# Darwin-Lagrangian Analysis for the Interaction of a Point Charge and a Magnet: Considerations Related to the Controversy Regarding the Aharonov-Bohm and Aharonov-Casher Phase Shifts

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## Abstract

The classical electromagnetic interaction of a point charge and a magnet is discussed by first calculating the interaction of a point charge with a simple model magnetic moment and then suggesting a multiparticle limit. The Darwin Lagrangian is used to analyze the electromagnetic behavior of the model magnetic moment (composed of two oppositely charged particles of different mass in an initially circular Coulomb orbit) interacting with a passing point charge. Considerations of force, energy, momentum, and center of energy are treated through second order in $1/c$. The changing magnetic moment is found to put a force back on a passing charge; this force is of order $1/c^2$ and depends upon the magnitude of the magnetic moment. The limit of a many-particle magnet arranged as a toroid is discussed. It is suggested that in the multiparticle limit, the electric fields of the passing charge are screened out of the body of the magnet while the magnetic fields of the passing charge penetrate into the body of the magnet. This is consistent with our understanding of the penetration of electromagnetic velocity fields into ohmic conductors. The proposed multiparticle limit is consistent with the conservation laws for energy and momentum, as well as constant motion of the center of energy, and Newton's third law for the net Lorentz forces on the magnet and on the point charge. The work corresponds to a classical electromagnetic analysis of the interaction which is basic to understanding the controversy over the Aharonov-Bohm and Aharonov-Casher phase shifts and represents a refutation of the suggestions of Aharonov, Pearle, and Vaidman.

## I. INTRODUCTION

The interaction of a point charge and a magnet is a complicated and controversial problem of electromagnetism. The problem is ignored by the classical physics textbooks and is discussed in the research literature in connection with the Shockley-James paradox,[1] and in connection with the Aharonov-Bohm[2] and Aharonov-Casher[3] phase shifts for particles. The problem in understanding arises because the interaction involves relativistic terms of order $1/c^2$ (where $c$ is the speed of light in vacuum) which are not nearly so familiar as nonrelativistic mechanics. Writing regarding the interaction of a point charge and a magnet in 1968, Coleman and Van Vleck remarked in an oft-cited article,[4] "Unfortunately, the equations which we have obtained are singularly resistant to a simple physical interpretation in terms of particles exchanging forces; ..." However, despite the complications and in line with the controversy, the problem is an important one which reflects back on our understanding of classical electromagnetism and on the connections between classical and quantum physics.

## II. THE PROBLEM AND THE CONTROVERSY

There are no electric or magnetic fields outside a long neutral solenoid or toroid when the currents are maintained constant. Therefore when a charged particle passes a long solenoid or a toroid, there are no electric or magnetic fields at the position of the passing charge due to the *unperturbed* charge and current densities of the magnet. On the other hand, there are clearly electric and magnetic fields due to the passing charge at the position of the magnet. The electric fields of the passing charge will cause accelerations of the charges which carry the currents which create the flux of the magnet. Also, the magnetic fields of the passing charge will cause a net Lorentz force on the magnet. Thus far the description would be approved by all physicists. However, the response of the multiparticle magnet seems so complicated that no one has calculated the magnet's response in detail.

Since it does not seem possible at present to carry out a complete multiparticle calculation starting from accepted theory, we are left with suggestive partial calculations and hence with competing points of view depending upon which aspects of the partial calculations are favored. At present, there are two competing interpretations for the behavior of a magnet and a passing point charge.

#### 1. *The No-Velocity-Change Point of View*

The supporters of the quantum topological view[2][3][5][6][7] of the Aharonov-Bohm phase shift claim that there are no velocity changes for the interacting charged particle or the magnet. Indeed, the supporters of this view say that there are no significant changes in the charge or current densities in the magnet. Therefore the passing charge never experiences a Lorentz force and never changes velocity. Furthermore, although the magnet does indeed experience a net Lorentz force due to the magnetic field of the passing charge, nevertheless the electric field of the passing charge penetrates into the magnet giving a "hidden momentum in magnets" whose change "cancels" the net magnetic Lorentz force on the magnet so that the center of energy of the magnet is never disturbed. In this point of view, the electromagnetic fields of the passing charge may cause confusion behind the scenes inside the magnet, but there is no change in the magnet's center of energy and there is no feedback signal sent to the passing charge which is causing the confusion in the magnet.[8]

#### 2. *The Classical-Lag Point of View*

The classical-lag point of view[9][10][11][12][13][14][15][16] takes a totally different perspective on the changes in the charge and current densities induced in the magnet. In this view, the induced densities lead to a Lorentz force back on the passing charge which is equal in magnitude and opposite in direction to the net magnetic Lorentz force which the magnetic field of the passing charge places on the magnet. The electric charges on the surface of the magnet screen the electric field of the passing charge out from the interior of the magnet, and therefore there is no significant change in the momentum of the electromagnetic fields. On the other hand, the magnetic field of the passing charge penetrates into the magnet, and it is the magnetic energy change associated with the overlapping magnetic fields which gives the magnitude of the energy change of the passing charge due to the back force. This view fits with what we know of the penetration of electric and magnetic velocity fields into ohmic conductors. In this scenario, we have explicit ideas concerning conservation of energy, linear momentum, and constant motion of the center of energy. We also have the validity of

Newton's third law for the net Lorentz forces between the magnet and the passing charge.

Both points of view predict the Aharonov-Bohm and Aharonov-Casher phase shifts. The no-velocity-change point of view claims that, in the light of their interpretation, the phase shifts represent completely new quantum topological effects occurring in the absence of classical forces, and there are no classical analogues. The classical-lag point of view claims that the phase shifts present classical velocity shifts analogous to those occurring when only one beam of light passes through a piece of glass before two coherent beams interfere. The conflict between the two points of view has existed for thirty years without ever being put to experimental test to determine whether or not there are velocity changes for the electrons passing through a toroid or past a long solenoid. The no-velocity-change point of view has been widely accepted because most physicists do not think of the possibility of induced charge and current densities in magnets; they do consider induced charge densities only in electrostatic situations. Furthermore, the proponents of the no-velocity-change point of view have declared that the lag point of view is impossible because i) the electromagnetic fields of the passing charge would not penetrate into a conductor surrounding a toroid or solenoid, and ii) the back electric field at the passing charge could not be of order $1/c^2$ and proportional to the magnetic flux of the magnet. The objection i) has been shown to be groundless.[13] Magnetic velocity fields do indeed penetrate into good conductors in exactly the required form which is completely different from the exponential skin-depth form taken by electromagnetic wave fields.[17]

The objection ii) is addressed in the present article. In 1968 Coleman and Van Vleck[4] discussed the interaction of a stationary point charge and a magnet using the Darwin Lagrangian. We will be following their approach in the following analysis. We will discuss the interaction of a passing point charge and a magnetic moment where the magnetic moment is modeled as a classical hydrogen atom and where the electromagnetic interactions are carried to order $1/c^2$ by using the Darwin Lagrangian. This is a well-defined classical electromagnetic system which is relativistic through order $1/c^2$. In order to separate out the electrostatic effects (which are independent of the magnetic moment) from magnetic effects dependent upon the magnetic moment, we will sometimes average over atoms and anti-atoms with the same magnetic moment. We will describe the motion and check all the conservation laws. We will find that in this case the induced currents are important and that there are electric Lorentz forces back on the passing charge which indeed are of order

$1/c^2$ and are proportional to the magnetic moment. There is also a displacement of the center of energy of the magnetic moment. This behavior contradicts the suggestions of the proponents of the no-velocity-change point of view.[5][6] Next we will discuss the passage to the limit of a multiparticle magnet. Finally, in this multiparticle limit, we discuss the conservation-law aspects which are mentioned above.

## III. THE DARWIN LAGRANGIAN AND ELECTROMAGNETIC FIELDS

The Darwin Lagrangian for particles of charge $e_a$, mass $m_a$, displacement $\mathbf{r}_a$, and velocity $\mathbf{v}_a$ is given by[4][18]

$$L = \sum_a \left( \frac{1}{2} m_a \mathbf{v}_a^2 + \frac{1}{8c^2} m_a \mathbf{v}_a^4 \right) - \frac{1}{2} \sum_a \sum_{b \neq a} \frac{e_a e_b}{r_{ab}} + \frac{1}{2} \sum_a \sum_{b \neq a} \frac{e_a e_b}{2c^2 r_{ab}} \left[ \mathbf{v}_a \cdot \mathbf{v}_b + \frac{(\mathbf{v}_a \cdot \mathbf{r}_{ab})(\mathbf{v}_b \cdot \mathbf{r}_{ab})}{r_{ab}^2} \right] \tag{1}$$

where $\mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b$ and $r_{ab} = |\mathbf{r}_a - \mathbf{r}_b|$. Lagrange's equations of motion give a canonical momentum

$$\mathbf{p}_a^{canonical} = \frac{\partial L}{\partial \mathbf{v}_a} = m_a \mathbf{v}_a \left( 1 + \frac{\mathbf{v}_a^2}{2c^2} \right) + \sum_{b \neq a} \frac{e_a e_b}{2c^2 r_{ab}} \left[ \mathbf{v}_b + \frac{\mathbf{r}_{ab}(\mathbf{r}_{ab} \cdot \mathbf{v}_b)}{r_{ab}^2} \right] \tag{2}$$

and a time derivative

$$\frac{d}{dt} \mathbf{p}_a^{canonical} = \frac{\partial L}{\partial \mathbf{r}_a} = \sum_{b \neq a} \frac{e_a e_b \mathbf{r}_{ab}}{2c^2 r_{ab}^3} - \sum_{b \neq a} \frac{e_a e_b \mathbf{r}_{ab}}{2c^2 r_{ab}^3} \left[ \mathbf{v}_a \cdot \mathbf{v}_b + \frac{3(\mathbf{v}_a \cdot \mathbf{r}_{ab})(\mathbf{v}_b \cdot \mathbf{r}_{ab})}{r_{ab}^2} \right] + \sum_{b \neq a} \frac{e_a e_b}{2c^2 r_{ab}^3} \left[ \mathbf{v}_a (\mathbf{v}_b \cdot \mathbf{r}_{ab}) + \mathbf{v}_b (\mathbf{v}_a \cdot \mathbf{r}_{ab}) \right] \tag{3}$$

The Darwin Lagrangian accurately reflects the classical electromagnetic interaction of charged particles through order $1/c^2$. To lowest order in $1/c^2$, the interaction among the charges is given by the Coulomb force and the nonrelativistic form of Newton's second law $\mathbf{F} = m\mathbf{a}$. This 0-order behavior can then be inserted back into the equations of motion to allow calculation of the higher-order corrections.

