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Radiation Damping in Classical Electrodynamics

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The problem of radiation from a charged particle in uniform acceleration is considered. It is shown that, by transforming from an inertial frame to the accelerated frame in which the particle is at rest, the magnetic field, and hence the radiation, is transformed away. Exact solutions of Maxwell's equations in the accelerated frame are obtained.

Dirac's classical equations of motion of a charged particle are rederived for the special case of one-dimensional motion. By doing the calculation in the permanent (noninertial) rest frame of the charged particle, it is shown that it is not necessary to use advanced fields as was done by Dirac. The calculation requires no modification of the energy-momentum tensor. The theory still contains a divergent term (as does Dirac's) but in a modified form.

A simple solution for one-dimensional motion is considered. By consideration of a case where an electron enters from a region of no field, passes through the region of the field and then again into a region of no field, it is shown that the conventional power radiation formula gives the same answer for the total power radiated as does Dirac's equation.

Hyperbolic motion is considered as a limiting case of motion through a finite region of space where there is a uniform electric field. The region where the field is defined is then allowed to become large compared to the electron radius.

I. INTRODUCTION

The most notable derivation of the equations of motion of a charged particle that include the effects of radiation damping is that of Dirac (*1*). Dirac used both retarded and advanced fields in calculating the flow of energy and momentum in the field. In this paper, it will be shown that the equations of motion can be obtained without the use of advanced fields by performing the calculation in the permanent rest frame of the particle. This is taken as an accelerated frame relative to which the particle is always at rest rather than an instantaneous rest frame. Since the advanced fields can be eliminated, an alternate interpretation of the equations of motion to that offered by Wheeler and Feynman (*2*) is possible. Due to the mathematical complications of using an accelerated frame, only one-dimensional motion is considered.

The problem of radiation from a uniformly accelerated charge is also considered. It is commonly accepted that an accelerated charge radiates energy;

in particular this is so when the acceleration is uniform and, indeed, Fulton and Rohrlich (*3*) derive the conventional radiation formula for this special case. There remains the question of a charged particle which is at rest in a permanent gravitational field. It clearly does not radiate. These two situations seem paradoxical. It is shown in this paper that radiation which is observed in an inertial frame relative to which a particle is uniformly accelerating is completely transformed away when a transformation is made to the permanent rest frame of the particle.

II. ACCELERATED FRAMES OF REFERENCE

In the treatment of the problem of an accelerated charged particle, it proves fruitful to consider it at rest at the origin of an accelerated frame. A transformation from an inertial to an accelerated frame with especially simple properties will be outlined. Since most of this material can be found in sections 96 and 97 of the book by Moller (*4*), only an outline of the results needed in this paper will be presented. The relativistic notation used is as follows: Greek letters run from 1 to 3, Latin from 1 to 4. The metric tensor has the diagonal form $(+1, +1, +1, -1)$. The accelerated frame will be denoted by R, the inertial frame by I. Only one-dimensional motion of R relative to I is considered. Generally, capital letters are employed for coordinates in I and small letters in R. Imaginary notation is not used. Thus $X^i = (X, Y, Z, cT)$ and $x^i = (x, y, z, ct)$. A dot will always be used for differentiation with respect to the proper time of the particle located at the origin of R.

It is convenient to introduce the function θ defined by

$$\tanh\theta = \frac{v}{c}, \qquad \sinh\theta = \frac{v}{c\sqrt{(1 - v^2/c^2)}}, \qquad \cosh\theta = \frac{1}{\sqrt{(1 - v^2/c^2)}} \tag{1}$$

where v is the velocity of the particle. Let $f^1(\tau)$ and $f^4(\tau)$, where τ is the proper time, denote the world line of the particle. The transformation is defined by means of

$$X = x\cosh\theta + f^1(\tau) \tag{2}$$

$$cT = x\sinh\theta + f^4(\tau) \tag{3}$$

with the auxiliary condition that the coordinate time t in R be the same as the proper time of the particle: $t = \tau$. Since the particle four velocity is

$$U^i = (c\sinh\theta, 0, 0, c\cosh\theta) \tag{4}$$

we can write (2) and (3) as

$$X = x\cosh\theta + \int_0^t c\sinh\theta\,dt \tag{5}$$

$$cT = x\sinh\theta + \int_0^t c\cosh\theta\,dt \tag{6}$$

The function θ depends only on t. The transformation of the differentials dx^i and dX^i are given by

$$dX = dx\cosh\theta + cdt(1 + x\dot\theta/c)\sinh\theta \tag{7}$$

$$cdT = dx\sinh\theta + cdt(1 + x\dot\theta/c)\cosh\theta \tag{8}$$

The inverse is

$$dx = \cosh\theta\,dX - \sinh\theta\,cdT \tag{9}$$

$$cdt = -\frac{\sinh\theta\,dX}{1 + x\dot\theta/c} + \frac{\cosh\theta\,cdT}{1 + x\dot\theta/c} \tag{10}$$

