Equilibrium of random classical electromagnetic radiation in the presence of a nonrelativistic nonlinear electric dipole oscillator*

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The scattering of random classical electromagnetic radiation by a nonrelativistic nonlinear electric dipole oscillator is calculated purely within classical physics using a perturbation expansion carried through second order in the nonlinear coupling constant. It is shown that the Rayleigh-Jeans radiation spectrum is an equilibrium distribution in the presence of such a nonlinear oscillator, while random radiation spectra given by Planck's law or by zero-point raditation are not equilibrium distributions but rather are altered by the scattering oscillator.

I. INTRODUCTION

The problem of the blackbody radiation spectrum representing the thermal equilibrium for random electromagnetic radiation has been discussed frequently within the present century. However, most treatments have included the use of statistical mechanics at some point in the analysis. We wish to avoid the *ad hoc* assumptions of statistical mechanics. In the present work we derive an equilibrium spectrum for random classical radiation by following detailed classical equations of motion for radiation scattering by a nonrelativistic nonlinear electric dipole oscillator.

A. Detailed balancing for particle systems

The point of view taken in our electromagnetic radiation calculation may be understood from an analogy involving classical particles in a box. The equilibrium velocity distribution for classical hard spherical particles may be obtained from statistical-mechanical assumptions about phase-space distributions; however, it may also be obtained through the Boltzmann equation by considering the exchange of energy and momentum in collisions between the particles and requiring that for the equilibrium distribution there is on the average no net transfer of particles from one velocity interval to another. The detailed balancing required for equilibrium has indeed been examined in detail from the nonrelativistic classical equations of motion for hard spheres.[1] The analogy with our subsequent calculation for random radiation is even closer if we consider all but one of the particles in the box as hard point particles which never collide with each other, but have one hard sphere of finite volume. The point masses collide with this sphere and so transfer energy and momentum until an equilibrium velocity distribution is attained.

B. Detailed balancing for radiation

The present writer is unaware of detailed-balancing arguments for classical electromagnetic radiation analogous to those just mentioned for particles, and so in the work below has provided a careful and detailed calculation which follows this point of view.

Random electromagnetic radiation enclosed in a box with perfectly conducting walls will never come to thermal equilibrium because the classical Maxwell equations are linear and there is no exchange of energy among the frequencies present. This situation is like that just described involving collisionless mass points moving in a box with perfectly reflecting walls. In the case of radiation, the attainment of equilibrium requires the introduction of some nonlinear mechanical system having electromagnetic interactions. This is analogous to the introduction of a particle of finite dimensions into the point-particle system. In discussions of thermal radiation, the scattering system for radiation is always chosen as a small, black particle which has negligible energy and entropy of its own, but which converts the energy spectrum of the radiation over to the spectrum of maximum entropy.[2] What is meant by maximum entropy for the radiation clearly depends upon the behavior of the black particle which determines the equilibrium radiation spectrum. In the particle case this is analogous to the attainment of the relativistic or nonrelativistic Boltzmann distribution for particles depending upon the use of relativistic or nonrelativistic mechanics for the particle collisions.[3] In the present electromagnetic calculation we insert not a black particle, which is too complicated for our mathematics, but rather a nonrelativistic nonlinear electric dipole oscillator. This system acts like a black particle in that it alters the initial distribution of random radiation. We will calculate the scattering by such a nonlinear system using a perturbation expansion through second order in the nonlinear coupling constant, and then will determine the radiation spectrum which is an equilibrium spectrum in the presence of such a scattering system.

C. Difference from familiar treatments

It should be emphasized that the present electromagnetic calculation is quite different from some earlier classical calculations[4] which consider a linear electric dipole which interacts both with random radiation through its dipole moment and with some mechanical heat bath through unspecified mechanical interactions. In these treatments the mechanical equilibrium of the oscillator is first discussed assuming the absence of electromagnetic interactions. In this way one predicts an average mechanical oscillator energy given by the equipartition value KT. Next, one ignores the mechanical interactions which make equilibrium possible and treats the interaction of a linear dipole oscillator with the random radiation field. Equilibrium between the linear oscillator and the radiation field requires that the radiation field have an average energy per normal mode at the natural frequency ω_0 of the oscillator which is equal to the average energy of the oscillator.[5] The familiar analysis now combines these two separate calculations and decides that the random radiation field in equilibrium with a mechanical system has an energy KT per normal mode, corresponding to the Rayleigh-Jeans distribution law.

In the electromagnetic calculation to follow, there is no discussion of mechanical equilibrium separate from the interaction with radiation. A single nonlinear oscillator is involved, and the criterion of radiation equilibrium is that the radiation spectrum should be unaltered by the radiation scattering of the nonlinear dipole oscillator.

D. Results

It is found that an equilibrium radiation spectrum in the presence of the nonrelativistic nonlinear electric dipole oscillator is just the Rayleigh-Jeans law. This will come as no surprise to those who are convinced that the Rayleigh-Jeans law is an inevitable result of classical as opposed to quantum physics. However, we also find that the nonrelativistic nonlinear oscillator attempts to convert the Lorentz-invariant spectrum of classical zero-point radiation into a non-Lorentz-invariant spectrum. Since zero-point radiation might be expected to be an equilibrium distribution at zero temperature, such a conversion seems surprising. Furthermore the present calculation naively seems to contradict the recent classical derivation[6] of the Planck radiation spectrum made by modifying the arguments of Einstein and Hopf[7] in the presence of zero-point radiation. The connections of the present calculation with the recent work in random electrodynamics will be discussed elsewhere. Here we will conjecture that the Rayleigh-Jeans law may arise not from the failure to use quantum mechanics but rather from the failure to use a relativistic electromagnetic system in interaction with the random radiation.

II. DESCRIPTION OF THE PHYSICAL SYSTEM

A. Initial random radiation

Free-field random classical electromagnetic radiation may be expressed as a superposition of transverse plane waves with vanishing scalar potential and vector potential

$$\vec{A}(\vec{r},t) = \mathrm{Re}\sum_{\lambda=1}^{2}\int d^3k\,\hat{\epsilon}(\vec{k},\lambda)\,\frac{c\mathfrak{h}(\vec{k},\lambda)}{i\omega}\times\exp\{i[\vec{k}\cdot\vec{r}-\omega t+\theta(\vec{k},\lambda)]\}, \quad (1)$$

where

$$\omega = ck, \quad (2)$$

the unit polarization vectors satisfy

$$\vec{k}\cdot\hat{\epsilon}(\vec{k},\lambda)=0, \quad (3)$$

$$\hat{\epsilon}(\vec{k},\lambda)\cdot\hat{\epsilon}(\vec{k},\lambda')=\delta_{\lambda\lambda'}, \quad (4)$$

$$\sum_{\lambda=1}^{2}\epsilon_i(\vec{k},\lambda)\epsilon_j(\vec{k},\lambda)=\delta_{ij}-k_ik_j/k^2, \quad (5)$$

and $\pi^2\mathfrak{h}^2(\vec{k},\lambda)$ is the spectral energy density per normal mode. The random character of the radiation[8] is expressed by the random phase $\theta(\vec{k},\lambda)$ which is distributed uniformly over $[0,2\pi]$, independently for each $\vec{k}$ and λ. The free electric and magnetic fields follow from (1) as

$$\vec{E}(\vec{r},t)=\sum_{\lambda=1}^{2}\int d^3k\,\hat{\epsilon}(\vec{k},\lambda)\mathfrak{h}(\vec{k},\lambda)\tfrac{1}{2}(\exp\{i[\vec{k}\cdot\vec{r}-\omega t+\theta(\vec{k},\lambda)]\}+\exp\{-i[\vec{k}\cdot\vec{r}-\omega t+\theta(\vec{k},\lambda)]\}), \quad (6)$$

$$\vec{B}(\vec{r},t)=\sum_{\lambda=1}^{2}\int d^3k\,\frac{\vec{k}\times\hat{\epsilon}(\vec{k},\lambda)}{k}\,\mathfrak{h}(\vec{k},\lambda)\tfrac{1}{2}(\exp\{i[\vec{k}\cdot\vec{r}-\omega t+\theta(\vec{k},\lambda)]\}+\exp\{-i[\vec{k}\cdot\vec{r}-\omega t+\theta(\vec{k},\lambda)]\}). \quad (7)$$

If the radiation is isotropic, then $\mathfrak{h}(\vec{k},\lambda)$ depends only upon $\omega = ck$.

B. Scattering system–Linear oscillator

The scattering system appearing most frequently in the literature[5] is that of a linear electric dipole oscillator which may be pictured as a point mass m with charge e attached to a small spring fastened to the origin and free to move along the x axis. The equations of motion for the nonrelativistic oscillator in the electromagnetic field are

$$m_0\ddot{x} = -m\omega_0^2 x + eE_x(\vec{r}, t), \tag{8}$$

$$\nabla^2\Phi = -4\pi\rho, \tag{9}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{-4\pi}{c}\vec{J}^{\perp}, \quad \nabla\cdot\vec{A} = 0, \tag{10}$$

where $\vec{E}(\vec{r}, t)$ is the full electric field including the self-fields evaluated at the position $\vec{r}$ of the charge. After renormalization,[9] so as to remove the self-fields and to provide a renormalized mass and radiation damping term, the nonrelativistic equation of motion (8) becomes

$$m\ddot{x} = -m\omega_0^2 x + m\tau\dddot{x} + eE_{\mathrm{in}x}(0,t), \tag{11}$$

where the forces appearing in Newton's second law are, respectively, the harmonic spring restoring force, the radiation damping force, and the initial random electric field incident on the oscillator. The dipole approximation is assumed for the small oscillator so that the electric field $E_{\mathrm{in}x}$ is evaluated at the origin $\vec{r} = 0$ rather than exactly at the position of the charge.

