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General connection between random electrodynamics and quantum electrodynamics for free electromagnetic fields and for dipole oscillator systems

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A general comparison is presented between random electrodynamics and quantum electrodynamics for the two systems which can be solved exactly in both theories, free electromagnetic fields and point dipole oscillators. The N-point correlation functions of the fields are computed in both theories and are found to differ in general because of the dependence upon the order of the quantum operators within products of operators. However, if all products of quantum operators are symmetrized by taking all permutations of the operator order, then the two theories give identical results for the correlation functions. Analogous results hold to all orders in the fine-structure constant for dipole oscillators in quantum and random electrodynamics. The theories agree only if the quantum operator products are symmetrized. In the limit that the oscillator couplings to the radiation fields vanish, the oscillators can be regarded as mechanical oscillators in quantum mechanics and in random mechanics. The theory of random mechanics is defined in terms of this limit which uncouples a mechanical oscillator from the radiation field. The average values of oscillator variables in random mechanics agree with those of symmetrized products in quantum mechanics. The question is then raised as to the physical significance of the many quantum operators which differ only in the order of their factors. It is pointed out that some operator products which are regarded as physically important, such as the square of the angular momentum, indeed involve unsymmetrized products of operators. On this account the average values of the angular momentum squared in the ground state of an isotropic three-dimensional harmonic oscillator differ between the random-mechanical and quantum-mechanical descriptions. However, there seems to be no case in which experiments have shown that the (unsymmetrized) quantum operator value is to be preferred to that provided by random mechanics. The presence of thermal radiation is next treated for free electromagnetic fields and for dipole-oscillator systems. Despite extraordinary differences in the points of view toward thermal radiation taken by the two theories, the conclusion is the same as that found for zero temperature; the two theories agree in their average values if all products of quantum operators are symmetrized. Finally, as a further example of the power of random electrodynamics to give an account of phenomena where Lorentz's classical electron theory failed, we investigate the diamagnetism of a charged three-dimensional isotropic oscillator. The mathematical descriptions at finite temperature are developed in full random electrodynamics and quantum electrodynamics and in second-order perturbation theory in quantum mechanics.

I. INTRODUCTION

In the past few years, there have been a number of classical electromagnetic calculations[1-11] which provide results in agreement with quantum electrodynamics. The results obtained include the blackbody radiation spectrum,[5] the fluctuations in thermal radiation,[6] the third law of thermodynamics,[7] rotator and oscillator specific heats,[5,8] the Van der Waals forces between macroscopic objects[3-4] and between polarizable particles.[9-11] The classical electromagnetic theory[12] involved in these calculations changes the boundary condition on Maxwell's equations to correspond to the presence of random classical electromagnetic zero-point radiation. The new theory, termed random electrodynamics, is Lorentz's classical electron theory[13,14] with this new boundary condition.

Clearly we would like to understand the general connections and contrasts between this classical theory and quantum electrodynamics. However, the mathematical difficulties in treating random radiation have limited the physical situations which have been solved thus far. In this paper we provide a general comparison between random and quantum electrodynamics for the two physical systems which can be solved exactly in both theories—free electromagnetic fields and charged point oscillator systems.

The work presented here seems of interest for several reasons. In the first place, random electrodynamics is a new, relatively unexplored theory, and hence the comparison with well-established quantum electrodynamics indicates something of the viability of the new theory. The second reason involves calculational facility. There are instances where it is easier to perform classical electromagnetic calculations in random electrodynamics rather than quantum perturbation theory calculations. Some recent work[9-11] on Van der Waals forces illustrates this point. By understanding the general connections between random and quantum electrodynamics, we make available the possibility of performing classical calculations

with assurance of agreement with the quantum results. Thirdly, here for the first time we find quantum electrodynamics confronted with a classical theory in which Planck's constant $\hbar$ does not vanish. This confrontation raises questions as to what are the crucial quantum aspects of quantum electrodynamics. The confrontation reopens questions on the meaning of quantum operator order, on the sharpness of energy values, on the assignment of angular momentum values, on the significance of excited states, and on the connections between quantum mechanics and quantum electrodynamics.

Following this Introduction, the paper is broken into four basic parts. The first compares the vacuum free-field theories of random and quantum electrodynamics. The second compares the theories when charged harmonic oscillators are coupled to radiation, and then explores the limit when this coupling vanishes. Here we find some interesting ideas on quantum operator order as related to energy and angular momentum eigenstates. The third part introduces thermal radiation, and then again compares the free-field and harmonic-oscillator descriptions of the random and quantum theories. Finally in the fourth part we show again that despite differences in the ideas of eigenstates and even in values for the angular momentum, the average energy values agree between the theories. In this case, we present a calculation for harmonic-oscillator systems in an external magnetic field and discuss the diamagnetic behavior in random and quantum electrodynamics.

II. FREE-FIELD CORRELATIONS IN RANDOM ELECTRODYNAMICS AND QUANTUM ELECTRODYNAMICS

A. Vacuum fluctuations in the free electromagnetic fields

The vacuum fluctuations in the quantum electromagnetic field are vaguely familiar to most physicists. Various textbooks assure us that these fluctuations are connected with the Heisenberg uncertainty principle for particles. Welton[15] has connected these fluctuations with the Lamb shift and low-energy Compton scattering. On the other hand, fluctuations in the classical electromagnetic field are present in random electrodynamics.[12] The fluctuations enter as one of the postulates on the homogeneous boundary conditions on Maxwell's equations. Here in part II we will explore the character of the fluctuations in the two theories by computing the correlation functions for the electromagnetic fields.

The connection between the two-point correlation functions in free-field quantum electrodynamics and in random electrodynamics was first presented by Marshall[2] in 1965. We will first review this work. Then we will break up the random phase part of the mathematical expression for the classical fields so as to introduce terms which stand as analogs to the annihilation and creation operators in quantum electrodynamics. Pursuing the analogy, we will evaluate the N-point correlation functions for the free fields in random electrodynamics and in quantum electrodynamics.

B. Plane-wave expansions for the free fields

The free electromagnetic fields satisfy Maxwell's homogeneous equations and hence allow expansions in terms of transverse plane waves. In random electrodynamics[12] the fields in empty space are

$$\vec{E}(\vec{r},t)=\sum_{\lambda=1}^{2}\int d^3k\,\hat{\epsilon}(\vec{k},\lambda)\,\mathfrak{h}(\vec{k},\lambda)\times\cos[\vec{k}\cdot\vec{r}-\omega t+\theta(\vec{k},\lambda)]\ , \tag{1}$$

$$\vec{B}(\vec{r},t)=\sum_{\lambda=1}^{2}\int d^3k\,\frac{\vec{k}\times\hat{\epsilon}(\vec{k},\lambda)}{k}\,\mathfrak{h}(\vec{k},\lambda)\times\cos[\vec{k}\cdot\vec{r}-\omega t+\theta(\vec{k},\lambda)]\ . \tag{2}$$

Here the polarization vectors satisfy

$$\vec{k}\cdot\hat{\epsilon}(\vec{k},\lambda)=0,\quad \hat{\epsilon}(\vec{k},\lambda)\cdot\hat{\epsilon}(\vec{k},\lambda')=\delta_{\lambda\lambda'}\ , \tag{3}$$

$$\sum_{\lambda=1}^{2}\epsilon_i(\vec{k},\lambda)\,\epsilon_j(\vec{k},\lambda)=\delta_{ij}-\frac{k_i k_j}{k^2}\ , \tag{4}$$

and the frequency ω is

$$\omega=ck$$
$$=c(k_x^{\,2}+k_y^{\,2}+k_z^{\,2})^{1/2}\ .$$

The quantity $\mathfrak{h}(\vec{k},\lambda)$ is a number giving the scale

$$\pi^2\mathfrak{h}^2(\vec{k},\lambda)=\tfrac{1}{2}\hbar\omega\ , \tag{5}$$

and $\theta(\vec{k},\lambda)$ is a random phase distributed uniformly on $[0,2\pi]$, independently distributed for each $\vec{k}$ and λ.

The free quantum electromagnetic field also satisfies Maxwell's homogeneous equations. Here it is convenient to write the expansion in transverse plane waves in the form[16]

$$\underline{\vec{E}}(\vec{r},t)=\sum_{\lambda=1}^{2}\int d^3k\,\hat{\epsilon}(\vec{k},\lambda)\frac{(\hbar\omega)^{1/2}}{2\pi}\,[\underline{a}(\vec{k},\lambda)\exp(-i\omega t+i\vec{k}\cdot\vec{r})+\underline{a}^{\dagger}(\vec{k},\lambda)\exp(i\omega t-i\vec{k}\cdot\vec{r})]\ , \tag{6}$$

$$\underline{\vec{B}}(\vec{r}, t) = \sum_{\lambda=1}^{2} \int d^3k \, \frac{\vec{k}\times\hat{\epsilon}(\vec{k}, \lambda)}{k} \, \frac{(\hbar\omega)^{1/2}}{2\pi} \left[\underline{a}(\vec{k}, \lambda) \exp(-i\omega t + i\vec{k}\cdot\vec{r}) + \underline{a}^{\dagger}(\vec{k}, \lambda) \exp(i\omega t - i\vec{k}\cdot\vec{r})\right] . \tag{7}$$

The polarization vectors here are just those listed above in (3) and (4). The operators $\underline{a}(\vec{k}, \lambda)$ and $\underline{a}^{\dagger}(\vec{k}, \lambda)$ are annihilation and creation operators on the Hilbert space, satisfying the commutation rules

$$[\underline{a}(\vec{k}, \lambda), \underline{a}(\vec{k}', \lambda')] = [\underline{a}^{\dagger}(\vec{k}, \lambda), \underline{a}^{\dagger}(\vec{k}', \lambda')] = 0 , \tag{8}$$

$$[\underline{a}(\vec{k}, \lambda), \underline{a}^{\dagger}(\vec{k}', \lambda')] = \delta_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k}') . \tag{9}$$

C. Evaluation of the two-point correlation function

The fluctuations in the electromagnetic field may be characterized in part by the field correlation functions at two different points in space and time. Thus in random electrodynamics, the correlation is obtained by averaging over the random phases

$$\begin{aligned}\langle E_i(\vec{r}_1, t_1) E_j(\vec{r}_2, t_2)\rangle &= \sum_{\lambda_1=1}^{2} \sum_{\lambda_2=1}^{2} \int d^3k_1 \int d^3k_2 \, \epsilon_i(\vec{k}_1, \lambda_1) \epsilon_j(\vec{k}_2, \lambda_2) \mathfrak{h}(\vec{k}_1, \lambda_1) \mathfrak{h}(\vec{k}_2, \lambda_2) \\ &\quad \times \langle \cos[\vec{k}_1\cdot\vec{r}_1 - \omega_1 t_1 + \theta(\vec{k}_1, \lambda_1)] \cos[\vec{k}_2\cdot\vec{r}_2 - \omega_2 t_2 + \theta(\vec{k}_2, \lambda_2)]\rangle \\ &= \sum_{\lambda=1}^{2} \int d^3k \, \epsilon_i(\vec{k}, \lambda) \epsilon_j(\vec{k}, \lambda) \mathfrak{h}^2(\vec{k}, \lambda) \tfrac{1}{2} \cos[\vec{k}\cdot(\vec{r}_1 - \vec{r}_2) - \omega(t_1 - t_2)] \\ &= \int d^3k \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \frac{\hbar\omega}{4\pi^2} \cos[\vec{k}\cdot(\vec{r}_1 - \vec{r}_2) - \omega(t_1 - t_2)] .\end{aligned} \tag{10}$$

Here we have used the averages

$$\begin{aligned}\langle \cos\theta(\vec{k}_1, \lambda_1) \cos\theta(\vec{k}_2, \lambda_2)\rangle &= \langle \sin\theta(\vec{k}_1, \lambda_1) \sin\theta(\vec{k}_2, \lambda_2)\rangle \\ &= \tfrac{1}{2}\delta_{\lambda_1\lambda_2} \delta^3(\vec{k}_1 - \vec{k}_2)\end{aligned} \tag{11}$$

and

$$\langle \cos\theta(\vec{k}_1, \lambda_1) \sin\theta(\vec{k}_2, \lambda_2)\rangle = 0 , \tag{12}$$

and we have introduced Eqs. (3)–(5). By similar calculations we obtain

$$\langle B_i(\vec{r}_1, t_1) B_j(\vec{r}_2, t_2)\rangle = \langle E_i(\vec{r}_1, t_1) E_j(\vec{r}_2, t_2)\rangle \tag{13}$$

and

$$\langle E_i(\vec{r}_1, t_1) B_j(\vec{r}_2, t_2)\rangle = \int d^3k \, \epsilon_{ijl} \frac{k_l}{k} \frac{\hbar\omega}{4\pi^2} \cos[\vec{k}\cdot(\vec{r}_1 - \vec{r}_2) - \omega(t_1 - t_2)] . \tag{14}$$

The analogous expressions can be obtained in quantum electrodynamics from the expansions (6)–(9). Thus we compute the vacuum expectation values

$$\langle 0 | \underline{E}_i(\vec{r}_1, t_1) \underline{E}_j(\vec{r}_2, t_2) | 0\rangle = \int d^3k \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \frac{\hbar\omega}{4\pi^2} \exp[i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2) - i\omega(t_1 - t_2)] , \tag{15}$$

$$\langle 0 | \underline{B}_i(\vec{r}_1, t_1) \underline{B}_j(\vec{r}_2, t_2) | 0\rangle = \langle 0 | \underline{E}_i(\vec{r}_1, t_1) \underline{E}_j(\vec{r}_2, t_2) | 0\rangle , \tag{16}$$

$$\langle 0 | \underline{E}_i(\vec{r}_1, t_1) \underline{B}_j(\vec{r}_2, t_2) | 0\rangle = \int d^3k \, \epsilon_{ijl} \frac{k_l}{k} \frac{\hbar\omega}{4\pi} \exp[i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2) - i\omega(t_1 - t_2)] . \tag{17}$$

D. Problem of operator order in quantum electrodynamics

The quantum results in (15)–(17) do not agree with the analogous correlations (10), (13), and (14) in random electrodynamics. However, the discrepancies can easily be seen to be related to a problem of quantum operator order. In random electrodynamics, the order of the fields is of no significance since these are classical fields,

$$\langle E_i(\vec{r}_1, t_1) E_j(\vec{r}_2, t_2)\rangle = \langle E_j(\vec{r}_2, t_2) E_i(\vec{r}_1, t_1)\rangle . \tag{18}$$

On the other hand, in quantum electrodynamics, the operators do not commute so that

$$\langle 0|\underline{E}_i(\vec{r}_1,t_1)\underline{E}_j(\vec{r}_2,t_2)|0\rangle = \langle 0|\underline{E}_j(\vec{r}_2,t_2)\underline{E}_i(\vec{r}_1,t_1)|0\rangle + \int d^3k\left(\delta_{ij}-\frac{k_i k_j}{k^2}\right)\frac{\hbar\omega}{2\pi^2}\, i\sin[\vec{k}\cdot(\vec{r}_1-\vec{r}_2)-\omega(t_1-t_2)]\,. \quad (19)$$

Physically, which is the correct operator order for the comparison with the classical theory? Quantum theory usually assigns Hermitian operators to physical observables. Thus if we take the Hermitian combination $\frac{1}{2}(\underline{E}_i\underline{E}_j+\underline{E}_j\underline{E}_i)$ of these Hermitian fields, then there is exact agreement between the correlations in quantum electrodynamics and random electrodynamics:

$$\langle 0|\tfrac{1}{2}[\underline{E}_i(\vec{r}_1,t_1)\underline{E}_j(\vec{r}_2,t_2)+\underline{E}_j(\vec{r}_2,t_2)\underline{E}_i(\vec{r}_1,t_1)]|0\rangle = \langle E_i(\vec{r}_1,t_1)E_j(\vec{r}_2,t_2)\rangle\,. \quad (20)$$

This agreement with the expectation value of the Hermitian combination of the quantum operators also holds for the other two-point functions (13) and (14). The problem encountered here of operator order in quantum electrodynamics will be a recurring one throughout this comparison between quantum electrodynamics and random electrodynamics.

