Example of mass–energy relation: Classical hydrogen atom accelerated or supported in a gravitational field

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(Received 20 November 1997; accepted 14 January 1998)

A simple example of the mass–energy relation is given by using electromagnetic calculations for a classical hydrogen atom, a point mass electron moving under the Coulomb force in a uniform circular orbit around a heavy point mass proton when the radiation-reaction self-force is neglected. The system mass is determined by calculating the ratio of the total external force to the system acceleration in the limit of small acceleration, and is shown to be equal to c^{-2} times the total system energy including the particle rest-mass energies, the particle kinetic energies, and the electrostatic potential energy. From the equivalence principle, the external forces can be regarded either as accelerating the atom relative to an inertial frame or else as supporting the atom in a gravitational field. The calculations are carried out for two different models. In the first case, the atom is regarded as accelerated by a frictionless surface which applies forces to both the electron and the proton. In the second case, the atom is accelerated by a single external force applied to the proton, and the electron orbit is displaced relative to the proton. In both cases the accelerating forces are constant forces and there are no external forces stabilizing the system.

I. INTRODUCTION

The connections between mass and energy within relativistic classical physics can be subtle and are sometimes misunderstood. On this account, it is helpful to have examples which illustrate these connections for simple cases.

In the past some simple examples have been given involving the acceleration of pairs of point charges q_1 and q_2 perpendicular to the direction of separation l.[1] In these cases, there is no work done by the external forces stabilizing the system and so one finds that the total force $\mathbf{F}_{\text{external, total}}$ necessary to slowly accelerate the configuration corresponds to a system mass M_{system} which is exactly c^{-2} times the total energy including the particle rest-mass energies and the system electrostatic potential energy, so

$$M_{\text{system}} \equiv \frac{\mathbf{F}_{\text{external, total}}}{\mathbf{a}} = \frac{1}{c^2}\left(m_1c^2 + m_2c^2 + \frac{q_1q_2}{l}\right). \quad (1)$$

Students are often fascinated to see this simple example of the mass–energy connection.

This ‘‘dumbbell’’ configuration has also been discussed[2] when the acceleration is in directions other than perpendicular to the line joining the charges. In these cases, the external stabilizing forces (which keep the dumbbell together) do work and Eq. (1) no longer holds. For example, in the case of acceleration parallel to the line l joining the charges, the system ‘‘mass’’ (external accelerating force divided by acceleration) includes a term $2q_1q_2/(lc^2)$ which is *twice* the electrostatic energy divided by c^2.

The appearance of a constant K other than $K=1$ for the term $Kq_1q_2/(lc^2)$ involving electrostatic energy on the right-hand side of Eq. (1) is an old problem which is famous in connection with the classical model of the electron where a factor of 4/3 occurs.[3] One of the several ways of understanding this surprising factor is in terms of a standard example in relativity texts[4] where a motor and conveyer belt transfer energy from one end of a plank to the other. Although there is no transfer of particles, the center of energy (mass) of the system shifts toward the end which receives the energy. In the case of a dumbbell configuration of two opposite charges q_1 and q_2 which are accelerated parallel to the line l connecting them, the external stabilizing forces do work and hence transfer energy. The stabilizing force $\mathbf{F}_b$ on the trailing (back) particle absorbs power at a rate $\mathbf{F}_b \cdot \mathbf{v} = (q_1 q_2 / l^2)\mathbf{v}$ while the external stabilizing force $\mathbf{F}_f$ on the leading (front) particle provides power at the same rate. This corresponds to a power flow over a distance l and hence involves a change in the center of mass; there is an associated momentum

$$\mathbf{P} = \frac{dm}{dt}\,\mathbf{l} = \frac{q_1 q_2 v}{l^2 c^2}\,\mathbf{l} = \frac{q_1 q_2}{l c^2}\,\mathbf{v}, \tag{2}$$

which is entirely separate from the momentum associated with particle mass. This is seen most vividly in the textbook example[4] where a continuous conveyer belt transfers the energy without any transfer of particles. When the charged dumbbell system is accelerated $\mathbf{a} = d\mathbf{v}/dt$, the external forces must provide this rate of change of momentum in addition to that associated with the acceleration of the rest masses and electrostatic energy of the system. Since the momentum in Eq. (2) associated with the energy transfer of the internal forces of constraint is equal to the momentum $(q_1 q_2 / l c^2)\mathbf{v}$ associated with the electrostatic potential energy, the result is that the total force needed to accelerate the system includes a term which looks like a mass of c^{-2} times twice the electrostatic potential energy. In the case of the classical model of the electron, one can think of the dumbbell configuration averaged over all directions with a resulting multiplying factor of 4/3, which is intermediate between the extreme values of 1 (where the forces of constraint do no work) and 2 (where the forces of constraint do maximum work) which appear for the perpendicular or parallel orientations of the dumbbell.

