NOTES AND DISCUSSIONS

Connecting linear momentum and energy for electromagnetic systems

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According to special relativity, the mass M of a system is related to its energy content E by $E=Mc^2$. Therefore, many physicists expect that the momentum $\mathbf{P}$ of an electromagnetic system of energy E and center-of-energy velocity $\mathbf{V}$ is given by $\mathbf{P}=(E/c^2)\mathbf{V}$ corresponding to the relation

$$\mathbf{p} = m\gamma\mathbf{v} = \left(\frac{E}{c^2}\right)\mathbf{v}, \tag{1}$$

which holds for a point mass m moving with velocity $\mathbf{v}$, where $E=m\gamma c^2$ and $\gamma=(1-v^2/c^2)^{-1/2}$. However, the presence of external forces of constraint can frustrate this expectation. The most famous example involves the classical model of the electron for which non-electromagnetic forces are required for stability, and the electromagnetic energy and momentum are related by

$$\mathbf{P}_{\text{em}} = \frac{4}{3}\left(\frac{U_{\text{em}}}{c^2}\right)\gamma\mathbf{V}, \tag{2a}$$

$$E_{\text{em}} = \gamma\left(\frac{U_{\text{em}}}{c^2}\right)\left(1+\frac{1}{3}V^2\right), \tag{2b}$$

where U_{em} is the electrostatic energy of the charge distribution in its own rest frame. The situation is sometimes referred to as the "4/3 problem" of the classical electron.[1]

Various examples have been given to illustrate the relation between momentum and energy in electromagnetic systems and to explain the departure from the expected form given in Eq. (1). Here we emphasize that the general connection between the energy E and momentum $\mathbf{P}$ of a relativistic electromagnetic system is given by[2]

$$\mathbf{P} = \frac{d}{dt}\left(\frac{E}{c^2}\mathbf{X}\right) - \sum_i (\mathbf{F}_{\text{ext},i}\cdot\mathbf{v}_i)\frac{\mathbf{r}_i}{c^2}, \tag{3}$$

where $\mathbf{X}$ is the system center of energy and the $\mathbf{F}_{\text{ext},i}$ are the external (non-electromagnetic) forces acting on the particles of mass m_i at position $\mathbf{r}_i$ and moving with velocity $\mathbf{v}_i$. Equation (3) follows from the generator of Lorentz transformations for classical electron theory in the presence of external forces.

If there is no *net* work done by the external forces, the system electromagnetic energy E is constant, and the time derivative in Eq. (3) becomes $d[(E/c^2)\mathbf{X}]/dt=(E/c^2)\mathbf{V}$, where $\mathbf{V}=d\mathbf{X}/dt$ is the velocity of the system center of energy. Thus if there is no net work done by the external forces, Eq. (3) becomes

$$\mathbf{P} = \left(\frac{E}{c^2}\right)\mathbf{V} - \sum_i (\mathbf{F}_{\text{ext},i}\cdot\mathbf{v}_i)\frac{\mathbf{r}_i}{c^2}. \tag{4}$$

There are several familiar cases:

(a) If no external forces are present, $\mathbf{F}_{\text{ext},i}=0$, the summation term on the right-hand side of Eq. (4) vanishes and the system momentum is indeed given by the expected relation (1). This case is discussed in Refs. 3–5 for charged particles with purely electromagnetic interactions, but only through order $1/c^2$ because of the complications related to radiation emission. This electromagnetic interaction between charged particles through order $1/c^2$ is described by the Darwin Lagrangian.[6] In this case, the momentum is given by

$$\mathbf{P} = \sum_i \left\{ m_i\left(1+\frac{1}{2}v_i^2/c^2\right)\mathbf{v}_i + \sum_{j\neq i} [e_ie_j/(2c^2r_{ij})][\mathbf{v}_j + (\mathbf{v}_j\cdot\mathbf{r}_{ij})\mathbf{r}_{ij}/r_{ij}^2] \right\}, \tag{5a}$$

and the energy is

$$E = \sum_i \left\{ m_i\left(c^2+\frac{1}{2}v_i^2\right) + \frac{1}{2}\sum_{j\neq i} e_ie_j/r_{ij} \right\}. \tag{5b}$$

(b) If external forces are present but there is no work done by the external forces, $\mathbf{F}_{\text{ext},i}\cdot\mathbf{v}_i=0$, then Eq. (4) again reduces to the form in Eq. (1). This case occurs when two charged particles move side-by-side along parallel frictionless rails; the external forces of constraint are perpendicular to the common velocity of the particles and so the external forces do no work. This situation is discussed in Refs. 4, 7, and 8. In this case the momentum is $\mathbf{P}=[m_1+m_2+e_1e_2/(c^2r_{12})]\gamma\mathbf{v}$ and the energy is $E=[m_1c^2+m_2c^2+e_1e_2/r_{12}]\gamma$.