E. Rewriting the random phase terms in analogy with quantum annihilation and creation operators

Although the form for the random electromagnetic field given in Eq. (1) has been used repeatedly in various calculations,[5-12] it will be convenient to rewrite the expression in a manner reminiscent of the annihilation and creation operators of quantum electrodynamics. Thus the random field (1) is exactly the same as

$$\vec{E}(\vec{r},t)=\sum_{\lambda=1}^{2}\int d^3k\,\hat{\epsilon}(\vec{k},\lambda)\frac{\hbar(\vec{k},\lambda)}{2}\left[a(\vec{k},\lambda)\exp(-i\omega t+i\vec{k}\cdot\vec{r})+a^*(\vec{k},\lambda)\exp(i\omega t-i\vec{k}\cdot\vec{r})\right]\,, \quad (21)$$

where we define

$$a(\vec{k},\lambda)\equiv\exp[i\theta(\vec{k},\lambda)] \quad (22)$$

and

$$a^*(\vec{k},\lambda)\equiv\exp[-i\theta(\vec{k},\lambda)]\,. \quad (23)$$

Then the random character of the radiation contained in the random phase $\theta(\vec{k},\lambda)$ appears only in the expressions $a(\vec{k},\lambda)$ and $a^*(\vec{k},\lambda)$. On averaging over the random phases, we see

$$\langle a(\vec{k},\lambda)a(\vec{k}',\lambda')\rangle = \langle a^*(\vec{k},\lambda)a^*(\vec{k}',\lambda')\rangle = 0\,, \quad (24)$$

$$\langle a(\vec{k},\lambda)a^*(\vec{k}',\lambda')\rangle = \langle a^*(\vec{k}',\lambda')a(\vec{k},\lambda)\rangle = \delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}')\,. \quad (25)$$

These expressions are to be compared with the vacuum expectation values for the quantum annihilation and creation operators,

$$\langle 0|\underline{a}(\vec{k},\lambda)\underline{a}(\vec{k}',\lambda')|0\rangle = \langle 0|\underline{a}^\dagger(\vec{k},\lambda)\underline{a}^\dagger(\vec{k}',\lambda')|0\rangle = 0\,, \quad (26)$$

$$\langle 0|\underline{a}(\vec{k},\lambda)\underline{a}^\dagger(\vec{k}',\lambda')|0\rangle = \delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}')\,, \quad (27)$$

$$\langle 0|\underline{a}^\dagger(\vec{k}',\lambda')\underline{a}(\vec{k},\lambda)|0\rangle = 0\,. \quad (28)$$

We notice in Eqs. (27) and (28) the importance of quantum operator order, something that does not arise in the classical random variables.

F. General N-point function in free-field random electrodynamics

Although it is interesting to find the agreement between random electrodynamics and quantum electrodynamics for the two-field correlation functions, we wish to investigate the general connection between the theories and so consider the correlations between products of many fields. In random electrodynamics this may be written in the form

$$\langle E_{i_1}(x_1)E_{i_2}(x_2)\cdots E_{i_{2n}}(x_{2n})\rangle = \sum_{\lambda_1=1}^{2}\sum_{\lambda_2=1}^{2}\cdots\sum_{\lambda_{2n}=1}^{2}\int d^3k_1\int d^3k_2\cdots\int d^3k_{2n}\,\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_{2n}}\frac{\hbar_1}{2}\frac{\hbar_2}{2}\cdots\frac{\hbar_{2n}}{2}$$
$$\times\langle[a_1\exp(-iK_1\cdot x_1)+a_1^*\exp(iK_1\cdot x_1)][a_2\exp(-iK_2\cdot x_2)+a_2^*\exp(iK_2\cdot x_2)]\cdots$$
$$\times[a_{2n}\exp(-iK_{2n}\cdot x_{2n})+a_{2n}^*\exp(iK_{2n}\cdot x_{2n})]\rangle\,. \quad (29)$$

For compactness we have introduced four-vector notation, so that here $E_{i_1}(x_1)$ stands for $E_{i_1}(\vec{r}_1,t_1)$, and

$K_1 \cdot x_1$ stands for $\omega_1 t_1 - \vec{k}_1 \cdot \vec{r}_1$. In order to evaluate the correlation function, we must average over the random phases as in (24) and (25). For higher products of random variables a, a^*, we find, for example,

$$\langle a_l a_m a_n^* \rangle = \langle \exp(i\theta_l) \exp(i\theta_m) \exp(-i\theta_n) \rangle = 0 , \tag{30}$$

$$\langle a_k a_l^* a_m a_n^* \rangle = \langle \exp(i\theta_k) \exp(-i\theta_l) \exp(i\theta_m) \exp(-i\theta_n) \rangle = \delta_{kl}\delta_{mn} + \delta_{kn}\delta_{lm} . \tag{31}$$

In averaging over the random phases $\theta(\vec{k}_1, \lambda_1), \ldots, \theta(\vec{k}_{2n}, \lambda_{2n})$ in Eq. (29), the nonzero contributions involve a complete pairing of all possible terms $a_l = \exp(-i\theta_l)$ with all possible terms $a_m^* = \exp(i\theta_m)$. Such a complete pairing is possible only if $2n$ is even. Thus the term in angular brackets in (29) becomes

$$\langle\ \rangle = \sum_\sigma \frac{1}{n!} \{\exp[-iK_{\sigma_1} \cdot (x_{\sigma_1} - x_{\sigma_2})] \delta_{\lambda(\sigma_1)\lambda(\sigma_2)} \delta^3(\vec{k}_{\sigma_1} - \vec{k}_{\sigma_2}) \times \exp[-iK_{\sigma_3} \cdot (x_{\sigma_3} - x_{\sigma_4})] \delta_{\lambda(\sigma_3)\lambda(\sigma_4)} \delta^3(\vec{k}_{\sigma_3} - \vec{k}_{\sigma_4}) \cdots \times \exp[-iK_{\sigma_{2n-1}} \cdot (x_{\sigma_{2n-1}} - x_{\sigma_{2n}})] \delta_{\lambda(\sigma_{2n-1})\lambda(\sigma_{2n})} \delta^3(\vec{k}_{\sigma_{2n-1}} - \vec{k}_{\sigma_{2n}})\} , \tag{32}$$

where the sum $\sum_\sigma$ is over all permutations of the integers $(1, 2, \ldots, 2n)$. The factor of $(1/n!)$ arises because the interchange of the n pairs $(\sigma_1\sigma_2)$ $(\sigma_3\sigma_4)\cdots(\sigma_{2n-1}\sigma_{2n})$ does not correspond to a new way of pairing the random phases. Hence the $n!$ permutations on these pairs must be removed as repetitive in the sum over all permutations. The presence of permutations involving the interchange of the pairs $(\sigma_1\sigma_2)$ into $(\sigma_2\sigma_1)$, etc. comes from the presence of both a and a^* as equal contributors in the expression (21) for $E_i(x)$. What is involved can be easily written out explicitly for the four-point function. Here there are 24 permutations of 1, 2, 3, 4. However, the pairs such as (12) (34) and (34) (12) do not give distinct contributions. Hence only 12 pairings of the random phases actually contribute.

G. General N-point function in free-field quantum electrodynamics

The result found by combining (29), (32), and (5) seems a complicated expression. What interests us, however, is not its value, but its connection with an analogous expression in free-field quantum electrodynamics. Thus we consider the vacuum expectation value of the quantum free fields

$$\langle 0 | \underline{E}_{i_1}(x_1) \underline{E}_{i_2}(x_2) \cdots \underline{E}_{i_{2n}}(x_{2n}) | 0 \rangle = \sum_{\lambda_1=1}^{2} \sum_{\lambda_2=1}^{2} \cdots \sum_{\lambda_{2n}=1}^{2} \int d^3k_1 \int d^3k_2 \cdots \int d^3k_{2n}\, \epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_{2n}} \frac{(\hbar\omega_1)^{1/2}}{2\pi} \frac{(\hbar\omega_2)^{1/2}}{2\pi} \cdots \frac{(\hbar\omega_{2n})^{1/2}}{2\pi} \times \langle 0 | [\underline{a}_1 \exp(-iK_1 \cdot x_1) + \underline{a}_1^\dagger \exp(iK_1 \cdot x_1)][\underline{a}_2 \exp(-iK_2 \cdot x_2) + \underline{a}_2^\dagger \exp(iK_2 \cdot x_2)] \cdots \times [\underline{a}_{2n} \exp(-iK_{2n} \cdot x_{2n}) + \underline{a}_{2n}^\dagger \exp(iK_{2n} \cdot x_{2n})] | 0 \rangle . \tag{33}$$

Here the operator $\underline{a}_1^\dagger$ acting to the left on the vacuum and $\underline{a}_{2n}$ acting to the right each give zero. Thus the end brackets involve only $\underline{a}_1$ and $\underline{a}_{2n}^\dagger$, respectively. Now we may move $\underline{a}_1$ through the other bracket over toward the vacuum state on the right. On moving $\underline{a}_1$ through the mth bracket, we pick up a contribution

$$\exp[-iK_1 \cdot (x_1 - x_m)] \delta_{\lambda_1\lambda_m} \delta^3(\vec{k}_1 - \vec{k}_m) .$$

When $\underline{a}_1$ acts on $|0\rangle$, then it gives zero. However, the expressions obtained on moving $\underline{a}_1$ to the right are products of $2n-2$ field operators, and we may repeat the same procedure with the left-hand-most annihilation and creation operators. Clearly if $2n$ is odd, the entire expression vanishes. If $2n$ is even, the resulting expression between the vacuum states in (33) is

$$\langle 0 | \cdots | 0 \rangle = \sum_\gamma \exp[-iK_{\gamma_1} \cdot (x_{\gamma_1} - x_{\gamma_2})] \delta_{\lambda(\gamma_1)\lambda(\gamma_2)} \delta^3(\vec{k}_{\gamma_1} - \vec{k}_{\gamma_2}) \exp[-iK_{\gamma_3} \cdot (x_{\gamma_3} - x_{\gamma_4})] \delta_{\lambda(\gamma_3)\lambda(\gamma_4)} \delta^3(\vec{k}_{\gamma_3} - \vec{k}_{\gamma_4}) \cdots \times \exp[-iK_{\gamma_{2n-1}} \cdot (x_{\gamma_{2n-1}} - x_{\gamma_{2n}})] \delta_{\lambda(\gamma_{2n-1})\lambda(\gamma_{2n})} \delta^3(\vec{k}_{\gamma_{2n-1}} - \vec{k}_{\gamma_{2n}}) . \tag{34}$$

Here γ is a permutation of the integers $(1, 2, \ldots, 2n)$, but not all such permutations are included. We must have that

$$\gamma_1 < \gamma_2, \quad \gamma_3 < \gamma_4, \quad \ldots, \gamma_{2n-1} < \gamma_{2n} \tag{35}$$

and also that

$$\gamma_1 < \gamma_3 < \gamma_5 < \cdots < \gamma_{2n-1} \tag{36}$$

because the order of the quantum operators was such that $\underline{E}_i(x_1)$ stood furthest to the left, and only the annihilation operator of $\underline{E}_i(x_1)$ contributed, etc.

Thus again we see that the corresponding values between random and quantum electrodynamics do not agree directly because of the appearance of distinct values associated with different operator orders in quantum theory. However, if we consider the fully symmetrized vacuum expectation value

$$\sum_\sigma \frac{1}{(2n)!} \langle 0 | \underline{E}_{i(\sigma_1)}(x_{\sigma_1}) \underline{E}_{i(\sigma_2)}(x_{\sigma_2}) \cdots \underline{E}_{i(\sigma_{2n})}(x_{\sigma_{2n}}) | 0 \rangle ,$$

where the sum is over all permutations σ, then the preferred ordering $\gamma_1 < \gamma_2$, etc., disappears. The number of permutations γ contributing in (34) is $(2n)!/(2^n n!)$ where the 2^n corresponds to removal of all terms, involving the interchange of γ_{2m-1} and γ_{2m}, and the $n!$ corresponds to removal of all interchanges of pairs of terms such as $(\gamma_1\gamma_2)(\gamma_3\gamma_4)\cdots(\gamma_{2n-1}\gamma_{2n})$ into $(\gamma_3\gamma_4)(\gamma_1\gamma_2)\cdots(\gamma_{2n-1}\gamma_{2n})$. The fully symmetrized expectation value contains repetitions of all these permutations which appeared in the initial unsymmetrized calculation (34). Thus

$$\begin{aligned}
&\langle 0 | \{\underline{E}_{i_1}(x_1) \underline{E}_{i_2}(x_2) \cdots \underline{E}_{i_{2n}}(x_{2n})\}_{\text{sym}} | 0 \rangle \\
&\quad = \sum_\sigma \frac{1}{(2n)!} \langle 0 | \underline{E}_{i(\sigma_1)}(x_{\sigma_1}) \underline{E}_{i(\sigma_2)}(x_{\sigma_2}) \cdots \underline{E}_{i(\sigma_{2n})}(x_{\sigma_{2n}}) | 0 \rangle \\
&\quad = \sum_{\lambda_1=1}^{2} \sum_{\lambda_2=1}^{2} \cdots \sum_{\lambda_{2n}=1}^{2} \int d^3k_1 \int d^3k_2 \cdots \int d^3k_{2n} \epsilon_{i_1} \epsilon_{i_2} \cdots \epsilon_{i_{2n}} \frac{(\hbar\omega_1)^{1/2}}{2\pi} \frac{(\hbar\omega_2)^{1/2}}{2\pi} \cdots \frac{(\hbar\omega_{2n})^{1/2}}{2\pi} \\
&\qquad \times \sum_\sigma \frac{1}{(2n)!} \frac{(2n)!}{2^n n!} \exp[-iK_{\sigma_1} \cdot (x_{\sigma_1} - x_{\sigma_2})] \delta_{\lambda(\sigma_1)\lambda(\sigma_2)} \delta^3(\vec{k}_{\sigma_1} - \vec{k}_{\sigma_2}) \\
&\qquad \times \exp[-iK_{\sigma_3} \cdot (x_{\sigma_3} - x_{\sigma_4})] \delta_{\lambda(\sigma_3)\lambda(\sigma_4)} \delta^3(\vec{k}_{\sigma_3} - \vec{k}_{\sigma_4}) \cdots \\
&\qquad \times \exp[-iK_{\sigma_{2n-1}} \cdot (x_{\sigma_{2n-1}} - x_{\sigma_{2n}})] \delta_{\lambda(\sigma_{2n-1})\lambda(\sigma_{2n})} \delta^3(\vec{k}_{\sigma_{2n-1}} - \vec{k}_{\sigma_{2n}}) .
\end{aligned} \tag{37}$$

The factors of $(2n)!$ cancel in the last form of (37) and the factor $2^n = (2^{1/2})^{2n}$ exactly compensates for the difference in normalization constants appearing in the random expression (21), (5), and the quantum expression (6). It is clear from the analogy between a and $\underline{a}$, and between a^* and $\underline{a}^\dagger$ that the arguments given here can be repeated for correlation functions involving the magnetic fields.