In the present discussion we wish to avoid any complications introduced by forces of constraint which do work. Indeed, we will eliminate the forces of constraint entirely. Accordingly, in our simple example we indeed find the expected mass–energy connection $E = mc^2$; both internal electrostatic energy and kinetic energy enter into the system energy with the correct coefficients.

We consider a classical model of the hydrogen atom where an electron of charge $-e$ and mass m is in a circular Coulomb orbit around a proton of charge $+e$ and mass M. The mass of the system is tested by applying accelerating forces in two cases. In the first case, the accelerating forces are applied to both the electron and the proton so as to accelerate them uniformly in the direction perpendicular to the orbit of the electron. We may think of the system as being accelerated by a frictionless plane which is oriented parallel to the orbit of the electron. In this case the electromagnetic calculations are carried out to all orders in c^{-1}, but to first order in the small acceleration a. In the second case, a single external accelerating force is applied to the proton alone; the proton accelerates and drags the electron along in an orbit a fixed distance behind it. In this case the electromagnetic calculations are carried to order c^{-2} in Gaussian units.

In both cases we can regard the system as accelerating in an inertial frame, or, by the equivalence principle, as being supported in a weak gravitational field.

II. CASE I: ACCELERATION OF A HYDROGEN ATOM BY A FRICTIONLESS PLANE

The hydrogen atom is treated classically as an electron in uniform circular motion around a massive proton neglecting radiation-reaction self-forces. The atom is assumed to lie in the xy plane and is accelerated in the z direction. This arrangement simplifies the situation to a steady state configuration where we can neglect the orbital motion of the proton. If the relativistic mass–energy connection holds, then the sum of the external forces must provide the change of momentum associated with the rest mass M of the proton, the rest mass m of the electron, the relativistic kinetic energy of the electron, and the system electrostatic potential energy $-e^2/r$. Thus we expect that in order to accelerate this hydrogen atom, we will need a sum of external forces of magnitude

$$\mathbf{F}_{\text{external, total}} = \frac{1}{c^2}\left(m\gamma c^2 + Mc^2 - \frac{e^2}{r}\right)\mathbf{a}. \tag{3}$$

We now go through the calculations to confirm this.

We imagine the atom being accelerated by a frictionless sheet providing external forces $\mathbf{F}_{M\,\text{ext}}$ on the proton and $\mathbf{F}_{m\,\text{ext}}$ on the electron, $\mathbf{F}_{\text{external, total}} = \mathbf{F}_{M\,\text{ext}} + \mathbf{F}_{m\,\text{ext}}$, with the direction of acceleration $\mathbf{a}$ normal to the frictionless sheet. For each particle the total force on the particle must provide the change of linear momentum $d\mathbf{p}/dt$ of the particle. When the atom comes instantaneously to rest at time $t=0$, the forces on the proton include the external accelerating force $\mathbf{F}_{M\,\text{ext}}$ and the electric force $e\mathbf{E}_m$ due to the electric field $\mathbf{E}_m$ of the electron, while the forces on the electron include the external force $\mathbf{F}_{m\,\text{ext}}$ and the electric force $-e\mathbf{E}_M$ due to the electric field of the proton,

$$\mathbf{F}_{M\,\text{ext}} + e\mathbf{E}_m = M\mathbf{a}, \quad \mathbf{F}_{m\,\text{ext}} - e\mathbf{E}_M = \frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}\right). \tag{4}$$

