[14]See Ref. 8, Eqs. (27)–(31).
[15]See Ref. 7, p. 197.
[16]See Ref. 7, p. 160 Eq. (12).
[17]The idea of connecting the nonrelativistic approximation for the energy, linear momentum, and angular momentum with the solenoid vector potential is taken from the work of G. T. Trammel, Phys. Rev. 134B, 1183 (1964). However, Trammel's calculation of the angular momentum is in error.

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Classical Electromagnetic Deflections and Lag Effects Associated with Quantum Interference Pattern Shifts: Considerations Related to the Aharonov-Bohm Effect

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Classical electromagnetic lag effects can give rise to quantum interference pattern shifts such as that observed experimentally in the Aharonov-Bohm effect involving electrons passing a solenoid. This paper presents an extensive comparison between interference pattern shifts based upon classical electromagnetic fields, and based upon classical electromagnetic potentials as suggested by Aharonov and Bohm. Stress is placed upon the difference between two types of interference pattern shifts: those involving deflection of the entire interference pattern and those involving a deflection of only the double-slit pattern while leaving the single-slit envelope undisplaced. The first type of shift is produced by a classical deflecting force. The second type of shift can be produced by classical electromagnetic lag effects, and is also the type of shift associated with the Aharonov-Bohm effect. The two types are confused in the literature. A new experiment is proposed which shows the relationship between a classical lag effect, due to electrostatic fields on electrons passing along different paths, and the associated quantum interference pattern shifts. The experiment is analyzed in detail using the WKB approximation in the Schrödinger equation and also semiclassical ideas. The classical limit for the situation illustrates the Bohr correspondence principle, showing the relative lag between the electron wave packets becoming a measurable classical lag with a disappearance of the interference pattern as the lag becomes large compared to the wave-packet dimensions. For small shifts, the phase change predicted for the new experiment is identical with the scalar potential effect proposed by Aharonov and Bohm for a slightly different, time-varying experimental arrangement. The theoretical and experimental differences for large phase shifts are noted. The possibility is raised that a new classical electromagnetic lag effect may occur for electrons passing a small solenoid. Using a particular model for energy conservation, the predicted lag effect can be calculated and is associated with a quantum interference pattern shift of the same magnitude as predicted by Aharonov and Bohm based upon the electromagnetic vector potential. Thus the possibility exists that the experiments of Chambers and of Möllenstedt and Bayh may not confirm the ideas of Aharonov and Bohm on the vector potential in quantum theory. Several experiments are suggested which allow confirmation that the Aharonov-Bohm effect indeed involves local effects of the classical electromagnetic potential, rather than local electromagnetic fields leading to a new classical lag effect and hence to the observed quantum interference pattern shift.

I. INTRODUCTION

A. The Need for an Analysis of Interference Pattern Shifts

The diffraction patterns produced by electrons passing through slits have formed a phenomenon[1] familiar to physicists for forty years. However, the shifts in these patterns due to electromagnetic effects have formed a subject of interest[2] within the last decade because they seem to present evidence for a new break between classical and quantum electrodynamics with regard to the role of the electromagnetic potentials. In the present paper, we will provide some new ideas and an extensive commentary on interference pattern shifts caused by classical electromagnetic fields, or, following the ideas of Aharonov and Bohm, by classical electromagnetic potentials.

It seems no surprise to physicists that classical electromagnetic fields lead to shifts in electron interference patterns. This influence of the classical upon the quantum aspects should be expected because of the close ties between classical and quantum electrodynamics. What seems unanticipated is that classical electromagnetic potentials, which apparently have no physical role in classical theory, should also affect the electron interference patterns in regions free of classical electromagnetic fields. This local effect of electromagnetic potentials appeared in 1948 in the work of Ehrenberg and Siday,[3] and was predicted clearly a decade later by Aharonov and Bohm.[4] Experiments by Chambers[5] and by Möllenstedt and Bayh[6] have confirmed the predictions for the interference pattern shift due to a solenoid which is now termed the Aharonov-Bohm effect.

The local influence of classical electromagnetic fields seems to represent a distinct and significant break from the ways of thinking involved in classical electromagnetism. This new role for the electromagnetic potential seems to have gradually slipped into the literature[7] of physics, often without sufficient emphasis on the theoretical significance of the break, and also without sufficient experimental proof that the break is indeed justified. Thus this paper represents an effort to analyze interference pattern shifts due to classical electromagnetic fields, and to suggest a number of experiments which should separate these interference pattern shifts from the new effects proposed by Aharonov and Bohm due to classical electromagnetic potentials.

B. Outline of the Paper

In the first section we draw a crucial distinction between two types of interference pattern shifts—between those shifts which involve a deflection of the entire interference pattern, and those shifts which involve a deflection of only the double-slit interference pattern, leaving the single-slit envelope undisplaced. The first type of shift involves a classical deflecting force, whereas the second does not. The quantum phase-shift analysis appropriate for a classical deflecting force is given.

Next we propose a classical electrostatic lag effect which is associated with a shift of only the double-slit electron interference pattern without involving a classical deflection. The effect is treated from semiclassical and quantum points of view, and later is reanalyzed in terms of Bohr's principle of complementarity.

Having shown how classical electromagnetic fields can give rise to quantum interference pattern shifts, we turn to Aharonov's and Bohm's proposal of a shift due to the classical electrostatic potential. For small deflections, this shift appears just like that due to a classical lag effect. However, for large deflections, we find the distinctive features of a shift based upon classical

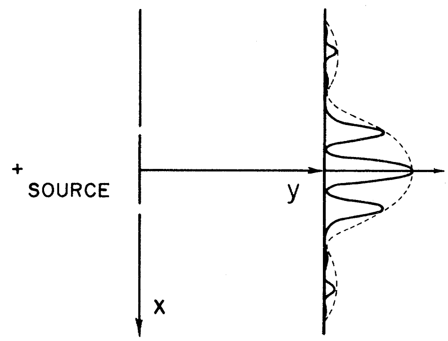

FIG. 1. The double-slit electron interference pattern showing the single-slit intensity envelope.

electromagnetic potentials; the interference pattern never breaks down no matter how large the electrostatic potentials are chosen, and no classical analog of the effect ever appears.

Now turning to the experimentally observed Aharonov-Bohm phase shift for electrons passing a solenoid, the possibility is raised that the shift may be due to an unanticipated classical electromagnetic lag effect and not due to the local effect of the classical vector potential. The analysis for a lag effect due to a solenoid is quite uncertain. However, it is concluded that there is as yet no firm evidence to support Aharonov's and Bohm's suggestion of interference pattern shifts due to classical electromagnetic potentials because no experiment performed thus far distinguishes between shifts due to lag effects and shifts due to electromagnetic potentials. Several experiments are mentioned, which should allow a conclusion as to whether or not the experimentally observed interference pattern shift from a solenoid is indeed due to local effects of the classical electromagnetic vector potential.

II. TYPES OF INTERFERENCE PATTERN SHIFTS—A FUNDAMENTAL DISTINCTION

A. The Undeflected Electron Interference Pattern

If a beam of electrons falls upon two slits cut in a plane surface, then the particle impacts form an interference pattern on a distant screen as sketched in Fig. 1. This wavelike behavior for particles lies entirely outside the realm of traditional classical physics. The interference pattern is analogous to that of wave optics with the wavelength λ given by the de Broglie relation

$$\lambda\!\!\!^{-} = \frac{\hbar}{p}, \qquad (1)$$

where $\lambda\!\!\!^{-}$ stands for $\lambda/2\pi$, $h = 2\pi\hbar$ is Planck's constant, and p is the electron momentum.