Thus provided the quantum operator order is fully symmetrized within quantum electrodynamics, we find precise agreement with random electrodynamics for the free-field correlations.

III. HARMONIC-OSCILLATOR SYSTEMS IN RANDOM AND QUANTUM ELECTRODYNAMICS

A. Quantum mechanics and random mechanics as derived theories

Within current physical theory, quantum behavior is regarded as fundamental, an inherent property of all systems regardless of the forces involved in their interactions. In particular, a harmonic oscillator may be a quantum operator, although it has no electromagnetic interactions. However, if the oscillator does have electromagnetic interactions, then the quantum behavior of the oscillator is tied neatly with the quantum behavior of the electromagnetic field.

This is not at all the view taken in random electrodynamics. Within this classical theory, all the random motion of particle systems at zero temperature is associated with the random forces due to the electromagnetic coupling of the system to the random zero-point radiation. Particles have no inherent random motion of their own.

In order to analyze the connections between these two quite different points of view apparent in the quantum and random theories, we will start with the case where the particle systems have electromagnetic interactions, and only then proceed to the limit where the particles are uncoupled from the radiation field. In this uncoupled limit, we find a description of mechanical harmonic oscillators in accord with quantum mechan-

ics in the quantum case, and we find a new theory called random mechanics in the random case.

B. Dipole-oscillator equations of motion

We consider a point electric dipole oscillator of dipole moment $ex\hat{i}$ located at the position $\vec{R}_0$ and oriented along the x axis. The Hamiltonian[17] for this system is given by

$$H = \frac{1}{2m_0}\left(p_x - \frac{e}{c}A_x(\vec{R}_0)\right)^2 + \tfrac{1}{2}m_0\omega_0^2 x^2 + \frac{1}{8\pi}\int (E^2+B^2)\,d^3R\ , \tag{38}$$

where m_0 is the bare mass, ω_0 is the natural frequency of the oscillator, and the coordinates in space are denoted by

$$\vec{R} = \hat{i}X + \hat{j}Y + \hat{k}Z\ .$$

The connections between random and quantum electrodynamics are seen most easily in the Heisenberg picture. In this case, the quantum operator equations of motion are identical with those of classical theory. In the Coulomb gauge, these are

$$m_0\ddot{x} = -m_0\omega_0^2 x + eE_x(\vec{R}_0, t)\ , \tag{39}$$

$$\nabla^2\Phi = -4\pi\rho\ , \tag{40}$$

$$\nabla^2\vec{A} - \frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J}^{\perp}\ , \tag{41}$$

with ρ the charge density associated with the point dipole, $\vec{J}^{\perp}$ the transverse current density

$$\vec{J}^{\perp}(\vec{R}, t) = \frac{1}{4\pi}\nabla\times\nabla\times\int d^3R'\,\frac{e\hat{i}\dot{x}\,\delta^3(\vec{R}'-\vec{R}_0)}{|\vec{R}-\vec{R}'|} \tag{42}$$

and

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\ . \tag{43}$$

Here the scalar potential is not regarded as a dynamical variable but stands for exactly

$$\Phi(\vec{R}) = \frac{exX}{|\vec{R}-\vec{R}_0|^3}\ . \tag{44}$$

The vector potential is given by

$$\vec{A}(\vec{R}, t) = \vec{A}^{\text{in}}(\vec{R}, t) + \frac{1}{c}\int d^3R'\,\frac{[\vec{J}^{\perp}(\vec{R}', t)]_{\text{ret}}}{|\vec{R}-\vec{R}'|}\ , \tag{45}$$

where $\vec{A}^{\text{in}}(\vec{R}, t)$ corresponds to the free electromagnetic field in quantum and random electrodynamics.

It is now possible to substitute the expression obtained for the electric field $\vec{E}$ in terms of x and $\vec{A}^{\text{in}}$ into the differential equation (39) for x. Moreover, since Eqs. (39)–(45) are linear in the quantum operators, it is possible to carry out the renormalization, which here involves only c numbers, exactly as in classical theory. Thus the resulting equation of motion for x in both classicial theory and in the Heisenberg picture of quantum electrodynamics[18] is

$$m\ddot{x} = -m\omega_0^2 x + \frac{2}{3}\frac{e^2}{c^3}\dddot{x} + eE_x^{\text{in}}(\vec{R}_0)\ , \tag{46}$$

where m is now the renormalized mass.

Because of the appearance of the damping term $\frac{2}{3}(e^2/c^3)\dddot{x}$, the dependence upon the initial conditions decays with time. The steady-state solutions of the equations of motion may be obtained by the Fourier transforms. Introducing the expansions for $\vec{E}$ and $\underline{\vec{E}}$ from the free-field expressions (21) and (6) which correspond here to the in-fields, we have in random electrodynamics

$$x = \frac{e}{m}\sum_{\lambda=1}^{2}\int d^3k\,\hat{\epsilon}(\vec{k}, \lambda)\frac{\mathfrak{h}(\vec{k}, \lambda)}{2}\left[\frac{a(\vec{k}, \lambda)}{C}\exp(-i\omega t + i\vec{k}\cdot\vec{R}_0) + \frac{a^*(\vec{k}, \lambda)}{C^*}\exp(i\omega t - i\vec{k}\cdot\vec{R}_0)\right]\ , \tag{47}$$

and in quantum electrodynamics

$$\underline{x} = \frac{e}{m}\sum_{\lambda=1}^{2}\int d^3k\,\hat{\epsilon}(\vec{k}, \lambda)\frac{(\hbar\omega)^{1/2}}{2\pi}\left[\frac{\underline{a}^{\dagger}(\vec{k}, \lambda)}{C}\exp(-i\omega t + i\vec{k}\cdot\vec{R}_0) + \frac{\underline{a}^{\dagger}(\vec{k}, \lambda)}{C^*}\exp(i\omega t - i\vec{k}\cdot\vec{R}_0)\right]\ , \tag{48}$$

where

$$C = -\omega^2 + \omega_0^2 - i\Gamma\omega^3\ , \tag{49}$$

$$\Gamma = \frac{2}{3}\frac{e^2}{mc^3}\ . \tag{50}$$

We also note that from (42) and (45), $\vec{A}$ is given in terms of $\vec{A}^{\text{in}}$ and x, so that $\vec{A}$ can also be expressed in terms of the free fields, as can the momentum

$$p = m\dot{x} + \frac{e}{c}A_x(\vec{R}_0)\ , \tag{51}$$

which is canonically conjugate to x. The relations hold in both the random and quantum theories.

Thus, in equilibrium, the behavior of the free electromagnetic field in random and in quantum electrodynamics completely determines the equilibrium behavior of a point dipole oscillator or indeed any collection of oscillators.

C. Ground-state average values including radiative corrections

Although Eqs. (38)–(51) take the same form in the classical and quantum theories, the symbols stand for somewhat different objects. In the classical case, x is the coordinate of the particle position, whereas in the quantum case $\underline{x}$ is the operator on Hilbert space corresponding to particle position. In order to compare the theories, we must compute the average values in the vacuum situation.

Thus we compute the vacuum expectation values $\langle 0|\underline{x}|0\rangle$, $\langle 0|\underline{p}|0\rangle$, $\langle 0|\underline{x}\,\underline{p}|0\rangle$, etc., in quantum electrodynamics, and the averages over the random phases $\langle x\rangle$, $\langle p\rangle$, $\langle xp\rangle$, etc., within random electrodynamics. From Eqs. (47), (48), and the comments connected with (51), we see that the agreement between the quantum vacuum expectation values and the random average values will be traced back essentially to averages over the free electromagnetic field expressions.

Clearly in quantum electrodynamics $\langle 0|\underline{x}|0\rangle$ and $\langle 0|\underline{p}|0\rangle$ involve $\langle 0|\underline{a}(\vec{k},\lambda)|0\rangle$, $\langle 0|\underline{a}^{\dagger}(\vec{k},\lambda)|0\rangle$, and so vanish. Analogously in random electrodynamics $\langle x\rangle$ and $\langle p\rangle$ vanish in the vacuum situation because they involve the averages over random phases $\langle a(\vec{k},\lambda)\rangle$ and $\langle a^*(\vec{k},\lambda)\rangle$. The mean-square oscillator displacement in the two cases is given by

$$\langle 0|\underline{x}^2|0\rangle = \frac{e^2}{m^2}\sum_{\lambda_1=1}^{2}\sum_{\lambda_2=1}^{2}\int d^3k_1\int d^3k_2\,\epsilon_{1x}\epsilon_{2x}\frac{(\hbar\omega_1)^{1/2}}{2\pi}\frac{(\hbar\omega_2)^{1/2}}{2\pi}$$
$$\times\left\langle 0\left|\left[\frac{\underline{a}_1}{C_1}\exp(-iK_1\cdot X_0)+\frac{\underline{a}_1^{\dagger}}{C_1^*}\exp(iK_1\cdot X_0)\right]\right.\right.$$
$$\left.\left.\times\left[\frac{\underline{a}_2}{C_2}\exp(-iK_2\cdot X_0)+\frac{\underline{a}_2^{\dagger}}{C_2^*}\exp(iK_2\cdot X_0)\right]\right|0\right\rangle, \tag{52}$$

and by

$$\langle x^2\rangle = \frac{e^2}{m^2}\sum_{\lambda_1=1}^{2}\sum_{\lambda_2=1}^{2}\int d^3k_1\int d^3k_2\,\epsilon_{1x}\epsilon_{2x}\frac{\hbar_1}{2}\frac{\hbar_2}{2}\left\langle\left[\frac{a_1}{C_1}\exp(-iK_1\cdot X_0)+\frac{a_1^*}{C_1^*}\exp(iK_1\cdot X_0)\right]\right.$$
$$\left.\times\left[\frac{a_2}{C_2}\exp(-iK_2\cdot X_0)+\frac{a_2^*}{C_2^*}\exp(iK_2\cdot X_0)\right]\right\rangle . \tag{53}$$

It is easy to show that

$$\langle 0|\underline{x}^2|0\rangle = \langle x^2\rangle . \tag{54}$$

Repeating the same procedures as in part II making use of the analogy between $\underline{a}$, $\underline{a}^{\dagger}$ and a, a^*, we conclude that there is exact agreement between the vacuum expectation values in quantum electrodynamics and the average values in random electrodynamics when the quantum operator products have been completely symmetrized.

D. Oscillators in the uncharged limit of random and quantum electrodynamics

In the limit that the charge e of the dipole oscillator goes to zero, the oscillator becomes uncoupled from the electromagnetic field and so is a pure mechanical system. In the quantum situation, we expect to recover the quantum mechanics of a harmonic oscillator in its ground state. And indeed we do. In the limit $e\to 0$, all the vacuum expectation values go over to the quantum-mechanical results. However, in the classical theory of random electrodynamics we note an analogous result. In the limit $e\to 0$ which decouples the oscillator from the fluctuating radiation, the fluctuations depending upon $\hbar$ remain. We do not obtain the traditional classical mechanical theory of an oscillator, but rather a particle theory which we will call random mechanics.

As examples of the transitions from full electrodynamic theories over to the mechanical theories, we will evaluate several average values beginning with $\langle 0|\underline{x}\,\underline{p}|0\rangle$ in the quantum case and $\langle x^2\rangle$ in the random case. We first note that in the required limit $e\to 0$ the term $eA_x(R_0)$ may be neglected in the canonical momentum,

$$\langle 0|\underline{x}\,\underline{p}|0\rangle = \langle 0|\underline{x}\,m\,\underline{\dot{x}}|0\rangle + O(e) . \tag{55}$$

Now

$$\langle 0|\underline{x}\, m\, \dot{\underline{x}}|0\rangle = \frac{e^2}{m^2}\sum_{\lambda_1=1}^{2}\sum_{\lambda_2=1}^{2}\int d^3k_1\int d^3k_2\,\epsilon_{1x}\,\epsilon_{2x}\,\frac{(\hbar\omega_1)^{1/2}}{2\pi}\,\frac{(\hbar\omega_2)^{1/2}}{2\pi}\,m$$
$$\times\left\langle 0\left|\left[\frac{a_1}{C_1}\exp(-iK_1\cdot X_0)+\frac{a_1^\dagger}{C_1^*}\exp(iK_1\cdot X_0)\right]\right.\right.$$
$$\left.\left.\times\left[-i\omega\frac{a_2}{C_2}\exp(-iK_2\cdot X_0)+i\omega\frac{a_2^\dagger}{C_2^*}\exp(iK_2\cdot X_0)\right]\right|0\right\rangle$$
$$=\frac{e^2}{m}\sum_{\lambda=1}^{2}\int d^3k\,\epsilon_x{}^2\,\frac{\hbar\omega}{4\pi^2}\,\frac{i\omega}{|C|^2}\ . \tag{56}$$

Using the expectation values (26)–(28), inserting (4) and (49), and carrying out the angular integrations,

$$\langle 0|\underline{x}\, m\, \dot{\underline{x}}|0\rangle = \frac{e^2}{m}\int_0^\infty dk\,k^2\,\frac{8\pi}{3}\left(\frac{\hbar\omega}{4\pi^2}\right)\frac{i\omega}{(-\omega^2+\omega_0^2)^2+(\Gamma\omega^3)^2}\ . \tag{57}$$

This expression is exact within quantum electrodynamics. It involves an integrand which is sharply peaked at the natural oscillator frequency ω_0. As the value of e appearing in the damping term

$$\Gamma=\frac{2}{3}\,\frac{e^2}{mc^3} \tag{58}$$

becomes smaller, the resonance behavior becomes increasingly sharp. Thus for small values of e, we may write $u=\omega-\omega_0$ and approximate

$$\langle 0|\underline{x}\, m\, \dot{\underline{x}}|0\rangle \cong i\,\frac{e^2}{mc^3}\,\frac{2\hbar}{3\pi}\int_{-\infty}^{\infty}du\,\frac{\omega_0{}^4}{(2\omega_0 u)^2+(\Gamma\omega_0{}^3)^2}\ . \tag{59}$$

The last integral is of the form

$$\int_{-\infty}^{\infty}dz\,\frac{1}{a^2z^2+b^2}=\frac{\pi}{ab}\ . \tag{60}$$

Thus

$$\langle 0|\underline{x}\, m\, \dot{\underline{x}}|0\rangle = i\,\frac{\hbar}{2}+O(e^2)\ . \tag{61}$$

In the limit $e\to 0$, this leaves exactly

$$\langle 0|\underline{x}\,\underline{p}|0\rangle = i\,\frac{\hbar}{2}\ . \tag{62}$$

This is precisely the value one obtains for the quantum harmonic oscillator.