It is sometimes revealing to rewrite the Lagrangian equations of motion in terms of the mechanical momentum

$$\mathbf{p}_a = m_a \mathbf{v}_a [1 + \mathbf{v}_a^2/(2c^2)] \tag{4}$$

Then Newton's second law

$$d\mathbf{p}_a/dt = \frac{d}{dt}\{m_a\mathbf{v}_a[1+\mathbf{v}_a^2/(2c^2)]\} = e_a\mathbf{E}(\mathbf{r}_a,t)+e_a(\mathbf{v}_a/c)\times\mathbf{B}(\mathbf{r}_a,t) \tag{5}$$

is obtained by carrying out the time derivative in the Darwin equations of motion (3) and recognizing the electric field as[19]

$$\mathbf{E}(\mathbf{r}_a,t) = \sum_{b\neq a}\left\{\frac{e_b\mathbf{r}_{ab}}{r_{ab}^3}\left[1+\frac{1}{2}\frac{\mathbf{v}_b^2}{c^2}-\frac{3}{2}\frac{(\mathbf{v}_b\cdot\mathbf{r}_{ab})^2}{c^2r_{ab}^2}\right]-\frac{e_b}{2c^2r_{ab}}\left[\mathbf{a}_b+\frac{\mathbf{r}_{ab}(\mathbf{r}_{ab}\cdot\mathbf{a}_b)}{r_{ab}^2}\right]\right\} \tag{6}$$

and the magnetic field as

$$\mathbf{B}(\mathbf{r}_a,t) = \sum_{b\neq a}\frac{e_b}{c}\frac{\mathbf{v}_b\times\mathbf{r}_{ab}}{r_{ab}^3} \tag{7}$$

where $\mathbf{a}_b$ is the acceleration of particle $b$. In Eq. (6), the terms of order $1/c^2$ provide the familiar effects of Faraday induction. We can also write the electromagnetic fields in terms of electromagnetic potentials as

$$\mathbf{E}(\mathbf{r}_a,t) = -\nabla_a\Phi(\mathbf{r}_a,t)-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}_a,t) \quad\text{and}\quad \mathbf{B}(\mathbf{r}_a,t)=\nabla_a\times\mathbf{A}(\mathbf{r}_a,t) \tag{8}$$

where[20]

$$\Phi(\mathbf{r}_a,t)=\sum_{b\neq a}\frac{e_b}{r_{ab}} \quad\text{and}\quad \mathbf{A}(\mathbf{r}_a,t)=\sum_{b\neq a}\frac{e_b}{2cr_{ab}}\left[\mathbf{v}_b+\frac{\mathbf{r}_{ab}(\mathbf{r}_{ab}\cdot\mathbf{v}_b)}{r_{ab}^2}\right] \tag{9}$$

We recognize from Eq. (2) and Eq. (9) that

$$\mathbf{p}_a^{canonical} = m_a\mathbf{v}_a[1+\mathbf{v}_a^2/(2c^2)]+(e_a/c)\mathbf{A}(\mathbf{r}_a,t) \tag{10}$$

where $\mathbf{A}(\mathbf{r}_a,t)$ is the vector potential due to all the other charges evaluated at the position $\mathbf{r}_a$ of the charge $e_a$.

## IV. TWO-PARTICLE MODEL FOR A MAGNETIC MOMENT

Our model for a magnetic moment will consist of two charge particles of different mass in Coulomb orbit around each other (a classical hydrogen atom). There is no electromagnetic radiation in the Darwin Lagrangian, and thus the orbiting charges do not lose energy in this $1/c^2$ approximation. Furthermore, for our model, we will average over the phases of orbital motion and also average over the configurations where the both the charges and the velocities of the charges are reversed in sign. In this fashion one maintains the magnetic moment behavior while averaging out the irrelevant electrostatic aspects.

In this article, the motion of the magnetic moment charges is considered extensively. Therefore, for simplicity of notation (and in contrast to the notation of Coleman and Van Vleck), the magnetic moment consists of a particle of charge $e$, small mass $m$, displacement $\mathbf{r}$, velocity $\mathbf{v}$, and acceleration $\mathbf{a}$ in orbit around a massive particle of charge $-e$, mass $M$ (with $M >> m$), displacement $\mathbf{R} \cong m\mathbf{r}/M \cong 0$, velocity $\mathbf{V} \cong m\mathbf{v}/M$, and acceleration $d\mathbf{V}/dt$. Since the mass $M$ is large compared to $m$, the displacement $\mathbf{R}$, velocity $\mathbf{V}$, and acceleration $d\mathbf{V}/dt$ are all small compared to $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ respectively. The distant point charge with which the magnetic moment interacts has charge $q$, mass $m_q$, displacement $\mathbf{r}_q$, velocity $\mathbf{v}_q$, and acceleration $d\mathbf{v}_q/dt$. Then from equations (4)-(7), our equations of motion for the charge $e$ in orbit, the massive particle $-e$, and the distant charge $q$ are respectively

$$\begin{aligned} \frac{d}{dt}\left[m\mathbf{v}\left(1+\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)\right] &= e\mathbf{E}_{-e}(\mathbf{r},t) + e\mathbf{E}_q(\mathbf{r},t) + e\frac{\mathbf{v}}{c}\times\mathbf{B}_q(\mathbf{r},t) \\ &= -\frac{e^2\mathbf{r}}{r^3} + \frac{eq\mathbf{r}_{eq}}{r_{eq}^3}\left[1+\frac{1}{2}\frac{\mathbf{v}_q^2}{c^2} - \frac{3}{2}\frac{(\mathbf{v}_q\cdot\mathbf{r}_{eq})^2}{c^2 r_{eq}^2}\right] + e\frac{\mathbf{v}}{c}\times\left(\frac{q}{c}\frac{\mathbf{v}_q\times\mathbf{r}_{eq}}{r_{eq}^3}\right) \end{aligned} \tag{11}$$

$$\begin{aligned} \frac{d}{dt}(M\mathbf{V}) &= -e\mathbf{E}_e(0,t) - e\mathbf{E}_q(0,t) \\ &= \frac{e^2\mathbf{r}}{r^3}\left(1+\frac{1}{2}\frac{\mathbf{v}^2}{c^2} - \frac{3}{2}\frac{(\mathbf{v}\cdot\mathbf{r})^2}{c^2r^2}\right) + \frac{e^2}{2c^2r}\left(\mathbf{a}+\frac{(\mathbf{a}\cdot\mathbf{r})\mathbf{r}}{r^2}\right) \\ &\quad - \frac{eq\mathbf{r}_q}{r_q^3}\left[1+\frac{1}{2}\frac{\mathbf{v}_q^2}{c^2} - \frac{3}{2}\frac{(\mathbf{v}_q\cdot\mathbf{r}_q)^2}{c^2r_q^2}\right] \end{aligned} \tag{12}$$

and

$$\begin{aligned} \frac{d}{dt}\left[m_q\mathbf{v}_q\left(1+\frac{1}{2}\frac{\mathbf{v}_q^2}{c^2}\right)\right] &= q\mathbf{E}_{-e}(\mathbf{r}_q,t) + q\mathbf{E}_e(\mathbf{r}_q,t) + q\frac{\mathbf{v}_q}{c}\times\mathbf{B}_e(\mathbf{r}_q,t) \\ &= q\frac{-e\mathbf{r}_q}{r_q^3} + q\frac{e\mathbf{r}_{qe}}{r_{qe}^3}\left(1+\frac{1}{2}\frac{\mathbf{v}^2}{c^2} - \frac{3}{2}\frac{(\mathbf{v}\cdot\mathbf{r}_{qe})^2}{c^2r_{qe}^2}\right) \\ &\quad - q\frac{e}{2c^2r_{qe}}\left(\mathbf{a}+\frac{(\mathbf{a}\cdot\mathbf{r}_{qe})\mathbf{r}_{qe}}{r_{qe}^2}\right) + q\frac{\mathbf{v}_q}{c}\times\left(\frac{e}{c}\frac{\mathbf{v}\times\mathbf{r}_{qe}}{r_{qe}^3}\right) \end{aligned} \tag{13}$$

where $\mathbf{r}_{qe} = \mathbf{r}_q - \mathbf{r} = -\mathbf{r}_{eq}$, and we have assumed that $\mathbf{V}^2/c^2 << 1$.

### A. Nonrelativistic Interaction

In order to understand the interaction represented by these equations of motion (11)-(13), we consider first the nonrelativistic approximation 0-order in $1/c^2$ where the equations become

$$m\mathbf{a} = -\frac{e^2\mathbf{r}}{r^3} + e\mathbf{E}_q^{(0)}(\mathbf{r}, t) \tag{14}$$

$$M\frac{d\mathbf{V}}{dt} = \frac{e^2\mathbf{r}}{r^3} - e\mathbf{E}_q^{(0)}(0, t) \tag{15}$$

and

$$m_q\frac{d\mathbf{v}_q}{dt} = q\frac{-e\mathbf{r}_q}{r_q^3} + q\frac{e\mathbf{r}_{qe}}{r_{qe}^3} \tag{16}$$

Here the small electrostatic field of the charge $q$ is essentially uniform across the magnetic moment

$$\mathbf{E}_q^{(0)}(\mathbf{r}, t) = \frac{q(\mathbf{r} - \mathbf{r}_q)}{|\mathbf{r} - \mathbf{r}_q|^3} \cong -\frac{q\mathbf{r}_q}{r_q^3} = \mathbf{E}_q^{(0)}(0, t) \tag{17}$$

since the charge $q$ is distant from the magnetic moment at the origin of coordinates, $r/r_q << 1$. The electrostatic field at the charge $q$ appearing on the right-hand side in Eq. (16) is an electric dipole field and is even smaller (for $q$ and $e$ of the same magnitude) because the magnetic moment is electrically neutral.

In this nonrelativistic approximation, the interaction of the distant point charge $q$ with this magnetic moment depends crucially upon the orientation of the magnetic moment. i)If the magnetic moment $\overrightarrow{\mu}$ at the origin is aligned parallel to the displacement $\mathbf{r}_q$ to the point charge, $\overrightarrow{\mu}||\mathbf{r}_q$, we find the stable electrostatic polarizability aspect. ii)If the magnetic moment $\overrightarrow{\mu}$ is aligned perpendicular to the displacement $\mathbf{r}_q$ to the point charge, $\overrightarrow{\mu}\perp\mathbf{r}_q$, then we find Solem's[21] unstable "strange polarizability" aspect. It is the second, unfamiliar aspect which is crucial for understanding the electric forces which are proportional to the magnetic moment.

#### 1. *Stable Electrostatic Polarizability*

If the distant charge $q$ lies along the axis perpendicular to the orbital motion and through its center, $\overrightarrow{\mu}||\mathbf{r}_q$, then the electric field $\mathbf{E}_q^{(0)}$ will cause a displacement $l$ of the orbital plane relative to the massive particle $M$. The equilibrium situation for the orbital motion with angular frequency $\omega$ corresponds to Newton's equations of motion in the radial and axial directions giving

$$m\omega^2 r = e^2 r\,(r^2 + l^2)^{-3/2} \qquad \text{and} \qquad eE_q^{(0)} = e^2 l\,(r^2 + l^2)^{-3/2} \tag{18}$$

Eliminating $r$ between the equations, we find $e^2/(m\omega^2)E_q^{(0)} = el = \mathfrak{p}$, where $\mathfrak{p}$ is the average electric dipole moment of the two-particle magnetic moment. Thus the magnetic moment in this orientation has an electrostatic polarizability

$$\alpha = e^2/(m\omega^2) \quad \text{where} \quad \overrightarrow{\mathfrak{p}} = \alpha \mathbf{E}_q^{(0)} \tag{19}$$

a form for $\alpha$ which is familiar for a dipole harmonic oscillator.[22] We notice that the polarizability is even in the charge $e$ and in the frequency $\omega$ and has no relation to the sign of the magnetic moment

$$\overrightarrow{\mu} = e\mathbf{L}/((2mc) = e\overrightarrow{\omega}r^2/(2c) \tag{20}$$