The interference pattern may be described roughly as a single-slit intensity pattern, due to the interference between contributions from different parts of the same slit, inside of which is fitted the double-slit pattern due to interference from contributions of the two distinct slits. See Fig. 1.

B. Interference Pattern Shifts

There are two types of interference pattern shifts which will be of interest to us, and which correspond to fundamentally different physical situations. In the first type, the entire interference pattern—double-slit pattern and single-slit envelope—is deflected as a whole as in Fig. 2. This corresponds to a classical deflection of the beam from the forward direction, and it arises from a classical deflecting force on the electron beam. The second type of interference pattern shift involves the deflection of only the double-slit pattern, while the single-slit envelope is undisplaced as in Fig. 3. Such an interference pattern shift can be produced by a classical electromagnetic lag effect. It is also the type of shift predicted theoretically and observed experimentally for the Aharonov-Bohm effect.

C. Type 1 Interference Pattern Shift: Deflection of the Entire Interference Pattern due to a Classical Deflecting Force

If a classical electric field $\vec{E}$ or a magnetic field $\vec{B}$ is perpendicular to an electron beam, there is a Lorentz force present which acts to deflect the beam. For example, if the situation is as indicated in Fig. 2, with an electric field E along the x axis present over the shaded region of width l, or else a magnetic field B along the z axis out of the paper, then there is a deflecting force as the beam passes through the shaded region. The electrons receive an impulse

$$\mathcal{I}_x = eE\left(\frac{l}{v}\right) = \frac{emEl}{p} \tag{2}$$

or

$$\mathcal{I}_x = \frac{evB}{c}\left(\frac{l}{v}\right) = \frac{eBl}{c}\ , \tag{3}$$

where l, m, and $v = p/m$ are the charge, mass, and initial speed of the electrons. This corresponds to deflection through an angle

$$\alpha = \frac{\Delta p}{p} = \frac{\mathcal{I}_x}{p}\ , \tag{4}$$

giving

$$\alpha = \frac{emEl}{p^2} \tag{5}$$

or

$$\alpha = \frac{eBl}{cp}\ . \tag{6}$$

If the electron beam gives rise to a quantum interference pattern, then we may expect to observe this same classical deflection of the entire interference pattern. The quantum analysis for the deflection can be obtained by use of the WKB approximation. Thus in the case of a transverse electric field, the electron beam crosses a region of length l in which there is a potential present:

$$V = -eEx. \tag{7}$$

In the wave function $\psi = \exp[i\hbar^{-1}(S - Et)]$, the change in phase $S/\hbar p$ for an electron of initial momentum $p = (2mE)^{1/2}$ entering a slit at the coordinate x is

$$\begin{aligned} S &\cong \int dy[2m(E - V(x))]^{1/2} \\ &\cong \int dy\,(2mE)^{1/2}\left(1 - \frac{1}{2}\,\frac{V}{E}\right) \\ &= py - \frac{mlV}{p} \\ &= py + \frac{emElx}{p}\ . \end{aligned} \tag{8}$$

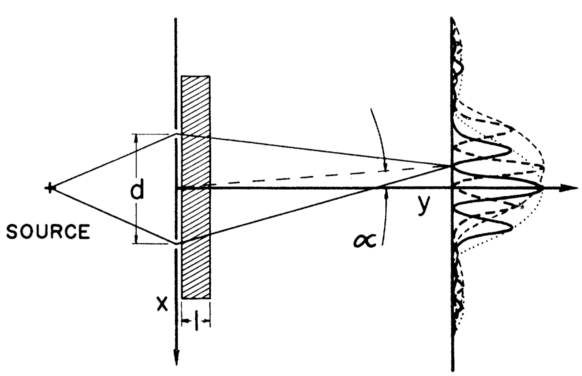

FIG. 2. Deflection of the entire interference pattern due to classical electromagnetic deflecting fields.

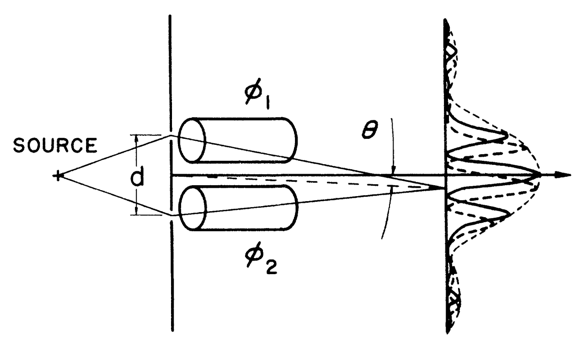

FIG. 3. Deflection of only the double-slit interference pattern. The single-slit envelope is undeflected.

This implies a phase change

$$\frac{\delta S}{\hbar}=\frac{emEl}{p\hbar}\delta x \tag{9}$$

between electrons which enter the slits a distance δx apart, δx not necessarily infinitesimal. This phase difference due to the electrostatic deflecting field can be compensated so as to bring the electrons back in phase provided the electrons are deflected through the angle α such that

$$p(\delta x\sin\alpha)-\frac{emEl\delta x}{p}=0 \tag{10}$$

or

$$\alpha=\frac{emEl}{p^2}\ , \tag{11}$$

which agrees with the result obtained from the classical analysis.

The case of deflection by a magnetic field along the z axis can be handled in similar fashion although the WKB approximation is less familiar for this case. Carrying out this classical approximation,[8] the phase may be obtained as

$$S=\int\left[(2mE)^{1/2}+\frac{e}{c}A_y\right]dy\cong py+\frac{e}{c}A_y l, \tag{12}$$

since A_y corresponds to a constant magnetic field in the z direction. The phase difference at the screen for two electrons which arrive at the slits δx apart (δx not necessarily infinitesimal) is

$$\delta S=\frac{e}{c}A_y(x+\delta x)l-\frac{e}{c}A_y(x)l$$
$$=\frac{e}{c}Bl\delta x, \tag{13}$$

since $\vec{B}=\nabla\times\vec{A}$ is a constant field in the z direction. (For example, take $\vec{A}=\hat{j}Bx$.) This phase is of the same form as was obtained for the case of electrostatic deflection, and again leads to the deflection of the entire interference pattern, this time by the angle

$$\alpha=\frac{eBl}{cp}\ . \tag{14}$$

D. Type 2 Deflection of Only the Double-Slit Interference Pattern

The second type of interference pattern shift involves the deflection of only the double-slit interference pattern, while the single-slit envelope is undisplaced. See Fig. 3. In this situation, the average beam is undeflected from the forward direction. Following Ehrenfest's theorem relating averages of quantum observables to classical observables, it follows that there is no classical deflection of the electron beam, and correspondingly no classical deflecting force. However, this does not mean that such an interference pattern shift cannot be due to classical forces; indeed much of this paper is devoted to investigating the connection between such shifts and classical electromagnetic lag effects. In this section, we have treated the type 1 phase shift in some detail because we will not deal with it further in this paper. All of the further analysis will be concerned with phase shifts of type 2.

The distinction between the two types of deflections cannot be overemphasized. There is confusion of the two types of shifts repeatedly in the author's correspondence and in the published literature.[9,10] The type 1 shift involving a classical deflection has even been incorrectly proposed[10] as a classical analog of the Aharonov-Bohm effect, which involves a type 2 deflection of only the double-slit pattern. The rule is as follows:

(i) A type 1 shift (deflection of the full interference pattern as in Fig. 2) involves an average deflection of the electron beam, and hence a classical deflection and a classical deflecting force.

(ii) A type 2 shift (deflection of only the double-slit pattern as in Fig. 3) involves no average deflection of the electron beam, and hence there is no classical deflection and no classical deflecting force.