In the steady-state situation this linear dipole oscillator may scatter radiation into directions different from that of the incident radiation. As shown in earlier work,[10] such a linear dipole oscillator will not change the energy spectrum and hence the entropy of any distribution of random radiation which is already isotropic.

C. Nonlinear oscillator

If we consider a nonlinear electric dipole oscillator, then we find that the frequency spectrum of random radiation may indeed be shifted; energy may be absorbed by the oscillator at one frequency and reradiated at another. Thus in considering the maximum entropy of a radiation spectrum, it is natural to place the radiation in contact with a nonlinear scattering system.

The simplest departure from the linear oscillator of Eq. (11) is to imagine that the restoring spring force includes a small nonlinearity $-\lambda x^2$ in addition to the harmonic restoring force $-m\omega_0^2 x$,

$$F_{\mathrm{spring}} = -m\omega_0^2 x - \gamma x^2. \tag{12}$$

The equation of motion for the nonrelativistic nonlinear electric dipole oscillator is then

$$m\ddot{x} = -m\omega_0^2 x - \gamma x^2 + m\tau\dddot{x} + eE_{\mathrm{in}x}(0,t). \tag{13}$$

The equation of motion (13) for the oscillator may be rewritten in terms of the electric dipole moment

$$p = ex \tag{14}$$

and a new coupling constant

$$\alpha = \frac{\gamma}{me}. \tag{15}$$

Dividing through by m and noting that

$$\tau = \frac{2}{3}\frac{e^2}{mc^3}, \tag{16}$$

we find the equation (13) then takes the form

$$\ddot{p} + \omega_0^2 p - \tau\dddot{p} - \alpha p^2 = \tfrac{3}{2}c^3\tau E_{\mathrm{in}x}(0,t). \tag{17}$$

The equation is seen to depend upon the parameters ω_0, τ, c, and the coupling constant α.

D. Radiation equilibrium in the presence of the oscillator

We now wish to make clear the physical basis for the radiation equilibrium criterion in the mathematical calculation to follow. We imagine a very large box filled with random electromagnetic radiation, and assume that initially the radiation is spatially homogeneous and isotropic. The equilibrium of the radiation spectrum is investigated by inserting a small nonlinear electric dipole oscillator as in (13), where the mechanical energy and entropy associated with the oscillator itself are assumed negligible. We wait until the oscillator motion has come to a steady state in the radiation, but not so long that any radiation is reflected from the walls of the large box and is scattered twice by the oscillator. We then examine the spectrum of the random radiation in the box. If the spectrum of random radiation is shifted by the oscillator, we conclude the original radiation spectrum was not in equilibrium and was not one of maximum entropy. Suppose, for example, that the large box is filled initially with a homogeneous, isotropic distribution of random radiation, all near the single special frequency ω_s. When the nonlinear oscillator of Eq. (13) is inserted, the random radiation drives the oscillator near frequency ω_s. However, the nonlinear forcing term introduces further oscillations at frequency $2\omega_s$, and accordingly the dipole oscillator radiates away some electromagnetic waves at frequency $2\omega_s$. Also, since the interaction between the oscillator and the radiation field is conservative, some of the radiation field energy at frequency ω_s must be absorbed to provide the energy radiated at $2\omega_s$. Thus the radiation pattern is not in equilibrium since the initial radiation spectrum involving energy near frequency ω_s is being altered to provide random radiation at $2\omega_s$. Specifically, if we consider the radiation crossing a spherical surface around the oscillator, we will find a time-average flow of energy into the surface at frequency ω_s and an average flow of energy out of the surface at frequency $2\omega_s$.

The situation for a general random radiation pattern is analogous to the example described above. The equilibrium of the radiation pattern is determined by considering the net flow of radiation at each frequency across a spherical surface surrounding the oscillator. Writing the time-average Poynting vector $\langle\vec{S}\rangle$ at the spherical surface as a function of the frequency of the radiation involved

$$\hat{n}\cdot\langle\vec{S}\rangle=\int_{\omega=0}^{\infty}\hat{n}\cdot\langle\vec{S}(\omega)\rangle d\omega, \tag{18}$$

we will, in general, find that $\hat{n}\cdot\langle\vec{S}(\omega)\rangle$ is positive for some frequencies and negative for others. A spectrum of radiation which is in equilibrium in the presence of the scattering system has

$$\hat{n}\cdot\langle\vec{S}(\omega)\rangle=0 \tag{19}$$

for all frequencies ω.

E. Outline of the calculation

Our electromagnetic calculation follows exactly the pattern discussed above. From the equation of motion (13), we may solve for the oscillator motion as a perturbation expansion in powers of the coupling constant $\alpha=\gamma/me$ for the nonlinear force. From the oscillator motion the electromagnetic fields in all space may be obtained, and hence the time-average Poynting vector $\langle\vec{S}\rangle$ may be decomposed according to the frequency of the radiation. The function $\langle\vec{S}(\omega)\rangle$ is then examined for various initial distributions of random electromagnetic radiation in order to find those distributions which give $\langle\vec{S}(\omega)\rangle=0$ even in the presence of the nonlinear dipole oscillator.

III. MOTION OF THE NONLINEAR OSCILLATOR

A. Expansion in the coupling constant

The steady-state motion of the dipole oscillator may be obtained from Eq. (13) as a perturbation expansion in the coupling constant α for the nonlinear force. We expand the displacement x as

$$x=x_0+x_1+x_2+\cdots, \tag{20}$$

where the subscripts refer to the order in α. Substituting from Eq. (20) into the differential equation (13) and then grouping all terms in the same order α, we find the following differential equations.

$$\text{order 0:}\ \ddot{x}_0+\omega_0^2x_0-\tau\dddot{x}_0=\frac{e}{m}E_{\text{in}x}(0,t), \tag{21}$$

$$\text{order 1:}\ \ddot{x}_1+\omega_0^2x_1-\tau\dddot{x}_1-\alpha e x_0^2=0, \tag{22}$$

$$\text{order 2:}\ \ddot{x}_2+\omega_0^2x_2-\tau\dddot{x}_2-2\alpha e x_0x_1=0, \tag{23}$$

etc.

It is clear that these equations may be solved successively by inserting the solutions of the lower-order equations into the higher-order equations. The electric field $\vec{E}_{\text{in}}(\vec{r},t)$ required in Eq. (21) corresponds to the random radiation field (6) in the absence of the oscillator.

B. Compact notations

In order to make the notation compact against the lengthy manipulations to follow, we will introduce the symbols μ, A, and F, where μ takes the values ±1 and

$$A=\exp\{-i[\mu\omega t+\mu\theta(\vec{k},\lambda)]\}, \tag{24}$$

$$F=\exp(i\mu\vec{k}\cdot\vec{r}). \tag{25}$$

The electric and magnetic fields (6), (7) may then be written

$$\vec{E}_{\text{in}}(\vec{r},t)=\sum_{\mu}\sum_{\lambda}\int d^3k\,\hat{\epsilon}\,\frac{\mathfrak{h}}{2}FA, \tag{26}$$

$$\vec{B}_{\text{in}}(\vec{r},t)=\sum_{\mu}\sum_{\lambda}\int d^3k\,\frac{\vec{k}\times\hat{\epsilon}}{k}\,\frac{\mathfrak{h}}{2}FA, \tag{27}$$

where the sum over $\mu=\pm1$ provides the needed complex conjugates for real fields.

C. Evaluation of x_0

The equation (21) (order zero in α) is simply the linear electric dipole oscillator equation which has been solved many times previously.[11] The steady-state solution in our compact notation is

$$x_0=\frac{e}{m}\sum_{\mu}\sum_{\lambda}\int d^3k\,\epsilon_x\,\frac{\mathfrak{h}}{2}\,\frac{A}{C}, \tag{28}$$

where

$$C=-(\mu\omega)^2+\omega_0^2-i\tau(\mu\omega)^3. \tag{29}$$

The symbol F does not appear in (28) because we are working in the dipole approximation

$$\vec{E}_{\text{in}}(\vec{r},t)\cong\vec{E}_{\text{in}}(0,t)$$

and for $\vec{r}=0$

$$F=\exp(i\mu\vec{k}\cdot\vec{0})=1. \tag{30}$$

D. Evaluation of x_1

Now, taking the steady-state solution (28) to the zero-order equation (21), we substitute it into the first-order equation (22) to obtain

$$\ddot{x}_1 + \omega_0^2 x_1 - \tau \dddot{x}_1 = \alpha \frac{e^3}{m^2} \sum_{\mu_1} \sum_{\mu_2} \sum_{\lambda_1} \sum_{\lambda_2} \int d^3k_1 \int d^3k_2 \epsilon_{x1} \epsilon_{x2} \frac{\mathfrak{h}_1}{2} \frac{\mathfrak{h}_2}{2} \frac{A_1}{C_1} \frac{A_2}{C_2}. \tag{31}$$

The subscripts refer to the different frequencies ω_1 and ω_2 involved. The time dependence in the right-hand side of (31) appears in the terms

$$A_1 A_2 = \exp\{-i[(\mu_1\omega_1 + \mu_2\omega_2)t + (\mu_1\theta_1 + \mu_2\theta_2)]\}. \tag{32}$$