The Lienard–Wiechert electric field for a point charge q is[5]

$$\mathbf{E} = q\left[\frac{(\hat{n} - \boldsymbol{\beta})(1-\beta^2)}{(1 - \hat{n}\cdot\boldsymbol{\beta})^3 R^2}\right]_{t_{\text{ret}}} + \frac{q}{c}\left[\frac{\hat{n}\times\{(\hat{n} - \boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}}{(1 - \hat{n}\cdot\boldsymbol{\beta})^3 R}\right]_{t_{\text{ret}}},$$
$$\mathbf{B} = \hat{n}_{\text{ret}} \times \mathbf{E}. \tag{5}$$

Here the proton position is given by

$$\mathbf{r}_M(t) = \hat{k}\tfrac{1}{2}at^2, \tag{6}$$

and that of the orbiting electron by

$$\mathbf{r}_m(t) = \hat{\imath}r\cos\omega t + \hat{\jmath}r\sin\omega t + \hat{k}\tfrac{1}{2}at^2 = \hat{r}r + \hat{k}\tfrac{1}{2}at^2, \tag{7}$$

where $v = r\omega = \beta c$. The expressions for velocity and acceleration follow directly by time differentiation in (6) and (7). The retarded time t_{ret} needed to evaluate the electric field $\mathbf{E}_m(\mathbf{0},0)$ due to the electron at the position $\mathbf{r}_M(0) = 0$ of the proton at $t=0$ corresponds to the time for light to travel across the displacement $\mathbf{r}_M(0)-\mathbf{r}_m(t_{\rm ref})$ from source point to field point. From (6) and (7) this is

$$t_{\rm ret}=-\frac{1}{c}\sqrt{r^2+\left(\frac{1}{2}at_{\rm ret}^2\right)^2}=-\frac{r}{c}+O(a^2). \tag{8}$$

Then the evaluation of $\mathbf{E}_m(\mathbf{0},0)$ from Eq. (5) requires through first order in the acceleration

$$\hat{n}_{\rm ret}=\frac{-\hat{r}_{\rm ret}r-\hat{k}\frac{1}{2}at_{\rm ret}^2}{\sqrt{r^2+(\frac{1}{2}at_{\rm ret}^2)^2}}=-\hat{r}_{\rm ret}-\hat{k}\,\frac{ar}{2c^2}+O(a^2), \tag{9}$$

$$\boldsymbol{\beta}_{\rm ret}=\frac{1}{c}\,(\hat{k}\times\hat{r}_{\rm ret}r\omega+\hat{k}at_{\rm ret})=\hat{k}\times\hat{r}_{\rm ret}\,\frac{v}{c}-\hat{k}\,\frac{ar}{c^2}+O(a^2), \tag{10}$$

$$\dot{\boldsymbol{\beta}}_{\rm ret}=-\hat{r}_{\rm ret}\,\frac{v^2}{rc}+\hat{k}\,\frac{a}{c}, \tag{11}$$

and

$$R_{\rm ret}=|\mathbf{r}_M(0)-\mathbf{r}_m(t_{\rm ret})|=\sqrt{r^2+(\tfrac{1}{2}at_{\rm ret}^2)^2}=r+O(a^2). \tag{12}$$

Substituting Eqs. (9)–(12) into Eq. (5), we find

$$\mathbf{E}_m(\mathbf{0},0)=q\,\frac{\left[-\hat{r}_{\rm ret}-\hat{k}\,\frac{ar}{2c^2}-\left(\hat{k}\times\hat{r}_{\rm ret}\,\frac{v}{c}-\hat{k}\,\frac{ar}{c^2}\right)\right]\left(1-\frac{v^2}{c^2}-\left(\frac{ar}{c^2}\right)^2\right)}{\left[1-\frac{1}{2}\left(\frac{ar}{c^2}\right)^2\right]^3r^2}+\frac{q}{c}\,\frac{\left(-\hat{r}_{\rm ret}-\hat{k}\,\frac{ar}{2c^2}\right)}{\left[1-\frac{1}{2}\left(\frac{ar}{c^2}\right)^2\right]^3r}$$
$$\times\left\{\left[-\hat{r}_{\rm ret}-\hat{k}\,\frac{ar}{2c^2}-\left(\hat{k}\times\hat{r}_{\rm ret}\,\frac{v}{c}-\hat{k}\,\frac{ar}{c^2}\right)\right]\times\left(-\hat{r}_{\rm ret}\,\frac{v^2}{cr}+\hat{k}\,\frac{a}{c}\right)\right\}. \tag{13}$$