(c) If two point charges are separated by a constant displacement parallel to their constant velocity, then the external forces of constraint indeed do work, but there is no *net* work done by the external forces because equal amounts of energy are introduced at one charge and removed at the other. In this case, there is no net work done by the external forces and thus Eq. (4) is valid. However, the energy is introduced at different points $\mathbf{r}_i$ and so the summation term in Eq. (4) does not vanish and accordingly the energy-momentum connection departs from the naive expectation given in Eq. (1). This summation term is given by $-\Sigma_{i=1}^2(\mathbf{F}_{\text{ext},i}\cdot\mathbf{v}_i)\mathbf{r}_i/c^2=e_1e_2\mathbf{v}/(\gamma\ell)$, where ℓ is the interparticle separation in the rest frame of the two charges. This situation is discussed in Refs. 4 and 8 where to order $1/c^2$

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the momentum is $\mathbf{P}=[m_1(1+\frac{1}{2}v^2/c^2)+m_2(1+\frac{1}{2}v^2/c^2)+2e_1e_2/(c^2\ell)]\mathbf{v}$; the energy is $E=[m_1(c^2+\frac{1}{2}v^2)+m_2(c^2+\frac{1}{2}v^2)+e_1e_2/\ell]$. There is a factor-of-two discrepancy in the electromagnetic field contributions corresponding to the presence of the summation term in Eq. (4).

(d) For the classical model of the electron, as in case (c), forces of constraint are present but do no *net* work so that Eq. (4) holds. The forces are applied at points with differing displacements $\mathbf{r}_i$ in the direction of motion and so the summation term in Eq. (4) is non-vanishing. The term involving the external forces gives an additional contribution so that we find Eq. (2), which is not in agreement with the form in Eq. (1). This case is discussed in Ref. 9.

The difference between the system (mechanical and electromagnetic) momentum $\mathbf{P}$ and the term $d[(E/c^2)\mathbf{X}]/dt$ involving the system (mechanical and electromagnetic) energy E is sometimes termed "hidden momentum."[10] We see in Eq. (3) that this hidden momentum is given by $-\Sigma_i(\mathbf{F}_{\text{ext},i}\cdot\mathbf{v}_i)\mathbf{r}_i/c^2$ and involves forces that are external to the electromagnetic system. The designation of this term as hidden momentum tells us little about the character of the non-electromagnetic energy and momentum flow associated with the external forces.

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[1] See, for example, J. Frenkel, "4/3 problem in classical electrodynamics," Phys. Rev. E **54**, 5859–5862 (1996).

[2] T. H. Boyer, "Illustrations of the relativistic conservation law for the center of energy," Am. J. Phys. **73**, 953–961 (2005); "Relativity, energy flow, and hidden momentum," **73**, 1184–1185 (2005).

[3] S. Coleman and J. H. Van Vleck, "Origin of 'hidden momentum forces' on magnets," Phys. Rev. **171**, 1370–1375 (1968); see Eq. (11).

[4] T. H. Boyer, "Lorentz-transformation properties for energy and momentum in electromagnetic systems," Am. J. Phys. **53**, 167–171 (1985).

[5] R. Coisson and G. Guidi, "Electromagnetic interaction momentum and simultaneity," Am. J. Phys. **69**, 462–463 (2001).

[6] See, for example, Ref. 3 or J. D. Jackson, *Classical Electrodynamics* (Wiley, New York 1999), 3rd ed., pp. 596–598.

[7] T. H. Boyer, "Electrostatic potential energy leading to an inertial mass change for a system of two point charges," Am. J. Phys. **46**, 383–385 (1978); "Electrostatic potential energy leading to a gravitational mass change for a system of two point charges," **47**, 129–131 (1979).

[8] D. J. Griffiths and R. E. Owen, "Mass renormalization in classical electrodynamics," Am. J. Phys. **51**, 1120–1126 (1983).

[9] H. Poincaré, "Sur la dynamique de l'electron," Rend. Circ. Mat. Palermo **21**, 129–173 (1906). This paper is translated with modern notation in H. M. Schwartz, "Poincaré's Rendiconti paper on relativity, Parts I, II, and III," Am. J. Phys. **39**, 1287–1294 (1971); **40**, 862–872 (1972); **40**, 1282–1287 (1972). F. Rohrlich, "Self-energy and stability of the classical electron," *ibid.* **28**, 639–643 (1960). F. R. Tangherlini, "Self-stress, covariance and the classical electron," *ibid.* **31**, 285–288 (1963). T. H. Boyer, "Classical model of the electron and the definition of electromagnetic field momentum," Phys. Rev. D **25**, 3246–3255 (1982); see also Ref. 6 in T. H. Boyer, "Lorentz-transformation properties for energy and momentum in electromagnetic systems," Am. J. Phys. **53**, 167–171 (1985). A historical review emphasizing modeling the electron has been given by F. Rohrlich, "The dynamics of a charged sphere and the electron," *ibid.* **65**, 1051–1057 (1997).

[10] V. Hnizdo, "Hidden momentum and the electromagnetic mass of a charge and current carrying body," Am. J. Phys. **65**, 55–65 (1997); "Hidden mechanical momentum and the field momentum in stationary electromagnetic and gravitational systems," **65**, 515–518 (1997).

Comment on "A generalized Helmholtz theorem for time-varying vector fields," by Artice M. Davis [Am. J. Phys. 74, 72–76 (2006)]

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In a recent paper Davis formulated the following generalization of the Helmholtz theorem for a time-varying vector field:[1]

$$\boldsymbol{F}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(\nabla\phi+\frac{\partial \boldsymbol{A}}{\partial t}\right)+\nabla\times(\nabla\times\boldsymbol{A}),\tag{1}$$

where ϕ and $\boldsymbol{A}$ are the Lorenz gauge retarded potentials. The purposes of this Comment are to point out that Davis's generalization is a version of the generalization of the Helmholtz theorem formulated some years ago by McQuistan[2] and Jefimenko,[3] and more recently by the present author,[4–6] and to show that Davis's expression for the field $\boldsymbol{F}$ is also valid for potentials in gauges other than the Lorenz gauge.