In the case of random mechanics, the value $\langle xp\rangle$ vanishes. This agrees with the symmetrized quantum expression,

$$\langle xp\rangle = \langle 0|\tfrac{1}{2}(\underline{x}\,\underline{p}+\underline{p}\,\underline{x})|0\rangle$$
$$=\frac{1}{2}\left(i\,\frac{\hbar}{2}-i\,\frac{\hbar}{2}\right)$$
$$=0\ . \tag{63}$$

The mean-square oscillator displacement in random electrodynamics is written out in (53). When we average over the random phases

$$\langle x^2\rangle = \frac{e^2}{m^2}\sum_{\lambda=1}^{2}\int d^3k\,\epsilon_x{}^2\,\hbar^2\,\frac{1}{2}\,\frac{1}{|C|^2}\ . \tag{64}$$

Except for a factor of $mi\omega$, the right-hand side of this expression agrees with that of (56). Following exactly the same arguments as presented above, we arrive at the $e\to 0$ limit corresponding to random mechanics

$$\langle x^2\rangle = \frac{1}{2}\,\frac{\hbar}{m\omega_0} \tag{65}$$

The general expression x^{2m} is obtained in analogous fashion,

$$\langle x^{2m}\rangle = \left(\frac{e}{m}\right)^{2m}\sum_{\lambda_1=1}^{2}\sum_{\lambda_2=1}^{2}\cdots\sum_{\lambda_{2m}=1}^{2}\int d^3k_1\int d^3k_2\cdots\int d^3k_{2m}\,\epsilon_{1x}\,\epsilon_{2x}\cdots\epsilon_{2mx}$$
$$\times\frac{\hbar_1}{2}\,\frac{\hbar_2}{2}\cdots\frac{\hbar_{2m}}{2}\left\langle\left(\frac{a_1}{C_1}+\frac{a_1^*}{C_1^*}\right)\left(\frac{a_2}{C_2}+\frac{a_2^*}{C_2^*}\right)\cdots\left(\frac{a_{2m}}{C_{2m}}+\frac{a_{2m}^*}{C_{2m}^*}\right)\right\rangle\ . \tag{66}$$

Here, for convenience we have set $X_0=0$ rather than noting that the space-time dependence in the factors $\exp(-iK\cdot X_0)$, $\exp(iK\cdot X_0)$ cancels out. The average over the random phases involves nonvanishing contributions from the sum of all pairings of the form

$$\left\langle\left(\frac{a_1}{C_1}+\frac{a_1^*}{C_1^*}\right)\left(\frac{a_2}{C_2}+\frac{a_2^*}{C_2^*}\right)\right\rangle\left\langle\left(\frac{a_3}{C_3}+\frac{a_3^*}{C_3^*}\right)\left(\frac{a_4}{C_4}+\frac{a_4^*}{C_4^*}\right)\right\rangle\cdots\left\langle\left(\frac{a_{2m-1}}{C_{2m-1}}+\frac{a_{2m-1}^*}{C_{2m-1}^*}\right)\left(\frac{a_{2m}}{C_{2m}}+\frac{a_{2m}^*}{C_{2m}^*}\right)\right\rangle\ , \tag{67}$$

including the pairing of the $\vec{k}_1$ parenthesis with the $\vec{k}_3$ parenthesis, etc. There are $(2m-1)!! = (2m)!/(m!\,2^m)$ such pairings, giving

$$\langle x^{2m}\rangle = \frac{(2m)!}{m!\,2^m}\left(\frac{e^2}{m^2}\sum_{\lambda=1}^{2}\int d^3k\,\epsilon_{x}{}^{2}\,\frac{\hbar^2}{2|C|^2}\right)^m$$

$$= \frac{(2m)!}{m!\,2^m}\left(\frac{1}{2}\,\frac{\hbar}{m\omega_0}\right)^m \tag{68}$$

from the evaluation of the integral above. The evaluation of $\langle p^{2n}\rangle$ involves analogous procedures. When treating $\langle x^{2m}p^{2n}\rangle$, we note first that we have contributions as in (67) from all separate pairings between the $\vec{k}_i$ parentheses for x, multiplying all separate pairings between the $\vec{k}_j$ for p. However, if even one parenthesis $(a/C + a^*/C^*)$ for x is paired with $(-im\omega a/C + im\omega a^*/C^*)$ for p, the term vanishes since from the expressions (24) and (25) for averages over random phases

$$im\omega\left\langle\left(\frac{a}{C}+\frac{a^*}{C^*}\right)\left(-\frac{a}{C}+\frac{a^*}{C^*}\right)\right\rangle = 0\,. \tag{69}$$

It follows that

$$\langle x^{2m}p^{2n}\rangle = \langle x^{2m}\rangle\langle p^{2n}\rangle$$

$$= \frac{(2m)!\,(2n)!}{m!\,n!\,2^{m+n}}\langle x^2\rangle^m\langle p^2\rangle^n\,. \tag{70}$$

The results in random mechanics (63), (65), (68), and (70) correspond to the $e\to 0$ limit of random electrodynamics when the dipole oscillator is uncoupled from the radiation field. The fluctuations depending upon $\hbar$ persist. We emphasize that the particle fluctuations in random electrodynamics and here in random mechanics are not intrinsic to the particle but rather are derived from the random zero-point radiation. The appearance of Planck's constant $\hbar$ is derived from its role setting the scale of zero-point radiation. However, it is easy to show that the same result (70) is obtained for any harmonic oscillator coupled to the electromagnetic field. One may imagine a point oscillator as the limit of an arbitrary static charge distribution at the end of a spring. Then the random motion of the oscillator is due to the fluctuations in the zero-point radiation involving a balance between the pickup and loss of energy. The balance in the limit of no coupling to the radiation field can be shown to be independent of the charge distribution of the oscillator.

E. Questions involving quantum operator order

The theory of random electrodynamics gives the same ground-state average values for the moments $\langle x^m p^n\rangle$ of harmonic-oscillator systems as does quantum electrodynamics provided the quantum operator products are completely symmetrized. However, it is also of interest to note the distinction between the random and quantum theories based upon the importance of quantum operator order. The comparison seems easiest in the uncoupled $e\to 0$ limit, giving random mechanics and quantum mechanics.

In order to illustrate the importance of operator order in quantum mechanics, we list in Table I some average values connected with the harmonic oscillator in random and quantum theories. The list of values in quantum mechanics is of necessity far longer than in the classical theory because changing the operator order changes the expectation value. However, does the quantum theory actually contain more physical information than the classical theory? Just what is the physical distinction between the Hermitian operators, and hence observables, $O_{x^2p^2} = \frac{1}{2}(\underline{x}^2\underline{p}^2 + \underline{p}^2\underline{x}^2)$ and $O_{xpxp} = \frac{1}{2}(\underline{x}\,\underline{p}\,\underline{x}\,\underline{p} + \underline{p}\,\underline{x}\,\underline{p}\,\underline{x})$? From Table I we see that these quantum operators have different values in the ground state, although in the classical theory there is no distinction between the average values of these expressions. How can this quantum difference be measured experimentally, at least in principle?

F. Sharp energy values and quantum operator order

Proceeding along this same line of inquiry, one may shift the question to ask where it is that experimental observations involve unsymmetrized

TABLE I. Average values for a mechanical oscillator at zero temperature.

Random mechanics	Quantum mechanics
$\langle x\rangle = 0$	$\langle 0\|\underline{x}\|0\rangle = 0$
$\langle p\rangle = 0$	$\langle 0\|\underline{p}\|0\rangle = 0$
$\langle x^2\rangle = \frac{1}{2}\frac{\hbar}{m\omega_0}$	$\langle 0\|\underline{x}^2\|0\rangle = \frac{1}{2}\frac{\hbar}{m\omega_0}$
$\langle p^2\rangle = \frac{1}{2}\hbar m\omega_0$	$\langle 0\|\underline{p}^2\|0\rangle = \frac{1}{2}\hbar m\omega_0$
$\langle xp\rangle = 0$	$\langle 0\|\underline{x}\,\underline{p}\|0\rangle = i\frac{\hbar}{2} = -\langle 0\|\underline{p}\,\underline{x}\|0\rangle$
$\langle x^4\rangle = 3\langle x^2\rangle^2$	$\langle 0\|\underline{x}^4\|0\rangle = 3\langle 0\|\underline{x}^2\|0\rangle^2$
$\langle p^4\rangle = 3\langle p^2\rangle^2$	$\langle 0\|\underline{p}^4\|0\rangle = 3\langle 0\|\underline{p}^2\|0\rangle^2$
$\langle x^2p^2\rangle = \langle x^2\rangle\langle p^2\rangle$	$\langle 0\|\underline{x}^2\underline{p}^2\|0\rangle = \langle 0\|\underline{p}^2\underline{x}^2\|0\rangle$ $= -\langle 0\|\underline{x}^2\|0\rangle\langle 0\|\underline{p}^2\|0\rangle$
	$\langle 0\|\underline{x}\,\underline{p}\,\underline{x}\,\underline{p}\|0\rangle = \langle 0\|\underline{p}\,\underline{x}\,\underline{p}\,\underline{x}\|0\rangle$ $= \langle 0\|\underline{x}^2\|0\rangle\langle 0\|\underline{p}^2\|0\rangle$
	$\langle 0\|\underline{x}\,\underline{p}^2\underline{x}\|0\rangle = \langle 0\|\underline{p}\,\underline{x}^2\underline{p}\|0\rangle$ $= 3\langle 0\|\underline{x}^2\|0\rangle\langle 0\|\underline{p}^2\|0\rangle$

operators. From the random-quantum connections obtained above, we will expect departures from the classical theory in these cases.

Now the Hamiltonian is an expression which already involves symmetrized products of operators, and there is no distinction between the average values of oscillator energy in the classical and quantum theories,

$$H = \frac{p^2}{2m} + \tfrac{1}{2} m \omega_0^2 x^2 , \tag{71}$$

$$\langle H \rangle = \tfrac{1}{2}\hbar\omega_0, \ \langle 0 | \underline{H} | 0 \rangle = \tfrac{1}{2}\hbar\omega_0 . \tag{72}$$

However, the average value of the square of the energy is different in the two theories. Thus

$$\langle H^2 \rangle = 2\langle H \rangle^2 , \tag{73}$$

whereas

$$\langle 0 | \underline{H}^2 | 0 \rangle = \langle 0 | \underline{H} | 0 \rangle^2 . \tag{74}$$

At first this may seem surprising. However, it follows directly from the importance of quantum operator order. Thus

$$\langle 0 | \underline{H}^2 | 0 \rangle = \left\langle 0 \left| \left[\frac{\underline{p}^4}{4m^2} + \frac{\omega_0^2}{4} (\underline{p}^2 \underline{x}^2 + \underline{x}^2 \underline{p}^2) + \tfrac{1}{4} m^2 \omega_0^4 \underline{x}^4 \right] \right| 0 \right\rangle , \tag{75}$$

where

$$\langle 0 | \underline{x}^4 | 0 \rangle = \langle x^4 \rangle$$

and (76)

$$\langle 0 | \underline{p}^4 | 0 \rangle = \langle p^4 \rangle ,$$

but $\langle 0 | \underline{p}^2 \underline{x}^2 + \underline{x}^2 \underline{p}^2 | 0 \rangle$ is not equal to $\langle x^2 p^2 \rangle$ because the quantum operator products have not been fully symmetrized to include $\underline{x}\underline{p}\underline{x}\underline{p}$, etc. Thus physically there is a distinction in our thinking regarding the results of the classical and quantum theories. In quantum mechanics, the energy is sharp, an eigenstate of energy with

$$\langle 0 | \underline{H}^n | 0 \rangle = \langle 0 | \underline{H} | 0 \rangle^n . \tag{77}$$

However, in the corresponding random mechanical system, the energy of the oscillator involves fluctuations. These fluctuations arise from the exchange of energy with the electromagnetic field, and they persist in the expressions even in the $e \to 0$ limit which decouples the classical oscillator from the electromagnetic field.

G. Angular momentum squared as an unsymmetrized quantum operator

Most experiments measure changes in the average energies of systems, and the discrepancy in the distribution of energies we have just found may not seem too serious. However, it turns out that the operator corresponding to the square of the angular momentum is also an unsymmetrized operator product. Since the angular momentum squared is frequently discussed in atomic physics experiments, it clearly should be investigated here. However, in order to do this we must first remark on three-dimensional harmonic-oscillator systems.

In Sec III B, we introduced a one-dimensional oscillator oriented along the x direction. The work can easily be extended to an isotropic three-dimensional oscillator. In this case we will have three independent equations of motion for x, y, and z, each analogous to (46), and the results go through just as before if we treat x, y, and z separately.

