The two-capacitor problem with radiation

Timothy B. Boykin, Dennis Hite, and Nagendra Singh
Department of Electrical and Computer Engineering, The University of Alabama in Huntsville, Huntsville, Alabama 35899

(Received 12 June 2001; accepted 8 November 2001)

We discuss the two-capacitor problem found in many introductory physics texts in which there appears to be missing energy in an ideal, zero-resistance circuit, following the sudden charging of one capacitor from another. The paradox of this missing energy is traditionally ascribed to finite-resistance wires, the initial assumption of an ideal circuit and the rapid nature of the charging notwithstanding. By treating radiative effects in the simplest approximation, we show that the paradox is really nothing more than an inappropriately applied lumped-parameter model. In particular, we show that in the zero-resistance circuit, radiation fully accounts for all of the energy lost. To explore radiative effects in more realistic circuits, we also discuss numerical examples that include a small resistance and inductance.
[DOI: 10.1119/1.1435344]

I. INTRODUCTION

The two-capacitor problem, in which an initially charged ideal capacitor is suddenly connected to an initially discharged ideal capacitor by an ideal wire (see Fig. 1), is found in many introductory physics texts.[1] The paradox of the problem centers around the missing energy. Specifically, for initial charges of (Q_1+Q_2) on C_1 and zero on C_2, we find, after closing the switch, charge Q_1 on C_1 and charge Q_2 on C_2 (with $Q_1/C_1=Q_2/C_2$), so that there is a missing or lost energy,

$$U_{\text{lost}}=U_{\text{initial}}-U_{\text{final}}=\frac{(Q_1+Q_2)^2}{2C_1}-\left(\frac{Q_1^2}{2C_1}+\frac{Q_2^2}{2C_2}\right)$$
$$=\frac{Q_2^2}{2}\frac{C_1+C_2}{C_1C_2}. \tag{1}$$

The customary explanation for the missing energy is that the wires making up the circuit are not ideal, and the energy is dissipated in their very small resistance. More recent treatments[2,3] have taken into account the self-inductance of the wires connecting the capacitors as well.

Although it is certainly true that the wires in a conventional circuit do have a small resistance, this explanation, which contradicts the initial assumption of an ideal circuit, as well as more recent discussions,[2,3] does not really answer the underlying question, for they all neglect radiation. Radiation is crucial to this problem because Kirchoff's voltage law (KVL), on which a lumped-parameter circuit description is based, *cannot* hold at high frequencies where it clearly contradicts Faraday's law. Without some lumped-parameter element included to model radiation, the circuit of Fig. 1 does not correctly represent the physical process of discharging and charging capacitors. This inadequacy should be immediately apparent, for the lumped-parameter description of Fig. 1 implies instantaneous charging/discharging, for which KVL cannot hold. After all, radiation does dissipate energy. From a pedagogical perspective, an investigation of radiative losses in the two-capacitor problem is therefore useful and, given the recent proliferation of wireless devices such as pagers and cellular telephones, timely.

Our approach will be to treat the radiating circuit in the simplest possible manner, as a magnetic dipole, from which we shall find an expression for the radiated power. We will then show that the radiation may be taken into account by a special, nonlinear resistor, permitting a lumped-parameter treatment with this new element to model radiative losses. That is, the KVL-violating radiation is confined to the new element, much as a conventional inductor contains all KVL violations in a simple *RLC* circuit. We will then apply this model to the ideal two-capacitor problem for which analytical solutions are available, as well as some more realistic examples of *RLC* circuits for which only numerical results are possible. In the former case we shall show that radiative losses alone prevent instantaneous charging/discharging and fully account for all of the missing energy. In the latter case, we will examine the point at which radiation becomes important in circuits with small, explicit inductances and resistances.

The paper is organized as follows. In Sec. II we give our treatment of radiation in the ideal two-capacitor problem. In Sec. III we present the more realistic (that is, nonideal), numerical results, and in Sec. IV we discuss our conclusions.