III. AN ELECTROSTATIC LAG EFFECT LEADING TO A QUANTUM INTERFERENCE PATTERN SHIFT

A. A New Experiment

We consider the following experimental situation involving an electron interference pattern. The two long conducting tubes shown in Figs. 3 and 4 are charged to different electrostatic potentials and are held at these potentials during the entire experiment. The coherent electron beam originating at the source is split into two parts and each is allowed to enter a conducting tube. The electrons drift down the tubes, and the beams are then recombined to give an interference pattern.

B. Quantum Phase Shift

An account of the interference pattern shift produced by this new experiment can be given in terms of the WKB approximation. The electrons passing through the different tubes are described in terms of plane waves:

$$\psi_1(y,t)=\exp\left[i\hbar^{-1}(S_1(y)-Et)\right],$$
$$\psi_2(y,t)=\exp\left[i\hbar^{-1}(S_2(y)-Et)\right], \tag{15}$$

which cross the potential wells

$$V_1 = e\phi_1 \text{ and } V_2 = e\phi_2 . \tag{16}$$

The time dependence $\exp[-i\hbar^{-1}Et]$ is the same for both beams, and, since the beams are compared at the same time, no phase shift occurs here.

The phases $\hbar^{-1}S_1, \hbar^{-1}S_2$ of the wave functions are given by

$$\begin{aligned} S_i &\cong \int [2m(E - V_i)]^{1/2} dy \\ &\cong \int (2mE)^{1/2}\left(1 - \frac{1}{2}\frac{V_i}{E}\right) dy \\ &= (2mE)^{1/2} y - \frac{mV_i l}{(2mE)^{1/2}} \\ &= py - \frac{mV_i l}{p}, \quad i = 1, 2 \end{aligned} \tag{17}$$

where p is the initial momentum of the electron and l is the length of each tube. It follows that the phase difference along the two paths of equal length is just

$$\begin{aligned} \frac{S_2 - S_1}{\hbar} &= \frac{e}{\hbar}\phi_1\frac{ml}{p} - \frac{e}{\hbar}\phi_2\frac{ml}{p} \\ &= \frac{e}{\hbar}(\phi_1 - \phi_2)\frac{ml}{p} \end{aligned} \tag{18}$$

or

$$\frac{S_2 - S_1}{\hbar} = \frac{e}{\hbar}(\phi_1 - \phi_2)t, \tag{19}$$

where

$$t = \frac{ml}{p} \tag{20}$$

is the approximate time of transit for a particle in either tube.

C. An Example of a Type 2 Shift of Only the Double-Slit Electron Interference Pattern

The phase difference in (18) can be viewed in terms of a deflection of the double-slit electron interference pattern as shown in Fig. 3. When the conducting tubes are at the same potential, the electron interference pattern (solid line) consists of a double-slit pattern inside a single-slit envelope, with the principal maximum in the forward direction. The shift in the double-slit pattern when the tubes are at different potentials is sketched as the broken curve in Fig. 3; the single-slit envelope is displaced, but the double-slit pattern has been deflected through the angle θ. The phase difference in Eq. (18) is compensated by the difference in path lengths of the trajectories as indicated in Fig. 5. Thus the waves of the two packets are in phase when

$$pd\sin\theta + e(\phi_1 - \phi_2)\frac{ml}{p} = 0 . \tag{21}$$

The angle of deflection is

$$\begin{aligned} \theta \sim \sin\theta &= \frac{e(\phi_2 - \phi_1)ml}{p^2 d} \\ &= \frac{e(\phi_2 - \phi_1)t}{pd} , \end{aligned} \tag{22}$$

which, incidentally, is independent of $\hbar$.

D. Classical Analysis Predicting a Lag

Under the experimental situation described above, the electrons experience classical electrostatic forces—upon entering and leaving the tubes. On entering the upper tube, an electron suffers a change in velocity:

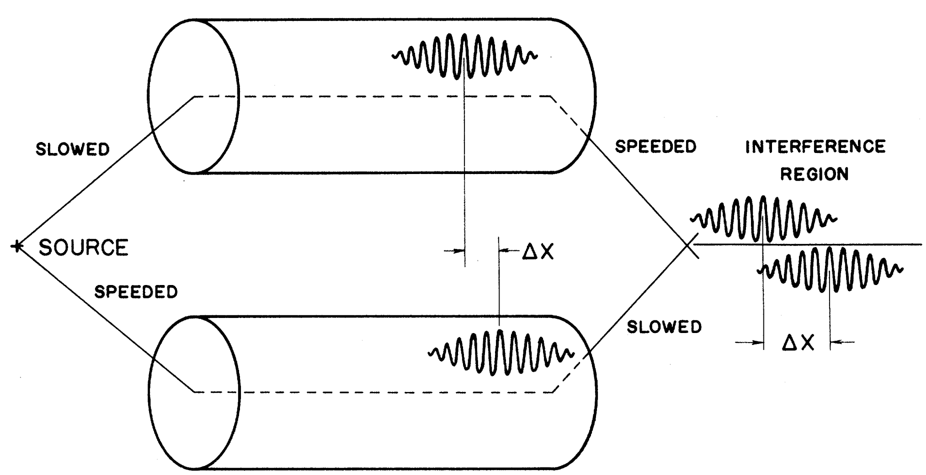

FIG. 4. The experimental arrangement for an electrostatic lag effect for wave packets. The lag effect is associated with an interference pattern shift.

$$\Delta v_1 \cong \frac{e\phi_1}{p}, \tag{23}$$

where p is its initial momentum, while entering the lower tube the change is

$$\Delta v_2 \cong -\frac{e\phi_2}{p}. \tag{24}$$

Thus, while the electrons drift down the tubes, there is a spatial lag introduced in the direction of motion which is just

$$\begin{aligned}\Delta y &= \int \Delta v_2 dt - \int \Delta v_1 dt \\ &= -\frac{e}{p}\int \phi_2 dt + \frac{e}{p}\int \phi_1 dt \\ &= \frac{e}{p}(\phi_1 - \phi_2)t \\ &= \frac{e(\phi_1 - \phi_2)ml}{p}.\end{aligned} \tag{25}$$

When the electrons emerge from the tubes, they are restored to their initial velocities, but are still relatively displaced by the Δy of Eq. (25). Thus the electrostatic forces produce no net change of momentum or energy for the electrons, but only a classical lag effect.

E. Phase and Group Velocities for the Electrons

It seems tantalizing to proceed from the classical lag to a phase difference by introducing the de Broglie wavelength λ. Thus we obtain a phase difference

$$\begin{aligned}\frac{2\pi\Delta y}{\lambda} &= \frac{p\Delta y}{\hbar} \\ &= \frac{e(\phi_1 - \phi_2)t}{\hbar},\end{aligned} \tag{26}$$

which agrees with Eq. (19). However, the connections between phase and group velocities in wave mechanics seem to introduce disconcerting complications here.

When an electron enters a tube and is slowed down, its phase velocity increases. The de Broglie wavelength λ_1 inside tube 1 is given by

$$\lambda_1 = \frac{h}{[2m(E - V_1)]^{1/2}}, \tag{27}$$

while the frequency remains

$$\omega = \frac{E}{\hbar}, \tag{28}$$

producing a phase velocity

$$v_{\text{phase}} = \lambda\!\!\!^{-}\omega = \frac{E}{[2m(E - V_1)]^{1/2}}, \tag{29}$$

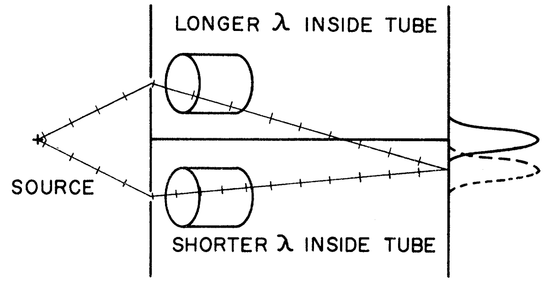

FIG. 5. Schematic indication of the electrostatic interference pattern deflection. The cross bars indicate wave lengths.

which is inversely proportional to the speed of the corresponding classical particle.