In this system it is possible to discuss the angular momentum in both quantum and random electrodynamics, and in the limits giving quantum and random mechanics. Specifically, we define

$$\vec{L} = \hat{i}(y p_z - z p_y) + \hat{j}(z p_x - x p_z) + \hat{k}(x p_y - y p_x) , \tag{78}$$

and the operator expression holds in quantum theory. Since the equations of motions for the different coordinate directions are uncoupled, it is easy to see that the average angular momentum vanishes in both theories,

$$\langle 0 | \underline{\vec{L}} | 0 \rangle = \langle \vec{L} \rangle = 0 . \tag{79}$$

However, the square of the angular momentum operator is

$$\begin{aligned} \underline{L}^2 = {} & (\underline{y}^2 \underline{p}_z^2 + \underline{z}^2 \underline{p}_y^2 - \underline{y}\,\underline{p}_y\,\underline{p}_z\,\underline{z} - \underline{p}_y\,\underline{y}\,\underline{z}\,\underline{p}_z) \\ & + (\underline{z}^2 \underline{p}_x^2 + \underline{x}^2 \underline{p}_z^2 - \underline{z}\,\underline{p}_z\,\underline{p}_x\,\underline{x} - \underline{p}_z\,\underline{z}\,\underline{x}\,\underline{p}_x) \\ & + (\underline{x}^2 \underline{p}_y^2 + \underline{y}^2 \underline{p}_x^2 - \underline{x}\,\underline{p}_x\,\underline{p}_y\,\underline{y} - \underline{p}_x\,\underline{x}\,\underline{y}\,\underline{p}_y) . \end{aligned} \tag{80}$$

We see that the operator product is not fully symmetrized; the term $\underline{y}\,\underline{p}_y\,\underline{p}_z\,\underline{z}$ appears, but there is no term $\underline{p}_y\,\underline{y}\,\underline{p}_z\,\underline{z}$. Corresponding to this lack of symmetrization, we find for the quantum isotropic oscillator ground state

$$\langle 0 | \underline{L}^2 | 0 \rangle = 0 , \tag{81}$$

while random mechanics gives

$$\begin{aligned} \langle L^2 \rangle &= \{\langle y^2 \rangle \langle p_z^2 \rangle + \langle z^2 \rangle \langle p_y^2 \rangle - \langle y p_y \rangle \langle p_z z \rangle - \langle p_y y \rangle \langle z p_z \rangle\} \\ &\quad + \{y \to z, z \to x\} + \{y \to x, z \to y\} \\ &= 3\left\{ \left(\frac{1}{2} \frac{\hbar}{m\omega_0} \right) (\tfrac{1}{2}\hbar m \omega_0) + \left(\frac{1}{2} \frac{\hbar}{m\omega_0} \right) (\tfrac{1}{2}\hbar m \omega_0) + 0 + 0 \right\} \\ &= \tfrac{3}{2}\hbar^2 . \end{aligned} \tag{82}$$

There is a well-defined difference in the physical descriptions of the systems which corresponds to these different average values. In the quantum theory, the system is pictured as associated with a sharp value (an eigenvalue), of the angular mo-

mentum $l=0$, and hence all higher moments of the angular momentum also vanish. However, in the classical picture of a particle being pushed about by random radiation, the particle sometimes is going around the force center rather than straight through. Thus classically, although the average angular momentum vanishes, the average of the square of the angular momentum is nonzero.

However, it is not wise to stop at this point. It is interesting to see what is the distinction between the quantum $\underline{L}^2$ and the symmetrized form $\underline{L}_{\rm sym}{}^2$ when we look at expectation values of quantum excited states. The symmetrized form of $\underline{y}\underline{p}_y\underline{p}_z\underline{z}$ is

$$\tfrac{1}{4}(\underline{y}\,\underline{p}_y\underline{p}_z\underline{z}+\underline{p}_y\underline{y}\,\underline{p}_z\underline{z}+\underline{y}\,\underline{p}_y\underline{z}\,\underline{p}_z+\underline{p}_y\underline{y}\,\underline{z}\,\underline{p}_z).$$

Thus the difference in the angular momentum operators is

$$\underline{L}^2-\underline{L}_{\rm sym}{}^2=\tfrac{1}{2}(\underline{p}_y\underline{y}-\underline{y}\underline{p}_y)(\underline{p}_z\underline{z}-\underline{z}\underline{p}_z)+\tfrac{1}{2}(y\to z)(z\to x)+\tfrac{1}{2}(y\to x)(z\to y). \quad (83)$$

However, the factors may be recognized as involving the basic quantum commutator

$$[\underline{x},\underline{p}_x]=i\hbar,\ \text{etc.}, \quad (84)$$

so that

$$\underline{L}^2-\underline{L}_{\rm sym}{}^2=-\tfrac{3}{2}\hbar^2. \quad (85)$$

But then in analyzing transitions between quantum eigenstates, the changes in the angular momentum squared are the same whether we use $\underline{L}^2$ or the symmetrized form $\underline{L}_{\rm sym}{}^2$ which agrees more closely with the classical theory of random mechanics.

At the present time, the author has not found an experimentally tested physical distinction between the symmetrized and unsymmetrized forms within quantum theory.

H. Three views of the uncharged harmonic oscillator

When considering an uncharged harmonic oscillator of average energy $\frac{1}{2}\hbar\omega_0$, three different views come to mind, corresponding to traditional classical mechanics, random mechanics, and quantum mechanics. A comparison between the physical descriptions takes on added interest in the light of physicists' repeated efforts to provide semiclassical models for quantum systems.

Within traditional classical mechanics the motion of a single harmonic oscillator of mass m, frequency ω_0, and average energy $\frac{1}{2}\hbar\omega_0$ is given by

$$x=\left(\frac{\hbar}{m\omega_0}\right)^{1/2}\sin(\omega_0 t+\phi). \quad (86)$$

If we consider a set of such oscillators which have random phase relationships to each other, then we can describe the set by a distribution on phase space confined to the ellipse

$$\frac{p^2}{2m}+\tfrac{1}{2}m\omega_0{}^2x^2=\tfrac{1}{2}\hbar\omega_0. \quad (87)$$

The average value of x^kp^l in this traditional classical view can be obtained as

$$\langle x^kp^l\rangle_{\rm traditional}=\frac{1}{2\pi}\int_{\theta=0}^{2\pi}d\theta\left[\left(\frac{\hbar}{m\omega_0}\right)^{1/2}\sin\theta\right]^k\times[(\hbar m\omega_0)^{1/2}\cos\theta]^l. \quad (88)$$

Clearly the average value vanishes if k or l is odd.

In random mechanics, defined as the limit from random electrodynamics which uncouples the mechanical system from the radiation field, we found

$$\langle x^{2m}p^{2n}\rangle=\frac{(2m)!(2n)!}{m!n!\,2^{m+n}}\left(\frac{1}{2}\,\frac{\hbar}{m\omega_0}\right)^m(\tfrac{1}{2}\hbar m\omega_0)^n. \quad (89)$$

Again $\langle x^kp^l\rangle$ vanishes if k or l is an odd integer. This behavior corresponds to a phase-space distribution

$$\rho(x,p)=\frac{1}{\pi\hbar}\exp\left(-\frac{m\omega_0x^2}{\hbar}-\frac{p^2}{m\omega_0\hbar}\right). \quad (90)$$

Clearly this is not a sharp energy distribution on phase space as was the ellipse above.

Finally we come to the quantum-mechanical oscillator which seems related to both of the above descriptions. The quantum ground state is specified as an eigenstate of energy, and so at first suggests the sharp distribution on phase space of traditional classical mechanics. On the other hand, we have noted that if the operator order is completely symmetrized, then the quantum expectation values agree with those of random mechanics. Thus, for example, if we compare the expectation value for x^{2n} for the three systems, we have

$$\langle x^{2n}\rangle_{\rm traditional}=\frac{(2n)!}{(n!)^2\,2^n}\left(\frac{\hbar}{2m\omega_0}\right)^n,$$

$$\langle x^{2n}\rangle_{\rm random}=\frac{(2n)!}{n!2^n}\left(\frac{\hbar}{2m\omega_0}\right)^n,$$

$$\langle 0|\underline{x}^{2n}|0\rangle_{\rm quantum}=\frac{(2n)!}{n!2^n}\left(\frac{\hbar}{2m\omega_0}\right)^n.$$

The expression $\langle x^{2n}\rangle_{\rm traditional}$ differs by a factor of $n!$ from the other two expressions.

Thus this comparison for the average values for x^{2n} suggests that random mechanics and quantum mechanics are closely related while the traditional classical view is quite different. However, on the other hand, both traditional classical mechanics and quantum mechanics give a sharp energy for the system, whereas random mechanics involves a distribution of energies. Thus although all three views have the same average energy $\frac{1}{2}\hbar\omega_0$, nevertheless the averages of the higher moments H^n differ. Traditional classical theory and quantum theory are in agreement that

$$\langle H^n\rangle_{\text{traditional}} = \langle 0|\underline{H}^n|0\rangle_{\text{quantum}} = (\tfrac{1}{2}\hbar\omega_0)^n ,$$

but random mechanics gives a different value. This was illustrated in Sec. III E for H^2. There we saw that despite the agreement between random and quantum theories for the values of $\langle x^4\rangle$ and $\langle p^4\rangle$, the quantum energy squared is in agreement with the traditional sharp-energy distribution on phase space. This is achieved in quantum theory by the value for the cross term

$$\langle 0|\tfrac{1}{4}\omega_0^2(\underline{x}^2\underline{p}^2+\underline{p}^2\underline{x}^2)|0\rangle = -\tfrac{1}{4}\hbar^2\omega_0^2 ,$$

which, surprisingly from a classical viewpoint, is negative. If the operator order in $\underline{x}^2\underline{p}^2$ were completely symmetrized, then the average value would agree with random mechanics and the energy would not be sharp.

IV. SYSTEMS AT FINITE TEMPERATURE IN RANDOM AND QUANTUM THEORIES

A. Presence of thermal radiation

In this section we turn to a comparison of random electrodynamics and quantum electrodynamics in the presence of thermal radiation. At finite temperatures, two new elements are introduced—the excitation of higher-energy levels of quantum systems and the tendency for quantum systems to approach the traditional classical mechanical description. As in the vacuum situation at $T=0$ considered previously, we will first consider the free electromagnetic fields, now including the presence of thermal radiation, and then will note the behavior of harmonic-oscillator systems which are coupled to the radiation.

B. Free-field correlations in thermal radiation for random electrodynamics

The difference between classical zero-point radiation and classical thermal radiation is immediate from a physical point of view. Zero-point radiation is universal, spreading throughout the universe with a homogeneous, isotropic, and Lorentz-invariant spectrum. The blackbody radiation spectrum, however, is not Lorentz-invariant. It has a preferred frame of references, and motion relative to this frame can be detected as retarding forces[5] upon systems with electromagnetic interactions. The preferred frame of reference of the thermal radiation corresponds to that of the container enforcing thermal equilibrium.

Nevertheless, within random electrodynamics there is no essential difference in the mathematical treatment of thermal radiation and of zero-point radiation. Both involve fluctuating classical electromagnetic radiation whose complete randomness is characterized by a Gaussian distribution of the amplitude for any Fourier coefficient. A derivation[5] of the thermal radiation spectrum entirely within random electrodynamics has been given; it arrives at the Planck spectrum, giving the average energy per normal mode:

$$\pi^2\mathfrak{h}^2(\vec{k},\lambda,T) = \tfrac{1}{2}\hbar\omega + \frac{\hbar\omega}{\exp(\hbar\omega/kT)-1} = \tfrac{1}{2}\hbar\omega\coth\left(\frac{\hbar\omega}{2kT}\right) . \tag{91}$$

As $T\to 0$, this goes over to the zero-point radiation.

Now the averages involved in random electrodynamics can be treated entirely with the random phase $\theta(\vec{k},\lambda)$ regarding $\mathfrak{h}(\vec{k},\lambda,T)$ as a constant number. Hence the calculation at finite temperature T gives for $\langle E_{i_1}(x_1)E_{i_2}(x_2)\cdots E_{i_n}(x_n)\rangle_T$ exactly the same result as before where $\mathfrak{h}(k,\lambda)$ is replaced by $\mathfrak{h}(k,\lambda,T)$. Thus, for example, in the two-point function, we have from (10) and (91)

$$\langle E_i(x_1)E_j(x_2)\rangle = \int d^3k\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\frac{\hbar\omega}{4\pi^2}\times\coth\left(\frac{\hbar\omega}{2kT}\right)\cos[K\cdot(x_1-x_2)] . \tag{92}$$

For convenience in comparing the correlation function with the quantum expression, we will rewrite this result as

$$\langle E_i(x_1)E_j(x_2)\rangle_T = \int d^3k\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\frac{\hbar\omega}{4\pi^2}\coth\left(\frac{\hbar\omega}{2kT}\right)\tfrac{1}{2}\{\exp[-iK\cdot(x_1-x_2)]+\exp[iK\cdot(x_1-x_2)]\} \tag{93}$$

and

$$\langle E_i(x_1)E_j(x_2)\rangle_T = \int d^3k\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\times\frac{1}{4\pi^2Z}\sum_{n=0}^{\infty}\hbar\omega\exp\left[-\frac{\hbar\omega(n+\frac{1}{2})}{kT}\right]\{(n+\tfrac{1}{2})\exp[-iK\cdot(x_1-x_2)]+(n+\tfrac{1}{2})\exp[iK\cdot(x_1-x_2)]\} . \tag{94}$$

In Eq. (94) we have made use of the expansion

$$\tfrac{1}{2}\coth\left(\frac{\hbar\omega}{2kT}\right) = \frac{\sum_{n=0}^{\infty}\hbar\omega(n+\frac{1}{2})\exp[-\hbar\omega(n+\frac{1}{2})/kT]}{\sum_{n=0}^{\infty}\exp[-\hbar\omega(n+\frac{1}{2})/kT]}, \tag{95}$$

where we have denoted

$$Z = \sum_{n=0}^{\infty}\exp\left[-\frac{\hbar\omega(n+\frac{1}{2})}{kT}\right]. \tag{96}$$

C. Free-field correlations in thermal radiation for quantum electrodynamics

The treatment of thermal radiation within free-field quantum electrodynamics takes quite a different form from the simple procedure given above in random electrodynamics. In the quantum case, we retain the same operator forms as used previously at $T=0$. However, we now have a state which is not the vacuum, but rather an incoherent assemblage of photons. The number of photons at a given wave vector $\vec{k}$ and polarization λ is determined by Boltzmann's factor $\exp[-\mathcal{E}_n(\vec{k},\lambda)/kT]$, where

$$\mathcal{E}_n(\vec{k},\lambda) = \hbar\omega_{\vec{k},\lambda}(n_{\vec{k},\lambda}+\tfrac{1}{2}), \tag{97}$$

and the discrete character of the photons is such as to give Planck's spectrum.