The generalized Helmholtz theorem states that a retarded vector field vanishing at infinity can be written as[4]

$$\boldsymbol{F}=-\nabla\int d^3x'\frac{[\nabla'\cdot\boldsymbol{F}]}{4\pi R}+\nabla\int d^3x'\frac{[\nabla'\times\boldsymbol{F}]}{4\pi R}+\frac{1}{c^2}\frac{\partial}{\partial t}\int d^3x'\frac{[\partial\boldsymbol{F}/\partial t]}{4\pi R},\tag{2}$$

where the square brackets denote the retardation symbol, $R=|\mathbf{x}-\mathbf{x}'|$, and the integrals are over all space. If we define the potentials Φ, $\mathbf{A}$, and $\mathbf{C}$ by

$$\Phi=\int d^3x'\frac{[\nabla'\cdot\boldsymbol{F}]}{4\pi R},\tag{3a}$$

$$\mathbf{A}=\int d^3x'\frac{[\nabla'\times\boldsymbol{F}]}{4\pi R},\tag{3b}$$

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[9] F. Rohrlich, "Causality, the Coulomb field, and Newton's law of gravitation," Am. J. Phys. **70**, 411–414 (2002).
[10] K.-H. Yang, "The physics of gauge transformations," Am. J. Phys. **73**, 742–751 (2005).
[11] J. A. Heras, "Instantaneous fields in classical electrodynamics," Europhys. Lett. **69**, 1–7 (2005).
[12] The author thanks V. Hnizdo for drawing his attention to the fact that both formulations of the generalized theorem are equivalent.

Comments on the tethered galaxy problem

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In a recent paper Davis, Lineweaver, and Webb make the counterintuitive assertion that a galaxy held "tethered" at a fixed distance from our own could emit blueshifted light. This effect was derived from the simplest Friedmann-Robertson-Walker (FRW) spacetimes and the $\Omega_M=0.3$, $\Omega_\Lambda=0.7$ case, which is believed to be a good late time model of our universe. In this paper, we recover their results in a more transparent way, revise their calculations, and propose a formulation of the tethered galaxy problem based on radar distance rather than comoving "proper" distance. This formulation helps to remove the coordinate-dependent nature of the tethered galaxy problem and establishes consistency between the empty FRW model and special relativity. In the general case, we see that, although the radar distance tethering reduces the redshift of a receding object, it does not do so sufficiently to cause the blueshift as found by Davis, Lineweaver, and Webb. We also discuss some important issues raised by this approach relating to the interpretation of the redshift, velocity, and distance in relativistic cosmology. © *2006 American Association of Physics Teachers.*
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I. INTRODUCTION

The homogeneous and isotropic expansion of the universe is described by Friedmann-Robertson-Walker (FRW) spacetimes.[1] In these spacetimes, we may construct a co-moving frame in which the spacetime manifold of general relativity is treated as expanding, while on average matter is at rest. If we wish to study independent dynamical objects within an expanding universe, we would like to quantify the effect of imposing such a cosmological background. We follow Davis *et al.*[2] by considering a galaxy endowed with a large peculiar velocity that characterizes the velocity deviation from the universal expansion or Hubble flow. The tethered galaxy problem considers the physics of the extreme case, where a galaxy is endowed with sufficient peculiar velocity so as to cancel the Hubble flow and remain, in some sense, at a fixed distance from a co-moving observer who follows the Hubble flow. We study how light from such a galaxy would be redshifted and propose modifications to the previous calculations[2] that suggest that a receding source could be significantly blueshifted.

By recasting the problem in a coordinate independent form, with the radar distance as our measure, we achieve results that are more intuitive than previous work.[2] With this construction, tethered galaxies in the empty FRW universe have zero redshift and the problem may be reconciled with the notion of constant spatial separation in special relativity.

We find that the reduction in redshift is much lower than that obtained by the previous analysis.[2] The effect of imposing the tether is not sufficient to cause the blueshift of an object in an expanding universe and is only significant at very large distances, at which we would not expect proportionally large peculiar velocities.

Throughout this paper, we shall refer to the unevaluated quantity $1+z$ as the redshift of a light source, but note that a value of $z<1$ will actually correspond to a blueshift. Similarly, the condition for light to be received with zero redshift implies that it is observed at its emission frequency, that is, is neither redshifted nor blueshifted.

In Sec. II, we review the tethered galaxy problem as posed in Ref. 2. We remove the explicit redshift dependence from their calculations and recover their results as a combination of cosmic redshift and the special relativistic Doppler shift. We are thus able to demonstrate why the peculiar velocity required to cancel the cosmic redshift does not correspond to that proposed to tether a galaxy against the Hubble flow.

We then discuss two problems with this approach. First, we note that the peculiar velocities do not correspond to a quantity we might regard as a worldline velocity except in the special relativistic limit. In Sec. III, we construct a general relativistic condition on the 4 velocity of a luminous particle in order that light is received at the fixed spatial origin without redshift. In Sec. IV, we discuss the limitations of the distance scale used in the original formulation. Using this measure, we show that for the Milne model under a coordinate transformation, the problem does not—as we might expect—agree with the analogous system in special relativity. More details of this example appear in Appendix A.

Motivated by this example, we propose recasting the tethered galaxy problem in terms of a theoretically observable quantity, the radar distance. We construct this new system of observers in Sec. V and propose a method for solving the system in terms of light signals. In Sec. VI, we compare the phenomenological results with a physical model. We see that the effects of the tethered galaxy problem persist although they are comparatively small below scales of 10^4-10^5 megaparsec (Mpc).