The group velocity properly reflects the slowing down predicted by classical theory. Here, the condition of stationary phase is

$$\begin{aligned}0 &= \frac{\partial\phi_1}{\partial E} \\ &= \frac{\partial}{\partial E}\left(\int [2m(E - V_1)]^{1/2} dy - Et\right),\end{aligned} \tag{30}$$

giving

$$0 = \int^y \frac{m}{[2m(E - V_1)]^{1/2}}\, dy - t \tag{31}$$

and

$$v_{\text{group}} = \frac{[2m(E - V_1)]^{1/2}}{m}. \tag{32}$$

Thus if the electron beam is chopped into wave packets, then a surprising situation seems to occur for this electrostatic experiment. The double-slit interference pattern is deflected in one direction as in Fig. 3, whereas the principal maximum (given by the region of overlap of the central parts of the wave packets) is deflected by an equal angle in the opposite direction as shown in Fig. 6.

F. Measurable Classical Lag

The experimental arrangement described above gives rise to an interference pattern shift and also to a classical lag effect. It is possible to make the classical lag into a measurable effect by chopping the electron beam into wave packets, and then making the potential differences between the tubes so large that the lag between the wave packets exceeds the spread of the wave packets. In this case the wave packets no longer overlap, and it is possible to obtain two distinct times of transit from the source depending upon which tube the electrons travel through. See Fig. 4.

IV. THE AHARONOV-BOHM EFFECT FOR THE ELECTROMAGNETIC SCALAR POTENTIAL

A. A New Mechanism for Interference Pattern Shifts

The interference pattern shifts based on classical electromagnetic deflecting forces form a familiar part of the physics of electromagnetic interactions. Also, the double-slit interference pattern shift associated with a classical lag effect adds nothing fundamentally new to our understanding, even if the effect is unfamiliar in the literature. It is the proposals of Aharonov and Bohm which represent a striking new suggestion in the physics of electromagnetism. Here a classical electromagnetic potential acts to shift the quantum interference pattern in a region free of electromagnetic fields.

B. Aharonov's and Bohm's Proposed Experiment

The following description is given by Aharonov and Bohm[4] for an experiment to "demonstrate the significance of potentials in the quantum theory." A coherent electron beam is split into two parts and each part is allowed to enter a long conducting cylinder as shown in Fig. 4. After the beams pass through the tubes, they are combined to interfere coherently. By means of time-determining electrical shutters, the beam is chopped into wave packets that are long compared with the de Broglie wavelength λ, but short compared with the lengths of the tubes. The electrostatic potential in each tube is determined by a time delay mechanism in such a way that the potential is zero until the packet is well inside the tube. The potential then grows to constant values ϕ_1 and ϕ_2 different for the two tubes. Finally the potential drops back to zero before the packet comes near the outer edge of the tube. Thus the potential is nonzero only while the electron is well inside the tube. The purpose of this arrangement is to ensure that the electron is in an electric potential without ever being in an electric field. The electron enters and leaves the tube when no electric fields are present and is shielded from the electric fields when it is inside the long conducting tubes.

The argument is now made that the phases of the two wave packets evolve differently in time due to the electrostatic energies $e\phi_1$ and $e\phi_2$, when the electron is inside one or the other of the tubes. The interference of the two parts of the beam will depend upon the phase difference between the two paths

$$-\frac{e}{\hbar}\int \phi_2 dt + \frac{e}{\hbar}\int \phi_1 dt = \frac{e}{\hbar}(\phi_1 - \phi_2)t, \tag{33}$$

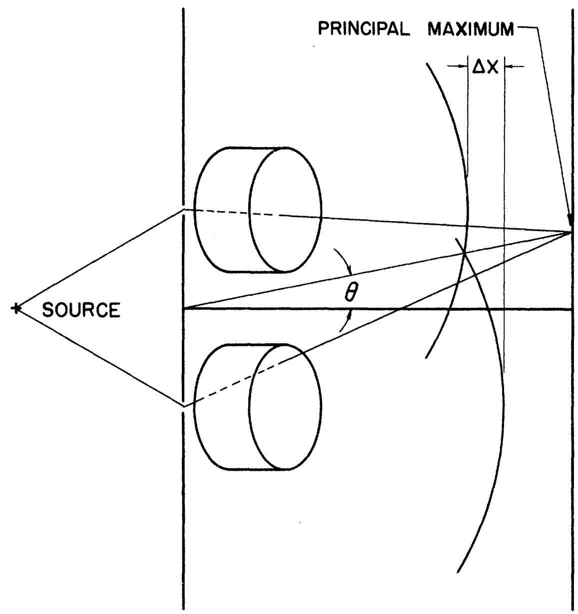

FIG. 6. Deflection of the principal maximum due to the electrostatic lag effect.

where here t is the time during which the potential exists. Interestingly enough, this phase shift between the electrons passing through different tubes is the same as that in (19) obtained for the experiment of Sec. III. Again the phase difference leads to a shift of only the double-slit interference pattern by the angle θ corresponding to Eq. (22):

$$\theta \sim \sin\theta = \frac{e(\phi_2 - \phi_1)t}{pd} \quad . \tag{34}$$

C. Comments on the Aharonov-Bohm Experiment

After discussing their proposed experiment, Aharonov and Bohm[4] conclude: "Thus there is a physical effect of the potentials even though no force is ever exerted on the electron. The effect is essentially quantum-mechanical in nature because it comes in the phenomenon of interference. We are therefore not surprised that it does not appear in classical mechanics."

The statement that the effect is essentially quantum-mechanical because it appears in connection with an interference pattern shift must be interpreted in the light of our previous analysis joining classical electromagnetic forces and quantum interference pattern shifts. However, Aharonov and Bohm are certainly correct in emphasizing the novelty of their ideas. Here we have the prediction of an interference pattern shift unassociated with any classical electromagnetic effect, a shift due to classical electromagnetic potentials acting in a region where there are no electromagnetic forces.

It seems curious that the phase shift of Eq. (33) for the Aharonov-Bohm proposal is so similar to that of Eq. (19) in Sec. III obtained for a phase shift based upon classical electrostatic forces. For small phase shifts, the interference pattern deflections have identical appearances. However, for large potential differences $(\phi_1 - \phi_2)$ between the two tubes, the behavior for Aharonov's and Bohm's proposed shift is entirely different from that associated with a classical lag effect.

In the deflection of Sec. III associated with a classical lag, large values of the potential difference will destroy the double-slit interference pattern when the relative lag between the waves becomes comparable to the coherence length of the electron beam. Alternatively, if the beam is chopped into wave packets, then increasing the potential difference gradually produces the classical electromagnetic lag effect when the relative lag becomes large compared to the wave-packet spatial dimensions. See Fig. 4. However, in the Aharonov-Bohm experiment, there is no change in the group velocity and no lag between the wave packets. Each wave of the packet in one of the tubes undergoes exactly the same phase shift $\exp[-ie\phi_1 t]$ or $\exp[-ie\phi_2 t]$ and this appears simply as an over-all multiplying phase factor. The potential difference $\phi_1 - \phi_2$ may be made as large as desired but the Aharonov-Bohm interference effect will never be washed out and no classical effect will ever appear.