The line element in R is

$$ds^2 = dx^2 + dy^2 + dz^2 - (1 + x\dot\theta/c)^2c^2\,dt^2 \tag{11}$$

The accelerated frame is thus time orthogonal. The spatial geometry is Euclidean and independent of time. Such reference frames are called rigid. The four-acceleration of the origin of R (hereafter denoted by O) relative to I is found from (4):

$$A^i = (g\cosh\theta, 0, 0, g\sinh\theta) \tag{12}$$

where $g = \dot\theta c$. We note that

$$G_{ij}A^iA^j = g^2 \tag{13}$$

where the special relativity metric has been denoted by G_{ij}. The four-acceleration (12) can be transformed to R by use of (9) and (10):

$$a^i = \frac{\partial x^i}{\partial X^k}A^k = (g, 0, 0, 0) \tag{14}$$

The gravitational scalar potential in R is given by

$$\chi(x, t) = g(t)x[1 + g(t)x/(2c^2)] \tag{15}$$

which shows that the gravitational field strength in the vicinity of O is g (*4*, section 92).

The force on the particle located at O necessary to maintain the acceleration is given by

$$F = \frac{d}{dT}(mc\sinh\theta) = mg \tag{16}$$

where m is the rest mass of the particle. Hyperbolic motion, or uniformly accelerated motion, is defined as that motion which is produced by the application of a constant force to the particle. Since the time dependence of the metric enters only through $\dot{\theta}$, we see that for hyperbolic motion, the metric in R is independent of time. If we were to solve Maxwell's equations for a point charge in a frame in which there is no time dependence, we would expect to find no radiation. That this is so is shown explicitly for the special case of the metric (11) when $\dot{\theta}$ is a constant.

The velocity of an arbitrary fixed point of R relative to I is found by setting $dx = 0$ in (7) and (8) and dividing one equation by the other:

$$dX/(cdT) = v/c = \tanh\theta \tag{17}$$

Thus the velocity of all points in R are the same provided that the measurements of velocity are carried out simultaneously in R. These measurements are not simultaneous in I.

A proper time interval of an arbitrary fixed point in R is given by

$$d\tau = dt\left(1 + \frac{gx}{c^2}\right) = dt\left(1 + \frac{x\dot{\theta}}{c}\right) = dT\sqrt{1 - \frac{v^2}{c^2}} \tag{18}$$

Note that at the origin of R, which is the location of the charged particle which we wish to consider, $d\tau = dt$. The dot refers to differentiation with respect to t. Thus the four-velocity and four-acceleration of an arbitrary fixed point of R are given by

$$U^i = (c\sinh\theta, 0, 0, c\cosh\theta) \tag{19}$$

$$A^i = \frac{dU^i}{d\tau} = \left(\frac{g\cosh\theta}{1 + gx/c^2}, 0, 0, \frac{g\sinh\theta}{1 + gx/c^2}\right) \tag{20}$$

Application of the transformations (9) and (10) to the four-vectors (19) and (20) gives

$$u^i = [0, 0, 0, c/(1 + gx/c^2)] \tag{21}$$

$$a^i = [g/(1 + gx/c^2), 0, 0, 0] \tag{22}$$

III. MAXWELL'S EQUATIONS

Maxwell's equations stated in an arbitrary space–time metric are

$$\frac{\partial F_{ik}}{\partial x^j} + \frac{\partial F_{kj}}{\partial x^i} + \frac{\partial F_{ji}}{\partial x^k} = 0 \tag{23}$$

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^k}(\sqrt{-g}F^{ik}) = s^i = \frac{\rho_0 u^i}{c} \tag{24}$$

where F_{ik} is the field tensor and ρ_0 is the charge density of the matter setting up the fields measured in a local rest system of inertia. Heaviside-Lorentz units are employed.

For the case of hyperbolic motion, characterized by the metric (11) with $g = \dot{\theta}c =$ const., it is possible to deduce a generalization of Poisson's equation. Introduction of the potentials

$$F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \tag{25}$$

into (24) gives

$$\nabla \times ([1 + gx/c^2]\nabla \times \mathbf{A}) = 0 \tag{26}$$

$$\nabla^2 A_4 - \frac{g}{c^2 + gx}\frac{\partial A_4}{\partial x} = \left(1 + \frac{gx}{c^2}\right)\rho_0 \tag{27}$$

where $\mathbf{A}$ is a vector denoting the three spatial components of the four potential. The charged matter is assumed to be at rest in R, i.e., its four-velocity is given by (21).