II. THE TETHERED GALAXY PROBLEM

We consider FRW spacetimes governed by the metric

$$ds^2 = -c^2dt^2 + a(t)^2[d\chi^2 + \eta^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)], \quad (1)$$

where $\eta = \sin\chi$, χ, or $\sinh\chi$ when the curvature is positively curved, flat, or negatively curved respectively. Field equations for such a universe dominated by matter and the cosmological constant reduce to the Friedmann equation[1] for $a(t)$. We express this equation in terms of the normalized mass density $\Omega_M = 8\pi G\rho_{M,0}/3H_0^2$ and the cosmological constant $\Omega_\Lambda = \Lambda/3H_0^2$ so that $k = (\Omega_\Lambda + \Omega_M - 1)$ characterizes the curvature

$$a' = \frac{da}{dt} = H_0\left[1 + \Omega_M\left(\frac{1}{a} - 1\right) + \Omega_\Lambda(a^2 - 1)\right]^{1/2}. \quad (2)$$

We follow Ref. 2 and consider radial motion in an FRW universe. The metric (1) reduces to

$$ds^2 = -c^2dt^2 + a(t)^2d\chi^2. \quad (3)$$

In Ref. 2, the proper distance $D = a\chi$ is defined as the metric distance along the surfaces of homogeneity t=constant. We henceforth refer to this measure as the co-moving distance because it is defined in terms of the co-moving coordinate system. Treating all galaxies as test particles, we invoke symmetry and choose our galaxy to reside at the origin of the co-moving radial coordinate χ. Other galaxies then move at a rate,

$$D' = a'\chi + a\chi', \quad (4)$$

where $'$ denotes differentiation with respect to the cosmic time t.

A comoving galaxy whose worldline is characterized by $\chi' = 0$ can thus be considered to retreat from us with velocity $v_{rec} = a'\chi$. This recession of co-moving observers is commonly referred to as the Hubble flow. Davis *et al.*[2] propose tethering the galaxy with a peculiar velocity v_{teth} to maintain a constant comoving distance, that is,

$$D' = v_{rec} + v_{teth} = 0. \quad (5)$$

The coordinate χ' is therefore chosen to satisfy $v_{rec} = -a\chi'$. The authors[2] then suggest that we might expect light emitted by this galaxy to arrive without a change in its wavelength, as would be the case in Newtonian cosmology. They proceed to rebut this conjecture by imposing a zero redshift condition on the worldline of the galaxy and showing that this condition is not, in general, consistent with the fixed comoving distance condition. Furthermore, they find nonlinear relations between the peculiar velocity required to cancel the cosmic redshift, v_{pec}, and the velocity that acts as a tether to the galaxy, v_{teth}, and parametrize both as a function of the redshift. We now recover their results by a more direct method and suggest alternatives to this approach.

In general relativity, the cosmological redshift is usually viewed as an effect of the expansion of space on the wavelength of light.[1] For two co-moving observers in an FRW universe the redshift of a light signal emitted at t_{em} is given by

$$1 + z_{rec} = \frac{a(t_{obs})}{a(t_{em})}. \quad (6)$$

We wish to endow the galaxy with a peculiar radial velocity sufficient to cancel this redshift. Davis *et al.*[2] invoke the special relativistic Doppler shift[1] characterized by

$$1 + z_{pec} = \left(\frac{c + v_{pec}}{c - v_{pec}}\right)^{1/2}, \quad (7)$$

and require the two effects to cancel,

$$(1 + z_{rec})(1 + z_{pec}) = 1. \quad (8)$$

If we combine Eqs. (6)–(8), we obtain

$$v_{pec} = c\left(\frac{a_{em}^2 - a_{obs}^2}{a_{em}^2 + a_{obs}^2}\right). \quad (9)$$

Equation (9) for v_{pec} does not, in general, agree with $v_{teth} = -a'\chi$ chosen to cancel v_{rec} by Eqs. (4) and (5). This result appears implicitly in Ref. 2. In Appendix A, we recover the relation $v_{pec} = f(v_{teth})$ analytically for the Milne cosmology ($\Omega_M = \Omega_\Lambda = 0$), flat matter ($\Omega_M = 1, \Omega_\Lambda = 0$), and cosmological constant dominated ($\Omega_M = 0, \Omega_\Lambda = 1$) FRW spacetimes.

There are two issues with the above argument. First, there is a technical problem with applying Eq. (7) to FRW spacetimes. The velocity v_{pec} in Eq. (9) does not correspond to a quantity we might treat as a worldline velocity. This fact can be seen from the general relativistic argument of Sec. III. Second, the comoving distance used here may not be the best measure we can employ to construct the tethered galaxy problem. In Sec. IV, we shall see that tethering a galaxy at a fixed value of D does not coincide with what we might expect from special relativity for an empty FRW spacetime. Hence, we suggest that this distance measure is misleading when used in the tethering problem, and we propose a more suitable alternative in Sec. V.

III. ZERO-REDSHIFT CONDITION

Equation (7) is derived for the Minkowski space of special relativity[1] in which v_{pec} can be interpreted as the velocity with respect to the inertial frame of a stationary observer. If we employ this result in a general relativistic spacetime, we must accept that v_{pec} will not, except in the Minkowski limit, correspond to the conjectured velocity $v_{teth} = -a\chi'$ in comoving coordinates. Therefore this method fails to give a meaningful description of zero-redshifted observers in FRW cosmology. We now outline the general relativistic version of this argument from which we can recover the 4 velocity of a zero-redshifted observer and hence the corresponding worldline.