*2. Solem's Unstable "Strange" Polarization*

If the magnetic moment is oriented perpendicular to the displacement to the distant charge $q$, $\overrightarrow{\mu}\perp\mathbf{r}_q$, then we find behavior which is mentioned only rarely in the physics literature.[21] It does not appear in Coleman and Van Vleck's article,[4] but it is crucial to understanding the classical electromagnetic interactions associated with the Aharonov-Bohm and Aharonov-Casher phase shifts. In this case when the angular momentum $\mathbf{L}$ of the orbit for the magnetic moment is perpendicular to the electric field $\mathbf{E}_q^{(0)}$ of the distant charge $q, \overrightarrow{\mu}\perp\mathbf{r}_q$, the initial circular orbit is transformed into an elliptical orbit of ever-changing ellipticity with its semi-major axis oriented perpendicular to both the angular momentum $\mathbf{L}$ and the electric field $\mathbf{E}_q^{(0)}$.[21] In order to analyze this motion, it is useful to introduce the Laplace-Runge-Lenz vector $\mathbf{K}$ for the Coulomb orbit of the charge $e$.[23] We assume that the much larger mass $M$ is at the origin, $\mathbf{R} \cong 0$, so that the charge $e$ moves with a displacement[21]

$$\mathbf{r} = \frac{3}{2}\frac{\mathbf{K}}{(-2mH_0)^{1/2}} + \frac{1}{4H_0}\frac{d}{dt}[m(\mathbf{r}\times\mathbf{v})\times\mathbf{r} + m\mathbf{v}r^2] \tag{21}$$

where $\mathbf{K}$ is the Laplace-Runge-Lenz vector[23]

$$\mathbf{K} = \frac{1}{(-2mH_0)^{1/2}}\left([\mathbf{r}\times(m\mathbf{v})]\times(m\mathbf{v}) + me^2\frac{\mathbf{r}}{r}\right) \tag{22}$$

and $H_0$ is the particle energy

$$H_0 = mv^2/2 - e^2/r \tag{23}$$

The equation (21) can be checked by carrying out the time derivative and then inserting the equation of motion $\mathbf{a} = -e^2\mathbf{r}/(mr^3)$ for every appearance of the acceleration $\mathbf{a} = d\mathbf{v}/dt = d^2\mathbf{r}/dt^2$. The Laplace-Runge-Lenz vector is constant in time for a Coulomb orbit, and the second term of (21) involving a time derivative shows how the displacement $\mathbf{r}$ varies in time. On time-averaging, the time derivative vanishes leaving

$$<\mathbf{r}>= \frac{3}{2}\frac{\mathbf{K}}{(-2mH_0)^{1/2}} \tag{24}$$

The average electric dipole moment $\overrightarrow{\mathfrak{p}}$ is given by

$$\overrightarrow{\mathfrak{p}} = e<\mathbf{r}>= \frac{3}{2}\frac{e\mathbf{K}}{(-2mH_0)^{1/2}} \tag{25}$$

We assume that initially the magnetic moment has a circular orbit for the charge $e$, and therefore initially the electric dipole moment vanishes, $\overrightarrow{\mathfrak{p}} = e<\mathbf{r}>= 0$ and $\mathbf{K} = 0$. However, in the presence of the electric field $\mathbf{E}_q^{(0)}$ of the distant charge $q$, the equation of motion for $e$ is given in Eq. (14). We assume that the field $\mathbf{E}_q^{(0)}$ is small so that the orbit remains Coulombic but now with a slowly changing Laplace-Runge-Lenz vector. The time rate of change of $\mathbf{K}$ can be obtained by differentiating both sides of equation (22) and the use of the equation of motion (14),

$$\begin{aligned}\frac{d\mathbf{K}}{dt} &= \frac{1}{(-2mH_0)^{1/2}}m\left\{\left[\mathbf{r}\times\left(\frac{-e^2\mathbf{r}}{r^3}+e\mathbf{E}_q\right)\right]\times\mathbf{v}+(\mathbf{r}\times\mathbf{v})\times\left(\frac{-e^2\mathbf{r}}{r^3}+e\mathbf{E}_q\right)+e^2\left[\frac{\mathbf{v}}{r}-\frac{\mathbf{r}(\mathbf{r}\cdot\mathbf{v})}{r^3}\right]\right\}\\ &= (-2mH_0)^{-1/2}me[-2\mathbf{r}(\mathbf{v}\cdot\mathbf{E}_q^{(0)})+\mathbf{E}_q^{(0)}(\mathbf{r}\cdot\mathbf{v})+\mathbf{v}(\mathbf{r}\cdot\mathbf{E}_q^{(0)})]\end{aligned} \tag{26}$$

We note again that the Laplace-Runge-Lenz vector would be constant in time were it not for the external electric field $\mathbf{E}_q^{(0)}$. Since we assume that the distant charge $q$ is causing a small perturbation, we may average the particle displacement $\mathbf{r}$ and velocity $\mathbf{v}$ over an orbit of the unperturbed motion. Now if $f(\mathbf{r},\mathbf{v})$ is any function of the displacement and velocity of the unperturbed orbit, then it is a periodic function in time with period given by the orbital period $T$. Therefore, the time average of the time derivative vanishes

$$\left\langle\frac{d}{dt}f(\mathbf{r},\mathbf{v})\right\rangle = \frac{1}{T}\int_0^{t=T} dt\,\frac{d}{dt}f(\mathbf{r},\mathbf{v}) = 0$$

In particular for $f(\mathbf{r},\mathbf{v}) = x_i x_j$ where $x_i$ and $x_j$ are the $i$th and $j$th components of $\mathbf{r}$, then we have

$$\left\langle \frac{d}{dt}(x_i x_j) \right\rangle = \langle x_i v_j \rangle + \langle x_j v_i \rangle = 0 \tag{27}$$

so that

$$\begin{aligned} \langle \mathbf{r}\cdot\mathbf{v}\rangle &= 0 \\ \left\langle \mathbf{r}\left(\mathbf{v}\cdot\mathbf{E}_q^{(0)}\right)\right\rangle &= \left\langle -\mathbf{v}\left(\mathbf{r}\cdot\mathbf{E}_q^{(0)}\right)\right\rangle = -\left\langle (\mathbf{r}\times\mathbf{v})\times\mathbf{E}_q^{(0)}\right\rangle/2 \end{aligned} \tag{28}$$

This result allows us to average over the unperturbed motion to obtain

$$\begin{aligned} (-2mH_0)^{1/2} d\mathbf{K}/dt &= \left\langle me[-2\mathbf{r}(\mathbf{v}\cdot\mathbf{E}_q^{(0)}) + \mathbf{E}_q^{(0)}(\mathbf{r}\cdot\mathbf{v}) + \mathbf{v}(\mathbf{r}\cdot\mathbf{E}_q^{(0)})]\right\rangle \\ &= (3/2)me[\langle\mathbf{r}\times\mathbf{v}\rangle\times\mathbf{E}_q^{(0)}] = (3/2)e\mathbf{L}\times\mathbf{E}_q^{(0)} = 3m(c\overrightarrow{\mu})\times\mathbf{E}_q^{(0)} \end{aligned} \tag{29}$$

where $\mathbf{L}$ is the angular momentum of the orbit and $\overrightarrow{\mu} = e\mathbf{L}/(2mc)$. Thus from Eqs. (25) and (29), the electric dipole moment is changing as

$$\frac{d\overrightarrow{\mathfrak{p}}}{dt} = \frac{9}{4}\frac{e^2}{(-2mH_0)}\mathbf{L}\times\mathbf{E}_q^{(0)} \tag{30}$$

This is a very strange polarization indeed. The initially unpolarized orbit does indeed develop an electrical polarization with time, but the predominant electric dipole moment depends upon the orbital angular momentum and is in a direction perpendicular to the applied electric field $\mathbf{E}_q^{(0)}$. Since the angular momentum $\mathbf{L}$ is related to the magnetic moment as $\overrightarrow{\mu} = e\mathbf{L}/(2mc)$, we have the developing polarization related to the magnetic moment. However, if we average over both the orbital positions and over both signs $\pm e$ of charge while maintaining the direction of the magnetic moment $\overrightarrow{\mu} = e\overrightarrow{\omega}r^2/(2c)$, then we see that the time rate of change of the Laplace-Runge-Lenz vector $\mathbf{K}$ does not average to zero while the average rate of change of polarization $\left\langle d\overrightarrow{\mathfrak{p}}/dt\right\rangle$ actually vanishes, since the direction of angular momentum in Eq. (20) reverses as the sign of the charge $e$ is reversed. We also notice that the rate of change of the Laplace-Runge-Lenz vector and of the electric dipole moment for an individual orbit depends upon the value of the field $\mathbf{E}_q^{(0)}$ alone and is independent of any rate of change of the electric field $\mathbf{E}_q^{(0)}$. This is completely different from the electrical polarization $\overrightarrow{\mathfrak{p}}$ found from the electrostatic polarizability in Eq. (19) where there is no *change* in the polarization unless the field $\mathbf{E}_q^{(0)}$ changes in time.

There are additional observations which should be made regarding the behavior of the magnetic moment under the action of the electric field $\mathbf{E}_q^{(0)}$ of the distant point charge $q$. The sum of the particle kinetic energy plus electrostatic potential energy is conserved. Indeed, while the average displacement $<\mathbf{r}>$ of the charge $e$ is initially zero and increases in time, the length of the semimajor axis of the orbit does not change and is oriented in a direction perpendicular to the electric field $\mathbf{E}_q^{(0)}$; the work done by the electric field $\mathbf{E}_q^{(0)}$ on the orbiting charge $e$ vanishes when averaged over the Coulomb orbit. The average position of the heavier mass $M$ with charge $-e$ also shifts slightly so as to maintain the position of the center of (rest) mass of the magnetic moment system at the origin; since the average electrostatic force on the magnetic moment (due to the uniform electric field $\mathbf{E}_q^{(0)}$ of the point charge $q$) vanishes, the position of the center of (rest) mass does not change. As the orbiting system develops an electric dipole moment $\overrightarrow{\mathfrak{p}}$, there are balancing electrostatic forces and torques on the magnetic moment due to the point charge and on the point charge due to the magnetic moment. However, when we average over magnetic moments carrying opposite charges $\pm e$ but the same magnetic moment $\overrightarrow{\mu} = e\overrightarrow{\omega}r^2/(2c)$, all of the dipole-associated electrostatic forces and torques vanish in the average.

## B. Electromagnetic Forces on the Distant Point Charge

### 1. *Force Associated with the Stable Electrostatic Polarization*

Having obtained the behavior of the magnetic moment in the 0-order nonrelativistic system, we now wish to consider the electromagnetic forces $\mathbf{F}_{on\,q}=q\mathbf{E}_\mu + q(\mathbf{v}_q/c)\times\mathbf{B}_\mu$ acting on the distant point charge $q$ due to the magnetic moment $\overrightarrow{\mu}$. The forces are different depending upon the orientation of the magnetic moment. When the magnetic moment $\overrightarrow{\mu}$ is parallel to the displacement $\mathbf{r}_q$ to the distant charge $q$, $\overrightarrow{\mu}||\mathbf{r}_q$, then we saw in Eq. (19) that the magnetic moment has an induced electric dipole moment $\overrightarrow{\mathfrak{p}} = \alpha\mathbf{E}_q$. Accordingly, the electrically polarized magnetic moment creates an electrostatic dipole field $\mathbf{E}_{\mathfrak{p}}(\mathbf{r}_q, t)$ which causes an electrostatic force $\mathbf{F}_{on\ q}$ on $q$ given by

$$\mathbf{F}_{on\ q} = q\mathbf{E}_{\mathfrak{p}}(\mathbf{r}_q, t) = q\{2\overrightarrow{\mathfrak{p}}\}r_q^{-3} = q\{2[e^2/(m\omega^2)]\mathbf{E}_q(0,t)\}r_q^{-3} = -\mathbf{r}_q q^2 e^2/(m\omega^2 r_q^7) \qquad (31)$$

The electrostatic force back at the charge $q$ is independent of the sign of the charge $q$, or of the sign of the charge $e$, or of the direction of rotation $\omega$. When averaged over the orbital motion and over both signs of charge $\pm e$ for the magnetic moment, the only force on $q$ is this electrostatic dipole force. There is no additional force of order $1/c^2$. As an aside, we note that for this orientation of the magnetic moment, $\overrightarrow{\mu}||\mathbf{r}_q$, the magnetic vector potential $\mathbf{A}_\mu$ vanishes along the axis through the magnetic moment parallel to the magnetic moment direction.