Thus noting the incoherent nature of the radiation, we have for the correlation of the electric fields

$$\langle|\underline{E}_i(x_1)\underline{E}_j(x_2)|\rangle_T$$
$$= \sum_{n_{\vec{k}_1\lambda_1}=0}^{\infty}\sum_{n_{\vec{k}_2\lambda_2}=0}^{\infty}\cdots\frac{1}{Z_1}\exp\left[-\frac{\mathcal{E}_n(\vec{k}_1,\lambda_1)}{kT}\right]\frac{1}{Z_2}\exp\left[-\frac{\mathcal{E}_n(\vec{k}_2,\lambda_2)}{kT}\right]$$
$$\times\cdots\langle n_{\vec{k}_1\lambda_1}\, n_{\vec{k}_2\lambda_2}\cdots|\underline{E}_i(x_1)\underline{E}_j(x_2)|n_{\vec{k}_1\lambda_1}\, n_{\vec{k}_2\lambda_2}\cdots\rangle, \tag{98}$$

where the state $|n_{\vec{k}_1\lambda_1}\, n_{\vec{k}_2\lambda_2}\cdots\rangle$ contains $n_{k_1\lambda_1}$ photons at wave vector $\vec{k}_1$ and polarization λ_1, $n_{\vec{k}_2\lambda_2}$ at $\vec{k}_2$ and λ_2, etc. Now from (6) and (7), the expectation value in the last line takes the form

$$\langle\{n\}|\underline{E}_i(x_1)\underline{E}_j(x_2)|\{n\}\rangle = \sum_{\lambda'=1}^{2}\sum_{\lambda''=1}^{2}\int d^3k'\int d^3k''\epsilon'_i\epsilon''_j$$
$$\times\frac{(\hbar\omega')^{1/2}}{2\pi}\frac{(\hbar\omega'')^{1/2}}{2\pi}\langle\{n\}|[\underline{a}'\exp(-iK'\cdot x_1)+\underline{a}'^{\dagger}\exp(iK'\cdot x_1)]$$
$$\times[\underline{a}''\exp(-iK''\cdot x_2)+\underline{a}''^{\dagger}\exp(iK''\cdot x_2)]|\{n\}\rangle$$
$$= \sum_{\lambda=1}^{2}\int d^3k\,\epsilon_i\epsilon_j\frac{\hbar\omega}{4\pi^2}\{(n_{\vec{k}\lambda}+1)\exp[-iK\cdot(x_1-x_2)]+n_{\vec{k}\lambda}\exp[iK\cdot(x_1-x_2)]\}. \tag{99}$$

We now substitute (99) into (98), interchange the order of integration and photon summation, and sum over all photons which are not of the type $n_{\vec{k}\lambda}$. This cancels all but one of the factors of Z^{-1}. Since the number of photons at the two polarizations $\lambda=1,2$ are the same in thermal radiation, $n_{\vec{k}_1}=n_{\vec{k}_2}$, we may sum over the polarizations as in (4), leaving

$$\langle|\underline{E}_i(x_1)\underline{E}_j(x_2)|\rangle_T = \int d^3k\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)$$
$$\times\frac{1}{4\pi^2 Z}\sum_{n=0}^{\infty}\hbar\omega\exp\left[-\frac{\hbar\omega(n+\frac{1}{2})}{kT}\right]\{(n+1)\exp[-iK\cdot(x_1-x_2)]+n\exp[iK\cdot(x_1-x_2)]\}. \tag{100}$$

Thus we find that there is a difference between the random and quantum theories for the correlation of the electric fields in thermal radiation. The difference is the discrepancy between $(n+\frac{1}{2})$ in the bracket of (94) and the $n+1$ or n in (100). We note once again that if we symmetrize the operator order, then the correlation involving the Hermitian operator

$$\tfrac{1}{2}[\underline{E}_i(x_1)\underline{E}_j(x_2)+\underline{E}_j(x_2)\underline{E}_i(x_1)]$$

will involve factors $\frac{1}{2}[(n+1)+n]=n+\frac{1}{2}$ and hence will agree exactly with the expression $\langle E_i(x_1)E_j(x_2)\rangle_T$ in random electrodynamics. We also see that at high temperatures when the terms with large n become important and we can neglect the difference between $n+1$ and $n+\frac{1}{2}$, and between n and $n+\frac{1}{2}$, then the operator order in quantum electrodynamics becomes immaterial.

The comparison between the theories can be carried through for correlations of the $\vec{E}$ and $\vec{B}$

fields in products of arbitrary length. The essential behavior follows what we have just found for the two-point function. Provided the quantum operator order is symmetrized, the quantum electrodynamic expectation value agrees exactly with the average in random electrodynamics. Moreover, at high temperatures the operator order within the quantum expressions becomes of proportionately less importance.

D. Charged harmonic oscillators in thermal radiation

The treatment of oscillators coupled to the electromagnetic field follows exactly as in Sec. III B for the behavior at zero temperature. Again we have the steady-state solutions (47) and (48) giving the position of the oscillator in terms of the incoming electrodynamic radiation fields. In the case of random electrodynamics the incoming radiation is regarded as including the thermal radiation, whereas in quantum electrodynamics the in operator make no reference to thermal radiation.

In both theories the average values for the oscillator behavior can be evaluated by procedures analogous to those used for the free-field correlation functions. Thus, for example, in random electrodynamics $\langle x^2\rangle_T$ is given exactly as in (53), with $\mathfrak{h}(\vec{k}, \lambda)$ replaced by $\mathfrak{h}(\vec{k}, \lambda, T)$,

$$\langle x^2\rangle_T = \frac{e^2}{m^2}\sum_{\lambda_1=1}^{2}\sum_{\lambda_2=1}^{2}\int d^3k_1\int d^3k_2\,\epsilon_{1x}\epsilon_{2x}\,\frac{\mathfrak{h}(\vec{k}_1,\lambda_1,T)}{2}\,\frac{\mathfrak{h}(\vec{k}_2,\lambda_2,T)}{2}$$
$$\times\left\langle\left[\frac{a_1}{C_1}\exp(-iK_1\cdot X)+\frac{a_1^*}{C_1^*}\exp(iK_1\cdot X)\right]\left[\frac{a_2}{C_2}\exp(-iK_2\cdot X)+\frac{a_2^*}{C_2^*}\exp(iK_2\cdot X)\right]\right\rangle, \tag{101}$$

where a and a^* are exactly as in (22) and (23). In quantum electrodynamics, we must average over the incoherent photon states

$$\langle|\underline{x}^2|\rangle_T = \frac{e^2}{m^2}\sum_{\lambda'=1}^{2}\sum_{\lambda''=1}^{2}\int d^3k'\int d^3k''\,\epsilon'_x\epsilon''_x\,\frac{(\hbar\omega')^{1/2}}{2\pi}\,\frac{(\hbar\omega'')^{1/2}}{2\pi}$$
$$\times\sum_{n_{\vec{k}_1\lambda_1}=0}^{\infty}\sum_{n_{\vec{k}_2\lambda_2}=0}^{\infty}\cdots\frac{1}{Z_1}\exp\left[-\frac{\mathcal{E}_n(\vec{k}_1,\lambda_1)}{kT}\right]\frac{1}{Z_2}\exp\left[-\frac{\mathcal{E}_n(\vec{k}_2,\lambda_2)}{kT}\right]\cdots$$
$$\times\left\langle n_{\vec{k}_1\lambda_1}n_{\vec{k}_2\lambda_2}\cdots\left|\left[\frac{\underline{a}'}{C'}\exp(-iK'\cdot X)+\frac{\underline{a}'^{\dagger}}{C'^*}\exp(iK'\cdot X)\right]\right.\right.$$
$$\left.\left.\times\left[\frac{\underline{a}''}{C''}\exp(-iK''\cdot X)+\frac{\underline{a}''^{\dagger}}{C''^*}\exp(iK''\cdot X)\right]\right|n_{\vec{k}_1\lambda_1}n_{\vec{k}_2\lambda_2}\cdots\right\rangle. \tag{102}$$

The comparison between the theories proceeds along lines analogous to those of Secs. IV B and IV C. The behavior of the free fields determines the average values for the oscillator behavior. We find that if and only if the quantum operator order $\underline{x},\underline{p}$ is symmetrized, the average values are identical between random and quantum electrodynamics. At high temperatures the operator order becomes proportionately less important within quantum electrodynamics.

E. Mechanical oscillators as the uncharged limit of random and quantum electrodynamics

The limit $e\to 0$ again decouples the dipole oscillator from the electromagnetic field. However, as in the vacuum situation, the effects of the radiation fluctuations still persist in this limit and completely determine the oscillator behavior.

The derivation of all average values $\langle x^k p^l\rangle$ in random mechanics follows directly from the work in Secs. III B–III D. We evaluate the integrals in the $e\to 0$ limit just as in Sec. III D, but replace the spectrum $\mathfrak{h}^2(\vec{k}, \lambda)$ by $\mathfrak{h}^2(\vec{k}, \lambda, T)$ as in Sec. IV B. Thus $\langle x^k p^l\rangle$ vanishes if k or l is an odd integer and

$$\langle x^{2m}p^{2n}\rangle_T = \frac{(2m)!\,(2n)!}{m!n!\,2^{m+n}}\langle x^2\rangle_T^m\langle p^2\rangle_T^n, \tag{103}$$

where now

$$\langle x^2\rangle_T = \frac{1}{2}\frac{\hbar}{m\omega_0}\coth\left(\frac{\hbar\omega_0}{2kT}\right), \tag{104}$$

$$\langle p^2\rangle_T = \tfrac{1}{2}\hbar m\omega_0\coth\left(\frac{\hbar\omega_0}{2kT}\right). \tag{105}$$

The distribution on phase space associated with these moments is

$$\rho(x,p)=\frac{1}{\pi\alpha}\exp\left(-\frac{m\omega_0 x^2}{\alpha}-\frac{p^2}{m\omega_0\alpha}\right), \tag{106}$$

where

$$\alpha=\hbar\coth\left(\frac{\hbar\omega_0}{2kT}\right). \tag{107}$$

The transition to the uncharged limit within quantum theory involves further complications which should allow the familiar interpretation of excited states within quantum mechanics. Again for the limit $e\to 0$, it is sufficient to take

$$p=m\dot{x}+O(e). \tag{108}$$

Then from (48) the quantum average value at temperature T for an expression involving products of position and momentum operators is

$$\langle|\underline{x}\,\underline{x}\,\underline{p}\cdots|\rangle_T=\sum_{n_{\vec{k}_1\lambda_1}=0}^{\infty}\sum_{n_{\vec{k}_2\lambda_2}=0}^{\infty}\cdots\frac{1}{Z_1}\exp\left[-\frac{\mathcal{E}_n(\vec{k}_1,\lambda_1)}{kT}\right]\frac{1}{Z_2}\exp\left[-\frac{\mathcal{E}_n(\vec{k}_2,\lambda_2)}{kT}\right]\cdots$$
$$\times\sum_{\lambda'=1}^{2}\sum_{\lambda''=1}^{2}\cdots\int d^3k'\int d^3k''\cdots\epsilon'_x\epsilon''_x\cdots\frac{(\hbar\omega')^{1/2}}{2\pi}\,\frac{(\hbar\omega'')^{1/2}}{2\pi}\cdots$$
$$\times\left\langle n_{\vec{k}_1\lambda_1}n_{\vec{k}_2\lambda_2}\cdots\left|\left(\frac{\underline{a}'}{C'}+\frac{\underline{a}'^{\dagger}}{C'^{*}}\right)\left(\frac{\underline{a}''}{C''}+\frac{\underline{a}''^{\dagger}}{C''^{*}}\right)\cdots\right|n_{\vec{k}_1\lambda_1}n_{\vec{k}_2\lambda_2}\cdots\right\rangle. \tag{109}$$

Here for convenience we have set $X=0$ in $\exp(-iK\cdot X)$ rather than showing that this space-time dependence cancels. We may think of the $\vec{k}',\lambda',\vec{k}'',\lambda''$, etc. in the integrand as being independent, and hence can break up the expectation value in the last line into a sum of product terms of the form

$$\left\langle\{n\}\left|\left(\frac{\underline{a}'}{C'}+\frac{\underline{a}'^{\dagger}}{C'^{*}}\right)\left(\frac{\underline{a}''}{C''}+\frac{\underline{a}''^{\dagger}}{C''^{*}}\right)\right|\{n\}\right\rangle\left\langle\{n\}\left|\left(\frac{\underline{a}'''}{C'''}+\frac{\underline{a}'''^{\dagger}}{C'''^{*}}\right)\left(\frac{\underline{a}^{\mathrm{iv}}}{C^{\mathrm{iv}}}+\frac{\underline{a}^{\mathrm{iv}\dagger}}{C^{\mathrm{iv}*}}\right)\right|\{n\}\right\rangle\cdots.$$

Here all pairings of the annihilation and creation operators are included in the sum with the order of the operators within each pair being maintained. Maintaining the order within each pair provides the distinction between

$$\left\langle\{n\}\left|\left(\frac{\underline{a}'}{C'}+\frac{\underline{a}'^{\dagger}}{C'^{*}}\right)\left(\frac{\underline{a}''}{C''}+\frac{\underline{a}''^{\dagger}}{C''^{*}}\right)\right|\{n\}\right\rangle=\frac{(2n'+1)}{|C'|^2}\,\delta_{\lambda'\lambda''}\,\delta^3(\vec{k}'-\vec{k}''), \tag{110}$$

$$\left\langle\{n\}\left|\left(\frac{\underline{a}'}{C'}+\frac{\underline{a}'^{\dagger}}{C'^{*}}\right)\left(\frac{\underline{a}''}{C''}-\frac{\underline{a}''^{\dagger}}{C''^{*}}\right)\right|\{n\}\right\rangle=-\frac{1}{|C'|^2}\,\delta_{\lambda'\lambda''}\,\delta^3(\vec{k}'-\vec{k}''), \tag{111}$$

and

$$\left\langle\{n\}\left|\left(\frac{\underline{a}'}{C'}-\frac{\underline{a}'^{\dagger}}{C'^{*}}\right)\left(\frac{\underline{a}''}{C''}+\frac{\underline{a}''^{\dagger}}{C''^{*}}\right)\right|\{n\}\right\rangle=\frac{1}{|C|^2}\,\delta_{\lambda'\lambda''}\,\delta^3(\vec{k}'-\vec{k}''). \tag{112}$$

In each case we have used the properties of annihilation and creation operators

$$\frac{\underline{a}^{\dagger}}{(n+1)^{1/2}}|n\rangle=|n+1\rangle \tag{113}$$

and

$$\frac{\underline{a}}{\sqrt{n}}|n\rangle=|n-1\rangle. \tag{114}$$

We find that half the integrations d^3k and sums $\sum_\lambda$ disappear from the δ functions arising in terms such as (110)–(112). The remaining integrals and sums are removed exactly as in Sec. III D where we made use of the sharply peaked integrand arising from the terms $|C|^{-2}$. The combinatorics seem quite complicated except in symmetric cases such as $\langle|\underline{x}^{2m}|\rangle_T$. Here it is not hard to see that after all the integrations in the $e\to 0$ limit

$$\langle|\underline{x}^{2m}|\rangle_T=\frac{(2m)!}{m!\,2^m}\left(\frac{\hbar}{2m\omega_0}\right)^m\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_m=0}^{\infty}(n_1+\tfrac{1}{2})(n_2+\tfrac{1}{2})\cdots(n_m+\tfrac{1}{2})$$
$$\times\exp\left[-\frac{\hbar\omega_0(n_1+\frac{1}{2})}{kT}\right]\exp\left[-\frac{\hbar\omega_0(n_2+\frac{1}{2})}{kT}\right]\cdots\exp\left[-\frac{\hbar\omega_0(n_m+\frac{1}{2})}{kT}\right]$$
$$=\frac{(2m)!}{m!\,2^m}\left[\frac{\hbar}{2m\omega_0}\coth\left(\frac{\hbar\omega_0}{2kT}\right)\right]^m \tag{115}$$

exactly as for $\langle x^{2m}\rangle_T$ in random mechanics. The interpretation of excited states within quantum mechanics requires that this also can be written as

$$\langle|\underline{x}^{2m}|\rangle_T = \frac{1}{Z}\sum_{n=0}^{\infty}\langle n|\underline{x}^{2m}|n\rangle\exp\left[-\frac{\hbar\omega(n+\frac{1}{2})}{kT}\right], \tag{116}$$

where $|n\rangle$ is the nth excited state of the oscillator. The agreement between the quantum mechanics and limit $e\to 0$ of quantum electrodynamics is easily checked for small numbers of operators $\underline{x}$ and $\underline{p}$.