II. RADIATION IN THE IDEAL TWO-CAPACITOR CIRCUIT

To model radiation in the simplest two-capacitor circuit (Fig. 1), in which the resistance and self-inductance of the circuit are neglected, we treat the radiating circuit as a magnetic dipole,[4] but leave the time dependence of the current unspecified (instead of assuming a sinusoidal form). We do not consider electric dipole radiation because to do so would necessitate delving into the details of the nature of the capacitors. We want to show that even with ideal capacitors, treated as lumped-parameter elements, there must still be a power loss due to radiation. Instead of including these details, we concentrate on the fundamental physics and therefore also assume that the current is a function of time only (and not the spatial coordinates). As shown in Fig. 2, we model our radiating circuit as a current loop of radius b centered at the origin. We measure the fields at point P (a distance $\mathbf{r}$ from the origin) and assume the loop to be small. That is, we take $2\pi b/c \ll \tau$, where c is the speed of light and τ is the characteristic time scale of the current, $\dot{I}(t) \sim I(t)/\tau$, $\ddot{I}(t) \sim I(t)/\tau^2$, etc., and we work in the far-field

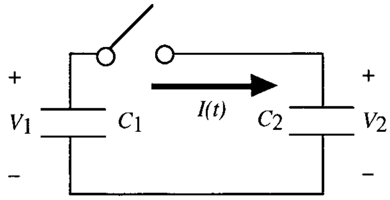

Fig. 1. The ideal two-capacitor circuit. Initially C_1 is charged and C_2 discharged, and the switch is closed at $t=0$. The wires connecting the capacitors have zero resistance.

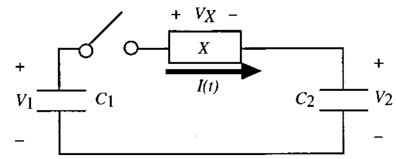

Fig. 3. Two-capacitor circuit with an equivalent lumped-parameter element, X, to model power loss due to radiation. The wires are still assumed to have zero resistance, so X is the only dissipative element in the circuit. The switch is closed at $t=0$.

region, $\tau \ll r/c$. For a 5 cm loop the assumption of uniform current implies that our treatment will be good for switching frequencies, $s=\tau^{-1}$, considerably less than $c/(2\pi b)$ $\approx 10^9\ \mathrm{s}^{-1} = 1\ (\mathrm{ns})^{-1}$.

Here we sketch our derivation; further details are given in Appendix A. In the expression for the retarded vector potential,[4]

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{I(t-\mathscr{r}/c)}{\mathscr{r}}[-\sin(\varphi')\mathbf{e}_x + \cos(\varphi')\mathbf{e}_y]b\,d\varphi', \quad \mathscr{r}=|\mathbf{r}-\mathbf{r}'| \tag{2}$$

at field point $\mathbf{r}$ and source point $\mathbf{r}'$, we expand the current to first-order in b/r,

$$\frac{1}{\mathscr{r}}I\left(t-\frac{\mathscr{r}}{c}\right)\approx\frac{1}{r}I\left(t-\frac{r}{c}\right) + \frac{b\sin(\theta)}{rc}\cos(\varphi-\varphi')\dot{I}\left(t-\frac{r}{c}\right), \tag{3}$$

where

$$\mathbf{r}=r\cos(\varphi)\sin(\theta)\mathbf{e}_x+r\sin(\varphi)\sin(\theta)\mathbf{e}_y+r\cos(\theta)\mathbf{e}_z, \tag{4}$$

$$\mathbf{r}'=b\cos(\varphi')\mathbf{e}_x+b\sin(\varphi')\mathbf{e}_y. \tag{5}$$

From Eq. (2) we obtain the far-field expressions for $\mathbf{E}$ and $\mathbf{B}$ in spherical coordinates:

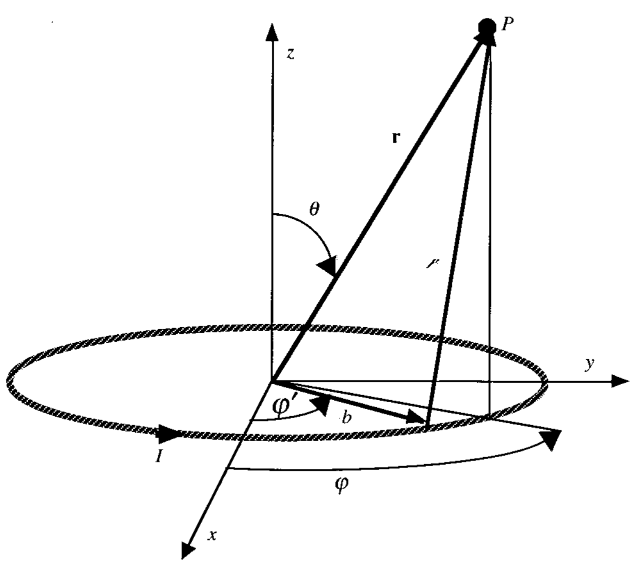

Fig. 2. Radiating loop of radius b (see text). The loop is located at the origin, far from the field point, P.