### 2. *Force Associated with Solem's Unstable "Strange" Polarization*

The situation is completely different when the magnetic moment is oriented perpendicular to the displacement $\mathbf{r}_q$, $\overrightarrow{\mu}\perp\mathbf{r}_q$. In this case we saw that after carrying out the averaging for the magnetic moment, there were no electric monopole or dipole contributions to a force back on the point charge $q$. Since these 0-order back forces vanish, the back forces in order $1/c^2$ caused by the 0-order changes of the magnetic moment are of considerable interest. The alteration in the shape of the Coulomb orbit leads to unbalanced accelerations $\mathbf{a}$ which lead to new contributions to the electric field according to Eq. (6). The vector potential in the Coulomb gauge of a point charge $e$ is given in Eq. (9), and we see that the last term in Eq. (6) corresponds to the electric field contribution from $-\partial \langle \mathbf{A}_e \rangle /\partial t = -\partial \mathbf{A}_\mu/\partial t$. Now the magnetic moment model corresponds to a magnetic moment given initially by $\overrightarrow{\mu} = e\overrightarrow{\omega}r^2/(2c)$ in Eq. (20), and therefore to a vector potential

$$\mathbf{A}_\mu(\mathbf{r},t) = \frac{\overrightarrow{\mu}\times\mathbf{r}}{cr^3} = \frac{e}{2mc^2}\frac{\mathbf{L}\times\mathbf{r}}{r^3} \tag{32}$$

Thus for our magnetic moment model, the average electric field $\mathbf{E}_\mu$ back at the charged particle $q$ will be related to the change in the angular momentum $\mathbf{L}$ of the orbit. Now the change in angular momentum $\mathbf{L}$ of the orbit of the charge $e$ is due solely to the presence of the external charge $q$ which gives $d\mathbf{L}/dt = \mathbf{r}\times e\mathbf{E}_q^{(0)}$, and, when averaged over one period of the motion, becomes from Eq. (24)

$$\frac{d\mathbf{L}}{dt} = \langle\mathbf{r}\rangle\times e\mathbf{E}_q^{(0)} = -e\mathbf{E}_q^{(0)}\times\langle\mathbf{r}\rangle = -e\mathbf{E}_q^{(0)}\times\frac{3}{2}\frac{\mathbf{K}}{(-2mH_0)^{1/2}} \tag{33}$$

Thus the electric field back at the charge $q$ is given by

$$\begin{aligned}\mathbf{E}_\mu(\mathbf{r}_q,t) &= -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_\mu(\mathbf{r}_q,t) = \frac{e}{2mc}\frac{\mathbf{r}_q}{r_q^3}\times\frac{d\mathbf{L}}{dt} \\ &= \frac{e}{2mc^2}\frac{\mathbf{r}_q}{r_q^3}\times\left(-e\mathbf{E}_q^{(0)}\times\frac{3}{2}\frac{\mathbf{K}}{(-2mH_0)^{1/2}}\right)\end{aligned} \tag{34}$$

Now our magnetic moment model is initially in a circular orbit with $\mathbf{K} = 0$, and $\mathbf{K}$ changed as in Eq. (29) only because of the presence of the electric field $\mathbf{E}_q^{(0)}$ due to the distant charge $q$. Thus the force $\mathbf{F}_{on\ q}$ on $q$ due to the electric field $\mathbf{E}_\mu$ of the magnetic moment is

$$\mathbf{F}_{on\ q} = q\mathbf{E}_\mu(\mathbf{r}_q, t) = q\frac{e}{2mc^2}\frac{\mathbf{r}_q}{r_q^3} \times \left( \frac{3}{2}\frac{-e\mathbf{E}_q^{(0)}(0,t')}{(-2mH_0)^{1/2}} \times \int_0^t dt' \left\{ 3m[c\overrightarrow{\mu}(t')] \times \mathbf{E}_q^{(0)}(0,t') \right\} \right) \quad (35)$$

where $\mathbf{E}_q^{(0)}$ is the electrostatic field in Eq. (17) of the distant charge $q$ acting on the magnetic moment. We notice that this force back on the charge $q$ due to the magnetic moment $\overrightarrow{\mu}$ is proportional to $q^3e^2\mu$; it changes sign with the external charge $q$, changes sign with the magnetic moment $\overrightarrow{\mu}$, but does not depend upon the sign of the charge $e$. Furthermore it changes sign with the reversal of the position $\mathbf{r}_q$ of the charge $q$. Finally, it does not depend upon any velocity of the charge $q$. It arises from the 0-order acceleration of the orbiting magnetic moment charge due to the *electrostatic* field $\mathbf{E}_q^{(0)}$ of the distant charge $q$. These properties are in total contrast with those found for electrostatic forces such as in Eq. (31).

## V. CONSERVATION LAWS

In our model, the (zero-order) electrostatic field of the passing charge causes a change in the magnetic moment which then produces an (order $1/c^2$) electric field back at the position of the passing charge. Since this back electric field is unanticipated by treatments (such as in the no-velocity-change point of view) which do not allow for changes in the charge and current densities of magnetic moments, it seems appropriate to discuss all the conservation laws associated with electromagnetic theory and to see how they are upheld by the present model.

### A. Linear Momentum in the Electromagnetic Field

The Darwin Lagrangian conserves linear momentum[24]. For our magnetic moment and passing charge, the total linear momentum is

$$\begin{aligned}\mathbf{P} &= \mathbf{P}_{\mu} + \mathbf{P}_{em\,\mu q} + \mathbf{p}_q \\ &= \left[M\mathbf{V}+m\mathbf{v}\left(1+\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right) - \frac{e^2}{2c^2 r}\left(\mathbf{v}+\frac{[\mathbf{v}\cdot\mathbf{r}]\mathbf{r}}{r^2}\right)\right] \\ &+ \left[\frac{qe}{2c^2|\mathbf{r}_q-\mathbf{r}|}\left(\mathbf{v}+\frac{[\mathbf{v}\cdot(\mathbf{r}_q-\mathbf{r})](\mathbf{r}_q-\mathbf{r})}{|\mathbf{r}_q-\mathbf{r}|^2}\right)\right] + \left[m_q\mathbf{v}_q\left(1+\frac{1}{2}\frac{\mathbf{v}_q^2}{c^2}\right)\right]\end{aligned} \tag{36}$$

Here we have grouped the total momentum into three terms which can be assigned to the magnetic moment, the electromagnetic fields between the magnetic moment and the charge $q$, and the mechanical momentum of the passing charge $q$. When averaged over the orbital motion of the magnetic moment, the system carries an average linear momentum in the electromagnetic field given by

$$\begin{aligned}\langle\mathbf{P}_{em\,\mu q}\rangle &= \left\langle\frac{1}{4\pi c}\int d^3r\,\mathbf{E}_q\times\mathbf{B}_{\mu}\right\rangle \\ &= \left\langle\frac{qe}{2c^2|\mathbf{r}_q-\mathbf{r}|}\left(\mathbf{v}+\frac{[\mathbf{v}\cdot(\mathbf{r}_q-\mathbf{r})](\mathbf{r}_q-\mathbf{r})}{|\mathbf{r}_q-\mathbf{r}|^2}\right)\right\rangle \\ &= \frac{q}{c}\frac{\overrightarrow{\mu}\times\mathbf{r}_q}{r_q^3} = \frac{q}{c}\mathbf{A}_{\mu}(\mathbf{r}_q,t)\end{aligned} \tag{37}$$

where, from Eq. (9), $\mathbf{A}_{\mu}(\mathbf{r}_q,t)$ is the vector potential in the Coulomb gauge due to the magnetic moment and evaluated at the position of the point charge $q$. Any contribution from the other electromagnetic field combination $\mathbf{E}_{\mu}\times\mathbf{B}_q$ is very small since the magnetic moment $\overrightarrow{\mu}$ has no net charge.

Now the time derivative of the electromagnetic field momentum $\langle\mathbf{P}_{em\,\mu q}\rangle$ in Eq. (37) involves changes connected with the particle position $\mathbf{r}_q$ and with the magnetic moment $\overrightarrow{\mu}$. We can write

$$\begin{aligned}\frac{d}{dt}\langle\mathbf{P}_{em\,\mu q}\rangle &= \frac{d}{dt}\left(\frac{q}{c}\mathbf{A}_{\mu}(\mathbf{r}_q,t)\right) \\ &= (\mathbf{v}_q\cdot\nabla_q)\left(\frac{q}{c}\mathbf{A}_{\mu}(\mathbf{r}_q,t)\right) + \frac{\partial}{\partial t}\left(\frac{q}{c}\mathbf{A}_{\mu}(\mathbf{r}_q,t)\right) \\ &= \nabla_q\left(\frac{q}{c}\mathbf{v}_q\cdot\mathbf{A}_{\mu}(\mathbf{r}_q,t)\right) - \frac{q}{c}\mathbf{v}_q\times[\nabla_q\times\mathbf{A}_{\mu}(\mathbf{r}_q,t)] + \frac{\partial}{\partial t}\left(\frac{q}{c}\mathbf{A}_{\mu}(\mathbf{r}_q,t)\right) \\ &= \nabla_q\left(\frac{q}{c}\mathbf{v}_q\cdot\frac{\overrightarrow{\mu}\times\mathbf{r}_q}{r_q^3}\right) - \frac{q}{c}\mathbf{v}_q\times\left[\nabla_q\times\left(\frac{\overrightarrow{\mu}\times\mathbf{r}_q}{r_q^3}\right)\right] + \frac{q}{c}\left(\frac{d\overrightarrow{\mu}}{dt}\times\frac{\mathbf{r}_q}{r_q^3}\right) \\ &= -\left\langle\mathbf{F}_{on\,\mu}^{Lorentz}\right\rangle - \left\langle\mathbf{F}_{on\,q}^{Lorentz}\right\rangle\end{aligned} \tag{38}$$

where

$$\left\langle \mathbf{F}^{Lorentz}_{on\,\mu} \right\rangle = \nabla_{\mathbf{r}} \left[ \overrightarrow{\mu} \cdot \mathbf{B}_q(\mathbf{r}, t) \right]_{r=0} = -\nabla_q \left( \frac{q}{c} \mathbf{v}_q \cdot \mathbf{A}_\mu(\mathbf{r}_q, t) \right) \tag{39}$$

and

$$\left\langle \mathbf{F}^{Lorentz}_{on\,q} \right\rangle = q\mathbf{E}_\mu(\mathbf{r}_q, t) + q\frac{\mathbf{v}_q}{c} \times \mathbf{B}_\mu(\mathbf{r}_q, t) \tag{40}$$

with

$$\mathbf{E}_\mu(\mathbf{r}_q, t) = -\frac{\partial}{\partial t}\mathbf{A}_\mu(\mathbf{r}_q, t) = -\frac{d\overrightarrow{\mu}}{dt} \times \frac{\mathbf{r}_q}{r_q^3} \tag{41}$$

and

$$\mathbf{B}_\mu(\mathbf{r}_q, t) = \nabla_q \times \mathbf{A}_\mu(\mathbf{r}_q, t) = \nabla_q \times \left( \frac{q}{c} \frac{\overrightarrow{\mu} \times \mathbf{r}_q}{r_q^3} \right) \tag{42}$$

Thus the average electromagnetic linear momentum $\langle \mathbf{P}_{em\,\mu q} \rangle$ in Eq. (37) changes with respect to time for two reasons: the change in $\overrightarrow{\mu}$ (due to the change in the orbital shape of the magnetic moment) and the change in the separation $\mathbf{r}_q$. As the shape changes for the orbit of the charge $e$ in the magnetic moment, the magnetic moment $\overrightarrow{\mu}$ changes creating an *electric* field at the position of the passing particle $q$. Thus due to this changing-$\mu$ effect, the linear momentum $\langle \mathbf{P}_{em\,\mu q} \rangle$ in the electromagnetic field decreases at the same rate that the linear momentum of the point charge $q$ increases due to the force from the electric field of the changing magnetic moment. The change in the electromagnetic linear momentum $\langle \mathbf{P}_{em\,\mu q} \rangle$ due to the changing position $\mathbf{r}_q$ is associated with the magnetic Lorentz forces on the magnetic moment and on the passing charge.