The first few values of $\langle x^k p^l\rangle_T$ and of $\langle|\underline{x}^k\underline{p}^l|\rangle_T$ are listed in Table II for a comparison between random and quantum mechanics. The symmetrized quantum operator forms, such as $\langle|\underline{x}^{2m}|\rangle_T$, $\langle|\underline{p}^{2m}|\rangle_T$, agree exactly with the random values. Also the nonvanishing averages in random mechanics increase as T increases. The discrepancy in the average values between the symmetrized forms (agreeing with random mechanics) and the unsymmetrized operator forms involves terms which do not increase with T as fast as the symmetrized form values. Thus for large values of T, the quantum operator order becomes immaterial in the sense that the fractional discrepancy between the values of the symmetrized and unsymmetrized forms becomes negligible.

TABLE II. Average values for a mechanical oscillator at finite temperature.

Random mechanics	Quantum mechanics
$\langle x\rangle_T = 0$	$\langle|\underline{x}|\rangle_T = 0$
$\langle p\rangle_T = 0$	$\langle|\underline{p}|\rangle_T = 0$
$\langle x^2\rangle_T = \frac{1}{2}\frac{\hbar}{m\omega_0}\coth\left(\frac{\hbar\omega_0}{2kT}\right)$	$\langle|\underline{x}^2|\rangle_T = \frac{1}{2}\frac{\hbar}{m\omega_0}\coth\left(\frac{\hbar\omega_0}{2kT}\right)$
$\langle p^2\rangle_T = \frac{1}{2}\hbar m\omega_0\coth\left(\frac{\hbar\omega_0}{2kT}\right)$	$\langle|\underline{p}^2|\rangle_T = \frac{1}{2}\hbar m\omega_0\coth\left(\frac{\hbar\omega_0}{2kT}\right)$
$\langle xp\rangle_T = 0$	$\langle|\underline{x}\,\underline{p}|\rangle_T = i\frac{\hbar}{2} = -\langle|\underline{p}\,\underline{x}|\rangle_T$
$\langle x^4\rangle_T = 3\langle x^2\rangle_T^2$	$\langle|\underline{x}^4|\rangle_T = 3\langle|\underline{x}^2|\rangle_T^2$
$\langle p^4\rangle_T = 3\langle p^2\rangle_T^2$	$\langle|\underline{p}^4|\rangle_T = 3\langle|\underline{p}^2|\rangle_T^2$
$\langle x^2p^2\rangle_T = \langle x^2\rangle_T\langle p^2\rangle$	$\langle|\underline{x}^2\underline{p}^2|\rangle_T = \langle|\underline{p}^2\underline{x}^2|\rangle_T = \langle|\underline{x}^2|\rangle_T\langle|\underline{p}^2|\rangle_T - \frac{1}{2}\hbar^2$
	$\langle|\underline{x}\,\underline{p}\,\underline{x}\,\underline{p}|\rangle_T = \langle|\underline{p}\,\underline{x}\,\underline{p}\,\underline{x}|\rangle_T = \langle|\underline{x}^2|\rangle_T\langle|\underline{p}^2|\rangle_T$
	$\langle|\underline{x}\,\underline{p}^2\underline{x}|\rangle_T = \langle|\underline{p}\,\underline{x}^2\underline{p}|\rangle_T = \langle|\underline{x}^2|\rangle_T\langle|\underline{p}^2|\rangle_T + \frac{1}{2}\hbar^2$

V. OSCILLATOR SYSTEMS IN EXTERNAL MAGNETIC FIELDS FOR RANDOM AND QUANTUM THEORIES: DIAMAGNETIC BEHAVIOR

A. Diamagnetic Behavior

During the preliminary work involved in the development of the theory of random electrodynamics, it was shown that sometimes physical behavior which was regarded as outside the province of classical physics could actually be described adequately by the introduction of classical electromagnetic zero-point radiation. The earlier calculations involved blackbody radiation, questions in statistical thermodynamics, and then van der Waals force calculations.

In the present paper summarizing the general connection between random and quantum electrodynamic theories for harmonic-oscillator systems, it is of interest to turn to the question of diamagnetism. This is a further example of physical behavior where classical theory seemed to fail and quantum theory was successful. However, the present paper gives us a different perspective. Since diamagnetic behavior is found in quantum harmonic-oscillator systems, we may be sure to find it also in random-mechanical systems.

The situation for diamagnetism within the classical electron theory of Lorentz is outlined by Rosenfeld.[19] Although the classical model of electrons moving about a nucleus does indeed suggest diamagnetic behavior, any attempt to apply Boltzmann statistical mechanics on phase space led to disastrous results—to the complete absence of diamagnetic behavior.

Within random electrodynamics, the situation is changed sharply. The classical model of point electrons in orbits around a nucleus is retained. However, as has been shown earlier,[6] the presence of classical zero-point radiation leads to a failure of the Boltzmann statistical mechanical ideas except in the limits of massive particles or high temperatures. The proper treatment within random electrodynamics consists in solving the classical electromagnetic equations of motion and then in extracting the average values of energy, angular momentum, etc.

The system of interest is that first treated by Marshall,[1] consisting of an external magnetic field present on an atom, taken as a three-dimensional isotropic dipole oscillator of mass m, natural frequency ω_0, and charge e. The presence of the magnetic field changes the ground state of the atom giving it net angular momentum and a net

magnetic moment. Again we will carry out parallel calculations within random and quantum theories.

B. Equations of motion in the presence of an external magnetic field

We assume that an external magnetic field $\vec{B} = B\hat{k}$ is present along the z axis and that the three-dimensional charged harmonic oscillator is located at the origin of coordinates. The equations of motion are identical in form for the coordinate position in random electrodynamics and for the operator representing the particle position in the Heisenberg picture of quantum electrodynamics. After renormalization as in Sec. III B, the particle equation of motion is

$$m\ddot{\vec{r}} = -m\omega_0^2\vec{r} + \frac{e}{c}\dot{\vec{r}}\times\vec{B} + \frac{2}{3}\frac{e^2}{c^3}\dddot{\vec{r}} + e\vec{E}^{\,\mathrm{in}} . \quad (117)$$

This vector differential equation represents three equations for the coordinates x, y, z. It is linear in the particle position with the incoming radiation as an inhomogeneous term, and hence the equations are soluble. The three coupled equations are

$$\ddot{x} + \omega_0^2 x - \Gamma\dddot{x} - 2\omega_L\dot{y} = \frac{e}{m}E_x^{\mathrm{in}} , \quad (118)$$

$$\ddot{y} + \omega_0^2 y - \Gamma\dddot{y} + 2\omega_L\dot{x} = \frac{e}{m}E_y^{\mathrm{in}} , \quad (119)$$

$$\ddot{z} + \omega_0^2 z - \Gamma\dddot{z} = \frac{e}{m}E_z^{\mathrm{in}} , \quad (120)$$

where ω_L is the Larmor frequency

$$\omega_L = \frac{eB}{2mc} . \quad (121)$$

The equilibrium solutions in random electrodynamics are

$$x = \frac{e}{m}\sum_{\lambda=1}^{2}\int d^3k\,\frac{\hbar}{2}\left[\frac{C\epsilon_x a + i2\omega\omega_L\epsilon_y a}{C^2-(2\omega\omega_L)^2}\exp(-i\omega t) + \frac{C^*\epsilon_x a^* - i2\omega\omega_L\epsilon_y a^*}{C^{*2}-(2\omega\omega_L)^2}\exp(i\omega t)\right] \quad (122)$$

$$y = \frac{e}{m}\sum_{\lambda=1}^{2}\int d^3k\,\frac{\hbar}{2}\left[\frac{C\epsilon_y a - i2\omega\omega_L\epsilon_x a}{C^2-(2\omega\omega_L)^2}\exp(-i\omega t) + \frac{C^*\epsilon_y a^* + i2\omega\omega_L\epsilon_x a^*}{C^{*2}-(2\omega\omega_L)^2}\exp(i\omega t)\right] , \quad (123)$$

$$z = \frac{e}{m}\sum_{\lambda=1}^{2}\int d^3k\,\frac{\hbar}{2}\left[\frac{\epsilon_z a}{C}\exp(-i\omega t) + \frac{\epsilon_z a^*}{C^*}\exp(i\omega t)\right] . \quad (124)$$

For quantum electrodynamics x, y, z become operators $\underline{x}, \underline{y}, \underline{z}$. We also replace $\hbar/2$ by $(\hbar\omega)^{1/2}/(2\pi)$ and the random variables a, a^* by the annihilation and creation operators $\underline{a}, \underline{a}^\dagger$ in order to convert (122)–(124) over to the solutions in quantum theory.

Once again the equilibrium solutions for the oscillator displacements are linear in the amplitudes of the incoming radiation. Hence the evaluation of average values for the oscillator variables is taken back to exactly the same procedures as employed in Secs. III C and IV D. Again, provided the quantum operator order is symmetrized in all products, there is agreement between the average values in quantum and random electrodynamics, even for finite temperatures. Since the quantum Hamiltonian is already a symmetrized form, there is agreement for the average energy of the system. Also, the magnetic moment operator

$$\underline{M}_z = -\frac{e}{2mc}\underline{L}_z$$

$$= -\frac{e}{2mc}(\underline{y}\underline{p}_x - \underline{p}_y\underline{x}) \quad (125)$$

involves products of commuting operators and hence may be regarded as symmetrized. The average magnetic moment $\langle M_z\rangle$ thus agrees between the two theories.

C. Evaluation of the magnetic moment in random mechanics

Although the general connection between quantum and random electrodynamics ensures agreement in the average magnetization, it is interesting to take the $e \to 0$ limit into quantum and random mechanics so as to see the contrast in the mathematical description between the two theories at various temperatures. From a physical point of view, the presence of the magnetic field induces a diamagnetic effect in the three-dimensional oscillator system. At high temperatures, the diamagnetic effect tends to be eliminated by the thermal fluctuations.

As yet there are no rules for working directly in random mechanics; rather, we must work in random electrodynamics where we have a valid classical theory, and then proceed to the $e \to 0$ limit. In this limit, we will evaluate the average angular momentum

$$\langle L_z\rangle = m\langle x\dot{y} - y\dot{x}\rangle , \tag{126}$$

and so obtain the average magnetic moment

$$\langle M_z\rangle = \frac{e}{2mc}\langle L_z\rangle . \tag{127}$$

When we introduce the expressions (122) and (123), differentiate y, and then average over the random phrases as in (24) and (25), the term $\langle x\dot{y}\rangle$ in (126) becomes

$$\langle x\dot{y}\rangle = \frac{e^2}{m}\sum_{\lambda=1}^{2}\int d^3k\,\frac{(\epsilon_x{}^2+\epsilon_y{}^2)(-2\omega^2\omega_L)(-\omega^2+\omega_0{}^2)\hbar^2}{|\Lambda_+|^2|\Lambda_-|^2} , \tag{128}$$

where

$$\Lambda_+ = C + 2\omega\omega_L , \tag{129}$$

$$\Lambda_- = C - 2\omega\omega_L . \tag{130}$$

When we sum over the polarizations as in (4), this becomes

$$\langle x\dot{y}\rangle = \int_{\omega=0}^{\infty} d\omega\,\frac{8}{3\pi}\,\frac{e^2}{m^2c^3}\,\omega_L\hbar\,\frac{\omega^5\coth(\hbar\omega/2kT)(\omega^2-\omega_0{}^2)}{|\Lambda_+|^2|\Lambda_-|^2} . \tag{131}$$

Now the denominator involves

$$|\Lambda_+|^2 = (-\omega^2+\omega_0{}^2+2\omega\omega_L)^2 + (\Gamma\omega^3)^2 , \tag{132}$$

$$|\Lambda_-|^2 = (-\omega^2+\omega_0{}^2-2\omega\omega_L)^2 + (\Gamma\omega^3)^2 . \tag{133}$$

In the limit of small e, the damping term $(\Gamma_\omega{}^3)^2$ which is of order e^4 vanishes, leaving poles at the positive frequencies

$$\omega_+ = \omega_L + (\omega_0{}^2+\omega_L{}^2)^{1/2} \cong \omega_0 + \omega_L + \frac{1}{2}\,\frac{\omega_L{}^2}{\omega_0} , \tag{134}$$

$$\omega_- = -\omega_L + (\omega_0{}^2+\omega_L{}^2)^{1/2} \cong \omega_0 - \omega_L + \frac{1}{2}\,\frac{\omega_L{}^2}{\omega_0} . \tag{135}$$

Here we have assumed that ω_L is small compared to ω_0. Thus the integral along the positive values of ω in (131) involves a sharply peaked integrand at $\omega=\omega_+$ and $\omega=\omega_-$. With the $e\to 0$ limit in mind, we follow the familiar procedure and replace all factors of ω by ω_+ or ω_- unless the combination $(\omega-\omega_+)$ or $(\omega-\omega_-)$ already appears:

$$\langle x\dot{y}\rangle \cong \frac{8}{3\pi}\,\frac{e^2}{m^2c^3}\,\omega_L\hbar\left\{\frac{\omega_+{}^5\coth(\hbar\omega_+/2kT)(\omega_+{}^2-\omega_0{}^2)}{(-\omega_+{}^2+\omega_0{}^2-2\omega_+\omega_L)^2}\int_{-\infty}^{\infty}d\omega\,\frac{1}{\{[\omega_+-(\omega_L-(\omega_0{}^2+\omega_L{}^2)^{1/2})]^2(\omega-\omega_+)^2+(\Gamma\omega_+{}^3)^2\}}\right.$$
$$\left.+\,\frac{\omega_-{}^5\coth(\hbar\omega_-/2kT)(\omega_-{}^2-\omega_0{}^2)}{(-\omega_-{}^2+\omega_0{}^2+2\omega_-\omega_L)^2}\int_{-\infty}^{\infty}d\omega\,\frac{1}{\{[\omega_--(-\omega_L-(\omega_0{}^2+\omega_L{}^2)^{1/2})]^2(\omega-\omega_-)^2+(\Gamma\omega_-{}^3)^2\}}\right\} . \tag{136}$$

The integrals are of the form (60).