Hence the steady-state solution to the first-order equation (31) is

$$x_1 = \alpha \frac{e^3}{m^2} \sum_{\mu_1} \sum_{\mu_2} \sum_{\lambda_1} \sum_{\lambda_2} \int d^3k_1 \int d^3k_2 \epsilon_{x1} \epsilon_{x2} \frac{\mathfrak{h}_1}{2} \frac{\mathfrak{h}_2}{2} \frac{A_1 A_2}{C_1 C_2 C_{1+2}}, \tag{33}$$

where, consistent with the earlier notation,

$$C_{1+2} = -(\mu_1\omega_1 + \mu_2\omega_2)^2 + \omega_0^2 - i\tau(\mu_1\omega_1 + \mu_2\omega_2)^3. \tag{34}$$

E. Evaluation of x_2

The second-order differential equation (23) now takes the form

$$\ddot{x}_2 + \omega_0^2 x_2 - \tau \dddot{x}_2 = 2\alpha e x_0 x_1,$$

$$= 2\alpha \frac{e^5}{m^3} \sum_{\mu_1} \sum_{\mu_2} \sum_{\mu_3} \sum_{\lambda_1} \sum_{\lambda_2} \sum_{\lambda_3} \int d^3k_1 \int d^3k_2 \int d^3k_3 \epsilon_{x1} \epsilon_{x2} \epsilon_{x2} \frac{\mathfrak{h}_1}{2} \frac{\mathfrak{h}_2}{2} \frac{\mathfrak{h}_3}{2} \frac{A_1 A_2 A_3}{C_1 C_2 C_3 C_{2+3}}, \tag{35}$$

with the steady-state solution

$$x_2 = 2\alpha \frac{e^5}{m^3} \sum_{\mu_1} \sum_{\mu_2} \sum_{\mu_3} \sum_{\lambda_1} \sum_{\lambda_2} \sum_{\lambda_3} \int d^3k_1 \int d^3k_2 \int d^3k_3 \epsilon_{x1} \epsilon_{x2} \epsilon_{x3} \frac{\mathfrak{h}_1}{2} \frac{\mathfrak{h}_2}{2} \frac{\mathfrak{h}_3}{2} \frac{A_1 A_2 A_3}{C_1 C_2 C_3 C_{2+3} C_{1+2+3}}, \tag{36}$$

$$C_{1+2+3} = -(\mu_1\omega_1 + \mu_2\omega_2 + \mu_3\omega_3)^2 + \omega_0^2 - i\tau(\mu_1\omega_1 + \mu_2\omega_2 + \mu_3\omega_3)^3. \tag{37}$$

IV. ELECTRIC FIELDS IN SPACE

A. Expansion of Poynting's vector in powers of α

The equilibrium of the random radiation pattern in the presence of the dipole oscillator is tested by evaluating the time average of the Poynting vector

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H}, \tag{38}$$

or in free space

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}. \tag{39}$$

The radiation $\vec{E}, \vec{B}$ present in space arises from two sources, the initial radiation $\vec{E}_{in}, \vec{B}_{in}$ and the radiation emitted by the dipole oscillator $\vec{E}', \vec{B}'$,

$$\vec{E} = \vec{E}_{in} + \vec{E}', \tag{40}$$

$$\vec{B} = \vec{B}_{in} + \vec{B}'. \tag{41}$$

The radiation fields emitted by the dipole are proportional to the dipole moment $p = ex$ and hence may be separated as to powers in the nonlinear coupling constants α in a fashion analogous to Eq. (20) for x,

$$p = p_0 + p_1 + p_2 + \cdots, \tag{42}$$

$$\vec{E}' = \vec{E}_0 + \vec{E}_1 + \vec{E}_2 + \cdots, \tag{43}$$

$$\vec{B}' = \vec{B}_0 + \vec{B}_1 + \vec{B}_2 + \cdots. \tag{44}$$

It follows that the Poynting vector may also be broken up into terms of zero order, first order, second order, etc., in the nonlinear coupling constant α,

$$\vec{S} = \frac{c}{4\pi}(\vec{E}_{in} + \vec{E}_0 + \vec{E}_1 + \vec{E}_2 + \cdots) \times (\vec{B}_{in} + \vec{B}_0 + \vec{B}_1 + \vec{B}_2 + \cdots), \tag{45}$$

$$\begin{aligned} \vec{S} = \frac{c}{4\pi} [&\vec{E}_{in} \times \vec{B}_{in} + (\vec{E}_{in} \times \vec{B}_0 + \vec{E}_0 \times \vec{B}_{in} + \vec{E}_0 \times \vec{B}_0) \\ &+ (\vec{E}_{in} \times \vec{B}_1 + \vec{E}_1 \times \vec{B}_{in} + \vec{E}_0 \times \vec{B}_1 + \vec{E}_1 \times \vec{B}_0) \\ &+ (\vec{E}_{in} \times \vec{B}_2 + \vec{E}_2 \times \vec{B}_{in} + \vec{E}_0 \times \vec{B}_2 + \vec{E}_2 \times \vec{B}_0 + \vec{E}_1 \times \vec{B}_1) + \cdots]. \end{aligned} \tag{46}$$

Now the first term $\vec{E}_{in}\times\vec{B}_{in}$ of Eq. (46) makes no reference to the presence of the oscillator; its average vanishes if the radiation is isotropic. The first bracket of three terms involves zero order in α, corresponding to a linear electric dipole oscillator. Although radiant energy may be shifted in direction by a linear oscillator, there is no shift of energy among the frequencies in steady-state oscillation; the oscillator radiates at the same frequency as the incident radiation. It was shown recently[10] that this term vanishes on the average provided the incident radiation is isotropic. The next bracket includes the four terms of first order in α. These terms all vanish individually when averaged over the random phases $\theta(\vec{k},\lambda)$ since $\vec{E}_1$ and $\vec{B}_1$ involve terms $\exp[\pm i(\theta_1+\theta_2)]$ whereas the other fields involve $\exp[\pm i\theta_3]$. On averaging over independent random phases

$$\langle\exp[\pm i(\theta_1\pm\theta_2)\pm i\theta_3]\rangle=0. \quad (47)$$

Hence it is the bracket of (46) containing second-order terms in α which first involves the nonlinear aspects of the oscillator,

$$\vec{S}_2=(\vec{E}_{in}\times\vec{B}_2+\vec{E}_2\times\vec{B}_{in}+\vec{E}_0\times\vec{B}_2+\vec{E}_2\times\vec{B}_0+\vec{E}_1\times\vec{B}_1). \quad (48)$$

B. Electric dipole fields

In order to evaluate the average $\langle\vec{S}_2\rangle$, we will need the fields $\vec{E}_{in}(\vec{r},t)$ and $\vec{B}_{in}(\vec{r},t)$ listed in Eqs. (26) and (27), and also the fields of an electric dipole oscillator at frequency ω which may be obtained from any standard text.[12] For compactness we introduce the notations

$$\vec{G}=k^3\exp(ikr)\left[(\hat{n}\times\hat{\imath})\times\hat{n}\left(\frac{1}{kr}\right)+[3\hat{n}(\hat{n}\cdot\hat{\imath})-\hat{\imath}]\left(\frac{1}{(kr)^3}-\frac{i}{(kr)^2}\right)\right], \quad (49)$$

$$\vec{H}=k^3\exp(ikr)\hat{n}\times\hat{\imath}\left(\frac{1}{kr}+\frac{i}{(kr)^2}\right), \quad (50)$$

where $\hat{\imath}$ is the unit vector in the x direction chosen as the direction of the dipole oscillator displacement. As before, we will use subscripts to denote the frequency or frequencies used in these expressions; for example, $\vec{H}_{1+2}$ means that $\mu_1k_1+\mu_2k_2$ is to be substituted for k in Eq. (50). It is clear that $\vec{H}_1^*=\vec{H}_{-1}$, $\vec{G}_1^*=\vec{G}_{-1}$, $A_1^*=A_{-1}$, $F_1^*=F_{-1}$, and $C_1^*=C_{-1}$, where, for example, $\vec{H}_{-1}$ means substituting $-\mu_1k_1$ for k in Eq. (50). Also we introduce the symbol

$$\beta=\frac{e^2}{m}. \quad (51)$$

With these notations, the fields $\vec{E}',\vec{B}'$ emitted by the dipole oscillator may be organized as to order in α as follows:

$$\vec{E}_0(\vec{r},t)=\beta\sum_\mu\sum_\lambda\int d^3k\,\epsilon_x\frac{\mathfrak{h}}{2}\frac{A\vec{G}}{C}, \quad (52)$$

$$\vec{E}_1(\vec{r},t)=\alpha\beta^2\sum_{\mu_1}\sum_{\mu_2}\sum_{\lambda_1}\sum_{\lambda_2}\int d^3k_1\int d^3k_2\,\epsilon_{x1}\epsilon_{x2}\frac{\mathfrak{h}_1}{2}\frac{\mathfrak{h}_2}{2}\frac{A_1A_2\vec{G}_{1+2}}{C_1C_2C_{1+2}}, \quad (53)$$

$$\vec{E}_2(\vec{r},t)=2\alpha^2\beta^3\sum_{\mu_1}\sum_{\mu_2}\sum_{\mu_3}\sum_{\lambda_1}\sum_{\lambda_2}\sum_{\lambda_3}\int d^3k_1\int d^3k_2\int d^3k_3\,\epsilon_{x1}\epsilon_{x2}\epsilon_{x3}\frac{\mathfrak{h}_1}{2}\frac{\mathfrak{h}_2}{2}\frac{\mathfrak{h}_3}{2}\frac{A_1A_2A_3\vec{G}_{1+2+3}}{C_1C_2C_3C_{2+3}C_{1+2+3}}, \quad (54)$$

and the magnetic fields are found by substituting $\vec{H}$ for $\vec{G}$ in each electric field expression (52)–(54). The frequency of the radiation involved is given from the factors of A present, or equally well may be read off from the subscript on $\vec{G}$. Thus $A\vec{G}$ in (52) involves radiation at frequency ω, whereas $A_1A_2\vec{G}_{1+2}$ in (53) involves radiation at frequency $|\mu_1\omega_1+\mu_2\omega_2|$ and $A_1A_2A_3\vec{G}_{1+2+3}$ in (54) involves radiation at frequency $|\mu_1\omega_1+\mu_2\omega_2+\mu_3\omega_3|$.