Simplifying and keeping only the z-component $\mathbf{E}_{mz}(\mathbf{0},0)$ relevant to the acceleration $\mathbf{a}$ of the entire atom, we have through first order in a with $q=-e$,

$$\mathbf{E}_{mz}(\mathbf{0},0)=\hat{k}\,\frac{ea}{2c^2r}. \tag{14}$$

Now we turn to the forces on the electron, which include the external force $\mathbf{F}_{m\ \rm ext}$ and the force $-e\mathbf{E}_M$ due to the electric field $\mathbf{E}_M$ at the electron due to the proton. Newton's second law for the electron gives

$$\mathbf{F}_{m\ \rm ext}-e\mathbf{E}_M=\frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}}\right)=m\gamma\left(-\hat{r}\,\frac{v^2}{r}+\hat{k}a\right). \tag{15}$$

Both the centripetal acceleration and the acceleration $\mathbf{a}=\hat{k}a$ are perpendicular to the instantaneous electron velocity $\mathbf{v}$ at $t=0$ and so allow us to take out the factor $\gamma=(1-v^2/c^2)^{-1/2}$ on the right-hand side of Eq. (15).

The electric field $\mathbf{E}_M$ of the proton at the position of the electron follows from Eq. (5) using the same retarded time in Eq. (8) substituted into Eq. (6) and its derivatives. Here the displacement from source point to field point is $\mathbf{r}_m(0)-\mathbf{r}_M(t_{\rm ret})$ and

$$\hat{n}_{\rm ret}=\frac{\hat{r}r-\hat{k}\frac{1}{2}at_{\rm ret}^2}{\sqrt{r^2+(\frac{1}{2}at_{\rm ret}^2)^2}}=\hat{r}-\hat{k}\,\frac{ar}{2c^2}+O(a^2), \tag{16}$$

$$\boldsymbol{\beta}_{\rm ret}=\frac{\hat{k}at_{\rm ret}}{c}=-\hat{k}\,\frac{ar}{c^2}+O(a^2), \tag{17}$$

$$\dot{\boldsymbol{\beta}}_{\rm ret}=\hat{k}\,\frac{a}{c}, \tag{18}$$

$$R_{\rm ret}=|\mathbf{r}_m(0)-\mathbf{r}_M(t_{\rm ret})|=\sqrt{r^2+(\tfrac{1}{2}at_{\rm ret}^2)^2}=r+O(a^2), \tag{19}$$

and from (5) the field is

$$\mathbf{E}_M(\hat{r}r,0)=\frac{q\left[\hat{r}-\hat{k}\,\frac{ar}{2c^2}-\left(-\hat{k}\,\frac{ar}{c^2}\right)\right]\left(1-\left(\frac{ar}{c^2}\right)^2\right)}{\left[1-\frac{1}{2}\left(\frac{ar}{c^2}\right)^2\right]^3r^2}+\frac{q}{c}\,\frac{\left(\hat{r}-\frac{\hat{k}ar}{2c^2}\right)\times\left\{\left[\hat{r}-\hat{k}\,\frac{ar}{2c^2}-\left(-\hat{k}\,\frac{ar}{c^2}\right)\right]\times\hat{k}\,\frac{a}{c}\right\}}{\left[1-\frac{1}{2}\left(\frac{ar}{c^2}\right)^2\right]^3r}. \tag{20}$$

Keeping terms only through first order in the acceleration and setting $q=e$, the field becomes

$$\mathbf{E}_M(\hat{r}r,0)=e\left(\frac{\hat{r}}{r^2}-\frac{\hat{k}a}{2c^2r}\right). \tag{21}$$

We use Eq. (21) in Eq. (15) and Eq. (14) in Eq. (4) to obtain in the z direction associated with the acceleration,

$$\mathbf{F}_{m\ \rm ext}+\hat{k}\,\frac{e^2a}{2c^2r}=\hat{k}m\gamma a, \tag{22}$$

$$\mathbf{F}_{M\ \rm ext}+\hat{k}\,\frac{e^2a}{2c^2r}=\hat{k}Ma. \tag{23}$$

The total external force needed to accelerate the hydrogen atom follows from adding Eqs. (22) and (23) giving

$$\mathbf{F}_{\text{external, total}} = \mathbf{F}_{m\ \text{ext}} + \mathbf{F}_{M\ \text{ext}} = \frac{1}{c^2}\left(m\gamma c^2 + Mc^2 - \frac{e^2}{r} \right)\mathbf{a}. \quad (24)$$