In this section, we employ the abstract index notation commonly used in general relativity (see, for example Sec. 2.4 of Ref. 3). For our purposes it is sufficient to recognize vectors with raised indices as covariant and those with lowered indices as contravariant.

The frequency of a light signal with wave vector k^a measured by an observer with 4 velocity u^a is given by[3]

$$\omega = -k_a u^a. \quad (10)$$

Therefore, in order that the signal is received at its emission frequency by a second observer whose 4 velocity is v^a we require $\omega_{obs} = \omega_{em}$, and hence

$$k_a u^a\big|_{t_{obs}} = k_a v^a\big|_{t_{em}}. \quad (11)$$

We again consider an FRW universe with the observer at the origin of the co-moving coordinates, $u^a=(c^{-1},0,0,0)$, and a radial light signal, $k^a=(a^{-1},-ca^{-2},0,0)$. Equation (11), with the normalization condition $v^a v_a=-1$, may be solved for the radial 4 vector $v^a=(\dot{t},\dot{\chi},0,0)$, where the dot denotes differentiation with respect to an affine parameter. We thus obtain the following equation for the gradient of the zero-redshift worldline in comoving coordinates;

$$\frac{d\chi}{dt}=\frac{v^\chi}{v^t}=\frac{c}{a_{\rm em}}\left(\frac{a_{\rm em}^2-a_{\rm obs}^2}{a_{\rm em}^2+a_{\rm obs}^2}\right). \tag{12}$$

For spacetimes in which the Friedmann equation (2) may be solved analytically for $a(t)$, we can integrate back along the light ray

$$\chi_{\rm em}=c\int_{t_{\rm em}}^{t_{\rm obs}}\frac{dt}{a(t)}, \tag{13}$$

and write $t_{\rm obs}=t_{\rm obs}(t_{\rm em},\chi_{\rm em})$. We may then write Eq. (12) as an ordinary differential equation in $(t_{\rm em},\chi_{\rm em})$ for the worldline of the zero-redshifted observer. This differential equation for the Milne case is derived in Appendix A.

IV. DISTANCE MEASURES IN FRW COSMOLOGY

When dealing with relativistic systems, it is often useful to think in terms of invariant quantities rather than those that depend on some particular observer, coordinate system, or frame of reference. The distance $D=a(t)\chi$ as defined in Sec. II is a logical choice when working with the comoving coordinates (t,χ). It is used widely in cosmology and is easily defined in terms of the FRW metric, Eq. (1), and physically well motivated because it relates to surfaces of homogeneity. The following argument also gives a strong indication that it relates to an observable quantity.

Weinberg[4] appealed to a chain of comoving galaxies lying close together on the line of sight between ourselves and a distant galaxy in an FRW universe. If we synchronize clocks to the cosmic time t, observers in each galaxy measure the distance to their neighboring galaxies. The comoving distance is the limit of the sum of these distances,

$$D(t)=\int_0^{\chi_1}\sqrt{g_{\chi\chi}}\,d\chi=\int_0^{\chi_1}a(t)d\chi=\chi_1 a(t). \tag{14}$$

Rindler[1] has suggested that we might view this theoretical measurement as the realization of many tiny rulers laid down at some instant in cosmic time. However, we should not allow ourselves to be carried away by this image. In Newtonian physics, the idea of laying down a ruler between a pair of points in space makes perfect sense, regardless of whether the scales involved make it physically practical. In general relativity, however, we cannot typically define such an idealized rigid body to act as a ruler (see for example Ref. 5); neither could we hope to obtain the infinite number of observers needed to obtain the limit in Weinberg's gedanken experiment. We suggest that the co-moving distance should be regarded as a measure motivated by the coordinate choice rather than an observable quantity.

Consider now the tethered galaxy problem in the Milne universe. This spacetime is the empty case of FRW with zero cosmological constant $(\Omega_M=\Omega_\Lambda=0)$ for which $a(t)=H_0t/t_0 \equiv A_0t$. The coordinate transformation

$$(T,X)=\left(ct\cosh\left(\frac{A_0\chi}{c}\right),ct\sinh\left(\frac{A_0\chi}{c}\right)\right) \tag{15}$$

reduces the line element to that of $(1+1)$ Minkowski space, $ds^2=-dT^2+dX^2$. In these coordinates, geodesics take the form of straight lines and special relativity holds at all scales. We would expect to define a tethered galaxy here as moving along one of the timelike geodesics $X=$constant. Light received from such a galaxy would be observed with zero redshift. However, the family of observers defined by $D'=0$ follow trajectories

$$A_0t\chi=(T^2-X^2)^{1/2}\tanh^{-1}\left(\frac{X}{T}\right)=\text{constant}. \tag{16}$$

Light from such observers is not received with zero redshift, and the tethering condition does not correspond to the fixing of spatial separation that we would expect from special relativity.

Another possible choice of distance scale would be to measure metric distance along spatial geodesics. Because our aim is to obtain a relation with the redshift of a light signal, we choose a distance measure that is calculated from null geodesics in the spacetime. As discussed in Sec. V, the radar distance fulfils this criterion as well as reproducing the expected results for the tethering problem in the Milne case.