V. EVALUATIONS OF POYNTING'S VECTOR

A. Evaluation of $\langle\vec{E}_1\times\vec{B}_1\rangle$

We now use the field expressions in Eqs. (52)–(54) to evaluate $\langle\vec{S}_2\rangle$. The term $\langle\vec{E}_1\times\vec{B}_1\rangle$ becomes

$$\langle\vec{E}_1(\vec{r},t)\times\vec{B}_1(\vec{r},t)\rangle=\alpha^2\beta^4\sum_{\mu_1}\cdots\sum_{\mu_4}\sum_{\lambda_1}\cdots\sum_{\lambda_4}\int d^3k_1\cdots\int d^3k_4\,\epsilon_{x1}\cdots\epsilon_{x4}\frac{\mathfrak{h}_1}{2}\cdots\frac{\mathfrak{h}_4}{2}\times\frac{\vec{G}_{1+2}\times\vec{H}_{3+4}}{C_1C_2C_{1+2}C_3C_4C_{3+4}}\langle A_1A_2A_3A_4\rangle. \quad (55)$$

We must now average over the random phases $\theta(\vec{k},\lambda)$ contained in A_1,A_2,A_3,A_4. The average involves[13]

$$\langle\exp[-i(\mu_1\theta_1+\mu_2\theta_2+\mu_3\theta_3+\mu_4\theta_4)]\rangle=\delta_{(1)(-2)}\delta_{(3)(-4)}+\delta_{(1)(-3)}\delta_{(2)(-4)}+\delta_{(1)(-4)}\delta_{(2)(-3)}, \tag{56}$$

where $\delta_{(1)(-2)}$ stands for

$$\delta_{(1)(-2)}=\delta_{\mu_1-\mu_2}\delta_{\lambda_1\lambda_2}\delta^3(\vec{k}_1-\vec{k}_2). \tag{57}$$

Then noting that the frequencies ω_i appearing in A cancel in the averaging of (56), we may sum over the δ functions to obtain

$$\langle\vec{E}_1\times\vec{B}_1\rangle=\alpha^2\beta^4\sum_{\mu_1}\sum_{\mu_2}\sum_{\lambda_1}\sum_{\lambda_2}\int d^3k_1\int d^3k_2\epsilon_{x1}{}^2\epsilon_{x2}{}^2\,\frac{\mathfrak{h}_1{}^2}{4}\,\frac{\mathfrak{h}_2{}^2}{4}\,\frac{2\vec{G}_{1+2}\times\vec{H}_{-1-2}}{C_1C_{-1}C_2C_{-2}C_{1+2}C_{-1-2}}\,. \tag{58}$$

The factor of 2 arises from equal contributions from $\delta_{(1)(-3)}\delta_{(2)(-4)}$ and $\delta_{(1)(-4)}\delta_{(2)(-3)}$. The term $\delta_{(1)(-2)}\delta_{(3)(-4)}$ in (56) makes no contribution here since

$$\vec{H}_{3-3}=0 \tag{59}$$

corresponding to the absence of radiation at zero frequency.

Now the functions $\vec{G}$ and $\vec{H}$ in (49) and (50) depend upon only the magnitude k and not the direction $\hat{k}$ or the polarization $\hat{\epsilon}$ of the incident radiation. Hence it is easy to sum over polarizations as in (5) and carry out the angular parts of the intergrations in $\int d^3k_1$ and $\int d^3k_2$. The integral needed is

$$\int d\Omega_{\vec{k}}\left[\sum_{\lambda=1}^{2}(\hat{\epsilon}\cdot\hat{\imath})^2\right]=\int d\Omega_{\vec{k}}\left[1-\left(\frac{\vec{k}\cdot\hat{\imath}}{k}\right)^2\right]$$
$$=\tfrac{8}{3}\pi. \tag{60}$$

We find

$$\begin{aligned}\langle\vec{E}_1(\vec{r},t)\times\vec{B}_1(\vec{r},t)\rangle&=\alpha^2\beta^4\sum_{\mu_1}\sum_{\mu_2}\int_{k_1=0}^{\infty}dk_1k_1{}^2\int_{k_2=0}^{\infty}dk_2k_2{}^2\,\frac{\mathfrak{h}_1{}^2}{4}\,\frac{\mathfrak{h}_2{}^2}{4}\left(\frac{8\pi}{3}\right)^2\frac{2\vec{G}_{1+2}\times\vec{H}_{-1-2}}{C_1C_{-1}C_2C_{-2}C_{1+2}C_{-1-2}}\\&=\alpha^2\beta^4\int_{k_1=0}^{\infty}dk_1k_1{}^2\int_{k_2=0}^{\infty}dk_2k_2{}^2\,\frac{\mathfrak{h}_1{}^2}{4}\,\frac{\mathfrak{h}_2{}^2}{4}\left(\frac{8\pi}{3}\right)^2\\&\quad\times\frac{1}{|C_1|^2|C_2|^2}\left(\frac{\vec{G}_{1+2}\times\vec{H}^*_{1+2}+\vec{G}^*_{1+2}\times\vec{H}_{1+2}}{|C_{1+2}|^2}+\frac{\vec{G}_{1-2}\times\vec{H}^*_{1-2}+\vec{G}^*_{1-2}\times\vec{H}_{1-2}}{|C_{1-2}|^2}\right).\end{aligned} \tag{61}$$

B. Evaluation of $\langle\vec{E}_{in}\times\vec{B}_2\rangle$

Next we evaluate $\langle\vec{E}_{in}(\vec{r},t)\times\vec{B}_2(\vec{r},t)\rangle$. Taking the expressions of Eqs. (26) and (54) with $\vec{H}$ substituted in the latter, we have

$$\begin{aligned}\langle\vec{E}_{in}(\vec{r},t)\times\vec{B}_2(\vec{r},t)\rangle&=2\alpha^2\beta^3\sum_{\mu_1}\cdots\sum_{\mu_4}\sum_{\lambda_1}\cdots\sum_{\lambda_4}\int d^3k_1\cdots\int d^3k_4\epsilon_{x1}\epsilon_{x2}\epsilon_{x3}\\&\quad\times\frac{\mathfrak{h}_1}{2}\cdots\frac{\mathfrak{h}_4}{2}\,\frac{F_4\hat{\epsilon}_4\times\vec{H}_{1+2+3}}{C_1C_2C_3C_{2+3}C_{1+2+3}}\langle A_1A_2A_3A_4\rangle.\end{aligned} \tag{62}$$

The average over the random phases $\theta(\vec{k},\lambda)$ remains as in (56). Summing over the δ functions gives

$$\langle\vec{E}_{in}\times\vec{B}_2\rangle=2\alpha^2\beta^3\sum_{\mu_1}\sum_{\mu_2}\sum_{\lambda_1}\sum_{\lambda_2}\int d^3k_1\int d^3k_2\epsilon_{x1}\epsilon_{x2}{}^2\,\frac{\mathfrak{h}_1{}^2}{4}\,\frac{\mathfrak{h}_2{}^2}{4}\left(\frac{2F_1\hat{\epsilon}_1\times\vec{H}_{-1}}{C_{-1}C_{-1}C_2C_{-2}C_{-1-2}}+\frac{F_1\hat{\epsilon}_1\times\vec{H}_{-1}}{C_0C_{-1}C_{-1}C_2C_{-2}}\right). \tag{63}$$

The factor of 2 arises from equal contributions of the $\delta_{(1)(-2)}\delta_{(3)(-4)}$ and $\delta_{(1)(-3)}\delta_{(2)(-4)}$ terms. The symbol C_0 means zero frequency in (28) for C

$$C_0=+\omega_0{}^2. \tag{64}$$

The sum over polarizations and evaluation of the angular integrals involves two different expressions. The sum on λ_2 and the angular integral in $\int d^3k_2$ is just as given above in Eq. (60). However, the sum over λ_1 and the integral over $\int d^3k_1$ involve

$$\int d\Omega_{\vec{k}}\left[\sum_{\lambda=1}^{2}\hat{\epsilon}(\hat{\epsilon}\cdot\hat{\imath})\right]\exp(+i\vec{k}\cdot\vec{r})=\int d\Omega_{\vec{k}}\left[\hat{\imath}-\frac{\vec{k}(\vec{k}\cdot\hat{\imath})}{k^2}\right]\exp(+i\vec{k}\cdot\vec{r}),$$
$$=\frac{4\pi}{k^3}\mathrm{Im}\vec{G}. \tag{65}$$

The expression $\exp(+i\vec{k}\cdot\vec{r})$ arises from F defined in (25). Hence

$$\langle\vec{E}_{in}(\vec{r},t)\times\vec{B}_2(\vec{r},t)\rangle=2\alpha^2\beta^3\sum_{\mu_1}\sum_{\mu_2}\int_{k_1=0}^{\infty}dk_1k_1^2\int_{k_2=0}^{\infty}dk_2k_2^2\,\frac{\mathfrak{h}_1^2}{4}\,\frac{\mathfrak{h}_2^2}{4}\,\frac{8\pi}{3}\,\frac{4\pi}{k_1^3}\times\left(\frac{2(\mathrm{Im}\vec{G}_1)\times\vec{H}_{-1}}{C_{-1}C_{-1}C_2C_{-2}C_{-1-2}}+\frac{(\mathrm{Im}\vec{G}_1)\times\vec{H}_{-1}}{C_0C_{-1}C_{-1}C_2C_{-2}}\right). \quad (66)$$