$$\mathbf{E}(\mathbf{r},t)\approx-\frac{\mu_0}{4}\frac{b^2\sin(\theta)}{rc}\ddot{I}\left(t-\frac{r}{c}\right)\mathbf{e}_\varphi, \tag{6}$$

$$\mathbf{B}(\mathbf{r},t)\approx\frac{\mu_0}{4}\frac{b^2\sin(\theta)}{rc^2}\ddot{I}\left(t-\frac{r}{c}\right)\mathbf{e}_\theta. \tag{7}$$

If we integrate the Poynting vector $\mathbf{S}=(\mathbf{E}\times\mathbf{B})/\mu_0$ over the spherical shell at r, we find the radiated power,

$$P_{\mathrm{rad}}=\frac{\pi b^4}{6\varepsilon_0 c^5}\left[\ddot{I}\left(t-\frac{r}{c}\right)\right]^2. \tag{8}$$

Equation (8) allows us to define an equivalent, nonlinear, lumped-parameter element that we shall call X, having the same power dissipation as the radiation:

$$V_X=\frac{P_{\mathrm{rad}}}{I}=K\frac{\ddot{I}^2}{I}, \quad K=\frac{\pi b^4}{6\varepsilon_0 c^5}. \tag{9}$$

We may then use this element in a lumped-parameter circuit to account for radiation (Fig. 3). We first write KVL for this circuit,

$$(V_2-V_1)+V_X=0, \tag{10}$$

and then note that the two capacitors may be combined into an equivalent series capacitance, C_s, by expressing each capacitor voltage in integral form via $I(t)=-C_1\dot{V}_1=C_2\dot{V}_2$,

$$V_c=(V_2-V_1)=-V_{1,0}+\frac{1}{C_s}\int_0^t I(t')dt', \tag{11a}$$

which implies that

$$I(t)=C_s\dot{V}_s, \quad C_s=\frac{C_1C_2}{C_1+C_2} \tag{11b}$$

for an initial voltage $V_{1,0}$ on C_1. Using Eqs. (9) and (11) to rewrite Eq. (10) in terms V_c, the equivalent series capacitor voltage, we obtain the nonlinear differential equation,

$$\ddot{V}_c^2+\frac{1}{KC_s}\dot{V}_cV_c=0 \tag{12}$$

which, rather surprisingly, has an analytical solution

$$V_c=Ae^{st}, \tag{13a}$$

where the eigenvalue equation

$$s\left[s^5+\frac{1}{KC_s}\right]=0 \tag{13b}$$

is found by substituting Eq. (13a) into Eq. (12). If we ignore the trivial case $(s=0)$, we have the nontrivial solutions,

$$s=\left(\frac{1}{KC_s}\right)^{1/5}e^{i(2n+1)\pi/5} \quad (n=0,1,2,3,4). \tag{14}$$

Observe that Eq. (12) is a nonlinear equation so that a linear combination of its solutions is *not* generally a solution (see Appendix B). We must therefore choose *one* of the solutions, Eq. (14), of which $n=2$ is the only physically admissible one, because the others give *complex* capacitor voltages. Thus, if we enforce the initial condition from Eq. (11), we obtain

$$V_c(t) = -V_{1,0} e^{st}, \tag{15a}$$

with

$$s = -\left(\frac{1}{KC_s}\right)^{1/5}. \tag{15b}$$

We emphasize that Eq. (15) demonstrates the finite time scale of the charging/discharging process. Contrary to the lumped-parameter description without radiation, the charging/discharging is *not* instantaneous, but rather is limited by the radiation resistance.