V. TETHERING WITH RADAR DISTANCE

The radar distance[1] is calculated by measuring the proper time between emission at $\tau_{\rm init}$ and observation at $\tau_{\rm obs}$ of a light signal reflected from a distant object

$$d_{\rm rad}=\frac{1}{2}c(\tau_{\rm obs}-\tau_{\rm init}). \tag{17}$$

For a co-moving observer $\dot{\chi}=0$ in an FRW universe, which is considered in the tethering problem, and cosmic time t is equivalent to proper time τ. We now proceed to investigate the dynamics of a galaxy tethered at a fixed radar distance in the comoving coordinates.

Following Jennison and McVittie[6] we consider a light signal sent from an observer at the spatial origin at time $t_{\rm init}$, received and immediately re-emitted by the second observer at $t_{\rm em}$. Our first observer then receives the reflected signal at $t_{\rm obs}$. The path of such a light ray in (t,χ) coordinates is illustrated in Fig. 1.

We integrate along the light path on the outward and return journey and obtain

$$\chi_{\rm em}=c\int_{t_{\rm init}}^{t_{\rm em}}\frac{dt}{a(t)}=c\int_{t_{\rm em}}^{t_{\rm obs}}\frac{dt}{a(t)}. \tag{18}$$

To tether the second observer, we require $ct_{\rm init}=ct_{\rm obs}-2d_{\rm rad}$, where $d_{\rm rad}$ is the constant radar distance. Equation (18) then provides a simultaneous equation in $t_{\rm em}$ that can be solved for given values of $t_{\rm obs}$ and $d_{\rm rad}$. We thus recover $\chi_{\rm em}$ as the solution to Eq. (18). For spacetimes in which Eq. (2) can be solved analytically we obtain a parametric expression for (χ,t) along the tethered worldline. Otherwise, this worldline can be approximated by interpolating the numerical solutions $\chi(t_{\rm obs},d_{\rm rad}),t(t_{\rm obs},d_{\rm rad})$ for fixed $d_{\rm rad}$ and variable $t_{\rm obs}$.

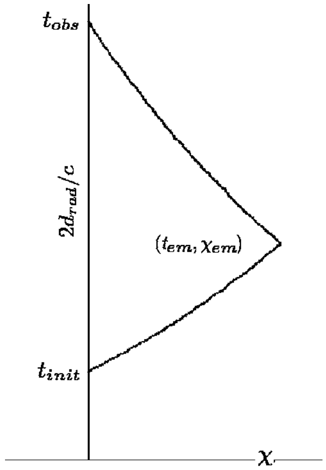

Fig. 1. Worldline of the light signal that characterizes an object's radar distance.

VI. QUANTIFYING THE EFFECT OVER INTERGALACTIC SCALES

We now have methods to compute the worldlines of zero-redshift observers and those tethered by comoving and radar distance. The next step is to address whether these worldlines are fundamentally different and how such differences manifest themselves on cosmic scales.

Our assumption in Sec. IV for preferring a radar distance tether was that it coincides with the zero-redshift condition for the Milne model whereas a co-moving tether does not. In general, however, all three conditions produce distinct worldlines. A simple example of this difference is the matter dominated universe, $a(t) \propto t^{2/3}$. We plot this case, along with the Milne model, in Fig. 2 to show the qualitative differences between the worldline trajectories.

Having established that there is a phenomenological difference, we now quantify this difference for the case of galaxies at typical astronomical distances tethered by the radar distance. We consider a flat universe dominated by the cosmological constant for which $a(t)=e^{H_0(t-t_0)}$ and an observer who receives a signal from a radially tethered galaxy at the current time, $t_{\text{obs}}=t_0$. Hence $a_{\text{obs}}=a_0=1$ and if we set $\beta \equiv a_{\text{em}}/a_{\text{obs}}$, we deduce from Eq. (18) that

$$a_{\text{em}} = \beta a_{\text{obs}} = \beta = \frac{c}{\chi_{\text{em}} H_0 + c} \tag{19}$$

in terms of the co-moving coordinate at emission, χ_{em}. We then follow the method of Sec. V to obtain the worldline $(t_{\text{em}}, \chi_{\text{em}})$ and hence the velocity

$$\frac{\dot{\chi}}{\dot{t}} = \frac{d\chi}{dt} = \frac{c(\beta - 1)}{\beta}. \tag{20}$$

If we solve the normalization condition $v^a v_a = -1$ for the 4 velocity of the galaxy $v^a = (\dot{t}, \dot{\chi}, 0, 0)$, we find

$$\dot{t} = \frac{1}{c}(\beta(2-\beta))^{-1/2}. \tag{21}$$

Finally we combine Eq. (21) with Eq. (10) and express the observed redshift as a function of the co-moving separation at emission,