Next we average the total system momentum in Eq. (36) over the orbital motion and differentiate with respect to time to find

$$\begin{aligned} \frac{d\mathbf{P}}{dt} = 0 = & \left[ \frac{d\left\langle \mathbf{P}_\mu \right\rangle}{dt} + \nabla_q \left( \frac{q}{c} \frac{\overrightarrow{\mu} \times \mathbf{r}_q}{r_q^3} \right) \right] \\ & + \left[ \frac{q}{c} \left( \frac{d\overrightarrow{\mu}}{dt} \right) \times \frac{\mathbf{r}_q}{r_q^3} - \frac{q}{c}\mathbf{v}_q \times \left[ \nabla_q \times \left( \frac{q}{c} \frac{\overrightarrow{\mu} \times \mathbf{r}_q}{r_q^3} \right) \right] + \frac{d\mathbf{p}_q}{dt} \right] \\ = & \left[ \frac{d\left\langle \mathbf{P}_\mu \right\rangle}{dt} - \left\langle \mathbf{F}^{Lorentz}_{on\,\mu} \right\rangle \right] + \left[ \frac{d\mathbf{p}_q}{dt} - \left\langle \mathbf{F}^{Lorentz}_{on\,q} \right\rangle \right] \end{aligned} \tag{43}$$

The equations of motion tell us that each of the quantities in square brackets vanishes. Note that the sum of the average Lorentz forces $\left\langle \mathbf{F}^{Lorentz}_{on\,\mu} \right\rangle + \left\langle \mathbf{F}^{Lorentz}_{on\,q} \right\rangle$ does not vanish, but rather (according to Eq. (38)) is equal to the negative rate of change of the electromagnetic field linear momentum $\langle \mathbf{P}_{em\,\mu q} \rangle$. Thus in the conservation law for linear momentum, the changing electromagnetic field momentum $\langle \mathbf{P}_{em\,\mu q} \rangle$ is partially balanced by the changing momentum of the magnetic moment and partially balanced by the changing momentum of the passing particle.

### B. Energy Conservation

The Darwin lagrangian conserves energy.[25] For our magnetic moment and passing charge, the total energy through order $1/c^2$ is

$$\begin{aligned} U &= U_\mu + U_{em\ \mu q} + U_q \\ &= \left[ Mc^2 + mc^2\left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}\right) - \frac{e^2}{r}\right] + [-\frac{eq}{r_q}+\frac{eq}{r_{eq}} \\ &+ \frac{eq}{2c^2 r_{eq}}\left(\mathbf{v}\cdot\mathbf{v}_q + \frac{(\mathbf{v}\cdot\mathbf{r}_{eq})(\mathbf{v}_q\cdot\mathbf{r}_{eq})}{r_{eq}^2}\right)] + \left[m_q c^2\left(1+\frac{1}{2}\frac{v_q^2}{c^2}+\frac{3}{8}\frac{v_q^4}{c^4}\right)\right] \end{aligned} \tag{44}$$

When averaged over the orbital motion of the magnetic moment, the electrostatic energy $-eq/r_q + eq/r_{eq}$ involves only quadrupole energies, which vanish when averaged $\pm e$, $\pm\omega$ with $\overrightarrow{\mu}$ held constant. The system carries an average magnetic energy in the electromagnetic field given by

$$\begin{aligned} \langle U_{em\ \mu q}\rangle &= \left\langle \frac{1}{8\pi}\int d^3r\,\mathbf{B}_q\times\mathbf{B}_\mu \right\rangle \\ &= \left\langle \frac{eq}{2c^2 r_{eq}}\left(\mathbf{v}\cdot\mathbf{v}_q + \frac{(\mathbf{v}\cdot\mathbf{r}_{eq})(\mathbf{v}_q\cdot\mathbf{r}_{eq})}{r_{eq}^2}\right)\right\rangle \\ &= \overrightarrow{\mu}\cdot\mathbf{B}_q(0,t) = \overrightarrow{\mu}\cdot\left(\frac{q}{c}\frac{\mathbf{v}_q\times(-\mathbf{r}_q)}{r_q^3}\right) \\ &= \frac{q}{c}\mathbf{v}_q\cdot\frac{\overrightarrow{\mu}\times\mathbf{r}_q}{r_q^3} = \frac{q}{c}\mathbf{v}_q\cdot\mathbf{A}_\mu(\mathbf{r}_q,t) \end{aligned} \tag{45}$$

The time derivative of the magnetic field energy $\langle U_{em\ \mu q}\rangle$ can be written using the calculations in Eq. (38) for $d\mathbf{A}_\mu/dt$

$$\begin{aligned} \frac{d}{dt}\langle U_{em\ \mu q}\rangle &= \frac{d}{dt}\left(\frac{q}{c}\mathbf{v}_q\cdot\mathbf{A}_\mu(\mathbf{r}_q,t)\right) = \mathbf{v}_q\cdot\frac{d}{dt}\left(\frac{q}{c}\mathbf{A}_\mu(\mathbf{r}_q,t)\right) \\ &= \mathbf{v}_q\cdot\left(-\left\langle\mathbf{F}^{Lorentz}_{on\,\mu}\right\rangle - \left\langle\mathbf{F}^{Lorentz}_{on\,q}\right\rangle\right) \end{aligned} \tag{46}$$

since $\langle U_{em\ \mu q}\rangle$ is already of order $1/c^2$ and any change in $\mathbf{v}_q$ due to the changing magnetic moment $\overrightarrow{\mu}$ is also of order $1/c^2$.

Since the total energy in Eq. (44) is constant in time, it follows from averaging over the orbital motion and differentiating with respect to time that

$$\begin{aligned}\frac{dU}{dt} &= 0 = \frac{d\langle U_\mu\rangle}{dt} + \frac{d}{dt}\langle U_{em\,\mu q}\rangle + \frac{dU_q}{dt} \\ &= \left[\frac{d\langle U_\mu\rangle}{dt} + \frac{q}{c}\mathbf{v}_q\cdot\left(\overrightarrow{\mu}\times\frac{d}{dt}\frac{\mathbf{r}_q}{r_q^3}\right)\right] + \left[\frac{q}{c}\mathbf{v}_q\cdot\left(\frac{d\overrightarrow{\mu}}{dt}\times\frac{\mathbf{r}_q}{r_q^3}\right) + \frac{dU_q}{dt}\right] \\ &= \left[\frac{d\langle U_\mu\rangle}{dt} - \mathbf{v}_q\cdot\left\langle\mathbf{F}_{on\,\mu}^{Lorentz}\right\rangle\right] + \left[\frac{dU_q}{dt} - \mathbf{v}_q\cdot\left\langle\mathbf{F}_{on\,q}^{Lorentz}\right\rangle\right]\end{aligned} \tag{47}$$

Here we have used the calculations in Eqs. (45) and (46); we also note that the dot product of $\mathbf{v}_q$ with the term involving $\mathbf{v}_q\times[\nabla_q\times\mathbf{A}_\mu]$ in Eq. (38) vanishes. The average energy in the magnetic field $\langle U_{em\,\mu q}\rangle$ changes because of the changing magnetic moment $\overrightarrow{\mu}$ and also due to the changing position $\mathbf{r}_q$ of the passing charge $q$. Just as above in Eq. (41), the changing magnetic moment is associated with an electric field $\mathbf{E}_\mu(\mathbf{r}_q,t)$ back at the passing charge which changes the kinetic energy of the passing charge.

$$\begin{aligned}\frac{dU_q}{dt} &= \mathbf{v}_q\cdot\left\langle\mathbf{F}_{on\,q}^{Lorentz}\right\rangle = q\mathbf{E}_\mu(\mathbf{r}_q,t)\cdot\mathbf{v}_q \\ &= -\mathbf{v}_q\cdot\frac{q}{c}\frac{\partial}{\partial t}\mathbf{A}_\mu(\mathbf{r}_q,t) = -\mathbf{v}_q\cdot\left[\frac{q}{c}\left(\frac{d}{dt}\overrightarrow{\mu}\right)\times\frac{\mathbf{r}_q}{r_q^3}\right]\end{aligned} \tag{48}$$

The change in the magnetic field energy associated with the changing position $\mathbf{r}_q$ of the passing charge is compensated by the change in the kinetic energy (in order $1/c^2$) of the orbiting charge of the magnetic moment. This energy change can be written in various forms

$$\begin{aligned}\frac{d\langle U_\mu\rangle}{dt} &= \mathbf{v}_q\cdot\left\langle\mathbf{F}_{on\,\mu}^{Lorentz}\right\rangle = -\frac{q}{c}\mathbf{v}_q\cdot\overrightarrow{\mu}\times\frac{d}{dt}\left(\frac{\mathbf{r}_q}{r_q^3}\right) = \overrightarrow{\mu}\cdot\frac{\partial}{\partial t}\mathbf{B}_q(0,t) \\ &= -\left(\mathbf{v}_q\cdot\nabla_q\right)\left(\frac{q}{c}\mathbf{v}_q\cdot\mathbf{A}_\mu(\mathbf{r}_q,t)\right) \\ &= \left(\mathbf{v}_q\cdot\nabla_q\right)\left\langle\frac{eq}{2c^2r_{eq}}\left(\mathbf{v}\cdot\mathbf{v}_q + \frac{(\mathbf{v}\cdot\mathbf{r}_{eq})(\mathbf{v}_q\cdot\mathbf{r}_{eq})}{r_{eq}^2}\right)\right\rangle \\ &= \left\langle e\mathbf{v}\cdot\frac{q\mathbf{r}_{eq}}{c^2r_{eq}^3}\left(\frac{-1}{2}\frac{v_q^2}{c^2} + \frac{3}{2}\frac{(\mathbf{v}_q\cdot\mathbf{r}_{eq})^2}{c^2r_{eq}^2}\right)\right\rangle = \left\langle e\mathbf{v}\cdot\left[\mathbf{E}_q(\mathbf{r},t) - \mathbf{E}_q^{(0)}(\mathbf{r},t)\right]\right\rangle\end{aligned} \tag{49}$$

and corresponds to energy delivered to a moving charge by the emf of the changing magnetic field of the passing charge. We notice that it is the relativistic $v_q^2/c^2$ terms in the electric field $\mathbf{E}_q$ which deliver the power to the orbiting charge. Thus in the energy conservation law, the changing magnetic field energy $\langle U_{em\,\mu q}\rangle$ is associated with the changing energy of both the magnetic moment and the passing charge.