Using Eq. (15) we can readily show that the radiation accounts for all of the missing energy. From Eqs. (15) and (11), we find the current, which, with Eq. (8), gives the radiated power. If we integrate the power crossing the spherical shell at r from $t=r/c$ (the arrival time of the switching signal) to infinity, we have

$$W_{\text{rad}} = \int_{r/c}^{\infty} P_{\text{rad}}\, dt = K \int_{r/c}^{\infty} \left(\frac{1}{KC_s}\right) C_s^2 V_{1,0}^2 \left(\frac{1}{KC_s}\right)^{1/5} \times \exp\left[-2\left(\frac{1}{KC_s}\right)^{1/5}\left(t - \frac{r}{c}\right)\right] dt, \tag{16}$$

$$W_{\text{rad}} = \frac{1}{2} C_s V_{1,0}^2 = \frac{1}{2} \frac{C_1 C_2}{C_1 + C_2} V_{1,0}^2. \tag{17}$$

On the other hand, from the initial and final states of the circuit,

$$V_{1,0}^2 = \left(\frac{Q_1 + Q_2}{C_1}\right)^2 = Q_2^2\left(\frac{1}{C_1} + \frac{1}{C_2}\right)^2 = Q_2^2\left(\frac{C_1 + C_2}{C_1 C_2}\right)^2, \tag{18}$$

so that from Eq. (1),

$$U_{\text{lost}} = \frac{Q_2^2}{2} \frac{C_1 + C_2}{C_1 C_2} = \frac{V_{1,0}^2}{2} \frac{C_1 C_2}{C_1 + C_2} = W_{\text{rad}}. \tag{19}$$

Hence, even in the simplest treatment, radiation accounts for all of the lost energy and the paradox is explained *without* invoking finite-resistance wires.

III. *RLC* CIRCUIT: NUMERICAL EXAMPLES

It is instructive to consider a more realistic, nonideal, circuit having in addition to the two capacitors and radiation resistance, a real resistor, R, and a real self-inductance, L (see Fig. 4). Collapsing the two capacitors into a single series equivalent so that $I(t) = C_s \dot{V}_c$ and applying KVL around the loop with V_X given by Eq. (9), we find the nonlinear differential equation

$$\dddot{V}_c^2 + \frac{L}{K} \ddot{V}_c \dot{V}_c + \frac{R}{K} \dot{V}_c^2 + \frac{1}{KC_s} \dot{V}_c V_c = 0, \tag{20}$$

which has exponential solutions as in Eq. (13a) yielding a corresponding characteristic polynomial

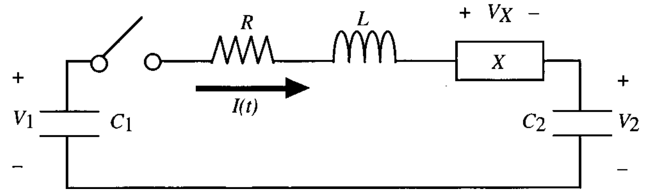

Fig. 4. A more realistic *RLC* circuit where *L* models the self-inductance of the loop and *R* models the resistance of all nonideal elements (wires, inductor, or capacitors). The switch is closed at $t=0$.

$$s^5 + \frac{L}{K} s^2 + \frac{R}{K} s + \frac{1}{KC_s} = 0. \tag{21}$$

All quantities have the same meanings as in Sec. II and as before, we omit the trivial case $s=0$. In our discussions it will be useful to distinguish some limiting cases: the *RLC* circuit,

$$L\ddot{V}_c + R\dot{V}_c + \frac{1}{C_s} V_s = 0, \tag{22a}$$

$$s^2 + \frac{R}{L} s + \frac{1}{LC_s} = 0, \tag{22b}$$

the nonlinear, radiation-limited *RL* circuit,

$$\dddot{V}_c^2 + \frac{L}{K} \ddot{V}_c \dot{V}_c = 0, \tag{23a}$$

$$s^3 + \frac{L}{K} = 0, \tag{23b}$$

and the simple *RC* circuit,

$$\dot{V}_c + \frac{1}{RC_s} V_c = 0, \tag{24a}$$

$$s + \frac{1}{RC_s} = 0, \tag{24b}$$

where the characteristic polynomial Eq. (23b) again omits the trivial $s=0$ case. We shall also find useful the concept of equivalent radiation resistance, $V_X = R_{\text{rad}} I$. Because the exponential solutions are as in Eq. (13a), Eq. (9) indicates that in the (generalized) frequency domain, the physically meaningful definition is

$$R_{\text{rad}} = K s^4. \tag{25}$$

Observe that this form is the same as for the radiation resistance of a sinusoidially driven magnetic dipole, $R_{\text{rad}} \sim (kb)^4$, where k is the wave number.[4] Finally, note that because the characteristic polynomial, Eq. (21), is fifth order, only numerical solutions are generally available (unlike the simple, trivially factorable, fifth-order polynomial of Sec. II).