C. Evaluation of $\langle\vec{E}_2\times\vec{B}_{in}\rangle$

The term $\langle\vec{E}_2(\vec{r},t)\times\vec{B}_{in}(\vec{r},t)\rangle$ is found in analogous fashion. Substituting from Eqs. (54) and (27), averaging over random phases as in (56), and summing over the δ functions, we find corresponding to (63)

$$\langle\vec{E}_2\times\vec{B}_{in}\rangle=2\alpha^2\beta^3\sum_{\mu_1}\sum_{\mu_2}\sum_{\lambda_1}\sum_{\lambda_2}\int d^3k_1\int d^3k_2\epsilon_{x1}\epsilon_{x2}{}^2\,\frac{\mathfrak{h}_1^2}{4}\,\frac{\mathfrak{h}_2^2}{4}\left(\frac{2F_1(-\hat{k}_1\times\hat{\epsilon}_1)\times\vec{G}_{-1}}{C_{-1}C_{-1}C_2C_{-2}C_{-1-2}}+\frac{F_1(-\hat{k}_1\times\hat{\epsilon}_1)\times\vec{G}_{-1}}{C_0C_{-1}C_{-1}C_2C_{-2}}\right). \quad (67)$$

The sum over λ_2 and the angular integration from $\int d^3k_2$ follow as in (60). The sum over λ_1 and the angular integration $\int d^3k_1$ involve

$$\int d\Omega_{\vec{k}}\left[\sum_{\lambda=1}^{2}\frac{\vec{k}\times\hat{\epsilon}}{k}(\hat{\epsilon}\cdot\hat{i})\right]\exp(+i\vec{k}\cdot\vec{r})=\int d\Omega_{\vec{k}}\left[\frac{\vec{k}\times\hat{i}}{k}\right]\exp(+i\vec{k}\cdot\vec{r})=\frac{-4\pi i}{k^3}\,\mathrm{Re}\,\vec{H}. \quad (68)$$

The term $\exp(+i\vec{k}\cdot\vec{r})$ in (68) comes from F_1 in (67) which involves $\exp(\pm i\vec{k}\cdot\vec{r})$ depending upon the sign of μ_1. As may be seen by taking complex conjugates in (68), the angular integral changes sign depending upon whether the + or − sign appears. Hence we must introduce an explicit factor of μ_1 to obtain

$$\langle\vec{E}_2(\vec{r},t)\times\vec{B}_{in}(\vec{r},t)\rangle=2\alpha^2\beta^3\sum_{\mu_1}\sum_{\mu_2}\int_{k_1=0}^{\infty}dk_1k_1^2\int_{k_2=0}^{\infty}dk_2k_2^2\,\frac{\mathfrak{h}_1^2}{4}\,\frac{\mathfrak{h}_2^2}{4}\,\frac{8\pi}{3}\,\frac{4\pi}{k_1^3}\times\left(\frac{2i\mu_1(\mathrm{Re}\vec{H}_1)\times\vec{G}_{-1}}{C_{-1}C_{-1}C_2C_{-2}C_{-1-2}}+\frac{i\mu_1(\mathrm{Re}\vec{H}_1)\times\vec{G}_{-1}}{C_0C_{-1}C_{-1}C_2C_{-2}}\right). \quad (69)$$

D. Sum $\langle\vec{E}_{in}\times\vec{B}_2\rangle+\langle\vec{E}_2\times\vec{B}_{in}\rangle$

In order to simplify the sum $\langle\vec{E}_{in}\times\vec{B}_2\rangle+\langle\vec{E}_2\times\vec{B}_{in}\rangle$, we note that

$$(\mathrm{Im}\vec{G})\times\vec{H}^*+i(\mathrm{Re}\vec{H})\times\vec{G}^*=\frac{1}{2i}(\vec{G}-\vec{G}^*)\times\vec{H}^*+\frac{i}{2}(\vec{H}+\vec{H}^*)\times\vec{G}^*,$$
$$=\frac{-i}{2}(\vec{G}\times\vec{H}^*+\vec{G}^*\times\vec{H}), \quad (70)$$

and by taking complex conjugates

$$(\mathrm{Im}\vec{G})\times\vec{H}-i(\mathrm{Re}\vec{H})\times\vec{G}=\frac{i}{2}(\vec{G}\times\vec{H}^*+\vec{G}^*\times\vec{H}). \quad (71)$$

Then adding Eqs. (66) and (69), always remembering that $\vec{G}_1^*=\vec{G}_{-1}$ and $\vec{H}_1^*=\vec{H}_{-1}$, we find

$$\langle\vec{E}_{in}(\vec{r},t)\times\vec{B}_2(\vec{r},t)\rangle+\langle\vec{E}_2(\vec{r},t)\times\vec{B}_{in}(\vec{r},t)\rangle=2\alpha^2\beta^3\int_{k_1=0}^{\infty}dk_1k_1^2\int_{k_2=0}^{\infty}dk_2k_2^2\,\frac{\mathfrak{h}_1^2}{4}\,\frac{\mathfrak{h}_2^2}{4}\,\frac{8\pi}{3}\,\frac{4\pi}{k_1^3}(\vec{G}_1\times\vec{H}_1^*+\vec{G}_1^*\times\vec{H}_1)\times\sum_{\mu_1}\sum_{\mu_2}\frac{-i\mu_1}{2}\left(\frac{2}{C_{-1}C_{-1}C_2C_{-2}C_{-1-2}}+\frac{1}{C_0C_{-1}C_{-1}C_2C_{-2}}\right). \quad (72)$$

E. Evaluation of $\langle\vec{E}_0\times\vec{B}_2\rangle$ and $\langle\vec{E}_2\times\vec{B}_0\rangle$

Next we turn to $\langle\vec{E}_0(\vec{r},t)\times\vec{B}_2(\vec{r},t)\rangle$ which we evaluate from Eqs. (52) and (54) with $\vec{H}$ substituted for $\vec{G}$ in the latter equation,

$$\langle\vec{E}_0(\vec{r},t)\times\vec{B}_2(\vec{r},t)\rangle=2\alpha^2\beta^4\sum_{\mu_1}\cdots\sum_{\mu_4}\sum_{\lambda_1}\cdots\sum_{\lambda_4}\int d^3k_1\cdots\int d^3k_4\epsilon_{x1}\cdots\epsilon_{x4}\frac{\mathfrak{h}_1}{2}\cdots\frac{\mathfrak{h}_4}{2}\times\frac{\vec{G}_4\times\vec{H}_{1+2+3}}{C_1C_2C_3C_{2+3}C_{1+2+3}C_4}\langle A_1A_2A_3A_4\rangle. \tag{73}$$

Averaging over the random phases as in (56) and then summing over the δ functions, we have

$$\langle\vec{E}_0\times\vec{B}_2\rangle=2\alpha^2\beta^4\sum_{\mu_1}\sum_{\mu_2}\sum_{\lambda_1}\sum_{\lambda_2}\int d^3k_1\int d^3k_2\epsilon_{x1}^2\epsilon_{x2}^2\frac{\mathfrak{h}_1^2}{4}\frac{\mathfrak{h}_2^2}{4}\left(\frac{2\vec{G}_1\times\vec{H}_{-1}}{C_{-1}C_{-1}C_2C_{-2}C_{-1-2}C_1}+\frac{\vec{G}_1\times\vec{H}_{-1}}{C_0C_{-1}C_{-1}C_2C_{-2}C_1}\right), \tag{74}$$

a form analogous to that of (63). The sum over the polarizations involves Eq. (60), giving

$$\langle\vec{E}_0(\vec{r},t)\times\vec{B}_2(\vec{r},t)\rangle=2\alpha^2\beta^4\sum_{\mu_1}\sum_{\mu_2}\int_{k_1=0}^{\infty}dk_1k_1^2\int_{k_2=0}^{\infty}dk_2k_2^2\frac{\mathfrak{h}_1^2}{4}\frac{\mathfrak{h}_2^2}{4}\left(\frac{8\pi}{3}\right)^2\left(\frac{2\vec{G}_1\times\vec{H}_{-1}}{C_{-1}C_{-1}C_2C_{-2}C_{-1-2}C_1}+\frac{\vec{G}_1\times\vec{H}_{-1}}{C_0C_{-1}C_{-1}C_2C_{-2}C_1}\right). \tag{75}$$

The final term $\langle\vec{E}_2(\vec{r},t)\times\vec{B}_0(\vec{r},t)\rangle$ is evaluated in parallel fashion giving just the result of (75) except that $\vec{G}_{-1}\times\vec{H}_1$ appears in place of $\vec{G}_1\times\vec{H}_{-1}$. Adding the expressions $\langle\vec{E}_0\times\vec{B}_2\rangle$ and $\langle\vec{E}_2\times\vec{B}_0\rangle$, we have

$$\langle\vec{E}_0(\vec{r},t)\times\vec{B}_2(\vec{r},t)\rangle+\langle\vec{E}_2(\vec{r},t)\times\vec{B}_0(\vec{r},t)\rangle=2\alpha^2\beta^4\int_{k_1=0}^{\infty}dk_1k_1^2\int_{k_2=0}^{\infty}dk_2k_2^2\frac{\mathfrak{h}_1^2}{4}\frac{\mathfrak{h}_2^2}{4}\left(\frac{8\pi}{3}\right)^2(\vec{G}_1\times\vec{H}_1^*+\vec{G}_1^*\times\vec{H}_1)\times\sum_{\mu_1}\sum_{\mu_2}\frac{2}{C_1C_{-1}C_{-1}C_2C_{-2}C_{-1-2}}+\frac{1}{C_0C_1C_{-1}C_{-1}C_2C_{-2}}. \tag{76}$$