IV. HYPERBOLIC MOTION

It is convenient to introduce the characteristic length $l = c^2/g$. With l so defined, the equation $g = \dot{\theta}c =$ const. gives $\theta = ct/l$. The equations of motion of the particle, i.e., the origin of frame R, are given by

$$X = l\sqrt{1 + (cT/l)^2} - l = l\cosh\theta - l \tag{28}$$

$$cT = l\sinh\theta \tag{29}$$

The four-velocity is

$$U^1 = c^2T/l, \qquad U^4 = c(X/l + 1) \tag{30}$$

The initial conditions chosen are such that $X = 0$ and $v = 0$ at $T = 0$ (Fig. 1).

The solution of Maxwell's equations for the four-potential in the inertial frame I is the familiar result

$$A_r(I) = \frac{e}{4\pi}\frac{U_r}{R_s U^s}, \quad R_k R^k = 0, \quad R^k = X_Q{}^k - X_P{}^k \tag{31}$$

where P refers to the field point and Q is the emission point (Fig. 1) on the particle world line. The fields in the inertial frame are given by Fulton and Rohrlich (3) who, however, use the initial conditions $v = 0$ and $X = l$.

In this paper, we are interested in obtaining the fields in the accelerated frame.

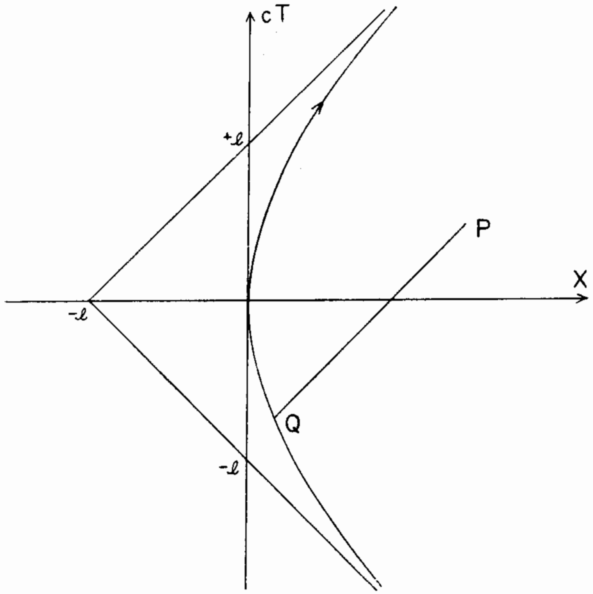

FIG 1. World line of particle in hyperbolic motion

Rather than attempt the solution of (27) directly, we will transform (31) from I to R. The integrated form of the transformation (5) and (6) is

$$X_P = (x + l) \cosh \theta_P - l \tag{32}$$

$$cT_P = (x + l) \sinh \theta_P \tag{33}$$

The constants of integration are chosen such that the equations of motion of the particle (28) and (29) are obtained for $x = 0$. The subscript P denotes the field point. The emission point in R is $x = 0$. The inversion of (32) and (33) gives

$$\begin{aligned} x + l &= \sqrt{(X_P + l)^2 - c^2T_P^2}, \qquad -l < x < \infty \\ &= -\sqrt{(X_P + l)^2 - c^2T_P^2}, \qquad x < -l \end{aligned} \tag{34}$$

$$\sinh \theta_P = \frac{cT_P}{\sqrt{(X_P + l)^2 - c^2T_P^2}}, \qquad \cosh \theta_P = \frac{X_P + l}{\sqrt{(X_P + l)^2 - c^2T_P^2}} \tag{35}$$

from which it is evident that the transformation is singular at $x = -l$ in R and along the asymptotes $X_P + l = \pm cT_P$ in I. Only one-quarter of the X, T plane is covered by the transformation. From (34) we see that if x is allowed to vary over the range of values $-l \leqq x < \infty$, the different points of R have trajectories which are hyperbolas that cover region A (Fig. 2). The transformation