As a first case, consider a simplified version of Fig. 4, a radiating *RC* circuit, with characteristic polynomial given by Eq. (21) with $L=0$. The loop radius is taken to be 5 cm and the series capacitance is 100 μF. In Fig. 5 we plot both the decay constant, s, and the ratio of radiation resistance to real resistance (which also is the ratio of radiated power to resistor power) versus the real resistance, R. Observe that for $R \geqslant 0.001\Omega$, $s \approx -1/RC$, that is, the usual *RC* decay. In this limit the decay rate is so slow that the nonlinear (that is,

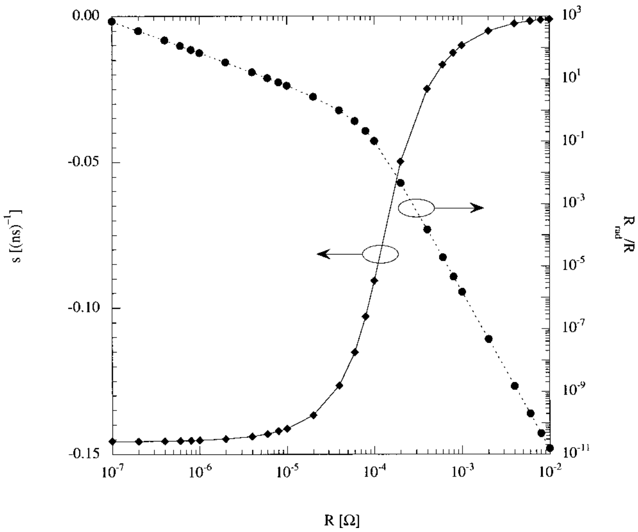

Fig. 5. Plot of the decay constant, s (linear scale), and the ratio of radiation resistance to real resistance (log scale) versus the real resistance, R, for an RC circuit; the loop radius is 5 cm and the series capacitance is 100 μF. Arrows indicate the vertical axis which is to be read for each plot. Note that the resistance ratio is also the ratio of radiated power to power dissipated by the real resistor. For larger R the circuit becomes essentially a conventional RC circuit, while for smaller R the radiation limits the decay rate.

radiative) term of Eq. (20) or Eq. (21) contributes little. At the opposite extreme, for $R \leqslant 1\ \mu\Omega$, the decay saturates out at its radiation-limited value, $s \approx -0.14\ (\text{ns})^{-1}$. [For our purpose of discerning qualitative behavior, $|s|$ is sufficiently smaller than $c/(2\pi b) \approx 1\ (\text{ns})^{-1}$ so that our approximations hold.] Perhaps the most important qualitative feature of the graph in Fig. 5 is that it shows that the switching does not happen instantaneously, because including the radiative power loss results in a nonlinear RC circuit.

Three other interesting cases involve radiating RLC circuits, for which the roots of the characteristic equation themselves are of the most importance. It is especially informative to see how the solutions of Eq. (21) can exhibit behavior characteristic of simpler circuits. Consider first the case of a 5 cm loop with $R=0$, $L=1\ \mu$H, $C=1\ \mu$F; the roots of Eq. (21) are

$$s=\{-18.712, -7.63\times 10^{-17} \pm i0.001, 9.36 \pm i16.20\}\ \ (\text{ns})^{-1}. \tag{26}$$

The large-magnitude complex-conjugate pair and the large-magnitude real root are approximate solutions of the nonlinear, radiation-limited RL circuit, Eq. (23). The complex-conjugate pair is physically inadmissible (each solution is complex) because Eq. (23) is nonlinear. The approximate real solution is not excited in our examples because we consider the case with nonzero initial capacitor voltage and zero initial inductor current. The small-magnitude complex-conjugate pair is simply the solution for a conventional RLC circuit with a resistance given by $R_{\text{rad}} = Ks^4 \approx K(1/\sqrt{LC})^4$. In other words, because the resulting circuit is severely underdamped, the larger (generalized) frequency is the oscillation, not the decay of the envelope, and the larger oscillation determines the radiation resistance. Because even the fastest phenomenon (here the oscillation) is rather slow, the nonlinear term is negligible, and we recover the limit of the simple RLC circuit, Eq. (22).