### C. Center-of-Energy Motion for the Magnetic Moment

In this section we will discuss the motion of the center of energy of the magnetic moment from two points of view. First we connect its motion to the motion of the passing charge using the conservation law for the constant motion of the system center of energy. Second we use the particle equations of motion to obtain what has been called "the equation of motion of the magnet," referring to the center of energy motion of the magnetic moment

The Darwin Lagrangian gives constant velocity to the system center of energy to order $1/c^2$, the same order to which the Darwin Lagrangian is invariant under Lorentz transformations.[4] The center of energy $\overrightarrow{\mathbf{X}}$ to order $1/c^2$ involves only rest mass energy and electrostatic energy

$$\begin{aligned}\frac{U}{c^2}\overrightarrow{\mathbf{X}} &= \frac{U_\mu}{c^2}\overrightarrow{\mathbf{X}}_\mu + \left[\frac{U_{em\ eq}}{c^2}\left(\frac{\mathbf{r}+\mathbf{r}_q}{2}\right) + \frac{U_{em\ -eq}}{c^2}\left(\frac{\mathbf{r}_q}{2}\right)\right] + \frac{U_q}{c^2}\mathbf{r}_q \\ &= \frac{1}{c^2}\left(U_\mu^{[q=0]} - e\mathbf{r}\cdot\mathbf{E}_q(0)\right)\left(\overrightarrow{\mathbf{X}}_\mu^{[q=0]} + \delta\overrightarrow{\mathbf{X}}_\mu\right) \\ &+ \left[\frac{eq}{r_{eq}}\left(\frac{\mathbf{r}+\mathbf{r}_q}{2}\right) + \frac{-eq}{r_q}\left(\frac{\mathbf{r}_q}{2}\right)\right] + \left[m_q\left(1+\frac{1}{2}\frac{\mathbf{v}_q^2}{c^2}\right)\right]\mathbf{r}_q \end{aligned} \tag{50}$$

where $U_\mu^{[q=0]}$ and $\overrightarrow{\mathbf{X}}_\mu^{[q=0]}$ correspond to the energy and center of energy of the magnetic moment when the passing charge is not present. When averaged over the orbital motion of the magnetic moment, the electromagnetic field contribution in Eq. (50) yields a quadrupole contribution, corresponding to the neutrality of the magnetic moment,

$$\begin{aligned}\left\langle\left[\frac{eq}{r_{eq}}\left(\frac{\mathbf{r}+\mathbf{r}_q}{2}\right) + \frac{-eq}{r_q}\left(\frac{\mathbf{r}_q}{2}\right)\right]\right\rangle &= \frac{-eq}{2}\left\langle\frac{\mathbf{r}}{r_q} - \frac{(\mathbf{r}\cdot\mathbf{r}_q)\mathbf{r}_q}{r_q^2} + O\left(\frac{r^2}{r_q^2}\right)\right\rangle \\ &= \frac{-eq}{2}\left\langle O\left(\frac{r^2}{r_q^2}\right)\right\rangle \end{aligned} \tag{51}$$

since $\langle\mathbf{r}\rangle = 0$, and this contribution vanishes entirely if we average over $\pm e$ and $\pm\omega$ so as to keep only the magnetic moment contribution. Furthermore, the magnetic momentum contribution in Eq. (50) can be averaged over the orbital motion to give (through first order in the interaction perturbation)

$$\begin{aligned}\left\langle\frac{U_\mu}{c^2}\overrightarrow{\mathbf{X}}_\mu\right\rangle &= \left\langle\frac{1}{c^2}\left(U_\mu^{[q=0]} - e\mathbf{r}\cdot\mathbf{E}_q(0)\right)\left(\overrightarrow{\mathbf{X}}_\mu^{[q=0]} + \delta\overrightarrow{\mathbf{X}}_\mu\right)\right\rangle \\ &= \frac{1}{c^2}U_\mu^{[q=0]}\left(\overrightarrow{\mathbf{X}}_\mu^{[q=0]} + \delta\overrightarrow{\mathbf{X}}_\mu\right) = \frac{\langle U_\mu\rangle}{c^2}\overrightarrow{\mathbf{X}}_\mu\end{aligned}$$

since $\langle\mathbf{r}\rangle = 0$. It follows that Eq. (50) becomes

$$\frac{U}{c^2}\overrightarrow{\mathbf{X}} = \frac{\langle U_\mu\rangle}{c^2}\overrightarrow{\mathbf{X}}_\mu + \frac{\langle U_q\rangle}{c^2}\mathbf{r}_q \tag{52}$$

Now differentiating twice with respect to time and noting that the energies $U$, $\langle U_\mu \rangle$, and $\langle U_q \rangle$ are all constant in time through 0-order in $1/c^2$, while $d^2\overrightarrow{\mathbf{X}}/dt^2 = 0$, we find

$$0 = \frac{\langle U_\mu \rangle}{c^2}\frac{d^2\overrightarrow{\mathbf{X}}_\mu}{dt^2} + \frac{\langle U_q \rangle}{c^2}\frac{d^2\mathbf{r}_q}{dt^2} \tag{53}$$

Thus the motions of the centers of energy of the magnetic moment and the passing charge are coupled together. Our equations (52) and (53) here correspond to Eqs. (14) and (15) in Coleman and Van Vleck's discussion of the interaction of a point charge and a magnet.

For the magnetic moment alone, the center of energy $\overrightarrow{\mathbf{X}}_\mu$ is defined as

$$\frac{U_\mu}{c^2}\overrightarrow{\mathbf{X}}_\mu = m\left(1 + \frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)\mathbf{r} + M\mathbf{R} - \frac{e^2}{c^2 r}\left(\frac{\mathbf{r}}{2}\right) \tag{54}$$

where the energy $U_\mu$ of the magnetic moment through 0-order in $1/c^2$ is

$$U_\mu = mc^2\left(1 + \frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right) + Mc^2 - \frac{e^2}{r} \tag{55}$$

and where we have taken the displacement $\mathbf{R}$ of the large mass $M$ as small compared to $\mathbf{r}$. In the nonrelativistic (0-order $1/c$) limit, the center of energy $\overrightarrow{\mathbf{X}}_\mu^{(0)}$ corresponds to the center of (rest) mass

$$(m + M)\overrightarrow{\mathbf{X}}_\mu^{(0)} = m\mathbf{r} + M\mathbf{R} \tag{56}$$

which in our example has been chosen so $\overrightarrow{\mathbf{X}}_\mu^{(0)} = 0$. Furthermore, the center of (rest) mass remains at rest since differentiating Eq. (56) with respect to time leads to the nonrelativistic statement regarding the momentum of the magnetic moment

$$(m + M)\frac{d}{dt}\overrightarrow{\mathbf{X}}_\mu^{(0)} = m\mathbf{v} + M\mathbf{V} = 0 \tag{57}$$

The 0-order (nonrelativistic) linear momentum of the magnetic moment indeed vanishes since the internal Coulomb forces within the magnetic moment satisfy Newton's third law and the nonrelativistic Coulomb forces on the two oppositely charged particles of the magnetic moment due to the distant point charge $q$ are equal and opposite in the approximation of Eq. (17).

If we differentiate Eq. (54) for the center of energy of the magnetic moment, we obtain,

$$\frac{U_\mu}{c^2}\frac{d\overrightarrow{\mathbf{X}}_\mu}{dt} + \frac{1}{c^2}\frac{dU_\mu}{dt}\overrightarrow{\mathbf{X}}_\mu = m\left(1 + \frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)\mathbf{v} + M\mathbf{V} - \frac{e^2}{2c^2 r}\left(\mathbf{v} - \frac{(\mathbf{r}\cdot\mathbf{v})\mathbf{r}}{r^2}\right) + \frac{m\mathbf{r}(\mathbf{v}\cdot\mathbf{a})}{c^2} \tag{58}$$

The acceleration **a** of the orbiting charge is given in Eq. (14) and the time derivative of the energy is related to the work done by the electric field of the passing charge $dU_\mu/dt = e\mathbf{v}\cdot\mathbf{E}_q$. Then averaging over the orbital motion, equation (58) becomes

$$\left\langle \frac{U_\mu}{c^2}\frac{d\overrightarrow{\mathbf{X}}_\mu}{dt} + \frac{1}{c^2}\frac{dU_\mu}{dt}\overrightarrow{\mathbf{X}}_\mu \right\rangle = \frac{\langle U_\mu\rangle}{c^2}\frac{d\overrightarrow{\mathbf{X}}_\mu}{dt} + \frac{1}{c^2}\left\langle e\mathbf{v}\cdot\mathbf{E}_q^{(0)}\right\rangle \overrightarrow{\mathbf{X}}_\mu = \frac{\langle U_\mu\rangle}{c^2}\frac{d\overrightarrow{\mathbf{X}}_\mu}{dt}$$
$$=< m\left(1+\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)\mathbf{v} + M\mathbf{V} - \frac{e^2}{2c^2 r}\left(\mathbf{v} - \frac{(\mathbf{r}\cdot\mathbf{v})\mathbf{r}}{r^2}\right)$$
$$+\frac{\mathbf{r}}{c^2}\left[\mathbf{v}\cdot\left(-\frac{e^2\mathbf{r}}{r^3} + e\mathbf{E}_q^{(0)}(\mathbf{r},t)\right)\right] > \tag{59}$$

where we have noted $\left\langle e\mathbf{v}\cdot\mathbf{E}_q^{(0)}\right\rangle = 0$. Now combining the terms involving $e^2$, and rewriting the average of $e\mathbf{r}\left[\mathbf{v}\cdot\mathbf{E}_q^{(0)}\right]/c^2$ as in Eq. (28), we have

$$\frac{\langle U_\mu\rangle}{c^2}\frac{d\overrightarrow{\mathbf{X}}_\mu}{dt} = \left\langle m\left(1+\frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right)\mathbf{v} + M\mathbf{V} - \frac{e^2}{2c^2 r}\left(\mathbf{v} + \frac{(\mathbf{r}\cdot\mathbf{v})\mathbf{r}}{r^2}\right)\right\rangle - \frac{1}{c}\overrightarrow{\mu}\times\mathbf{E}_q^{(0)}(0,t)$$
$$= \langle \mathbf{P}_\mu\rangle - \frac{1}{c}\overrightarrow{\mu}\times\mathbf{E}_q^{(0)}(0,t) \tag{60}$$

This result (60) corresponds to Eq. (26) of the work by Coleman and Van Vleck.[4] Next differentiating Eq. (60) with respect to time so as to obtain a second derivative of $\overrightarrow{\mathbf{X}}_\mu$

$$\frac{\langle U_\mu\rangle}{c^2}\frac{d^2\overrightarrow{\mathbf{X}}_\mu}{dt^2} = \frac{d}{dt}\langle \mathbf{P}_\mu\rangle - \frac{d}{dt}\left(\frac{1}{c}\overrightarrow{\mu}\times\mathbf{E}_q^{(0)}(0,t)\right) \tag{61}$$

This equation is sometimes called "the equation of motion for a magnetic moment."[5][6]

### D. The Argument over Hidden Momentum in Magnets

Because the interaction of a magnet and a passing point charge is so poorly understood, there can arise certain notions which are used as "explanations" but are not explored in detail. "Hidden momentum in magnets" is such a notion. We will illustrate the situation using our calculations for the interaction of a point charge and a hydrogen-atom magnetic moment which we have calculated above.

