbf{P}' = \mathbf{I}'$ between the actual momentum (28) of the electrostatic system as seen in the S' frame, and the momentum expression (30) following from Lorentz transformation of the energy and momentum in S. Everything is consistent. The discrepancy in the mechanical and electromagnetic momentum between the actual value and the value obtained from Lorentz transformation from the static frame is actually *required* for the validity of the impulse-momentum relation in both the S and S' frames.

III. SUMMARY

The mechanical energy and momentum of a single point mass behave as a four-vector under Lorentz transformation. Apparently the naive reaction of many students is to expect that for all systems or parts of systems the energy and momentum should form a four-vector, and they are surprised when they do not. Indeed our discussion at the beginning of this article regarding the Lorentz transformation of the electromagnetic energy and momentum of a capacitor involves just such an anticipation of four-vector transformation properties for a part of a system. In this paper we have explored the Lorentz transformation behavior of energy and momentum for some simple electromagnetic systems consisting of point charges. We find that the total energy and momentum of a system with no external forces indeed form a four-vector under Lorentz transformation. But the considerations which play a role in this result may seem subtle. Time nonsimultaneity and the forces of one part of the system upon the other are elements which are crucially required. On the other hand, if one considers only part of the total energy of a system, such as just the electrostatic energy in some configuration, or if one considers a system which experiences explicit external forces, then the energy and momentum considered will be altered by sources outside of the part under consideration. These alterations look different in different inertial frames precisely because of some of the subtle aspects of relativity, and hence the four-vector character for the energy and momentum considered is lost. The simple examples of this paper should serve as reminders for the basic patterns of behavior to be anticipated for the energy and momentum of more complicated electromagnetic systems.[6]

[1]See, for example, W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, MA, 1955), Chap. 17.

[2]See, also, J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed., Chap. 11; L. Eyges, *The Classical Electromagnetic Field* (Addison-Wesley, Reading, MA, 1972), Chap. 12; J. R. Reitz and F. J. Milford, *Foundations of Electromagnetic Theory* (Addison-Wesley, Reading, MA, 1967), 2nd ed., Chap. 17.

[3]See, also, E. M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1965), Chap. 6; R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1962), Vol. II, Sec. 13-6.

[4]See Ref. 1, Secs. 10.5 and 10.6.

[5]See, for example, L. Page and N. I. Adams, Am. J. Phys. **13**, 141 (1945); or L. Page and N. I. Adams, *Electrodynamics* (Van Nostrand, New York, 1940), p. 175, Eqs. (56–9), (56–10).

[6]The perspective provided by these examples has been applied to illuminate the famous factor of $\frac{4}{3}$ in the classical model of the electron; see T. H. Boyer, Phys. Rev. D **25**, 3246 (1982). The analysis given there can be extended beyond the reported connection between the momenta so as to include the connection between the energies U and U' seen in the two inertial frames S and S'. The work W' done by the external forces stabilizing the sphere as measured in the S' frame is

$$W'(t'_i) = \int_{t'=-\infty}^{t'=t'_i} V \int d^3x' f'_{\text{ext}\,x}(t',\mathbf{r}') = \mathbf{V}\cdot\mathbf{I}'(t'_i).$$

Hence, $\Delta U' = W' = \frac{1}{3}V^2\gamma U_{\text{em}}/c^2$, which is precisely the discrepancy from four-vector behavior which is found for the energy U'.