Another interesting case is that of a 10 cm loop with $R=10\ \mu\Omega$, $L=1$ nH, $C=1\ \mu$F, for which the roots are

$$s=\{-0.743, -6.22\times 10^{-6} \pm i3.16\times 10^{-2}, 0.372 \pm i0.643\}\ \ (\text{ns})^{-1}. \tag{27}$$

As before we see the inadmissible, complex solution pair, and the real root from the radiation-limited RL circuit. Here, while the circuit is again highly underdamped (the oscillation $1/\sqrt{LC}$ largely determines s), the radiation and real resistances are comparable, $R_{\text{rad}} \approx K(1/\sqrt{LC})^4 \approx 2.44\times 10^{-6}\ \Omega$, so that the decay constant is given by $-(R+R_{\text{rad}})/(2L)$. (Note the very small values of R and L needed to achieve comparable radiation and real resistances.) If in this same loop we instead use $R=10\ \Omega$, $L=1$ nH, $C=1\ \mu$F, we recover the usual RC circuit, Eq. (24), because the roots are

$$s=\{-1.00\times 10^{-4}, -1.01 \pm i0.955, 1.01 \pm i1.06\}\ \ (\text{ns})^{-1}, \tag{28}$$

where we see that the real solution corresponds quite closely to $1/RC$. (The radiation resistance here is truly negligible and the conventional lumped-parameter treatment is fully adequate.)

IV. CONCLUSIONS

Motivated by the realization that a conventional, lumped-parameter description is inadequate for any circuit with rapidly varying currents and voltages, we have added a simplified treatment of radiation losses to the two-capacitor problem found in many introductory physics and electromagnetism texts. With our model we have shown that power loss due to radiation prevents instantaneous charging/discharging and in fact accounts for all of the missing energy in the textbook problem, thus resolving the paradox *without* invoking nonideal, finite-resistance wires. Finally, we have also investigated some more realistic circuits having small resistances and self-inductances along with the radiative equivalent circuit element, and have recovered the correct lumped-parameter (that is, low-frequency) limits.

APPENDIX A: DETAILED DERIVATION OF THE RADIATED POWER

We begin with the expression for the retarded vector potential,[4]

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_0^{2\pi} \frac{I(t-\mathscr{r}/c)}{\mathscr{r}}[-\sin(\varphi')\mathbf{e}_x + \cos(\varphi')\mathbf{e}_y] b\, d\varphi', \quad \mathscr{r} = |\mathbf{r}-\mathbf{r}'| \tag{A1}$$

at field point $\mathbf{r}$ and source point $\mathbf{r}'$. Next, we find the expansions for $\mathscr{r}$ and $1/\mathscr{r}$ to first-order in b/r using Eqs. (4) and (5):

$$\begin{aligned}\mathscr{r} &= \sqrt{r^2 - 2\mathbf{r}\cdot\mathbf{r}' - r'^2} \\ &= r\sqrt{1 - \frac{2b}{r}\sin(\theta)\cos(\varphi-\varphi') + b^2} \\ &\approx r - b\sin(\theta)\cos(\varphi-\varphi'),\end{aligned} \tag{A2}$$

$$\frac{1}{\mathscr{r}} \approx \frac{1}{r} + \frac{b}{r^2}\sin(\theta)\cos(\varphi-\varphi'), \tag{A3}$$

so that to first order in b/r,

$$\frac{1}{\not r}I\left(t-\frac{\not r}{c}\right)\approx\frac{1}{r}I\left(t-\frac{r}{c}\right)+\frac{b\sin(\theta)}{rc}\cos(\varphi-\varphi')\dot{I}\left(t-\frac{r}{c}\right)$$
$$+\frac{b}{r^2}I\left(t-\frac{r}{c}\right)\sin(\theta)\cos(\varphi-\varphi'). \quad \text{(A4)}$$