V. BOHR'S PRINCIPLE OF COMPLEMENTARITY AND QUANTUM INTERFERENCE PATTERN SHIFTS

A. The Role of the Complementarity Principle

Electron interference patterns involve the wavelike properties of particles. The type 2 interference pattern shifts of only the double-slit pattern are not a familiar part of the textbook literature of quantum mechanics, and hence it seems appropriate to comment upon the consistency of the wave and particle interpretations. The Bohr complementarity of the wave and particle aspects requires that the classical idea of a particle trajectory, when subjected to the Heisenberg uncertainty principle, shall not allow a violation of the ideas of wave mechanics. In the present situation, the principle requires that if it is possible to determine through which slit each electron passes on its way to the screen, then the electron interference pattern must vanish.

The complementarity principle can be tested for all the classical measurements possible on the experiment. The analysis given by Furry and Ramsey[11] checks the consistency for certain electromagnetic measurements. Because of our interest in classical lag effects, we will sketch the verification of the principle for time-of-flight measurements.

B. Time Delay and Complementarity for the Traditional Double-Slit Interference Pattern

We consider a double-slit interference pattern, as in Fig. 7, which is predicted by the wave mechanical view of physics. From the particle point of view, the pattern is formed by particles passing through the slits and following classical trajectories to the screen. We now chop the electron beam into wave packets and attempt to time the period elapsed between the emission of a wave packet from the source and its arrival at the screen. In order to make the measurements accurate, the width a of each individual slit is made quite small, and accordingly the single-slit diffraction envelope, governed by $\frac{1}{2} a \sin\theta = (n + \frac{1}{2})\lambda$ for a minimum, is so broad that it can be ignored compared to the double-slit pattern. The times required for transit from the source to either slit are the same, but the times from the slits to the screen are different for the two slits. The time required to go from the screen to the first maximum at the side of the principal maximum at the center of the pattern is r_1/v for passage from the first slit and r_2/v for passage from the second. For small angles of deflection, the time difference Δt between the paths is

$$\Delta t \cong \frac{d \sin\theta}{v} = \frac{\lambda}{v} = \frac{h}{vp} , \tag{35}$$

where the de Broglie relation for the particle has been used. However, the uncertainty Δt in the time required for a particle to cross the region from the slits to the screen follows from the Heisenberg uncertainty relation for the wave packet

$$h \lessapprox \Delta p \Delta y \lessapprox pv\Delta t \tag{36}$$

or

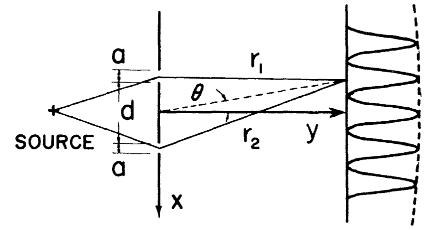

FIG. 7. Bohr's correspondence principle tested by time-of-flight measurements.

$$\Delta t \gtrsim \frac{h}{vp} . \qquad (37)$$

Thus the time difference Δt for transit from the different slits will be lost in the timing uncertainties Δt associated with the Heisenberg uncertainty relation applied to the wave packet.

However, if we are determined to find the time lag Δt, then we should not restrict ourselves to the first interference maximum in the region of the principal maximum as in Fig. 7, but should put our detector at the nth maximum from the central maximum, where n is 10 or 100. Then by the arguments above, we would have $\Delta t \sim n\Delta t$, and we could indeed pick out the transit time difference Δt for particles coming from the different slits above the time uncertainty Δt. It may seem that we have violated the principle of complementarity, but this is not the case. We must reckon with the uncertainties in phase in the wave packet when the particle momentum is known only to an accuracy Δp. Thus away from the principal maximum at the nth fringe, $n = 10$ or 100, the interference pattern will be washed out to give a uniform classical pattern corresponding to electrons rebounding from the edges of the slits. In this case we can indeed measure time differences corresponding to arrival from the different slits, but now the wave interference pattern is not present. The complementarity view is confirmed.

C. Complementarity and Interference Pattern Shifts

The shift of an interference pattern may seem to introduce new aspects for the complementarity principle. This is not the case, however, for the nonspecific tests which make no reference to the actual interference pattern shift. Thus, for example, the electromagnetic measurements suggested by Furry and Ramsey [11] make no use of the interference pattern shift, and are valid for any double-slit interference pattern. The time-of-flight test given in Sec. V B involves the principal interference maximum formed by the overlap of the centers of the wave packets passing through the different slits, and will require reconsideration only if the principal maximum is deflected through a large angle. For the Aharonov-Bohm experiment, the principal maximum never departs by more than one fringe from the forward direction, and the time-of-flight analysis is exactly as for the undeflected pattern. Thus it appears that the Aharonov-Bohm phase shift is irrelevant to considerations of the complementarity principle.

For the interference pattern shift described in Sec. III, there is a deflection of the principal interference maximum in the opposite direction from the double-slit interference pattern, and it is of interest to see how this deflection is connected through the correspondence principle with the classical electromagnetic lag effect. The double-slit interference pattern slit is as in Fig. 3, but the deflection of the principal maximum is as in Fig. 6. If the beam is chopped into wave packets in order to carry out time-of-flight measurements, then the interference pattern survives in the region of the principal maximum. However, the time of transit to the principal maximum in Fig. 6 involves two compensating effects. The electron passing through the upper tube is slowed down on entering the tube so that when it emerges, it lags behind the electron through the lower tube by the amount

$$\Delta y = \frac{e(\phi_1 - \phi_2) m l}{p^2} . \qquad (38)$$

However, the path length to the principal maximum is longer for the electron passing through the lower tube by the amount $d \sin\theta$. Thus the difference in transit time from the source to the principal maximum is

$$\Delta t = \frac{d \sin\theta}{v} - \frac{e(\phi_1 - \phi_2) m l}{vp^2} = 0 , \qquad (39)$$

since the principal maximum appears at the angle

$$\sin\theta = \frac{e(\phi_1 - \phi_2) m l}{p^2 d} . \qquad (40)$$

Thus the deflection of the principal maximum is connected with the classical time delay in precisely such a way that it is impossible by time-of-flight measurements to determine through which slit the electron passed on its way to the interference pattern. The correspondence principle is again confirmed.

VI. THE POSSIBILITY OF A CLASSICAL LAG EFFECT DUE TO A SOLENOID

A. Energy Considerations for a Lag Effect

When an electron passes alongside a constant-current solenoid, there is a change in the energy of the electromagnetic field due to the overlap between the magnetic field of the passing electron and the magnetic field inside the solenoid. This energy change has been emphasized by Liebowitz,[12] and has been treated in detail recently by the present author.[13] In connection with this energy change, the possibility is raised that a physical solenoid might occasion a classical lag for particles passing on opposite sides of the solenoid, in a manner analogous to the electrostatic lag discussed in Sec. III. Such a classical lag might produce the interference pattern shift observed experimentally by Chambers[5] and by Möllenstedt and Bayh.[6] See Fig. 8.