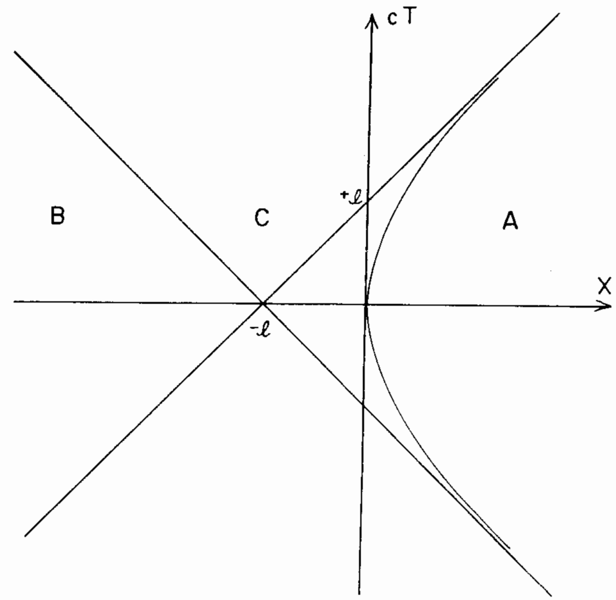

FIG. 2. Region of X, T plane covered by transformation to a uniformly accelerated frame of reference.

for $x < -l$ gives region B in which there are no fields. However, fields appear in C, which is not covered by the transformation. These fields are not accessible to observers in R.

The transformation of the potentials from I to R is fairly straightforward and leads to the result

$$A^1(R) = -\frac{e}{4\pi} \frac{1}{x + l} \tag{36}$$

$$A^4(R) = \frac{el}{4\pi(x + l)^2} \frac{\cosh(\theta_P - \theta_Q)}{\sinh(\theta_P - \theta_Q)} \tag{37}$$

In order to get the potentials expressed entirely in terms of the field point P, we use the causality condition $R_sR^s = 0$. The transformation of this condition into the accelerated frame leads to

$$\cosh(\theta_P - \theta_Q) = \frac{l^2 + \rho^2 + (x + l)^2}{2l(x + l)} \tag{38}$$

$$\sinh(\theta_P - \theta_Q) = \frac{\xi}{2l(x + l)} \tag{39}$$

where

$$\xi^2 = 4l^2\rho^2 + [l^2 - \rho^2 - (x + l)^2]^2 \tag{40}$$

The potential (37) is

$$A^4(R) = \frac{el}{4\pi\xi}\frac{l^2 + \rho^2 + (x + l)^2}{(x + l)^2} \tag{41}$$

The covariant component of A^4 is

$$A_4 = g_{44}A^4 = -\frac{e}{4\pi\xi l}[l^2 + (x + l)^2 + \rho^2] \tag{42}$$

By direct substitution, we can show that (36) and (42) satisfy the differential equations (26) and (27).

The electric and magnetic fields are to be calculated from (25):

$$E_x = F_{14} = \frac{\partial A_4}{\partial x}, \qquad E_y = \frac{\partial A_4}{\partial y}, \qquad E_z = \frac{\partial A_4}{\partial z} \tag{43}$$

$$B_x = B_y = B_z = 0 \tag{44}$$

Expressed in cylindrical coordinates, the electric field is

$$E_x = \frac{e}{4\pi}\frac{4l(x + l)}{\xi^3}[(x + l)^2 - l^2 - \rho^2] = \frac{e}{4\pi}\frac{x}{r^3}\left[1 - \frac{\rho^2}{2xl} + \cdots\right] \tag{45}$$

$$E_\rho = \frac{e}{4\pi}\frac{8\rho l(x + l)^2}{\xi^3} = \frac{e}{4\pi}\frac{\rho}{r^3}\left[1 + \frac{x}{2l} + \cdots\right] \tag{46}$$

The above fields provide an exact solution of Maxwell's equations for a stationary point charge in a uniform gravitational field. For $g = 980$ cm/sec^2, the characteristic length l is about one light year. A direct tensor transformation of the fields given by Fulton and Rohrlich (*3*) leads to identical results. The magnetic field is absent; the terms in the energy-momentum tensor which would be associated with a flow of energy and momentum vanish. There is no radiation in the accelerated frame even though radiation is observed in the inertial frame.

V. ARBITRARY MOTION IN ONE DIMENSION

Assuming an external electric field E in the X-direction of the inertial frame, we will solve simultaneously Maxwell's equations and the equations of motion of the particle. The fields, as well as the motion of the particle, are determined by the single invariant function $\theta(t)$. The electromagnetic field will be obtained in the accelerated frame as an expansion in the function $\theta(t)$, the first order terms appearing as $r\dot{\theta}/c$ and second order terms as $(r\dot{\theta}/c)^2$ or $r\ddot{\theta}/c^2$. By calculation of the flow of energy and momentum from the particle, its equation of motion,

i.e., the function $\theta(t)$, is determined. In order for the theory to be valid, it must be true that $r\dot{\theta}/c \ll 1$ when r is much larger than the classical radius of the electron. The expressions for the fields are needed to second order. Higher order terms give no contribution to the flow of energy and momentum in the limit $r \to 0$.