F. Summing contributions to $\langle\vec{S}_2\rangle$

The four terms in $\langle\vec{S}_2\rangle$ which involve a zero-order field times a second-order field may be written from (72) and (76) as

$$\langle\vec{E}_{in}\times\vec{B}_2\rangle+\langle\vec{E}_2\times\vec{B}_{in}\rangle+\langle\vec{E}_0\times\vec{B}_2\rangle+\langle\vec{E}_2\times\vec{B}_0\rangle$$
$$=2\alpha^2\beta^4\int_{k_1=0}^{\infty}dk_1k_1^2\int_{k_2=0}^{\infty}dk_2k_2^2\frac{\mathfrak{h}_1^2}{4}\frac{\mathfrak{h}_2^2}{4}\left(\frac{8\pi}{3}\right)^2(\vec{G}_1\times\vec{H}_1^*+\vec{G}_1^*\times\vec{H}_1)$$
$$\times\left\{\sum_{\mu_1}\sum_{\mu_2}\frac{1}{C_0|C_1|^4|C_2|^2|C_{1+2}|^2}\left(\frac{3}{2k_1^3\beta}\right)[-i\mu_1C_0C_1C_1C_{1+2}-\tfrac{1}{2}i\mu_1C_1C_1|C_{1+2}|^2+2C_0C_1C_{1+2}\omega_1^3\tau+C_1|C_{1+2}|^2\omega_1^3\tau]\right\}, \tag{77}$$

where we have noted

$$\tfrac{2}{3}k_1^3\beta=\omega_1^3\tau. \tag{78}$$

Now from the definition (29), we see that

$$C_1-C_{-1}=-2i\tau\omega_1^3, \tag{79}$$

and indeed

$$\mu_1C_1+\mu_{-1}C_{-1}=-2i\tau\omega_1^3. \tag{80}$$

Thus if we replace $\omega_1^3\tau$ by (80) in the last two terms of (77), we have a partial cancellation in which the expression in the final brackets becomes

$$[\,]=[iC_0|C_1|^2C_{1+2}\mu_{-1}+\tfrac{1}{2}i|C_1|^2|C_{1+2}|^2\mu_{-1}]. \tag{81}$$

If we now group the four terms of the sums $\sum_{\mu_1}\sum_{\mu_2}$ as ($\mu_1=1$, $\mu_2=1$; $\mu_1=-1$, $\mu_2=-1$) and ($\mu_1=1$, $\mu_2=-1$; $\mu_1=-1$, $\mu_2=1$) while recalling

$$\mu_{-1}=-\mu_1 \tag{82}$$

and

$$C_{1+2}-C_{-1-2}=-2i\tau(\omega_1+\omega_2)^2, \tag{83}$$

$$C_{1-2}-C_{-1+2}=-2i\tau(\omega_1-\omega_2)^3, \tag{84}$$

we find that the last term in (81) vanishes and the expression in the curly brackets of (77) becomes

$$\{\,\}=\left\{\frac{1}{|C_1|^2|C_2|^2\omega_1^3}\left[\frac{-2(\omega_1+\omega_2)^3}{|C_{1+2}|^2}+\frac{-2(\omega_1-\omega_2)^3}{|C_{1-2}|^2}\right]\right\}. \tag{85}$$

The expression $\vec{G}\times\vec{H}^*+\vec{G}^*\times\vec{H}$ may be evaluated from equations (49) and (50). The real part vanishes for

$$\left(\frac{1}{(kr)^3}+\frac{i}{(kr)^2}\right)\left(\frac{1}{kr}+\frac{i}{(kr)^2}\right),$$

and hence the ezpression simplifies to

$$\vec{G}^*\times\vec{H}+\vec{G}\times\vec{H}^*=\frac{k^4\hat{n}}{r^2}\,|\hat{n}\times\hat{i}\,|^2. \tag{86}$$

The terms contributing to $\langle\vec{S}_2\rangle$ may be rewritten by combining (77), (85), (86), (61), and recalling

$$k_{1+2}=k_1+k_2 \tag{87}$$

in $\vec{G}_{1+2}$ and $\vec{H}_{1+2}$,

$$\langle\vec{E}_{in}(\vec{r},t)\times\vec{B}_2(\vec{r},t)+\vec{E}_2(\vec{r},t)\times\vec{B}_{in}(\vec{r},t)+\vec{E}_0(\vec{r},t)\times\vec{B}_2(\vec{r},t)+\vec{E}_2(\vec{r},t)\times\vec{B}_0(\vec{r},t)\rangle$$

$$=\frac{\hat{n}|\hat{n}\times\hat{i}\,|^2}{r^2}2\alpha^2\beta^4\int_{k_1=0}^{\infty}dk_1k_1^2\int_{k_2=0}^{\infty}dk_2k_2^2\,\frac{\mathfrak{h}_1^2}{4}\,\frac{\mathfrak{h}_2^2}{4}\left(\frac{8\pi}{3}\right)^2\frac{1}{c^4|C_1|^2C_2|^2}\left[\frac{-2(\omega_1+\omega_2)^3\omega_1}{|C_{1+2}|^2}+\frac{-2(\omega_1-\omega_2)^3\omega_1}{|C_{1-2}|^2}\right], \tag{88}$$

$$\langle\vec{E}_1(\vec{r},t)\times\vec{B}_1(\vec{r},t)\rangle=\frac{\hat{n}|\hat{n}\times\hat{i}\,|^2}{r^2}2\alpha^2\beta^4\int_{k_1=0}^{\infty}dk_1k_1^2\int_{k_2=0}^{\infty}dk_2k_2^2\,\frac{\mathfrak{h}_1^2}{4}\,\frac{\mathfrak{h}_2^2}{4}\left(\frac{8\pi}{3}\right)^2\frac{1}{c^4|C_1|^2C_2|^2}\left[\frac{(\omega_1+\omega_2)^4}{|C_{1+2}|^2}+\frac{(\omega_1-\omega_2)^4}{|C_{1-2}|^2}\right], \tag{89}$$

with $\langle\vec{S}_2\rangle$ found by taking the sum of Eqs. (88) and (89),

$$\langle\vec{S}_2(\vec{r},t)\rangle=\langle\vec{E}_{in}\times\vec{B}_2+\cdots+\vec{E}_2\times\vec{B}_0\rangle+\langle\vec{E}_1\times\vec{B}_1\rangle. \tag{90}$$

The sum over μ no longer appears in these expressions (88), (89), and the subscripts now refer to $\mu=+1$ in the previous definitions. The frequency of the radiation appearing in Poynting's vector was indicated in our earlier equations by the subscripts on the terms $\vec{G}$ and $\vec{H}$. In equations (88) and (89) these subscripts have disappeared, and we must therefore recall that the radiation frequency in (88) corresponded to ω_1 whereas in (89) the terms in the final brackets refer to frequencies $\omega_1+\omega_2$ and $\omega_1-\omega_2$.

VI. EVALUATING RADIATION SPECTRUM EQUILIBRIUM

A. Energy conservation

Because the calculations leading to $\langle\vec{S}_2\rangle$ have been somewhat complicated, it seems wise to run some simple checks on the result. Thus, for example, we expect the total energy in the radiation field to be conserved. In particular we expect that when integrated over all frequencies $\langle\vec{S}_2\rangle=0$, even though $\langle\vec{S}_2(\omega)\rangle\neq0$, corresponding to the net radiation at frequency ω entering or leaving a spherical surface around the oscillator.

For example, suppose the initial radiation were homogeneous in space, isotropic in direction, but were all at a single special frequency ω_s so that the spectrum of $\vec{E}_{in}$, $\vec{B}_{in}$ is given by

$$\mathfrak{h}^2(\vec{k},\lambda)=\mathfrak{A}\delta(\omega-\omega_s),\qquad \omega=ck. \tag{91}$$

With such a spectrum, it is easy to evaluate the second-order contribution $\langle\vec{S}_2\rangle$ to the Poynting vector. Substituting (91) into (88) and (89), and carrying out the integrations over the δ functions

$$\langle\vec{S}_2(\vec{r},t)\rangle=\frac{\hat{n}|\hat{n}\times\hat{i}\,|}{r^2}2\alpha^2\beta^4k_s^4\,\frac{\mathfrak{A}^2}{16}\left(\frac{8\pi}{3}\right)^2\frac{1}{c^4|C_s|^4}\times\left\{\left[\frac{(2\omega_s)^4}{|C_{2s}|^2}+0\right]+\left[\frac{-2(2\omega_s)^3\omega_s}{|C_{2s}|^2}+0\right]\right\}=0. \tag{92}$$

Thus indeed there is no net energy flow across the surface surrounding the oscillator in second order in α. However, the expression (92) gives more information than this. The term $(2\omega_s)^4/|C_{2s}|^2$ in $\langle\vec{S}_2\rangle$ corresponds to radiation at frequency $2\omega_s$ flowing out of the surface. On the other hand, the term $-2(2\omega_s)^3\omega_s/|C_{2s}|^2$ corresponds to an equal flow of energy into the surface, but it is at frequency ω_s. This is exactly what we expect physically. The incident radiation energy is at frequency ω_s. The oscillator absorbs some of this energy leading to a net flux of energy toward the oscillator at frequency ω_s, and the oscillator then reradiates the energy leading to a net flux of energy away from the oscillator at frequency $2\omega_s$. Higher-order terms in the nonlinear coupling constant α would lead to absorption at ω_s and to radiation emitted at higher multiples of ω_s with energy conservation holding to every order in α. Clearly this radiation spectrum, with random radiation initially all near the frequency ω_s, is not in equilibrium since the nonlinear oscillator converts some of the radiation to frequency $2\omega_s$ which was not present in the initial spectrum.