This is exactly the required mass–energy connection involving a system mass

$$\frac{\mathbf{F}_{\text{external, total}}}{\mathbf{a}} \equiv M_{\text{system}} = \frac{1}{c^2}\left(m\gamma c^2 + Mc^2 - \frac{e^2}{r} \right). \quad (25)$$

Here we recognize the electrostatic potential energy $-e^2/r$ as well as the total particle energies $m\gamma c^2$, Mc^2 (rest energy and kinetic energy) of the electron and proton.

III. CASE II: ACCELERATION OF A HYDROGEN ATOM BY A FORCE ON THE PROTON

Our second and related calculation treats the situation when a single external force accelerating the system is applied to the proton, but not to the electron. One can picture the external force as a string applied to the proton. In this case the electron is accelerated solely by the electric field of the proton and must lag behind (or below) the proton. We will carry our calculation through to order c^{-2}.

If our calculations are carried through order c^{-2} only, we can replace the Lienard–Wiechert field expressions by their Page and Adams[6] approximations involving the present (unretarded) time,

$$\mathbf{E} = q\frac{\hat{n}}{r^2}\left(1 + \frac{1}{2}\frac{v^2}{c^2} - \frac{3}{2}\frac{(\mathbf{v}\cdot\hat{n})^2}{c^2} \right) - \frac{q}{2c^2 r}(\mathbf{a} + (\mathbf{a}\cdot\hat{n})\hat{n}), \quad (26)$$

$$\mathbf{B} = \frac{q\mathbf{v}\times\hat{n}}{cr^2}. \quad (27)$$

In this case the equation of motion for the electron involves no external force but only the electromagnetic force due to the proton,

$$(-e)\mathbf{E}_M = m\gamma\mathbf{a} = m\left(1 + \frac{1}{2}\frac{v^2}{c^2} \right)\left(-\hat{r}\frac{v^2}{r} + \hat{k}a \right) + O(c^{-3}), \quad (28)$$

since the acceleration is perpendicular to the orbital velocity $\mathbf{v}$. While the proton coordinate is still given by $\mathbf{r}_M(t) = \hat{k}\frac{1}{2}at^2$ as in Eq. (6), here the electron orbit lags a distance h behind the proton at a position

$$\mathbf{r}_m(t) = \hat{i}r\cos\omega t + \hat{j}r\sin\omega t + \hat{k}(\tfrac{1}{2}at^2 - h)$$
$$= \hat{r}r + \hat{k}(\tfrac{1}{2}at^2 - h). \quad (29)$$

The electric field $\mathbf{E}_M$ is due to the uniformly accelerating proton which is instantaneously at rest at the origin at $t=0$. The field follows from (26) and (29) with the insertion of the field coordinates $\mathbf{r}_m(0) = \hat{r}r - \hat{k}h$, the source position $\mathbf{r}_M(0) = 0$, velocity $\mathbf{v}_M = 0$, and acceleration $\mathbf{a}_M = a$,

$$\mathbf{E}_M(\hat{r}r - \hat{k}h, 0) = \frac{e(\hat{r}r - \hat{k}h)}{(r^2+h^2)^{3/2}} - \frac{e}{2c^2(r^2+h^2)^{1/2}} \times \left(\hat{k}a - \frac{ah(\hat{r}r - \hat{k}h)}{r^2+h^2} \right). \quad (30)$$

Now we expect that the difference h between the z coordinates of the electron and proton should be first order in the acceleration a, vanishing in the limit $a \to 0$. Thus, just as we retained only first-order terms in the acceleration a, so we retain only first-order terms in h. Therefore the electron equation of motion (28) with the electric field (30) becomes

$$-e^2\left(\frac{\hat{r}}{r^2} - \frac{\hat{k}h}{r^3} \right) + \frac{e^2 a}{2c^2 r}\hat{k} = m\left(1 + \frac{1}{2}\frac{v^2}{c^2} \right)\left(-\hat{r}\frac{v^2}{r} + \hat{k}a \right). \quad (31)$$