In the discussion of hyperbolic motion, we assumed at the start a special form for θ, viz., $\dot{\theta}$ = const. It will be shown that this condition occurs as a limiting case when a particle is placed in a constant electric field and the region in which the field is defined is allowed to become infinite.

As in the case of hyperbolic motion, the vector potential can be transformed by the four-vector transformation (9) and (10):

$$A^i = \frac{ec}{4\pi R_s U^s(Q)}\left[-\sinh(\theta_P - \theta_Q), 0, 0, \frac{\cosh(\theta_P - \theta_Q)}{1 + x_P\dot{\theta}_P/c}\right] \tag{47}$$

The integrated transformation equations (5) and (6) can be applied to the field point P and the emission point Q. For $x = 0$, these equations are

$$X_Q = \int_0^{t_Q} c \sinh\theta\, dt \tag{48}$$

$$cT_Q = \int_0^{t_Q} c \cosh\theta\, dt \tag{49}$$

These equations represent the equations of motion of the particle. The quantity R^i, which represents the relative coordinates of emission point and field point in I, is

$$R^1 = X_Q - X_P = -\left(x_P\cosh\theta_P + c\int_Q^P \sinh\theta\, dt\right) \tag{50}$$

$$R^4 = cT_Q - cT_P = -\left(x_P\sinh\theta_P + c\int_Q^P \cosh\theta\, dt\right) \tag{51}$$

Of course, R^i is not a four-vector under the transformation being used here. The causality condition $R_sR^s = 0$ can be expressed

$$\left(x_P\cosh\theta_P + c\int_Q^P \sinh\theta\, dt\right)^2 + \rho^2 - \left(x_P\sinh\theta_P + c\int_Q^P \cosh\theta\, dt\right)^2 = 0 \tag{52}$$

$$\rho^2 = Y^2 + Z^2 = y^2 + z^2$$

For $R_sU^s(Q)$ we find

$$R_sU^s(Q) = x_P c\sinh(\theta_P - \theta_Q) + c^2\int_Q^P \cosh(\theta - \theta_Q)\, dt \tag{53}$$

We define $\delta = t_P - t_Q$. For retarded fields, $\delta > 0$. The vector potential (47) is to be obtained as an expansion in δ, which is then eliminated by use of the causality condition (52). By means of a Taylor's series expansion about the point P, the following results are obtained:

$$\sinh(\theta - \theta_Q) = \dot{\theta}\delta - \tfrac{1}{2}\ddot{\theta}\delta^2 + \tfrac{1}{6}(\dddot{\theta} + \dot{\theta}^3)\delta^3 \tag{54}$$

$$\cosh(\theta - \theta_Q) = 1 + \tfrac{1}{2}\dot{\theta}^2\delta^2 - \tfrac{1}{2}\dot{\theta}\ddot{\theta}\delta^3 \tag{55}$$

$$\int_P^Q \sinh\theta\, dt = -\delta\sinh\theta + \tfrac{1}{2}\dot{\theta}\delta^2\cosh\theta - \tfrac{1}{6}\delta^3(\ddot{\theta}\cosh\theta + \dot{\theta}^2\sinh\theta) \tag{56}$$

$$\int_P^Q \cosh\theta\, dt = -\delta\cosh\theta + \tfrac{1}{2}\dot{\theta}\delta^2\sinh\theta - \tfrac{1}{6}\delta^3(\ddot{\theta}\sinh\theta + \dot{\theta}^2\cosh\theta) \tag{57}$$

$$\int_Q^P \cosh(\theta - \theta_Q)\, dt = \delta + \tfrac{1}{6}\dot{\theta}^2\delta^3 \tag{58}$$

Since θ and its derivatives are all evaluated at P in the above expressions, the subscript P has been dropped. Application of (54) and (58) to (53) gives

$$R_s U^s(Q) = c^2\delta[1 + (1/c)x\dot{\theta} - (\tfrac{1}{2}c)x\ddot{\theta}\delta + (\tfrac{1}{6})\dot{\theta}^2\delta^2] \tag{59}$$

The causality condition (52) can be written

$$r^2 - c^2\delta^2 + \tfrac{1}{4}c^2\dot{\theta}^2\delta^4 - xc\dot{\theta}\delta^2 + \tfrac{1}{3}xc\delta^3\ddot{\theta} - \tfrac{1}{3}c^2\delta^4\dot{\theta}^2 = 0 \tag{60}$$