If we are not interested in the spectral properties of the radiation but only in checking that energy conservation indeed holds for our expressions (88), (89), and (90), then we may regard k_1 and k_2 as dummy variables of integration and may symmetrize in the subscripts 1 and 2 throughout Eq. (88). The only terms in (88) which are not already symmetric are in the brackets which become

$$[\;]=\left[\frac{-2(\omega_1+\omega_2)^3}{|C_{1+2}|}\,\frac{(\omega_1+\omega_2)}{2}+\frac{-2}{|C_{1-2}|^2}\,\frac{[(\omega_1-\omega_2)^3\omega_1+(\omega_2-\omega_1)^3\omega_2]}{2}\right]$$

$$=\left[\frac{-(\omega_1+\omega_2)^4}{|C_{1+2}|^2}+\frac{-(\omega_1-\omega_2)^4}{|C_{1-2}|^2}\right]. \qquad (93)$$

This is the same form as the negative of the last bracket in (89). Thus, indeed, equations (88) and (89) add to zero; energy is conserved.

B. Spectral decomposition for $\langle\vec{S}_2\rangle$

In order to obtain the spectral distribution of the second-order term $\langle\vec{S}_2\rangle$ of the Poynting vector, and so discuss the equilibrium of the radiation pattern, we must change the variables of integration in (89) to correspond to the frequencies of the radiation involved. Since these frequencies are $\omega_1+\omega_2$ and $|\omega_1-\omega_2|$, we introduce

$$k_p=k_1+k_2 \qquad (94)$$

and

$$k_m=k_1-k_2. \qquad (95)$$

The Jacobian of the transformation (94), (95) has a magnitude of 2, and the limits of integration may be obtained by superimposing the k_pk_m-coordinate axes upon the k_1k_2-coordinates axes. We find from (89)

$$\langle\vec{E}_1\times\vec{B}_1\rangle=\frac{\hat{n}|\hat{n}\times\hat{i}|^2}{r^2}2\alpha^2\beta^4\tfrac{1}{2}\int_{k_p=0}^{\infty}dk_p\int_{k_m=-k_p}^{k_m=k_p}dk_m\frac{(k_p+k_m)^2}{4}\,\frac{(k_p-k_m)^2}{4}\left(\frac{\mathfrak{h}_{(p+m)/2}}{2}\right)^2\left(\frac{\mathfrak{h}_{(p-m)/2}}{2}\right)^2\times\left(\frac{8\pi}{3}\right)^2\frac{k_p^4}{|C_p|^2|C_{(p+m)/2}|^2|C_{(p-m)/2}|^2}$$
$$+\frac{\hat{n}|\hat{n}\times\hat{i}|^2}{r^2}2\alpha^2\beta^4\tfrac{1}{2}\int_{-\infty}^{\infty}dk_m\int_{k_p=|k_m|}^{\infty}dk_p\frac{(k_p+k_m)^2}{4}\,\frac{(k_p-k_m)^2}{4}\left(\frac{\mathfrak{h}_{(p+m)/2}}{2}\right)^2\left(\frac{\mathfrak{h}_{(p-m)/2}}{2}\right)^2\times\left(\frac{8\pi}{3}\right)^2\frac{k_m^4}{|C_m|^2|C_{(p+m)/2}|^2|C_{(p-m)/2}|^2}. \qquad (96)$$

Now because the terms in $p+m$ and $p-m$ enter symmetrically and $|C_p|^2=|C_{-p}|^2$, $|C_m|^2=|C_{-m}|^2$, we may drop all the negative parts of the integrals while doubling the positive contributions. Thus in the first integral of (96) we may replace

$$\tfrac{1}{2}\int_0^\infty dk_p\int_{-k_p}^{k_p}dk_m \quad\text{by}\quad \int_0^\infty dk_p\int_0^{k_p}dk_m$$

and in the second integral of (96) replace

$$\tfrac{1}{2}\int_{-\infty}^\infty dk_m\int_{|k_m|}^{\infty}dk_p \quad\text{by}\quad \int_0^\infty dk_m\int_{k_m}^{\infty}dk_p.$$

Now we interchange the labels m and p in the second integrand. The only terms which change form are $\mathfrak{h}_{(p-m)/2}$ which becomes $\mathfrak{h}_{(m-p)/2}$, and $|C_{(p-m)/2}|^2$ which becomes $|C_{(m-p)/2}|^2=|C_{(p-m)/2}|^2$. Thus, provided we take the absolute value writing $\mathfrak{h}_{|p-m|/2}$, the interchange will make no difference. But now the two double integrals become

$$\int_0^\infty dk_p\int_0^{k_p}dk_m \quad\text{and}\quad \int_0^\infty dk_p\int_{k_p}^{\infty}dk_m,$$

which may be combined into a single double integral

$$\int_0^\infty dk_p\int_0^\infty dk_m.$$

Hence, changing the notation so that the subscript p becomes a while m becomes b, we have the result

$$\langle\vec{E}_1(\vec{r},t)\times\vec{B}(\vec{r},t)\rangle=\frac{\hat{n}|\hat{n}\times\hat{i}|^2}{r^2}2\alpha^2\beta^4\int_0^\infty dk_a\int_0^\infty dk_b\frac{(k_a+k_b)^2}{4}\,\frac{(k_a-k_b)^2}{4}\left(\frac{\mathfrak{h}_{(a+b)/2}}{2}\right)^2\left(\frac{\mathfrak{h}_{|a-b|/2}}{2}\right)^2\times\left(\frac{8\pi}{3}\right)^2\frac{k_1^4}{|C_a|^2|C_{(a+b)/2}|^2|C_{(a-b)/2}|^2}, \qquad (97)$$

where $\omega_a=ck_a$ corresponds to the frequency of the radiation emitted.

The full spectral decomposition of the second-order term in the Poynting vector may be written from

(88) and (97) as

$$\langle \vec{S}_2(\vec{r},t)\rangle = \frac{\hat{n}|\hat{n}\times\hat{i}|^2}{r^2}2\alpha^2\beta^4\left(\frac{8\pi}{3}\right)^2\frac{1}{c^{10}} \times \int_0^\infty d\omega_a\left\{\frac{\omega_a^3}{|C_a|^2}\left(\frac{\mathfrak{h}_a}{2}\right)^2\int_0^\infty d\omega_b\frac{\omega_b^2}{|C_b|^2}\left(\frac{\mathfrak{h}_b}{2}\right)^2\left[\frac{-2(\omega_a+\omega_b)^3}{|C_{a+b}|^2}+\frac{-2(\omega_a-\omega_b)^3}{|C_{a-b}|^2}\right] + \frac{\omega_a^4}{|C_a|^2}\int_0^\infty d\omega_b\frac{(\omega_a+\omega_b)^2}{4|C_{(a+b)/2}|^2}\,\frac{(\omega_a-\omega_b)^2}{4|C_{(a-b)/2}|^2}\left(\frac{\mathfrak{h}_{(a+b)/2}}{2}\right)^2\left(\frac{\mathfrak{h}_{|a-b|/2}}{2}\right)^2\right\} \tag{98}$$

where ω_a gives the frequency of the radiation.

C. Equilibrium criterion

Our criterion for the equilibrium of a random radiation spectrum in the presence of a nonrelativistic nonlinear electric dipole oscillator may now be stated in mathematical terms. Equilibrium requires a spectral distribution $\mathfrak{h}^2$ for the initial radiation such that the curly brackets in (98) vanishes for every frequency ω_a. Physically this indicates that there is no frequency ω_a at which energy is being absorbed from the incoming radiation and reradiated at some other frequency. The oscillator makes no shift of the radiation energy among the various frequencies.

To emphasize the purely mathematical nature of our criterion, we replace $\mathfrak{h}^2$ in the bracket by f and introduce $x=\omega_a$, $y=\omega_b$. Then we wish to find all real functions f such that for $x>0$

$$0=-2f(x)\int_{y=0}^\infty dy\frac{y^2}{|C(y)|^2}f(y)\left[\frac{(x+y)^3}{|C(x+y)|^2}+\frac{(x-y)^3}{|C(x-y)|^2}\right] +x\int_{y=0}^\infty dy\frac{[\frac{1}{2}(x+y)]^2}{|C(\frac{1}{2}(x+y))|^2}\,\frac{[\frac{1}{2}(x-y)]^2}{|C(\frac{1}{2}(x-y))|^2}f\left(\frac{x+y}{2}\right)f\left(\left|\frac{x-y}{2}\right|\right), \tag{99}$$

where

$$|C(x)|^2=(-x^2+\delta^2)^2+\epsilon^2x^6, \tag{100}$$

for fixed real constants δ and ϵ.

D. Rayleigh-Jeans Law as an equilibrium spectrum

Some idea of what is required of solutions f for (99) may be obtained by integrating numerically with various choices for f, and values for δ and ϵ. In this way it was quickly found that the Rayleigh-Jeans spectrum was indeed an equilibrium distribution, whereas spectra corresponding to the Planck law with or without zero-point radiation were not. We conjecture, but have not proved, that in the presence of our nonlinear oscillator the Rayleigh-Jeans law is the unique equilibrium distribution of random radiation.