Because the proponents of the no-velocity-change point of view are so sure that there is no force back on a charged particle passing a magnet, they also feel sure that there must be no change in the center of energy of the magnet. Thus if the center of energy of the magnet did change position, then according to our Eq. (53) (and according to Coleman and Van Vleck's Eq. (15)), the passing charge must accelerate. Moreover, there is clearly a possibility of acceleration for the magnet's center of energy since there is an obvious magnetic Lorentz force on the magnet given by $\mathbf{F}_{on\,\mu} = \nabla(\overrightarrow{\mu} \cdot \mathbf{B}_q)$. Now fundamental classical theorems connect the force and changes in system momentum so that we must have $\mathbf{F}_{on\,\mu} = \nabla(\overrightarrow{\mu} \cdot \mathbf{B}_q) = d\langle \mathbf{P}_\mu \rangle /dt$. But our equation (61) gives an escape from motion for the center of energy of the magnet because there is a second term in the expression for the acceleration of the center of energy. Thus the proponents of the no-velocity-change point of view decide that the quantity $-(1/c)\overrightarrow{\mu} \times \mathbf{E}_q$ represents a "hidden momentum in magnets" whose change "cancels" the classical applied force. Indeed, a mechanical momentum of the required form is mentioned in a footnote in Coleman and Van Vleck's work[4] and now appears in an electromagnetism text book.[26] However, no one who speaks of "hidden momentum in magnets" has ever given any relativistic calculation which shows how this momentum carries out this cancellation without continuing changes in the charge and current densities of the magnet. "Hidden momentum in magnets" (as used by the proponents of the no-velocity-change point of view) seems to be an idea which exists simply to prevent the motion of the center of energy of a magnet. As we see above in our explicit model of a hydrogen-atom magnetic moment and a point charge, there is indeed a force back on the passing charge and there is indeed motion of the center of energy of the magnet. Both of these results are contrary to the claims of the proponents of the no-velocity-change point of view.

## VI. TRANSITION TO A MULTIPARTICLE MAGNET

Experimental observation of the interaction of a magnet and a point charge (such as in the Aharonov-Bohm phase shift) involves not two-particle magnetic moments but rather multiparticle magnets. We are interested in understanding the experimental situation based upon the insight gained from the fundamental interaction involving a two-particle magnetic dipole moment.

Within classical electromagnetism, the transition to a multiparticle system is most familiar for the electrostatics of polarizable particles. In our calculation above, we found that our magnetic moment oriented in the direction of the displacement $\mathbf{r}_q$, $\overrightarrow{\mu}||\mathbf{r}_q$, acted like a polarizable particle producing a back force of magnitude $F_{on\ q} = q^2e^2/(m\omega^2 r_q^6)$ back on the point charge $q$. When the polarizability is larger (for example, $m$ is smaller for fixed $\omega$), then the force back on the distant charge is larger. Also, when we have many polarizable particles present, the force back on the distant particle does not disappear but rather increases to a well-defined limit. Thus if we consider a dielectric wall formed by polarizable particles, then the mutual interaction among the polarizable particles changes the functional dependence of the force over toward $F_{on\ q} = q^2/(2r_q)^2$, which holds for a conducting wall where the force is independent of the polarizability in the limit of large polarizability. This occurs because polarizable particles which are next to each other in the wall form electric dipole moments which tend to cancel the external electric field $\mathbf{E}_q$ at the position of the other electric dipoles in the wall.

In an analogous fashion, we expect multiparticle interactions within a magnet to alter the back force on a passing charge found in Eq. (35). We note that the force back at the passing charge $q$ due to our model magnetic moment can be varied by changing the mass of the orbiting charge while keeping the magnetic moments fixed. When the magnetic moment involves a small mass $m$ (and thus is easily influenced by the external electric field $\mathbf{E}_q$), the force back at the passing charge is larger, just as is true for a polarizable particle in the electrostatic situation. The most symmetrical multiparticle arrangement of magnetic moments involves $N$ magnetic moments arranged around a circle as a toroid with the distant charged particle $q$ located along the axis of the toroid. The 0-order (nonrelativistic) electrostatic force on each of the orbiting charges of the toroid due to the charge $q$ is $e\mathbf{E}_q^{(0)}$ just as before, while the back force on the charge $q$ is now $N$ times as large. Again, in analogy with the electrostatic situation, we expect that due to multiparticle interactions within the toroid the back force on a passing charge will not disappear but rather will increase to a limit. Now there will be nonrelativistic electrostatic forces between the charges of the $N$ magnetic moments. Also, each of the orbiting charges $e$ produces acceleration fields of order $1/c^2$ which act on all of the other orbiting charges of the magnetic moment. Since the $1/c^2$-acceleration fields act on each of the other orbiting charges of the toroid, the back force on *each* orbiting charge $e$ increases as the number $N$ of two-particle magnetic moments increases. These acceleration fields always cause forces such as to oppose any change in the currents of the toroid. This corresponds to a self-inductance effect which increases as $N^2$ when there are $N$ current-carrying loops.

It is important to notice that the present situation does *not* correspond to the elementary mutual-inductance problem of electromagnetism texts. In mutual inductance effects, the self-induced emf is such as to oppose any change in magnetic flux introduced externally and the magnitude of the back emf is independent of the current which is flowing in the toroid winding. In our case here, the initial accelerations tending to change the magnetic flux through the toroid do not arise from any induced emf through the toroid. Indeed in the limit $\mathbf{v}_q = 0$ there is no emf at all in the toroid. Furthermore, the back force on the charge $q$ does not behave as in Lenz's law. Rather, the tendency to change the currents of the toroid arises from Solem's strange polarization associated with the *electrostatic* field of the external charge $q$ treated as a *uniform* electric field across each magnetic moment; the change in the magnetic moment is proportional to the magnetic moment and changes sign with the sign of $q$ as seen in Eq. (35).

We expect that in the multiparticle limit, the electrostatic interactions within the toroid will tend to screen the field of the passing charge $q$ out of the toroid and the back force on the passing charge will be limited by the magnetic energy of interaction. Indeed, calculations for ohmic conductors suggest that the electric fields of a passing charge are screened out of the body of the conductor by surface charges while the magnetic fields of the passing charge penetrate into the body of the conductor.[17] We note that if the point charge is held at rest outside a conductor, then the electric fields of the point charge are screened out of the body of the conductor by surface charges. If the charged particle is moving, we do not expect this electric-field screening to suddenly disappear. On the other hand, it has been shown that magnetic fields due to moving charges penetrate into an ohmic conductor giving a time-integral of the magnetic field which is independent of the conductivity of the materials.[17] As was suggested earlier, this is precisely the result which is needed to account for the Aharonov-Bohm phase shift as a classical lag associated with energy-related classical forces.[13]

### A. Energy, Momentum, and Forces in the Multiparticle Limit

Let us now consider the momentum, energy, and forces when a charged particle $q$ moves with velocity $\mathbf{v}_q$ down the axis of a magnet in the form of a toroid which is initially at rest. The screening of the electric field of the passing charge out of the body of the magnet implies that the electric field vanishes inside the toroid and therefore there is no significant contribution to momentum from the electromagnetic field of the form $\mathbf{P}_{em\,\mu q}$ discussed above, and no significant energy flow across the magnet. It follows from Eq. (36), that now the total system momentum consists of only two contributions, one each from the magnet and the passing charge

$$\mathbf{P} = \mathbf{P}_\mu + m_q\mathbf{v}_q\left(1+\frac{1}{2}\frac{\mathbf{v}_q^2}{c^2}\right) \quad \text{multiparticle limit} \tag{62}$$

The Lorentz forces on the magnet and on the passing charge then satisfy Newton's third law

$$\begin{aligned} 0 &= \frac{d\mathbf{P}}{dt} = \frac{d\mathbf{P}_\mu}{dt} + \frac{d}{dt}\left[m_q\mathbf{v}_q\left(1+\frac{1}{2}\frac{\mathbf{v}_q^2}{c^2}\right)\right] \\ &= \mathbf{F}^{Lorentz}_{on\,\mu} + \mathbf{F}^{Lorentz}_{on\,q} \qquad \text{multiparticle limit} \end{aligned} \tag{63}$$

Furthermore, since the electric field is screened out of the body of the magnet, the center of energy motion of the magnet in Eq. (61) becomes the familiar Newton's second law connecting the center of mass motion with the net Lorentz force

$$\frac{\langle U_\mu\rangle}{c^2}\frac{d^2\overrightarrow{\mathbf{X}}_\mu}{dt^2} = \frac{d}{dt}\langle \mathbf{P}_\mu\rangle = \mathbf{F}^{Lorentz}_{on\,\mu} \tag{64}$$

The net Lorentz force on the magnet is exactly the original standard classical magnetic Lorentz force[28] on the magnet due to the magnetic fields of the passing charge,

$$\begin{aligned} \mathbf{F}^{Lorentz}_{on\,\mu} &= \left[\nabla_{\mathbf{r}}\left\{\overrightarrow{\mu}\cdot\mathbf{B}_q(\mathbf{r},t)\right\}\right]_{\mathbf{r}=0} = -\nabla_q\left\{\overrightarrow{\mu}\cdot\left(q\frac{\mathbf{v}_q}{c}\times\frac{(-\mathbf{r}_q)}{r_q^3}\right)\right\} \\ &= -\nabla_q\left\{\frac{q}{c}\mathbf{v}_q\cdot[\overrightarrow{\mu}\times\frac{\mathbf{r}_q}{r_q^3}]\right\} = -\frac{q}{c}\left(\mathbf{v}_q\cdot\nabla_q\right)\mathbf{A}_\mu(\mathbf{r}_q) \end{aligned} \tag{65}$$

where we have written the magnetic field of the charged particle evaluated at the origin as $\mathbf{B}_q = q\mathbf{v}\times(-\mathbf{r}_q)c^{-1}r_q^{-3}$, have used standard vector identities, have recognized the magnetic vector potential $\mathbf{A}_\mu(\mathbf{r}_q) = \overrightarrow{\mu}\times\mathbf{r}_q/r_q^3$ of the magnet at the position of the charged particle $q$, and have dropped the magnetic Lorentz force $(q/c)\mathbf{v}_q\times\mathbf{B}_\mu$ which vanishes for a point charge $q$ on the axis of a toroid. Newton's third law in Eq. (63) for the forces between the toroid and the passing charge requires that

$$\mathbf{F}^{Lorentz}_{on\,q} = \frac{q}{c}\left(\mathbf{v}_q\cdot\nabla_q\right)\mathbf{A}_\mu(\mathbf{r}_q) \tag{66}$$

While the electric velocity field of a passing charge is screened out of a good conductor, the magnetic field penetrates into a good conductor with a time integral which is independent of the conductivity of the ohmic material of the conductor.[17] Thus the magnetic field energy $U_{em}$ associated with the overlap of the toroid magnetic field and the point charge magnetic field[27] is

$$U_{em\,\mu q} = \frac{1}{8\pi}\int d^3r\, 2\mathbf{B}_q \cdot \mathbf{B}_\mu = q\frac{\mathbf{v}_q}{c} \cdot \mathbf{A}_\mu(\mathbf{r}_q) \tag{67}$$

just what was given for $\langle U_{em\,\mu q}\rangle$ in Eq. (45). Let us assume that this magnetic field energy is equal to the change in kinetic energy of the passing charge due to the electric fields from the changing charge and current densities of the magnet. Since the change in magnetic field energy is of order $1/c^2$, we need to consider only the nonrelativistic approximation to the passing particle kinetic energy. Then we find

$$\begin{aligned} \frac{1}{2}m_q\mathbf{v}_q^2 - \frac{1}{2}m_q\mathbf{v}_{q0}^2 &= U_{em\,\mu q} \\ m_q\mathbf{v}_{q0}\cdot\Delta\mathbf{v}_q &= \frac{q}{c}\mathbf{v}_{q0}\cdot\mathbf{A}_\mu(\mathbf{r}_q) \end{aligned} \tag{68}$$

where $\mathbf{v}_{q0}$ is the velocity of the charged particle $q$ when far from the magnet where $\mathbf{A}_\mu(\mathbf{r}_q)$ vanishes, and $\Delta\mathbf{v}_q$ is the change in the velocity of the passing charge. Thus we find

$$m_q\Delta\mathbf{v}_q = (q/c)\mathbf{A}_\mu(\mathbf{r}_q) \tag{69}$$

and the force on the passing charge is therefore

$$\mathbf{F}_{on\,q} = m_q d\mathbf{v}_q/dt = m_q d(\Delta\mathbf{v}_q)/dt = (q/c)\left(\mathbf{v}_q\cdot\nabla_q\right)\mathbf{A}_\mu(\mathbf{r}_q) \tag{70}$$

exactly as found in Eq. (66) from Newton's third law. Thus there is a certain consistency between our momentum and energy considerations. However, it should be noted that the kinetic energy change for the passing charge is assumed to be of the same sign as the change in energy of the magnetic field. Energy conservation thus requires that the charges carrying the currents of the toroid must absorb twice the kinetic energy change of the passing charge. If the currents of the toroid act in a fashion analogous to a battery in magnetic systems involving mechanical work, then such an energy balance is consistent with what is found for familiar magnetic systems.[29] We note that the energy absorbed by the center of mass motion of the magnet is of order $1/c^4$ and hence is negligible, since the recoil velocity of the center of energy of the toroidal magnet (which was initially at rest) is of the order of $1/c^2$ from Eq. (65).