In order to eliminate δ (and hence t_Q) from the expressions for the potentials, it is necessary to solve (60) for δ. Approximate solutions are $c\delta = \pm r$, the positive root corresponding to retarded potentials and the negative root to advanced potentials. To obtain a more accurate solution of (60) the approximate value $c\delta = +r$ can be substituted into the fourth order terms and the resulting equation solved to give

$$c^2\delta^2 = r^2\left[1 - \frac{x\dot{\theta}}{c} + \left(\frac{x\dot{\theta}}{c}\right)^2 + \frac{xr\ddot{\theta}}{3c^2} - \frac{\dot{\theta}^2 r^2}{12c^2}\right] \tag{61}$$

$$c\delta = r\left[1 - \frac{x\dot{\theta}}{2c} + \frac{3x^2\dot{\theta}^2}{8c^2} + \frac{xr\ddot{\theta}}{6c^2} - \frac{\dot{\theta}^2 r^2}{24c^2}\right] \tag{62}$$

$$\frac{1}{c\delta} = \frac{1}{r}\left[1 + \frac{x\dot{\theta}}{2c} - \frac{x^2\dot{\theta}^2}{8c^2} - \frac{xr\ddot{\theta}}{6c^2} + \frac{\dot{\theta}^2 r^2}{24c^2}\right] \tag{63}$$

By use of the above equations, the retarded potentials (47) can be expressed in the form

$$A^4 = \frac{e}{4\pi r}\left[1 - \frac{3x\dot{\theta}}{2c} + \frac{15x^2\dot{\theta}^2}{8c^2} + \frac{xr\ddot{\theta}}{3c^2} + \frac{3r^2\dot{\theta}^2}{8c^2}\right] \tag{64}$$

$$A_4 = -\frac{e}{4\pi r}\left[1 + \frac{x\dot{\theta}}{2c} - \frac{x^2\dot{\theta}^2}{8c^2} + \frac{xr\ddot{\theta}}{3c^2} + \frac{3r^2\dot{\theta}^2}{8c^2}\right] \tag{65}$$

$$A^1 = -\frac{e}{4\pi r}\left[\frac{r\dot{\theta}}{c} - \frac{\ddot{\theta}r^2}{2c^2} - \frac{xr\dot{\theta}^2}{c^2} + \frac{5xr^2\dot{\theta}\ddot{\theta}}{4c^3} + \frac{\dddot{\theta}r^3}{6c^3} + \frac{rx^2\dot{\theta}^3}{c^3}\right] \tag{66}$$

$$A^2 = A^3 = 0 \tag{67}$$

The expression for A^1 is accurate to third order.

It is convenient to express the fields in polar coordinates, $x'^i = (r, \psi, \phi, ct)$, (Fig. 3). This is done by means of the vector transformation

$$A_\mu' = P_\mu^{\ \nu} A_\nu, \qquad A_4 = A_4', \tag{68}$$

$$P_1^{\ 1} = \cos\psi, \qquad P_2^{\ 1} = -r\sin\psi, \qquad P_3^{\ 1} = 0 \tag{69}$$

Since there is no dependence of the fields on ϕ, the transformation has been carried out in the plane $\phi = 0$. The resulting potentials are

$$A_1' = -\frac{e}{4\pi r}\left[\left(\frac{r\dot{\theta}}{c} - \frac{r^2\ddot{\theta}}{2c^2} + \frac{r^2\dddot{\theta}}{6c^3}\right)\cos\psi - \left(\frac{r^2\dot{\theta}^2}{c^2} - \frac{5r^3\dot{\theta}\ddot{\theta}}{4c^3}\right)\cos^2\psi + \frac{r^3\dot{\theta}^3}{c^3}\cos^3\psi\right] \tag{70}$$

$$A_2' = \frac{e}{4\pi}\left[\left(\frac{r\dot{\theta}}{c} - \frac{r^2\ddot{\theta}}{2c^2} + \frac{r^2\dddot{\theta}}{6c^3}\right)\sin\psi - \left(\frac{r^2\dot{\theta}^2}{c^2} - \frac{5r^2\dot{\theta}\ddot{\theta}}{4c^3}\right)\cos\psi\sin\psi + \frac{r^3\dot{\theta}^3}{c^3}\cos^2\psi\sin\psi \tag{71}$$