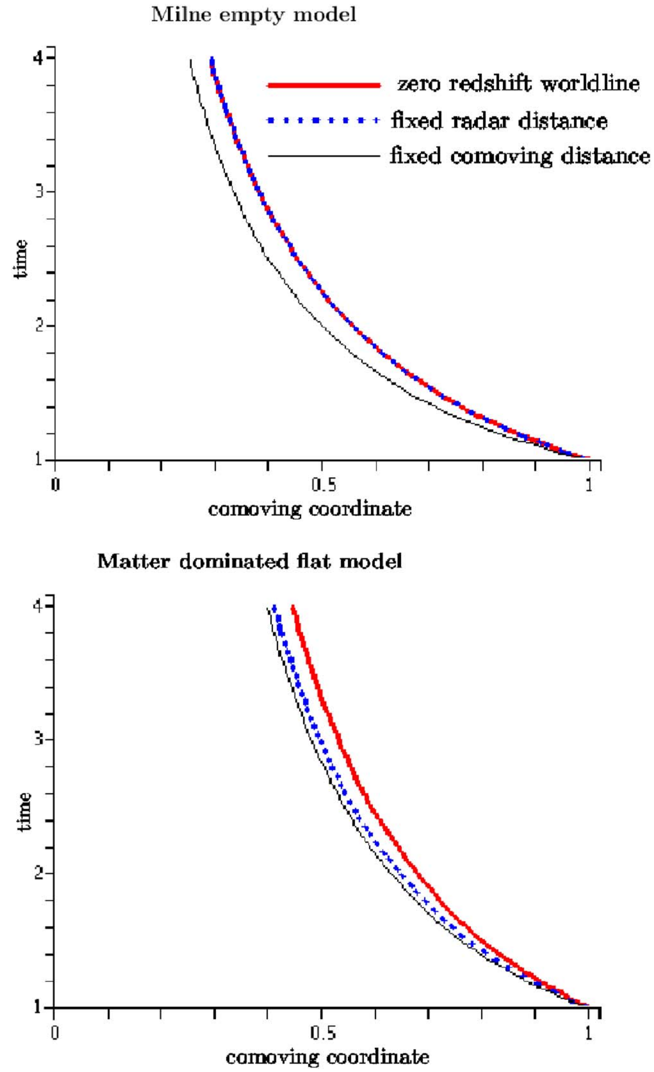

Fig. 2. Worldlines demonstrating the phenomenological differences between the zero-redshift condition and the two notions of tethering. The cosmological parameters c, H_0, and t_0 are set to unity. In the Milne model, the zero redshift and fixed radar distance worldlines coincide.

$$\frac{\omega_{\text{em}}}{\omega_{\text{obs}}} = \frac{c a_{\text{obs}} \dot{t} + a_{\text{em}} a_{\text{obs}} \dot{\chi}}{a_{\text{em}}}, \tag{22}$$

$$z = \frac{\omega_{\text{em}}}{\omega_{\text{obs}}} - 1 = \frac{1}{\sqrt{\left(\frac{c}{H_0 \chi_{\text{em}}} + c\right)\left(2 - \frac{c}{H_0 \chi_{\text{em}}} - c\right)}}. \tag{23}$$

In Fig. 3, we plot this redshift compared to that of a particle following the Hubble flow at the same emission point. We see that the redshift difference is comparatively small below an initial separation of around 10^4 Mpc. We also note that the difference between the tethering condition and zero redshift (horizontal axis) becomes apparent at this scale. This behavior is typical of the general case where we include suitable values for the cosmological parameters. In particular this occurs for the FRW spacetime satisfying $\Omega_M=0.3$, $\Omega_\Lambda=0.7$, which closely agrees with current observations for the matter content of our universe (see, for example, Ref. 7). Note that in Fig. 3 the co-moving tether is coincident with the radar distance tether. This is a feature of the $\Omega_M=0$,

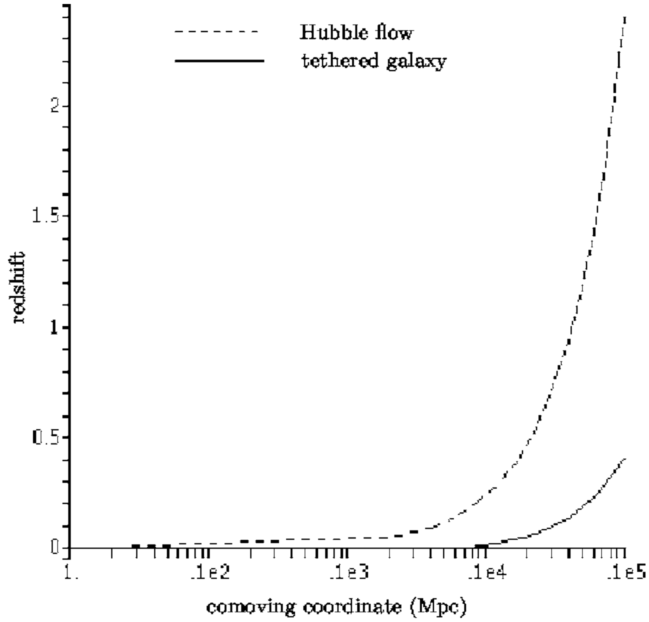

Fig. 3. Redshift of a tethered galaxy compared to that of a galaxy following the Hubble flow in a universe dominated by the cosmological constant. Initial separations range from 1 Mpc to 10^5 Mpc; χ is plotted logarithmically.

$\Omega_\Lambda=1$ spacetime and is not generic to FRW spacetimes. For example, in the Milne case, tethered particles starting together at $t_0=10^4$ Mpc would differ in their comoving position by around 16% after 5 Gyr.

VII. SUMMARY

We have recast the tethered galaxy problem in terms of the radar distance and a redshift derived from the 4 velocity. As a consequence, we now obtain a result for the empty FRW spacetime that agrees with special relativity. Because all of our results are expressed in terms of velocities on the galactic worldlines, we can directly relate these to the redshift effect. We show that the act of tethering a galaxy against the expansion of an FRW universe results in a greatly reduced redshift compared to a coincident comoving source. This effect is only apparent at large astronomical scales and thus is unlikely to influence nearby systems for which we might expect to observe peculiar velocities comparable to those resulting from the Hubble flow. We should note that this formulation of the tethered galaxy problem does not predict the blueshift of tethered sources in a expanding FRW universe.