It turns out that an analytic proof of equilibrium can be provided for the solution $f(x)=$ constant, corresponding to the Rayleigh-Jeans radiation spectrum

$$\pi^2\mathfrak{h}^2(\vec{k},\lambda)=KT \tag{101}$$

Thus if f is a constant, we may remove it from all terms in (99). Also both integrals are then even in y, and so may be extended from $-\infty$ to ∞ and multiplied by $\frac{1}{2}$. Thus if $f(x)=$constant is indeed a solution, then (99) becomes

$$0=-\int_{-\infty}^\infty dy\frac{y^2}{|C(y)|^2}\left[\frac{(x+y)^3}{|C(x+y)|^2}+\frac{(x-y)^3}{|C(x-y)|^2}\right] +\frac{x}{2}\int_{-\infty}^\infty dy\frac{[\frac{1}{2}(x+y)]^2}{|C(\frac{1}{2}(x+y))|^2}\,\frac{[\frac{1}{2}(x-y)]^2}{|C(\frac{1}{2}(x-y))|^2}. \tag{102}$$

Now we break the integral in the first line into two pieces, one involving $(x+y)^3$ and the other involving $(x-y)^3$. In the first we change the variable of integration to u where $y=\frac{1}{2}(u-x)$ and in the second we change to u where $y=\frac{1}{2}(u+x)$. Then the first line of (102) becomes

$$-\int_{-\infty}^\infty\frac{du}{2}\frac{[\frac{1}{2}(u-x)]^2}{|C(\frac{1}{2}(u-x))|^2}\,\frac{[\frac{1}{2}(u+x)]^3}{|C(\frac{1}{2}(u+x))|^2} -\int_{-\infty}^\infty\frac{du}{2}\frac{[\frac{1}{2}(u+x)]^2}{|C(\frac{1}{2}(u+x))|^2}\,\frac{[\frac{1}{2}(x-u)]^3}{|C(\frac{1}{2}(x-u))|^2}. \tag{103}$$

Now writing these two integrals under a single integration symbol, noting that

$$\tfrac{1}{2}(u+x)+\tfrac{1}{2}(x-u)=x,$$

and that C in (100) is an even function of the argument, and changing the dummy variable of integration from u to y, we find that the expression in (103) is just the negative of the second line of (102). Indeed f equals to a constant function is a solution of (99).

VII. DISCUSSION

Our result that the Rayleigh-Jeans law provides an equilibrium distribution of random classical radiation in the presence of a nonrelativistic nonlinear electric dipole oscillator will come as no surprise to most physicists. We seem to have provided a particularly sharp proof of what most physics textbooks indicate: the equilibrium (thermal) distribution of random radiation within classical physics is inevitably the Rayleigh-Jeans law. Our calculations indeed illustrate this for a nonrelativistic nonlinear mechanical system interacting with radiation.

However, our numerical calculations also show something else. They show that the Lorentz-invariant spectrum of classical electromagnetic zero-point radiation is not in equilibrium in the presence of a nonrelativistic nonlinear electric dipole oscillator. Rather, the computer calculations show that the oscillator acts to shift the energy spectrum; it gives net absorption of electromagnetic radiation at all frequencies above some frequency ω_c and gives a net emission of radiation at all frequencies below ω_c. In a crude approximation the oscillator is acting to push the classical zero-point radiation toward the Rayleigh-Jeans law of thermal radiation.

Now this result is paradoxical in any physical description which regards classical electromagnetic zero-point radiation as an equilibrium distribution of random radiation at zero temperature. It is easy to show that our nonlinear oscillator, together with filters and a movable mirror, may be coupled to the infinite reservoir of zero-point radiation energy so as to provide an unlimited source of energy to do useful work. Past experience suggests that such perpetual-motion machines of the second kind do not occur in nature, and hence we must ask whether they may be avoided within the present analysis.

Some physicists, no doubt, would avoid our problem by concluding that zero-point radiation is not a permissible concept within classical electromagnetism. However, there is another conceivable explanation. We conjecture that the appearance of the Rayleigh-Jeans spectrum as an equilibrium spectrum for random radiation is connected with the use of nonrelativistic instead of relativistic physics for the oscillator system which determines the equilibrium. We conjecture that classical zero-point radiation is an equilibrium distribution for any relativistic scattering system.

In the first place, it is clear that the use of nonrelativistic or relativistic mechanics leads to different equilibrium distributions for particles. Thus consider two groups of ideal gas particles, one of mass m_1 for which $m_1c^2/KT \gg 1$ and the other of mass m_2 for which $m_2c^2/KT \ll 1$. If nonrelativistic physics is used, both sets of particles will come to equilibrium at the Maxwell-Boltzmann distribution.[1] On the other hand, if relativistic physics is used, both sets of particles will come to equilibrium at Jüttner's relativistic distribution.[3] Now for the massive particles, the Maxwell-Boltzmann distribution is a very good approximation to the Jüttner distribution for all but a few high-velocity particles. However, for the light particles, the Maxwell-Boltzmann distribution gives an entirely false picture of the distribution. Thus any argument which attemps naively to find the actual velocity distribution for the light particles by using the Maxwell-Boltzmann result for heavy particles plus nonrelativistic mechanics is doomed to failure.

It may be that most classical analyses for thermal radiation involve an analogous error. Radiation is an intrinsically relativistic system, and attempts to deduce the equilibrium distribution by combining with a nonrelativistic mechanical system may be fundamentally wrong.

As a natural extension of these conjectures, we suggest the possibility that classical electromagnetic zero-point radiation may be an equilibrium distribution for every relativistic scattering system. Such a result is required if zero-point radiation is to be regarded as an equilibrium distribution for random radiation at zero temperature. Moreover such a result seems a natural extension of heuristic ideas of entropy and disorder.

Although quantitative ideas of disorder often involve *ad hoc* assumptions of statistical mechanics, there are certain situations which from considerations of symmetry are immediately recognizable as involving greatest possible disorder. Thus, for example, two spatially homogeneous radiation distributions involving the same energy per unit frequency interval but only one of which is isotropic are immediately recognized as differing in their disorder; the radiation distribution which is isotropic has the greater disorder.

In earlier work[10] it was proved that any isotropic distribution of random radiation is an equilibrium distribution for a linear dipole oscillator. This result fits perfectly with our heuristic idea of physical interactions increasing the disorder of the universe. Thus, in general, the linear oscillator can not shift the energy among the frequencies of radiation, but can change only the direction of radiation. However, if the initial distribution of radiation is isotropic, then it is already at maximum disorder with respect to direction and the linear oscillator can not decrease this disorder with respect to direction. We should note that a linear dipole oscillator fixed at the origin and oriented along say the x axis indeed has a preferred axis in space. Nevertheless an initially isotropic distribution of random radiation indeed remains isotropic, as proved by detailed calculation.[10] Moreover from these ideas of disorder we conjecture that not only a dipole oscillator but any linear oscillator scatterer, involving a quadrupole or higher multipole for example, will leave the radiation distribution isotropic.

Further we suggest that a heuristic idea of disorder may also be applied for the zero-point radiation spectrum. Suppose one were asked to specify the "most disordered" spectrum of random radiation having a given spectral density of energy per unit volume at a given special frequency ω_s but with no restriction on the total energy content. Despite the heuristic nature of the term "most disordered," we believe that within a relativistic universe there is no ambiguity in the answer. The most disordered spectrum is one which is homogeneous and isotropic in space in every inertial frame, and such that the spectral dinsity takes on the required value in every inertial frame. It has been proved[6] there is only one such spectrum of random radiation, zero-point radiation with the multiplicative constant of the spectrum set by the required spectral energy density value at ω_s. If we now consider a relativistic system scattering zero-point radiation, then we find that any such scatterer must leave the zero-point radiation spectrum invariant if it is not to decrease the maximum disorder of the initial radiation spectrum. We conjecture that all physical systems increase the disorder of the universe, and hence that every relativistic electromagnetic scattering system will leave the zero-point radiation spectrum invariant.

The conjectures made here will be studied in future work.

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[1]See for example, Y. P. Terletskii, *Statistical Physics* (American Elsevier, New York, 1971), Sec. 32, 33.

[2]See for example, M. Planck, *Theory of Heat Radiation* (Dover, New York, 1959).

[3]In Ref. 1, Terletskii uses nonrelativistic mechanics to obtain the Maxwell-Boltzmann velocity distribution. Relativistic mechanics is discussed by S. R. de Groot, in *The Boltzmann Equation*, proceedings of the international symposium "100 Years Boltzmann Equation," Vienna, 1972, edited by E. G. D. Cohen and W. Thirring (Springer, Berlin, 1973) [Acta Phys. Austriaca Suppl. 10 (1973)], p. 529. de Groot obtains the relativistic ideal gas distribution of F. Jüttner, Ann. Phys. (Leipz.) 34, 856 (1911).

[4]See for example, the traditional treatment presented by M. Born, in *Atomic Physics* (Hafner, New York, 1966), 7th edition, pp. 253–255.

[5]See Ref. 4. Also, for a different approach, M. Abraham and R. Becker, *Theorie der Elektrizität* (B. G. Teubner, Leipzig, 1933), Vol. II, 6th edition, pp. 373–375; M. Jammer, *The Conceptual Development of Quantum Mechanics* (McGraw-Hill, New York, 1966), Appendix A.

[6]T. H. Boyer, Phys. Rev. 182, 1374 (1969); *ibid*. 186, 1304 (1969).

[7]A. Einstein and L. Hopf, Ann. Phys. (Leipz.) 33, 1105 (1910).

[8]See Refs. 2 and 7. Also S. O. Rice, in *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax (Dover, New York, 1954), p. 133.

[9]See for example, S. Coleman, Rand Corp. report, 1961 (unpublished).

[10]T. H. Boyer, Phys. Rev. D 11, 790 (1975), Appendix B.

[11]See Ref. 5. Also T. H. Boyer, Phys. Rev. A 6, 314 (1975), Sec. II.

[12]See for example, J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), p. 271.

[13]T. H. Boyer, Phys. Rev. D 11, 809 (1975), Sec. II F.