It seems helpful here to make a comparison with the electrostatic lag effect described in Sec. III. The account of the electrostatic case seems transparent. A particle passing through one of the slits in Fig. 3 enters one of the tubes and has its velocity changed on entrance due to the electrostatic field of the charges on the tube. One way to do the calculation for the change in velocity as the electron enters the upper tube is to compute the change of energy in the electromagnetic fields. Starting with

$$\mathcal{E} = \frac{1}{8\pi}\int (\vec{E}^2 + \vec{B}^2)\, d^3x\,, \tag{41}$$

where $\vec{E}$ includes the fields of the electron and the charge on the conducting tube which causes the potential ϕ_1, the energy change $\Delta\mathcal{E}$, when the electron is in the tube, is just

$$\Delta\mathcal{E} = e\phi_1\,. \tag{42}$$

The energy change of the electric field is compensated by the change in the kinetic energy of the electron:

$$0 = \Delta(\tfrac{1}{2}mv^2) + \Delta\mathcal{E}$$

$$= p\Delta v + e\phi_1\,, \tag{43}$$

which is just Eq. (23). The particle is slowed by the electrostatic field when entering the tube and is speeded up when leaving. The relative lag introduced as the particles pass through the two tubes at different potentials is associated with the shift in interference pattern.

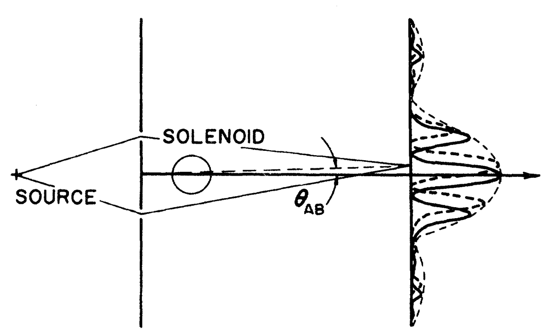

FIG. 8. The Aharonov-Bohm interference pattern deflection.

Precisely analogous arguments may be presented for the case when a long thin solenoid is placed between the slits of the diffraction screen as in Fig. 8. Now starting with the energy in the electromagnetic field in (41), the energy in the magnetic field is found to change due to interference between the solenoid magnetic field and the magnetic field of the passing electron, the energy shift in the nonrelativistic approximation being[14]

$$\Delta\mathcal{E} = \frac{e}{c}\vec{v}\cdot\vec{A}, \tag{44}$$

where $\vec{A}$ is the vector potential (in the Coulomb gauge) of the solenoid evaluated at the position of the particle. Let us assume that the change in energy in the electromagnetic field is compensated by a change in the kinetic energy of the passing electron:

$$0 = \Delta(\tfrac{1}{2}mv^2) + \Delta\mathcal{E}$$

$$= p\Delta v + \frac{e}{c}\vec{v}\cdot\vec{A}. \tag{45}$$

The electron is slowed when approaching the solenoid and then speeded up when leaving, and on the other side of the solenoid, the order is reversed. The relative lag between the particles passing on opposite sides of the solenoid parallels Eq. (25):

$$\Delta y = \int \Delta v_2\, dt - \int \Delta v_1\, dt$$

$$= \frac{e}{cp}\int_2 \vec{A}\cdot\vec{v}\,dt - \frac{e}{cp}\int_1 \vec{A}\cdot\vec{v}\,dt$$

$$= \frac{e}{cp}\oint \vec{A}\cdot d\vec{s}$$

$$= \frac{e\Phi}{cp}\,, \tag{46}$$

where the closed line integral goes along path 2 from the slit to the screen and then back along path 1. Here Φ is the enclosed flux of the solenoid.

Exactly parallel to Eq. (26), we might associate a relative phase change due to the relative displacement of the packets

$$\frac{2\pi\Delta y}{\lambda} = \frac{p\Delta x}{\hbar}$$

$$= \frac{e\Phi}{c\hbar}\,. \tag{47}$$

The interference pattern shift may be calculated as a deflection of the double-slit pattern analogous to Eqs. (21) and (22), with

$$\theta \sim \sin\theta = \frac{e\Phi}{cpd} , \tag{48}$$

which is independent of $\hbar$. Just such a shift in the double-slit interference pattern has been observed in experiments by Chambers and by Möllenstedt and Bayh, and is known as the Aharonov-Bohm effect. However, none of the experimentalists has interpreted the phase shift as associated with a new classical electromagnetic lag effect.

B. Force Considerations for a Lag Effect

Although the *energy* calculations above provide a tantalizing and transparent suggestion of a lag effect, any attempt to give a realistic account of the electromagnetic *forces* seems to involve enormous complications. In the electrostatic lag effect of Sec. III, it was possible to compute both the force on the electron and the energy in the electromagnetic field, while pretending that the charge distribution on the conducting tube is exactly as it was before the electron approached the tube. If we try this same approximation of an unaffected source for the case of the solenoid, then there is absolutely no force on the passing electron, but there is a large change of energy in the electromagnetic field. Clearly, unless we imagine external forces on each of the individual charges on the solenoid, the approximation of an unaffected source cannot be maintained. An exactly similar situation holds for the momentum changes. In the electrostatic lag case, the electron and the charge distribution on the tube exert electrostatic forces on each other which satisfy Newton's third law for nonrelativistic velocities. However, this is not true of an electron passing a solenoid. In this case the electron exerts forces on the currents of the solenoid (associated with a large change in momentum in the electromagnetic field), yet in the approximation of unaffected solenoid currents there is no force back on the passing electron.

These observations reveal that the approximation of an unaffected source is suitable for the electrostatic case, but would violate the conservation of energy and of momentum in the solenoid case. The currents in the solenoid must change in some fashion so as to provide conservation of both energy and momentum for the total system. However, calculating the changes in the solenoid currents means dealing with a multiparticle electromagnetic system which is enormously complicated.

In an effort to deal with the multiparticle behavior, the author has carried out computer calculations entirely within classical electromagnetism for the interaction of a passing charged particle with a model solenoid consisting of four layers of eight charges each, the charges confined to move in circles. The mutual interactions between all the particles were followed when a passing charged particle was introduced outside the solenoid. It was found that the fields of the passing charge produced small accelerations of the particles in the solenoid which in turn produced electric and magnetic fields back at the position of the passing charge. These return fields on the passing charged particle indeed gave rise to a relative lag effect for particles passing on opposite sides of the solenoid. Unfortunately, the computer analysis of all interparticle classical electromagnetic forces became enormously cumbersome and lengthy as the number of particles in the solenoid was increased. It was impossible to go to a more realistic solenoid consisting of a large number of particles. However, for this model solenoid involving relatively few particles, the classical electromagnetic lag effect was unambiguous.

VII. THE AHARONOV-BOHM EFFECT FOR A SOLENOID

A. The Aharonov-Bohm Analysis

As part of proposals that quantum interference patterns could be shifted by classical electromagnetic potentials in regions free of classical electromagnetic fields, Aharonov and Bohm[4] suggested that the vector potential outside a long solenoid would cause a shift in an electron double-slit diffraction pattern. The experiment, involving the insertion of a solenoid between the two slits as shown in Fig. 8, is identical to that mentioned in Sec. VI in terms of a possible classical electromagnetic lag effect. However, the theoretical interpretation is distinctly different, precisely along those lines in which the Aharonov-Bohm proposals for the scalar potential in Sec. IV differ from the shift associated with the electrostatic lag effect in Sec. III.