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APPENDIX A: ANALYTICAL EXAMPLES

The following calculations show that the conditions for the Milne universe for an observer to move at fixed radar distance from the origin and to emit light with zero redshift are equivalent. This conclusion agrees with the argument of Sec. II, where we observed that the Milne universe may be transformed into a subset of Minkowski space in which the equivalence can be seen easily. We then recover the results of Sec. III in Ref. 2. This calculation may be compared with Appendix C of Ref. 2, where the original assertion that tethered galaxies could appear significantly blueshifted arises.

1. Fixed radar distance in the Milne universe

As in Sec. IV, we consider the empty FRW spacetime in which the scale factor may be written $a(t)=H_0t/t_0=A_0t$. A photon is emitted at t_{init}, reflected by the tethered galaxy at t_{em}, and observed at t_{obs}. Along the null geodesics $ds^2=0$ we characterize the path of a photon by

$$\chi = c\int_{t_{\text{init}}}^{t_{\text{em}}} \frac{dt}{a(t)} = c\int_{t_{\text{em}}}^{t_{\text{obs}}} \frac{dt}{a(t)}. \tag{A1}$$

If we require the radar distance to remain fixed for all times, then

$$c(t_{\text{obs}} - t_{\text{init}}) = 2d. \tag{A2}$$

We combine Eqs. (A1) and (A2) and obtain the relation

$$t_{\text{em}}^2 = t_{\text{obs}}\left(t_{\text{obs}} - \frac{2d}{c}\right), \tag{A3}$$

which we substitute into the right-hand side of Eq. (A1),

$$\chi = \frac{c}{A_0}[\ln(t_{\text{obs}}) - \ln(t_{\text{em}})]. \tag{A4}$$

If we solve Eq. (A4) for $t_{\text{obs}}=e^{\chi A_0/c}t_{\text{em}}$ and combine it with Eq. (A3), we find an expression $t(\chi)$ for the worldline for the galaxy tethered by a constant proper distance,

$$t_{\text{em}} = \frac{2d}{\sinh(A_0\chi/c)}. \tag{A5}$$

The form of Eq. (A5) is expected from Eq. (15) where we hold X constant and set $2d=X$.

2. Zero redshift in the Milne universe

Equation (12) tells us that

$$\frac{d\chi}{dt} = \frac{c}{A_0t_{\text{em}}}\left[\frac{(A_0t_{\text{em}})^2 - (A_0t_{\text{obs}})^2}{(A_0t_{\text{em}})^2 + (A_0t_{\text{obs}})^2}\right]. \tag{A6}$$

Equation (13) then gives

$$\chi = c\int_{t_{\text{em}}}^{t_{\text{obs}}} \frac{dt}{a(t)} = \frac{c}{A_0}[\ln(t_{\text{obs}}) - \ln(t_{\text{em}})], \tag{A7}$$

and hence

$$t_{\text{obs}} = t_{\text{em}}e^{A_0\chi/c}. \tag{A8}$$

If we combine these two relations, we find

$$\frac{d\chi}{dt} = \frac{c}{A_0t_{\text{em}}}\frac{1 - e^{2A_0\chi/c}}{1 + e^{2A_0\chi/c}} = \frac{c\tanh(A_0\chi/c)}{A_0t_{\text{em}}}, \tag{A9}$$

which agrees with what we find by differentiating Eq. (A5).

3. Analytic forms of $v_{\rm teth}=f(v_{\rm pec})$ for simple FRW spacetimes

From Sec. II, we consider $v_{\rm teth}=-a'\chi=A_0\chi$. We again use Eq. (A4) and see that

$$t_{\rm em}=t_{\rm obs}e^{v_{\rm teth}/c}. \tag{A10}$$

If we substitute Eq. (A10) into Eq. (9), we obtain the relation

$$v_{\rm pec}=c\frac{1-e^{v_{\rm teth}/c}}{1+e^{v_{\rm teth}/c}}=-c\tanh\left(\frac{v_{\rm teth}}{c}\right). \tag{A11}$$

The results for the flat matter-dominated and cosmological constant-dominated universes can be recovered by the same technique:

$$v_{\rm pec}=\frac{1-(1-v_{\rm teth}/c)^2}{1-(1+v_{\rm teth}/c)^2}. \tag{A12}$$

and

$$v_{\rm pec}=-\frac{16-(2+v_{\rm teth}/c)^4}{16+(2+v_{\rm teth}/c)^4}. \tag{A13}$$

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[1]W. Rindler, *Relativity: Special, General and Cosmological* (Oxford University Press, Oxford, 2001).
[2]T. M. Davis, C. H. Lineweaver, and J. K. Webb, “Solutions to the tethered galaxy problem in an expanding universe and the observation of receding blueshifted objects,” Am. J. Phys. **71**, 358–364 (2003).
[3]R. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[4]S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[5]A. Trautman, “Foundations and current problems of general relativity,” in *Brandeis Summer Institute in Theoretical Physics*, edited by S. Deser and K. W. Ford (Prentice–Hall, Englewood Cliffs, N.J., 1964), Vol. 1.
[6]R. C. Jennison and G. C. McVittie, “Radar distance in Robertson Walker spacetimes,” Proc. R. Soc. Edinburgh **1**, 1–17 (1975).
[7]E. J. Copeland, M. Sami, and S. Tsujikawa, “Dynamics of dark energy,” arXiv:hep-th/0603057.

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