Separating this into $\hat{r}$ and $\hat{k}$ components gives

$$\frac{e^2}{r^2} = m\left(1 + \frac{1}{2}\frac{v^2}{c^2} \right)\frac{v^2}{r} \quad (32)$$

and

$$\frac{e^2 h}{r^3} = \left[m\left(1 + \frac{1}{2}\frac{v^2}{c^2} \right) - \frac{e^2}{2c^2 r} \right] a. \quad (33)$$

Turning to the equation of motion for the proton, we have this particle at rest at $t=0$, so Newton's second law takes the form

$$\mathbf{F}_{M\ \text{ext}} + (+e)\mathbf{E}_m = M\mathbf{a}. \quad (34)$$

The electric field $\mathbf{E}_m(\mathbf{0},0)$ due to the electron at the position $\mathbf{r}_M(0) = 0$ of the proton at time $t=0$ is found from (26) with the electron source located at

$$\mathbf{r}_m(0) = \hat{r}r - \hat{k}h \quad (35)$$

with velocity

$$\mathbf{v}_m(0) = \hat{k}\times\hat{r}v \quad (36)$$

and acceleration

$$\mathbf{a}_m(0) = -\hat{r}\frac{v^2}{r} + \hat{k}a, \quad (37)$$

so that

$$\mathbf{E}_m(\mathbf{0},0) = (-e)\frac{(-\hat{r}r + \hat{k}h)}{(r^2+h^2)^{3/2}}\left(1 + \frac{1}{2}\frac{v^2}{c^2} + 0 \right) - \frac{(-e)}{2c^2(r^2+h^2)^{1/2}}\left[-\hat{r}\frac{v^2}{r} + \hat{k}a + \left(-\hat{r}\frac{v^2}{r} + \hat{k}a \right)\cdot\frac{(-\hat{r}r + \hat{k}h)(-\hat{r}r + \hat{k}h)}{r^2+h^2} \right]. \quad (38)$$

Retaining only terms through first order in a and second order in c^{-1}, Eq. (34) with $\mathbf{E}_m$ from (38) becomes

$$\mathbf{F}_{M\ \text{ext}} - e^2\left(-\frac{\hat{r}}{r^2} + \frac{\hat{k}h}{r^3} \right)\left(1 + \frac{1}{2}\frac{v^2}{c^2} \right) + \frac{e^2}{2c^2 r}\left[-\hat{r}\frac{2v^2}{r} + \hat{k}\left(a + \frac{hv^2}{r^2} \right) \right] = M\mathbf{a}. \quad (39)$$

Taking the z components and simplifying, this is

$$F_{M\ \text{ext}} - \frac{e^2 h}{r^3} + \frac{e^2 a}{2c^2 r} = Ma. \quad (40)$$

Now substituting from Eq. (33) to eliminate h, we find Eq. (40) takes the form

$$F_{M\ \mathrm{ext}}=\frac{1}{c^2}\left(Mc^2+mc^2+\frac{1}{2}mv^2-\frac{e^2}{r}\right)a. \qquad (41)$$

Thus the single external force acts as though it were accelerating (or supporting against gravity) a system mass

$$M_{\mathrm{system}}=\frac{1}{c^2}\left(Mc^2+mc^2+\frac{1}{2}mv^2-\frac{e^2}{r}\right). \qquad (42)$$

Thus indeed the system mass M_{system} is associated with the system energy content by the famous equation $E=mc^2$.

IV. CLOSING SUMMARY

Einstein's dictum that the mass of an object is a measure of its energy content becomes comprehensible when one can see the mechanism which connects forces accelerating a system to the binding energy and internal kinetic energy of a system. This can be illustrated easily using electromagnetic theory for a classical hydrogen atom where radiation is ignored. In this case one finds that external forces which accelerate the atom or which support the atom against gravity account for the masses of the proton and of the electron as well as the kinetic energy of the electron and the electric potential energy of the configuration.

ACKNOWLEDGMENT

I wish to thank Dr. Daniel C. Cole for proofreading the manuscript and suggesting improvements.

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