Aharonov and Bohm remark that in a suitable WKB approximation, the phase of the electron wave function is changed due to the vector potential by an amount $(e/c\hbar)\int \vec{A}\cdot d\vec{r}$. This integral has a different value for electrons passing around opposite sides of a solenoid as in Fig. 8, and so occasions a phase difference:

$$\frac{S_2 - S_1}{\hbar} = \frac{e}{c\hbar}\int_2 \vec{A}\cdot d\vec{r} - \frac{e}{c\hbar}\int_1 \vec{A}\cdot d\vec{r}$$

$$= \frac{e}{c\hbar}\oint \vec{A}\cdot d\vec{r}$$

$$= \frac{e\Phi}{c\hbar}. \tag{49}$$

This phase difference can be observed as a shift in the double-slit interference pattern as in Fig. 8. The single-slit envelope involves the interference between electrons passing through a single slit and past the same side of the solenoid. Accordingly there is no new phase shift due to the solenoid for the single-slit envelope which is undisplaced. The phase difference for electrons passing on opposite sides of the solenoid leads to a deflection of only the double-slit pattern. Following the familar argument of wave optics,

$$\frac{p d \sin\theta_{AB}}{\hbar} - \frac{e\Phi}{c\hbar} = 0, \tag{50}$$

and the deflection angle is

$$\theta_{AB} \sim \sin\theta_{AB} = \frac{e\Phi}{c p d}. \tag{51}$$

It seems worthwhile to list a few aspects of the Aharonov-Bohm deflection which are sometimes misunderstood in the literature:

(i) The average beam is undeflected from the forward direction. This follows from the absence of any shift in the single-slit envelope, and corresponds to the absence of any classical deflection.

(ii) The principal maximum of the double-slit interference pattern never departs from the forward direction by more than one interference fringe.

(iii) The deflection angle θ_{AB} can be written in terms of classical physical quantities and does not depend upon the quantum of action $\hbar$.

The first remark about the absence of any average deflection of the electron beam is a particularly sore point in the literature. The reader who has difficulty with this idea should see the experimental photographs such as Fig. 7 in the article[15] by Möllenstedt and Bayh. The double-slit interference maxima are shifted with the field in the solenoid, but the average intensity of the pattern, which is given by the single-slit envelope, is invariant. The double-slit fringes appear at one side of the pattern and disappear at the other as they move with the solenoid field.

B. An Apparent Inconsistency in the Aharonov-Bohm Analysis

The classical analysis of Ref. 13 for the energy in the electromagnetic field as a charged particle passes a constant-current solenoid seems to reveal an apparent inconsistency in the analysis of Aharonov and Bohm for the effects of the vector as compared with the scalar electromagnetic potential. Thus in the case of the analysis for the scalar potential outlined in Sec. IV, the electron wave functions ψ_1 and ψ_2 were said to reflect the change in energy in the electromagnetic fields associated with the electrostatic fields carried by the electron, even though the electron kinetic energy was unaltered. It was this change in the electromagnetic field energy (energy furnished by the external forces charging the conducting tube) which led to the Aharonov-Bohm phase shift for the scalar potential.

However, when the vector potential case was analyzed by Aharonov and Bohm under the assumption of constant solenoid currents, the change in energy in the electromagnetic field (energy furnished by the external forces maintaining the constant solenoid currents) associated with the magnetic fields carried by the electrons was excluded from any contribution to the phases of the wave functions ψ_1 and ψ_2. The situations seem quite analogous for the two cases, and there does not seem to be any reason for the apparent inconsistency in the analysis.

It is interesting to note what happens if we attempt to be consistent by including the phase change associated with the electromagnetic field energy for the case of particles passing a constant-current solenoid. Following the pattern of the Aharonov-Bohm analysis for the scalar potential case, we expect a phase difference associated with the electromagnetic field energy of Eq. (44):

$$-\int_2 \mathcal{E}\,dt + \int_1 \mathcal{E}\,dt = -\frac{e}{c}\int_2 \vec{v}\cdot\vec{A}\,dt + \frac{e}{c}\int_1 \vec{v}\cdot\vec{A}\,dt$$

$$= -\frac{e}{c}\int_2 \vec{A}\cdot d\vec{r} + \frac{e}{c}\int_1 \vec{A}\cdot d\vec{r}$$

$$= -\frac{e}{c}\oint \vec{A}\cdot d\vec{r}$$

$$= -\frac{e\Phi}{c}, \tag{52}$$

which is just equal in magnitude but opposite in direction from the phase difference (49) associated with the momentum part of the wave function.

Thus applying the Aharonov-Bohm analysis for the phase shift due to energy changes and combining it with their analysis for the vector potential sketched above, we apparently conclude that Aharonov and Bohm should predict no net phase shift for an electron beam passing around a constant-current solenoid. It would seem that some further clarification of the analysis is needed.

Note added in proof. In private conversation Professor D. Bohm suggested that quantum phase changes for particles should be associated with only electrostatic energy changes and not with the full energy changes in the electromagnetic fields as evaluated in this section. Nevertheless it should be emphasized that the quantum calculations for the Aharonov-Bohm effect treat the problem within a one-particle Hamiltonian, in effect ignoring the reactions of the solenoid particles. In Sec. VI B we saw that within classical electromagnetism, the reactions of the solenoid particles are crucial to the conservation of energy and momentum, providing effects in the Hamiltonian as large as those based upon a single particle passing a solenoid with constant (external) currents.

VIII. EXPERIMENTS TO CONFIRM THE ROLE OF CLASSICAL ELECTROMAGNETIC POTENTIALS IN QUANTUM INTERFERENCE PATTERN SHIFTS

A. The Present Experimental Situation

In this section we summarize the present experimental status of the various interference pattern shifts suggested from theory. In the succeeding sections, we propose some further experiments to verify that the observed Aharonov-Bohm effect for a solenoid is indeed caused by local effects of the vector potential and is not associated with a classical electromagnetic lag effect.

(i) The type 1 shift of the entire electron interference pattern due to a classical magnetic deflecting field appears in the experimental work of Chambers[5] and of Bayh,[16] and seems to agree with the theoretical analysis.

(ii) To the author's knowledge, no one has observed a type 2 shift of only the double-slit electron interference pattern due to classical electrostatic fields and associated with a classical lag effect. The theoretical analysis for such a shift seems to fit within the traditional connections between classical and quantum theory, and one would be surprised if the analysis were contradicted by experiment.

Note added in proof. During a private conversation, Professor G. Möllenstedt suggested that the interference pattern shift associated with a classical electrostatic lag effect had in essence been observed at the Institut für Angewandte Physik in Tübingen. The situation corresponds to Sec. III with the two tubes at the same electrostatic potential but of different lengths. In the experiments an electron beam was split and the two parts of the beam were passed through thin sheets of aluminum, the thickness in one beam being ~100 Å and in the other ~200 Å. Since the average electrostatic potential inside the material was different from that of the surrounding space, the electrons changed velocities on entering and leaving the material. The different length of material in the two beams produced a relative classical displacement of the electrons emerging from the different paths, and hence an interference pattern shift. This interference pattern shift was observed experimentally.

(iii) The Aharonov-Bohm phase shift due to the classical electromagnetic scalar potential has never been observed. Experimental confirmation of this effect would be convincing evidence for the role of classical electromagnetic potentials in causing quantum interference pattern shifts in regions free of classical electromagnetic fields.

(iv) Type 2 shifts of only the double-slit electron interference pattern due to a magnetic whisker and due to a solenoid have been observed experimentally by Chambers[5] and by Möllenstedt and Bayh.[6] However, there seems to be no evidence which allows one to decide as to whether the shift is associated with a classical electromagnetic lag effect such as described in Sec. VI, or is due to a local effect of the classical magnetic vector potential such as suggested by Aharonov and Bohm.[4] All of the experiments suggested below are related to the observed interference pattern shift due to a solenoid.

B. Attempt to Break Down the Interference Pattern with a Large Solenoid Field

If the interference pattern deflection is associated with a classical electromagnetic lag effect such as described in Sec. VI, then one can break down the double-slit electron diffraction pattern by making the relative lag large compared to the coherence length for the electron beam. The single-slit pattern depends upon the coherence of the electron beam across each separate slit and will be unaffected by the solenoid field.