Now in our later work, we will be interested in comparing our results here with those of familiar quantum-mechanical perturbation theory carried to the lowest nonvanishing order. Hence we will need the value for $\langle x\dot{y}\rangle$ only through first order in ω_L/ω_0. Thus retaining only the necessary low-order terms, Eq. (136) becomes

$$\langle x\dot{y}\rangle \cong \frac{8}{3\pi}\,\frac{e^2}{m^2c^3}\,\omega_L\hbar\left[\frac{\omega_+{}^5}{16\omega_+{}^2\omega_L{}^2}\coth\left(\frac{\hbar\omega_+}{2kT}\right)\frac{(2\omega_0\omega_L+2\omega_L{}^2)}{2\omega_0\Gamma\omega_+{}^3}\,\pi + \frac{\omega_-{}^5}{16\omega_-{}^2\omega_L{}^2}\coth\left(\frac{\hbar\omega_-}{2kT}\right)\frac{(-2\omega_0\omega_L+2\omega_L{}^2)}{2\omega_0\Gamma\omega_-{}^3}\,\pi\right] . \tag{137}$$

We now expand all expressions involving ω_+ and ω_- and retain terms contributing through first order to $\langle x\dot{y}\rangle$. This gives

$$\langle x\dot{y}\rangle \cong -\frac{1}{3}\,\frac{e^2}{m^2c^3}\,\frac{\omega_L\hbar}{\Gamma\omega_0}\left[\coth\left(\frac{\hbar\omega_0}{2kT}\right)+\omega_0\frac{d}{d\omega_0}\coth\left(\frac{\hbar\omega_0}{2kT}\right)\right] . \tag{138}$$

The contribution from $\langle -y\dot{x}\rangle$ is equal to that from $\langle x\dot{y}\rangle$, giving average angular momentum

$$\langle L_z\rangle = -\frac{\omega_L}{\omega_0}\hbar\left[\coth\left(\frac{\hbar\omega_0}{2kT}\right)+\omega_0\frac{d}{d\omega_0}\coth\left(\frac{\hbar\omega_0}{2kT}\right)\right] \tag{139}$$

or

$$\langle L_z\rangle = -\frac{\omega_L}{\omega_0}\hbar\left(\frac{1+\epsilon}{1-\epsilon}-\frac{\hbar\omega_0}{kT}\,\frac{2\epsilon}{(1-\epsilon)^2}\right) , \tag{140}$$

where

$$\epsilon = \exp\left(-\frac{\hbar\omega_0}{kT}\right) . \tag{141}$$

In the limit of zero temperature, $T \to 0$, we have $\epsilon \to 0$ in (140). Thus at absolute zero

$$\langle L_z \rangle = -\frac{\omega_L}{\omega_0}\hbar\,, \tag{142}$$

and the diamagnetic effect gives an average magnetic moment

$$\langle M_z \rangle = -\frac{e}{2mc}\left(\frac{\omega_L}{\omega_0}\hbar\right). \tag{143}$$

At high temperatures $kT \gg \hbar\omega$, we can use the expansion of $\coth z$ for small z

$$\coth z = \frac{1}{z} + \frac{z}{3} - \frac{z^3}{45} + \cdots . \tag{144}$$

From (139) this gives

$$\langle L_z \rangle \cong -\frac{\hbar^2\omega_L}{3kT} \tag{145}$$

and

$$\langle M_z \rangle \cong -\frac{e}{2mc}\left(\frac{\hbar^2\omega_L}{3kT}\right) \text{ as } T \to \infty\,. \tag{146}$$

The differential change in energy associated with an external magnetic field is simply the magnetic moment dotted into the magnetic field

$$d(\Delta\mathcal{E}) = -\vec{\mathrm{M}} \cdot d\vec{\mathrm{B}}\,. \tag{147}$$

Thus the energy change here associated with the diamagnetism is

$$\Delta\mathcal{E} = -\tfrac{1}{2}\langle M_z \rangle B$$

$$= -\frac{1}{2}\,\frac{e^2}{2mc}\langle L_z \rangle B$$

where $\langle L_z \rangle$ is as in (139). Introducing the value of the Larmor frequency ω_L from (121), the energy change can be written as

$$\Delta\mathcal{E} = \frac{c^2 B^2 \hbar}{8m^2c^2\omega_0}\left[\coth\left(\frac{\hbar\omega_0}{2kT}\right) + \omega_0 \frac{d}{d\omega_0}\coth\left(\frac{\hbar\omega_0}{2kT}\right)\right]. \tag{148}$$

D. Evaluation of the magnetic moment in quantum-mechanical perturbation theory

Since the quantum electrodynamic and random electrodynamic expressions for the energy agree exactly for all values of $e \neq 0$, we can be assured that the quantum and random theories agree in the limit $e \to 0$. Nevertheless, the exact quantum electrodynamic solution presented here is so different in appearance from its $e \to 0$ limit quantum mechanics, that it is of interest to present the quantum-mechanical point of view explicitly.

In quantum mechanics the external magnetic field $\vec{\mathrm{B}} = B\hat{k}$ changes the mechanical oscillator Hamiltonian over to

$$\underline{H} = \frac{\underline{p}^2}{2m} - \frac{e\,\vec{\underline{p}}}{mc} \cdot \vec{\underline{A}} + \frac{e^2}{2mc^2}\,\vec{\underline{A}}^2 + \tfrac{1}{2}m\omega_0^2\,\vec{\underline{r}}^2\,, \tag{149}$$

where the vector potential can be chosen as

$$\vec{\underline{A}} = \tfrac{1}{2}(\underline{x}\,\hat{j} - \underline{y}\,\hat{i})B. \tag{150}$$

The diamagnetic energy change appears in second order in e. The energy change of the state $|n_x n_y n_z\rangle$ is in lowest order

$$\Delta\mathcal{E}^{(1)}_{n_x n_y n_z} = \langle n_x n_y n_z | \underline{H'} | n_x n_y n_z \rangle\,, \tag{151}$$

where the perturbation is

$$\underline{H'} = -\frac{e}{mc}\,\vec{\underline{p}} \cdot \vec{\underline{A}} + \frac{e^2}{2mc^2}\,\vec{\underline{A}}^2\,. \tag{152}$$

Thus

$$\Delta\mathcal{E}^{(1)}_{n_x n_y n_z} = \left\langle n_x n_y n_z \left| \frac{e^2}{2mc^2}\,\underline{A}^2 \right| n_x n_y n_z \right\rangle$$

$$= \frac{e^2 B^2}{8mc^2}\,\langle n_x n_y n_z | (\underline{x}^2 + \underline{y}^2) | n_x n_y n_z \rangle$$

$$= \frac{e^2 B^2}{8mc^2}\,\frac{\hbar}{2m\omega_0}(2n_x + 1 + 2n_y + 1)\,. \tag{153}$$

Second-order perturbation theory also contributes to order e^2 as

$$\sum_I \frac{|\langle n_x n_y n_z | \underline{H'} | I \rangle|^2}{\mathcal{E}_0 - \mathcal{E}_I}$$

$$= \sum_I \frac{e^2}{mc^2}\,\frac{|\langle n_x n_y n_z | \vec{\underline{p}} \cdot \vec{\underline{A}} | I \rangle|^2}{\mathcal{E}_0 - \mathcal{E}_I} + O(e^4)\,. \tag{154}$$

The matrix element involved is

$$\langle n_x n_y n_z | \underline{x}\,\underline{p}_y - \underline{y}\,\underline{p}_x | n'_x n'_y n'_z \rangle\,.$$

However, the symmetry of the unperturbed system between x and y ensures that this vanishes in the sum (154). Thus $\Delta\mathcal{E}^{(1)}_{n_x n_y n_z}$ is the full energy change through order e^2.

The average energy at temperature T is found by summing over all the excited states of the atom where the probability of occupation of the state is given by the Boltzmann factor

$$\langle |\mathcal{E}| \rangle_T = \frac{\sum_{n_x=0}^{\infty}\sum_{n_y=0}^{\infty}\sum_{n_z=0}^{\infty}\mathcal{E}_{n_x n_y n_z}\exp(-\mathcal{E}_{n_x n_y n_z}/kT)}{\sum_{n_x=0}^{\infty}\sum_{n_y=0}^{\infty}\sum_{n_z=0}^{\infty}\exp(-\mathcal{E}_{n_x n_y n_z}/kT)}\,, \tag{155}$$

where, from (153),

$$\mathcal{E}_{n_x n_y n_z} = \hbar\omega_0\left(1+\frac{e^2B^2}{8m^2c^2\omega_0^2}\right)(n_x+\tfrac{1}{2}) + \hbar\omega_0\left(1+\frac{e^2B^2}{8m^2c^2\omega_0^2}\right)(n_y+\tfrac{1}{2}) + \hbar\omega_0(n_z+\tfrac{1}{2}) + O(e^4)\,. \tag{156}$$

The sums are easily evaluated as the traditional series involving simple harmonic oscillators. The sums over n_x and n_y involve $\hbar\omega_0(1+e^2B^2/8m^2c^2\omega_0^2)$ while that over n_z involves $\hbar\omega_0$. Thus

$$\langle|\mathcal{E}|\rangle_T = 2\times\tfrac{1}{2}\hbar\omega_0\left(1+\frac{e^2B^2}{8m^2c^2\omega_0^2}\right)\coth\left[\frac{\hbar\omega_0}{2kT}\left(1+\frac{e^2B^2}{8m^2c^2\omega_0^2}\right)\right] + \tfrac{1}{2}\hbar\omega_0\coth\left(\frac{\hbar\omega_0}{2kT}\right). \tag{157}$$

Expanding and retaining terms through order e^2, this gives exactly the diamagnetic change of energy $\Delta\mathcal{E}$ computed in (148) as the $e\to 0$ limit of random electrodynamics. The values for $\langle|\underline{M}_z|\rangle_T$ and $\langle|\underline{L}_z|\rangle_T$ are also in agreement.

VI. CLOSING SUMMARY

Quantum electrodynamics is the theory par excellence of contempory physics. Its calculations seem unambiguous and its predictions have been verified experimentally with extraordinary accuracy. Now it is clear that quantum electrodynamics has close connections with nineteenth century classical electromagnetism. These connections are frequently referred to, but rarely are they exhaustively explored because of the extreme shift in physical concepts involved—from continuity to quanta—and because traditional classical electrodynamics makes no pretense of treating physical systems where Planck's constant $\hbar$ seems to play a role.

However, a new classical electromagnetic theory, random electrodynamics, has been proposed, based upon Lorentz's classical electron theory with a new homogeneous boundary condition on Maxwell's equations. The new theory involves purely classical concepts for particles, forces, and fields.

In this paper we confront quantum electrodynamics with this new classical theory for a limited class of physical systems, those for which exact calculations can be carried out in both theories, namely free electromagnetic fields and point dipole-oscillator systems. And in this comparison, random electrodynamics seems to provide predictions which are presently as acceptable experimentally as those of quantum electrodynamics. However, the predictions of the theories are not the same. The comparison reflects interest back on quantum theory in raising questions regarding the meanings of operator order in quantum systems.

In general we find that the average values for physical quantities which are calculated in random electrodynamics agree with those calculated in quantum mechanics provided the order of the operators is symmetrized in all products of quantum operators. Thus, for example, the classical expression x^2p^2 for the position and momentum of a point dipole operator corresponds exactly in its average value at all temperatures and to all orders in the fine-structure constant with the symmetrized quantum operator

$$\tfrac{1}{6}(\underline{x}^2\underline{p}^2+\underline{x}\,\underline{p}\,\underline{x}\,\underline{p}+\underline{x}\,\underline{p}^2\,\underline{x}+\underline{p}^2\,\underline{x}^2+\underline{p}\,\underline{x}\,\underline{p}\,\underline{x}+\underline{p}\,\underline{x}^2\,\underline{p})\,.$$

If an arbitrary unsymmetrized form involving two $\underline{x}$'s and two $\underline{p}$'s were involved, its average values would not agree with those of random electrodynamics. This connection between random electrodynamics and symmetrized quantum electrodynamics persists in the limit as the dipole oscillator is uncoupled from the electromagnetic radiation field so as to give a mechanical oscillator in random mechanics and in quantum mechanics.

It is pointed out that some quantum operators which are regarded as physically meaningful are presently used in their unsymmetrized forms. This is true, for example, of the angular momentum squared $\underline{L}^2$. Accordingly, the ground state of an atom has zero angular momentum, including $\langle 0|\underline{L}^2|0\rangle = 0$. In a semiclassical description of this quantum result, we must picture the electron as moving straight through the force center and never around it. However, in the classical picture within random electrodynamics, the average of the angular momentum squared is not zero, precisely because sometimes the electron is moving around the nucleus. And the average value in random electrodynamics agrees with the averaged value of the symmetrized operator form $\underline{L}_{\rm sym}^{\ 2}$ in quantum theory. We find, however, that

$$\underline{L}_{\rm sym}^{\ 2} = \underline{L}^2 + \tfrac{3}{2}\hbar^2\,,$$

so that this same discrepancy holds for all the excited states and so does not seem to be physically measurable in atomic spectra.

The theories of random electrodynamics and quantum electrodynamics are also compared at finite temperature, where the mathematical descriptions are widely divergent, and, lastly, in the presence of a magnetic field yielding diamagnetic behavior. In all cases the average values of the theories agree exactly provided the quantum

operator products are symmetrized. Clearly it will be of interest to compare the two theories for more general physical systems.

ACKNOWLEDGMENTS

Throughout this paper I have taken advantage of work by T. W. Marshall along closely related lines presented in two striking papers in 1963 and 1965, listed here as Refs. 1 and 2. Although the mathematical description in the present work looks rather different and is carried to higher order, and the physics proceeds in somewhat different directions from Marshall's work, nevertheless the debt is considerable. I wish to thank Professor Harry Soodak for raising stimulating questions which led to some developments in this paper.

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$$\underline{a} = e^{i\pi/2}\underline{a}_{BD} = i\underline{a}_{BD} ,$$
$$\underline{a}^{\dagger} = e^{-i\pi/2}\underline{a}^{\dagger}_{BD} = -i\underline{a}^{\dagger}_{BD} .$$

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