One should note the difference in perspectives between the analysis given here in the classical-lag point of view and that suggested by proponents of the no-velocity-change point of view (those who support the quantum topological interpretation of the Aharonov-Bohm phase shift). It was pointed out by Coleman and Van Vleck,[4] and repeated above in Eq. (53), that the accelerations of the centers of energy for the toroid and the passing charge must be related as in Newton's third law. We have assumed that the electric field of the passing charge is screened out of the magnet, have obtained the force on the passing charge $q$ by assuming that it is the third law partner of the usual magnetic Lorentz force on the toroidal magnet, and then have shown that this force is directly related to the energy change in the magnetic fields which penetrate into the magnet. The no-velocity-change point of view claims that there is no force back on the passing charge, that the magnetic moment of the magnet does not change, and that the changing electromagnetic field momentum is associated with "hidden momentum in magnets" whose change "cancels" the magnetic Lorentz force on the magnet. This requires that the electric field of the passing charge should penetrate into the magnet so as to give the "hidden momentum," a penetration which seems contrary to the screening of electric fields by conductors. Furthermore, this point of view tells us nothing about magnetic energy changes between the passing charge and a toroid.

## VII. DISCUSSION

Although the Aharonov-Bohm phase shift is well known and is now standard in all the recent quantum mechanics texts, most physicists seem unaware of the long-standing controversy regarding the interpretation of the phase shift. In 1959, Aharonov and Bohm[2] solved the Schroedinger equation and predicted their phase shift. The phase shift has been observed experimentally.[30] Aharonov and Bohm attracted attention to their phase shift by claiming that their predicted phase shift occurred in the absence of classical electromagnetic forces and velocity changes and represented a new quantum topological effect with no analogue in classical theory. There is no experimental evidence for this claim. Indeed, the interpretation has aroused controversy. Most of the initial controversy regarding the Aharonov-Bohm phase shift centered on a distraction, whether or not the shift was caused by stray magnetic fields outside the solenoid or toroid. This aspect of the controversy has been removed by the toroidal experiments of Tonomura[31] which allow very little stray magnetic flux.

The suggestion that the Aharonov-Bohm phase shift might be based upon a classical lag effect involving classical electromagnetic forces and velocity changes (the suggestion repeated here) depends upon our understanding of classical electromagnetism. The conventional attitude regarding the Aharonov-Bohm phase shift is best stated by Aharonov, Pearle, and Vaidman:[5] "In the Aharonov-Bohm effect it is obvious that the electron is not subject to any electromagnetic force, because the magnetic field lies wholly within the filament and so is zero at the electron's location." This naive statement omits the crucial possibility of induced charge or current densities in the magnet leading to forces back on the passing charge. Indeed, induced currents do lead to forces back on passing charges; the phase shifts may well arise from classical lag effects.

In the 1970s, it was suggested that the possible influence of the electromagnetic fields of the passing charge could be removed by surrounding the solenoid or toroid by a conductor which would screen out the electromagnetic fields.[32] Experiment showed that the phase shift persisted even when the solenoid was surrounded by a conductor.[31] However, it was realized that although electric fields are indeed well screened by a conductor, magnetic velocity fields penetrate into an ohmic conductor (and also into superconductors at high frequencies) in a form which is completely different from the skin-depth behavior of wave fields, and indeed there is an invariant time integral which has precisely the correct form to account for the Aharonov-Bohm phase shift as an energy-related lag effect based on classical forces.[17] The experiments to date do no not remove the possibility of a classical electromagnetic basis for the Aharonov-Bohm phase shift.[13] In addition, it was pointed out that electrostatic forces can give interference pattern shifts which take exactly the same form as the Aharonov-Bohm phase shift.[10] Matteucci and Pozzi confirmed this experimentally in 1985.[33]

In 1984, Aharonov and Casher[3] suggested a second phase shift, this time for a magnetic moment passing a line charge, which they claimed was the duel of the Aharonov-Bohm phase shift and again occurred in the absence of classical forces and velocity changes. However, it was pointed out that conventional classical electromagnetic theory clearly predicted a force on a passing magnetic moment treated as a current loop, and Newton's second law suggested a lag effect.[12] To counter this observation, Aharonov, Pearle and Vaidman[5] introduced a new analysis for the interaction of a magnetic moment and a point charge, and claimed that the magnetic moment, although indeed experiencing a net Lorentz force, nevertheless moved as though it experienced no forces whatsoever, because of changes in "hidden momentum in magnets" cancelling the applied Lorentz force.

For the Aharonov-Bohm phase shift, the Aharonov-Casher phase shift, and the Shockley-James paradox, the heart of the controversy and paradox involves the interaction between a point charge and a magnetic moment through order $1/c^2$. Although the literature of the Aharonov-Bohm phase shift is full of statements about the interaction which claim to exclude any possibility of an explanation based upon classical electromagnetic forces[7], the claims often depend upon nonrelativistic models[34] or point to familiar effects, such as aspects of mutual inductance, which indeed will not give the desired behavior,[35] but overlook the 0-order forces on the charges of the magnet because the magnet is neutral. Coleman and Van Vleck have treated the interaction consistently relativistically using the Darwin Lagrangian. In the present work, we have followed the Darwin Lagrangian analysis. We have modeled the magnetic moment as a classical hydrogen atom interacting with the passing charge through the Darwin Lagrangian, and have noted particularly the nonrelativistic behavior of the magnetic moment pointed out by Solem. The model is unambiguous in its prediction of classical electromagnetic forces, energies, and changes of the center of energy. It is the 0-order accelerations which cause electric fields in order $1/c^2$ which act strongly on the passing charge.

The transition to a multiparticle limit still allows ambiguities. However, the assumption that in this limit the electric fields are screened out of the magnet while the magnetic fields penetrate into the magnet both fits with what is known for ohmic conductors and also allows for a consistent treatment of the conservation laws of relativistic theory. The discussion given here represents a refutation of the suggestions of Aharonov, Pearle, and Vaidman regarding the role of "hidden momentum in magnets" and confirms the semiclassical calculations of both the Aharonov-Bohm and Aharonov-Casher phase shifts based upon classical lag effects.[11][12] What is needed now are experiments to test whether or not the Aharonov-Bohm and Aharonov-Casher phase shifts occur in the presence or absence of velocity changes for the passing particles.[36]

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[20] See, for example, Jackson in ref. 8, p. 594.

[21] J. C. Solem, "The strange polarization of the classical atom," Am. J. Phys.**55**, 906-909 (1987); L. C. Biedenharn, L. S. Brown, and J. C. Solem, "Comment on 'The strange polarization of the classical atom,'" Am. J. Phys. **56**, 661-663 (1988). See also, J. C. Solem, "Variations on the Kepler Problem," Found. Phys. **27**, 1291-1306 (1997).

[22] See, for example, Jackson in ref. 8, p. 155.

[23] See, for example, H. Goldstein, *Classical Mechanics*, 2nd edn. (Addison-Wesley, Reading,

Massachusetts 1981), pp. 102-104. The sign and normalization have been chosen as in the work by Biedenharn, Brown, and Solem listed in ref. 21.

[24] The constancy of the system momentum **P** with respect to time can be checked by differentiating Eq. (36) with respect to time and then using the equations of motion (11)-(13).

[25] The constancy of the system energy $U$ with respect to time can be checked by differentiating Eq. (44) with respect to time and then using the equations of motion (11)-(13).

[26] "Origin of 'Hidden Momentum Forces' on Magnets" is the title of the work of Coleman and Van Vleck in ref. 4, and "hidden momentum in magnets" is elaborated in work by Aharonov, Pearle, and Vaidman in ref. 5, and later by Vaidman by ref.6. Indeed, "hidden momentum in magnets" has now reached the textbook literature; see D. J. Griffiths, *Introduction to Electrodynamics,* 3rd edn. (Prentice Hall, Upper Saddle River, New Jersey 1999), pp. 520-521. Our equation (61) is the same as equation (19) in Vaidman's article. It is not the equation appearing in the work of Aharonov, Pearle, and Vaidman, since these proponents of the no-velocity-change point of view are sure from the start that the magnetic moment $\overrightarrow{\mu}$ must not change, and therefore they do not allow the last time derivative in our Eq. (61) to fall on $\overrightarrow{\mu}$.

[27] T. H. Boyer, "Classical Electromagnetic Interaction of a Charged Particle with a Constant-Current Solenoid," Phys. Rev. D **8**, 1667-1679 (1973).

[28] See, for example, Jackson in ref. 8, p. 185.

[29] See, for example, T. H. Boyer, "Electric and magnetic forces and energies for a parallel-plate capacitor and a flattened, slip-joint solenoid," Am. J. Phys. **69**, 1277-1279 (2001).

[30] The phase shift was first observed by R. G. Chambers, "Shift of an electron interference pattern by enclosed magnetic flux," Phys. Rev. Lett. **5**, 3-5 (1960). M. Peshkin and A. Tonomura in ref. 7 give extensive references to the experimental tests.

[31] A. Tonomura, N. Osakabe, T. Matsuda, T. Kawasaki, J. Endo, S. Yano, and H. Yamada, "Evidence for the Aharonov-Bohm Effect with Magnetic Field Completely Shielded from Electron Wave," Phys. Rev. Lett. **56**, 792-795 (1986).

[32] I proposed such experiments in preprints in the early 1970s and at a meeting with Professor Moellenstedt in Tubingen during the summer of 1972. However, by the time my manuscripts were accepted for publication in 1973, I realized that the penetration situation for electromagnetic velocity fields was quite different from that for wave fields.

[33] G. Matteucci and G. Pozzi, "New diffraction experiment on the electrostatic Aharonov-Bohm

effect," Phys. Rev. Lett. **54**, 2469-2470 (1985).

[34] See, for example, M. Peshkin, I. Talmi, and L. J. Tassie, "The quantum mechanical effects of magnetic fields confined to inaccessible regions," Ann. Phys. (NY) **12**, 426-435 (1961), especially Sec. V.

[35] The changing magnetic flux through the magnetic moment (or toroid) due to the point charge's magnetic field causes a back force which is of the order of $1/c^4$, (since it does not begin with accelerations due to the Coulomb force), is independent of the magnetic moment of the toroid, and (following Lenz's law) is always such as to try to slow the charge. The force discussed in Eq. (35) here is totally different, always acting to both accelerate and decelerate the point charge with the time order of the acceleration and deceleration depending upon the orientation of the magnetic moment and the sign of the charge.

[36] See the preceding article, T. H. Boyer, "The Paradoxical Forces for the Classical Electromagnetic Lag Associated with the Aharonov-Bohm Phase Shift," submitted for publication.