In the far-field region the propagation delay to the field point is much longer than the switching time, τ, that is, $\dot{I}(t)\sim I(t)/\tau$ with $\tau \ll r/c$. The last term of Eq. (A4) is negligible compared to the second, because its size is b/r^2 and that of the second is $b/(rc\tau)$; thus their ratio is $c\tau/r\ll 1$. If we drop the last term and substitute Eq. (A4) into Eq. (A1), we have in spherical coordinates

$$\mathbf{A}(\mathbf{r},t)\approx\frac{\mu_0}{4}\frac{b^2\sin(\theta)}{rc}\dot{I}\left(t-\frac{r}{c}\right)\mathbf{e}_\varphi. \quad \text{(A5)}$$

Equations (6) and (7) follow directly from using Eq. (A5) in $\mathbf{E}=-\partial\mathbf{A}/\partial t$ and $\mathbf{B}=\nabla\times\mathbf{A}$, retaining only the slowly decaying terms of order $1/r$. From these equations we obtain the Poynting vector

$$\mathbf{S}(\mathbf{r},t)=\frac{1}{\mu_0}\mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)$$
$$=\frac{1}{16\varepsilon_0}\frac{b^4}{r^2c^5}\left[\sin(\theta)\ddot{I}\left(t-\frac{r}{c}\right)\right]^2\mathbf{e}_r, \quad \text{(A6)}$$

which, when integrated over the spherical shell at radius r, yields the power *radiated* by the loop

$$P_{\text{rad}}=\int_0^{2\pi}\int_0^{\pi}\mathbf{S}\cdot\mathbf{e}_r r^2\sin(\theta)d\theta\, d\varphi=\frac{\pi b^4}{6\varepsilon_0 c^5}\left[\ddot{I}\left(t-\frac{r}{c}\right)\right]^2. \quad \text{(A7)}$$

We emphasize that Eq. (A7) refers to the power *radiated* (by the loop), while the usual expression

$$P=-\oint_{\Delta V}\mathbf{S}\cdot d\mathbf{a} \quad \text{(A8)}$$

refers to the power *absorbed* (by the charges within a volume V bounded by the closed surface ΔV).

APPENDIX B: NONLINEAR EQUATIONS AND COMPLEX-CONJUGATE PAIR SOLUTIONS

Our nonlinear homogeneous differential equations all have real coefficients so it follows that their solutions are either real functions or complex-conjugate pairs, because if V is a solution so is V^*. Although a *single* complex solution is obviously inadmissible as *real* capacitor voltage, it might be argued that a *real* linear combination of complex-conjugate solutions could be taken as a solution of a nonlinear differential equation. (Were this the case, it would imply that the real and imaginary parts of a complex solution were *separately* solutions of the nonlinear differential equation.) Here we show for the two nonlinear equations which admit analytical solutions, Eqs. (12) and (23a), this is *not* the case, so that complex solutions must be discarded altogether.

Direct substitution shows that Eqs. (12) and (23a) admit analytical solutions of the form

$$V_c(t)=A\exp[s_n t]. \quad \text{(B1)}$$

The solutions of Eq. (12) are

$$s_n=\left(\frac{1}{KC_s}\right)^{1/5}e^{i(2n+1)\pi/5}\quad(n=0,1,2,3,4) \quad \text{(B2)}$$

with complex-conjugate pairs $n=0,4$ and $n=1,3$. The solutions of Eq. (23) are

$$s_n=\left(\frac{L}{K}\right)^{1/3}e^{i(2\pi+1)\pi/3}\quad(n=0,1,2) \quad \text{(B3)}$$

with complex-conjugate pair $n=0,2$. The pairings follow by shifting phases by 2π:

$$e^{i9\pi/5}=e^{-i\pi/5}e^{i10\pi/5}=e^{-i\pi/5}e^{i2\pi}=e^{-i\pi/5}, \quad \text{(B4)}$$

$$e^{i7\pi/5}=e^{-i3\pi/5}, \quad \text{(B5)}$$

$$e^{i5\pi/3}=e^{-i\pi/3}. \quad \text{(B6)}$$

An arbitrary complex solution $V_\pm(t)$ of Eq. (12) or (23a) therefore has the form

$$V_\pm(t)=Ae^{\pm i\beta}\exp[s_\pm t],\quad V_-(t)=V_+^*(t), \quad \text{(B7)}$$

$$s_\pm=\varphi\pm i\omega,\quad s_-=s_+^*, \quad \text{(B8)}$$

where the constants A, β, σ, and ω are real and s_+ is, respectively, given by Eq. (B2) with $n=0$, 1 or Eq. (B3) with $n=0$.