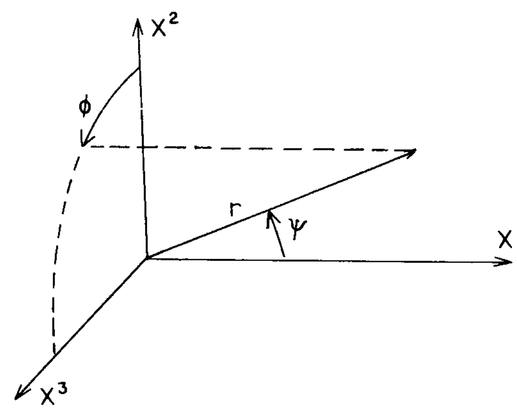

FIG. 3. Polar coordinates

$$A_3' = 0 \tag{72}$$

$$A_4' = -\frac{e}{4\pi}\left[\frac{1}{r} + \left(\frac{\dot{\theta}}{2c} + \frac{\ddot{\theta}r}{3c^2}\right)\cos\psi + \frac{r\dot{\theta}^2}{8c^2}(3 - \cos^2\psi)\right] \tag{73}$$

Quantities expressed in polar coordinates will either be denoted by a prime or the fact will be explicitly noted.

The fields are calculated by means of

$$F'_{ij} = \frac{\partial A_j'}{\partial x'^i} - \frac{\partial A_i'}{\partial x'^j} \tag{74}$$

In the following expressions, all quantities are expressed in polar coordinates and the prime is dropped. The fields are

$$F_{12} = \frac{e}{4\pi r}\left[\left(-\frac{r^2\ddot{\theta}}{2c^2} + \frac{r^3\dddot{\theta}}{3c^3}\right)\sin\psi + \frac{5r^3\dot{\theta}\ddot{\theta}}{4c^3}\cos\psi\sin\psi\right] \tag{75}$$

$$F_{14} = \frac{e}{4\pi r^2}\left[1 + \frac{2\ddot{\theta}r^2}{3c^2}\cos\psi + \frac{\dot{\theta}^2r^2}{8c^2}(\cos^2\psi - 3)\right] \tag{76}$$

$$F_{24} = \frac{e}{4\pi r}\sin\psi\left[\frac{r\dot{\theta}}{2c} - \frac{2r^2\ddot{\theta}}{3c^2} - \frac{r^2\dot{\theta}^2}{4c^2}\cos\psi\right] \tag{77}$$

$$F_{i3} = 0 \tag{78}$$

The contravariant components are found by means of

$$F^{ij} = g^{ik}g^{jm}F_{km} \tag{79}$$

with the metric tensor given by

$$g_{11} = 1, \qquad g_{22} = r^2, \qquad g_{33} = r^2\sin^2\psi, \qquad g_{44} = -(1 + x\dot{\theta}/c)^2 \tag{80}$$

The result is

$$F^{14} = -\frac{e}{4\pi r^2}\left[1 - \frac{2r\dot{\theta}}{c}\cos\psi + \frac{\dot{\theta}^2r^2}{c^2}\left(\frac{25}{8}\cos^2\psi - \frac{3}{8}\right) + \frac{2\ddot{\theta}r^2}{3c^2}\cos\psi\right] \tag{81}$$

$$F^{24} = -\frac{e}{4\pi r^3}\sin\psi\left[\frac{r\dot{\theta}}{2c} - \frac{2\ddot{\theta}r^2}{3c^2} - \frac{5r^2\dot{\theta}^2}{4c^2}\cos\psi\right] \tag{82}$$

$$F^{12} = \frac{e}{4\pi r^3}\left[\left(\frac{r^3\dddot{\theta}}{3c^3} - \frac{r^2\ddot{\theta}}{2c^2}\right)\sin\psi + \frac{5r^3\dot{\theta}\ddot{\theta}}{4c^3}\cos\psi\sin\psi\right] \tag{83}$$

Notice that F^{12} and F_{12} are accurate to third order.

The expressions for the fields can be tested by substitution into Maxwell's equations (24), the nontrivial components of which are

$$\frac{\partial}{\partial\psi}(\sqrt{-g}\,F^{12}) + \frac{1}{c}\frac{\partial}{\partial t}(\sqrt{-g}\,F^{14}) = 0 \tag{84}$$

$$\frac{\partial}{\partial r}(\sqrt{-g}\,F^{21}) + \frac{1}{c}\frac{\partial}{\partial t}(\sqrt{-g}\,F^{24}) = 0 \tag{85}$$

$$\frac{\partial}{\partial r}(\sqrt{-g}\,F^{41}) + \frac{\partial}{\partial\psi}(\sqrt{-g}\,F^{42}) = 0 \tag{86}$$

where

$$\sqrt{-g} = r^2\sin\psi(1 + x\dot{\theta}/c) \tag{87}$$

The fields we have obtained do indeed satisfy Maxwell's equations. Since differentiation with respect to t makes second order terms go to third order, we have troubled to find F^{12} to third order as an additional check on the method.