In the Aharonov-Bohm analysis, the phase shift is exactly the same for all electrons which pass the solenoid no matter what their momentum is. The solenoid field may be made arbitrarily large and the double-slit interference pattern will never break down but will simply continue to move through the single-slit envelope.

C. Attempt to Observe Differences in Time of Flight

In the analysis based upon a classical electromagnetic lag effect, the lag due to the solenoid currents may be made sufficiently large as to give a measurable time delay for electrons passing on opposite sides of the small solenoid. In this case the electron beam would have to be chopped into wave packets to allow the time-of-flight measurements. In the Aharonov-Bohm analysis there is no lag effect and hence there will be no difference in the time of flight for electrons passing on opposite sides of the solenoid.

D. Attempts to Observe Changes in Velocity for the Passing Electrons

In the classical lag analysis, the electrons passing the solenoid are actually moving faster on one side and slower on the other. If an electrostatic deflection is arranged, then the deflections should depend upon the particle velocities and any velocity change between the two sides of the solenoid should be made apparent. No velocity changes are suggested in the Aharonov-Bohm analysis.

IX. CONCLUSION

Shifts in quantum interference patterns can be caused by classical electromagnetic fields. The familiar type 1 shift of the entire pattern is associated with a classical electromagnetic deflecting force. A type 2 shift of only the double-slit pattern can arise in conjunction with a classical lag effect produced by classical electromagnetic fields.

Aharonov and Bohm have suggested the possibility of interference pattern shifts associated with classical electromagnetic potentials in regions free of classical electromagnetic fields, and their view seems widely accepted in the literature. At present there seems no conclusive experimental evidence to support Aharonov's and Bohm's idea of this new role for the classical electromagnetic potentials. The suggestion of a scalar potential effect has not been tested, and the experiments of Chambers and of Möllenstedt and Bayh are ambiguous because they do not rule out an interpretation based upon a classical electromagnetic lag effect. Crucial experimental tests for the Aharonov-Bohm hypothesis seem feasible but have not yet been performed.

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[1]For example, excellent electron diffraction patterns from gratings have been obtained experimentally by C. Jönsson, Z. Phys. 161, 454 (1961).

[2]The recent interest in interference pattern shifts has all been in connection with the Aharonov-Bohm effect. It seems helpful to provide a listing of some of the literature:

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(ii) Experimental Confirmation. F. G. Werner and D. R. Brill, Phys. Rev. Lett. 4, 344 (1960); R. G. Chambers, *ibid.* 5, 3 (1960); G. Möllenstedt, in *Proceedings of the European Regional Conference on Electron Microscopy*, edited by A. L. Hourvink and B. J. Spits (Academic, New York, 1960); H. Boersch, H. Hamisch, K. Grohmann, and D. Wohleben, Z. Phys. 165, 79 (1961); H. Boersch, H. Hamisch, and K. Grohmann, *ibid.* 169, 263 (1962); G. Möllenstedt and W. Bayh, Naturwissenschaften 49, 81 (1962); Phys. Bl. 18, 299 (1962); W. Bayh, Z. Phys. 169, 492 (1962); G. Schaal, C. Jönsson, and E. F. Krimmel, Optik 24, 529 (1966/67); H. Wahl, *ibid.* 28, 417 (1968/69); 30, 508 (1970); 30, 577 (1970). A review of the experimental literature is given by J. Woodilla and H. Schwarz, Am. J. Phys. 39, 111 (1971).

(iii) Theoretical Interpretations. (a) Part of the secondary literature is devoted to a controversy as to whether the Aharonov-Bohm effect should be interpreted as implying that the electromagnetic potentials have a new importance in quantum theory. M. Peshkin, I. Talmi, and L. J. Tassie, Ann. Phys. (N.Y.) 12, 426 (1961); L. Tassie and M. Peshkin, *ibid.* 16, 177 (1961); B. DeWitt, Phys. Rev. 125, 2189 (1962); Y. Aharonov and D. Bohm, *ibid.* 125, 2192 (1962); E. L. Feinberg, Usp. Fiz. Nauk. 78, 53 (1962) [Sov. Phys.—Usp. 5, 753 (1963)]; W. Franz, Z. Phys. 184, 85 (1965); D. Bohm (letter of 1968) printed in review by H. Erlich-

son listed below; L. Jánossy, Acta Phys. 29, 419 (1970).

(b) The question of the consistency with the principle of complementarity is raised by W. H. Furry and N. F. Ramsey [Phys. Rev. 118, 623 (1960)].

(c) A summary appears in the work of R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, Mass., 1964), Vol. 1, Sec. 15-5. A partial review has been made by H. Ehrlichson, Am. J. Phys. 38, 162 (1970). A clarification of some misinterpretations is given by T. H. Boyer, Am. J. Phys. 40, 56 (1972).

(d) Many papers in the secondary literature accept the Aharonov-Bohm interpretation and extend the ideas to new realms—to Yang-Mills field theories, to gravity. D. Wisnivesky and Y. Aharonov, Ann. Phys. (N.Y.) 45, 479 (1967); M. S. Cohen, J. Appl. Phys. 38, 4966 (1967). Gravity: J. S. Dowker, Nuovo Cimento 52B, 129 (1967); G. Papini, *ibid.* 52B, 136 (1967); D. Greenberger, Ann. Phys. (N.Y.) 47, 116 (1968); K. Kraus, *ibid.* 50, 102 (1968).

(e) There is secondary literature which investigates the effect from the point of view of traditional classical electrodynamics or of unconventional new physical ideas. B. Liebowitz, Bull. Am. Phys. Soc. 8, 355 (1963); 9, 401 (1964); 10, 47 (1965); Nuovo Cimento 38, 932 (1965); 46B, 125 (1966); Z. Phys. 207, 20 (1967); G. T. Trammel, Phys. Rev. 134, B1183 (1964); P. Hrasko, Nuovo Cimento 44B, 452 (1966); E. Kasper, Z. Phys. 196, 415 (1966); 207, 24 (1967); T. H. Boyer, preceding paper, Phys. Rev. D 8, 1667 (1973).

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[9]M. Peshkin, S. Talmi, and L. J. Tassie, Ann. Phys. (N.Y.) 12, 426 (1961).

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[12]B. Liebowitz, Nuovo Cimento 38, 932 (1965).

[13]T. H. Boyer, preceding paper, Phys. Rev. D 8, 1667 (1973).

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[15]G. Möllenstedt and W. Bayh, Phys. Bl. 18, 299 (1962).

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Ground State of Atoms and Molecules in a Superstrong Magnetic Field

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In superstrong magnetic fields, of the order of magnitude supposed to exist in neutron stars, atoms are characterized by enormous binding energies, small dimensions, and a very elongated distribution of charge having approximately the shape of a concave-ended cylinder. Their ionization energies have a monotonic dependence on the atomic number, and tend to level off at high Z; the very peculiar distribution of charge favors the formation of negative ions. When the magnetic field increases, nuclei tend to align and share their electrons, forming linear molecules. These features are very different from those of ordinary matter, and may have important astrophysical consequences, e.g., on the emission of charged particles from the surface of a magnetic neutron star, or the structure of its crust.

I. INTRODUCTION

In superstrong magnetic fields, of the order of magnitude supposed to exist at the surface of neutron stars (10^{12}–10^{13} G), the cyclotron frequency of an electron becomes much larger than its Bohr frequency in a hydrogenlike ion of not too high an atomic number Z. The transverse motion (with respect to the magnetic field) of the electron bound in such an ion is then determined essentially by the magnetic field alone; the Coulomb field of the nucleus binds the electron along the magnetic lines of force. In this "adiabatic approximation" the electron moves in an one-dimensional Coulomb potential, truncated at some small distance of the order of the cyclotron radius of