We next construct a general real candidate solution from Eq. (B7) by taking a linear combination:

$$V_c(t)=V_+(t)+V_-(t), \quad \text{(9)}$$

where the generality follows from the fact that A and β are arbitrary. It is useful to write the derivatives of Eq. (B9) in two different forms

$$\frac{d^nV_c}{dt^n}=\frac{d^nV_+}{dt^n}+\frac{d^nV}{dt^n}=(s_+)^nV_+(t)+(s_+^*)^nV_+^*(t). \quad \text{(B10)}$$

Using this expression, we find for the first term of either Eq. (12) or Eq. (23a)

$$\dddot{V}_c^2=\dddot{V}_+^2+\dddot{V}_-^2+2\dddot{V}_+\dddot{V}_-=\dddot{V}_+^2+\dddot{V}_-^2+2A^2e^{\sigma t}|s_+|^6. \quad \text{(B11)}$$

If we substitute Eqs. (B8)–(B11) into Eq. (12), we find

$$\dddot{V}_c^2+\frac{1}{KC_s}\dot{V}_cV_c=\left\{\dddot{V}_+^2+\frac{1}{KC_s}\dot{V}_+V_+\right\}$$
$$+\left\{\dddot{V}_-^2+\frac{1}{KC_s}\dot{V}_-V_-\right\}+A^2e^{2\sigma t}\left(\frac{1}{KC_s}\right)$$
$$\times\left[2\left(\frac{1}{KC_s}\right)^{1/5}+s_++s_+^*\right]. \quad \text{(B12)}$$

Now because V_+ and V_- are separately solutions of Eq. (12), it follows that the terms in curly braces above are each zero. Further simplifying Eq. (B12), we conclude that a real linear combination of complex-conjugate solutions of Eq. (12) is *not* itself a solution because

$$\dddot{V}_c^2+\frac{1}{KC_s}\dot{V}_cV_c=2A^2e^{2\sigma t}\left(\frac{1}{KC_s}\right)^{6/5}\left[1+\cos\left(\frac{2n+1}{5}\pi\right)\right]$$
$$\neq 0\quad(n=0,1). \quad \text{(B13)}$$

Therefore only the real solution, $n=2$ in Eq. (B2), is allowed. Similarly, following the procedures above for Eq. (23a) leads to

$$\dddot{V}_c^2 + \frac{L}{K}\ddot{V}_c\dot{V}_c = \left\{ \dddot{V}_+^2 + \frac{L}{K}\ddot{V}_+\dot{V}_+ \right\} + \left\{ \dddot{V}_-^2 + \frac{L}{K}\ddot{V}_-\dot{V}_- \right\} + 3A^2 e^{2\sigma t}\left(\frac{L}{K}\right)^2, \tag{B14}$$

where once again the terms in curly braces are zero. As above, we see that a real linear combination of complex-conjugate solutions is *not* itself a solution because

$$\dddot{V}_c^2 + \frac{L}{K}\ddot{V}_c\dot{V}_c = 3A^2 e^{2\sigma t}\left(\frac{L}{K}\right)^2 \neq 0. \tag{B15}$$

Thus, only the real solution, $n=1$ in Eq. (B3), is allowed.

[1]See, for example, D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics*, 4th ed. (Wiley, New York, 1993), p. 750.

[2]R. A. Powell, ''Two capacitor problem: A more realistic view,'' Am. J. Phys. **47**, 460–462 (1979).

[3]K. Mita and M. Boufaida, ''Ideal capacitor circuits and energy conservation,'' Am. J. Phys. **67**, 737–739 (1999).

[4]D. J. Griffiths, *Introduction to Electrodynamics*, 2nd ed. (Prentice-Hall, Englewood Cliffs, NJ, 1989), pp. 396–399, 407–411. (Griffiths assigns the general magnetic dipole derived here as Problem 9.14 but does not treat it in the text.)