The accelerated particle is being driven by an external field. For motion in one dimension, the only possible type of driving field is an electric field E in the X-direction of the inertial frame I. This field can be an arbitrary function of X. In order to find what the external field is in the accelerated frame R, it is necessary to make a tensor transformation of

$$F^{14}(I) = -E, \qquad F_{14}(I) = E \tag{88}$$

The transformation is done by means of the coefficients found in Eqs. (7) through (10). The result expressed in rectangular coordinates is

$$F^{14} = -\frac{E}{1 + x\dot{\theta}/c}, \qquad F_{14} = \left(1 + \frac{x\dot{\theta}}{c}\right)E \tag{89}$$

A further transformation to polar coordinates using the coefficients (69) yields

$$F_{24} = -r[1 + (r\dot{\theta}/c)\cos\psi]E\sin\psi \tag{90}$$

$$F_{14} = [1 + (r\dot{\theta}/c)\cos\psi]E\cos\psi \tag{91}$$

$$F^{24} = \frac{E\sin\psi}{r[1 + (r\dot{\theta}/c)\cos\psi]} \tag{92}$$

$$F^{14} = -\frac{E\cos\psi}{1 + (r\dot{\theta}/c)\cos\psi} \tag{93}$$

The calculation of the equation of motion requires the expression for the energy momentum tensor. Actually, in frame R we are not calculating the equation of motion of the particle, but rather the forces necessary to maintain it stationary at the origin. The expression for the energy-momentum tensor is

$$T^i_j = F_{jk}F^{ik} - \tfrac{1}{4}\delta^i_jF_{km}F^{km} \tag{94}$$

The fields are the sum of the fields (75) through (83) of the particle itself and the external fields (90) through (93). The following formulas for $T_j{}^i$ are accurate to second order and are expressed in polar coordinates:

$$\begin{aligned} T_1^1 = {} & \frac{1}{2}E^2\sin^2\psi - \frac{1}{2}\left[\frac{e}{4\pi r^2} + E\cos\psi\right]^2 \\ & + \frac{1}{2}\left[\frac{e^2\cos\psi}{8\pi^2 r^4} + \frac{eE}{4\pi r^2}(2\cos^2\psi - \sin^2\psi)\right]\frac{r\dot\theta}{c} \\ & - \frac{1}{2}\left[\left(\frac{e}{4\pi r^2}\right)^2\left(\frac{7}{2}\cos^2\psi - 1\right) + \frac{eE}{4\pi r^2}\left(\frac{15}{4}\cos^2\psi - \frac{9}{4}\right)\cos\psi\right]\frac{r^2\dot\theta^2}{c^2} \\ & - \frac{1}{2}\left[\frac{e^2}{12\pi^2 r^4}\cos\psi - \frac{eE}{3\pi r^2}(\sin^2\psi - \cos^2\psi)\right]\frac{\ddot\theta r^2}{c^2} \end{aligned} \tag{95}$$

$$\begin{aligned} T_1^2 = {} & \left[\frac{e}{4\pi r^2} + E\cos\psi\right]\frac{E\sin\psi}{r} - \left[\frac{3eE}{8\pi r^3}\cos\psi\sin\psi + \frac{e^2}{32\pi^2 r^5}\sin\psi\right]\frac{r\dot\theta}{c} \\ & + \frac{\sin\psi}{r}\left[\frac{5}{4}\left(\frac{e}{4\pi r^2}\right)^2\cos\psi + \frac{3eE}{8\pi r^2}\left(\frac{5}{4}\cos^2\psi - \frac{1}{4}\right)\right]\frac{r^2\dot\theta^2}{c^2} \\ & + \left[\frac{e^2}{24\pi^2 r^5}\sin\psi + \frac{eE}{3\pi r^3}\sin\psi\cos\psi\right]\frac{r^2\ddot\theta}{c^2} \end{aligned} \tag{96}$$

$$T_1^3 = 0, \qquad T_2^3 = 0, \qquad T_2^2 = -T_1^1, \qquad T_4^4 = -T_3^3 \tag{97}$$

$$T_1^4 = \frac{eE}{4\pi r^2}\frac{r^2\ddot\theta}{2c^2}\sin^2\psi \tag{98}$$

$$T_2^4 = \frac{e}{4\pi r}\frac{r^2\ddot\theta}{2c^2}\sin\psi\left[\frac{e}{4\pi r^2} + E\cos\psi\right] \tag{99}$$

$$\begin{aligned} T_3^3 = {} & \frac{1}{2}\left[\frac{e}{4\pi r^2} + E\cos\psi\right]^2 + \frac{1}{2}E^2\sin^2\psi \\ & - \frac{1}{2}\left[\frac{e^2\cos\psi}{8\pi^2 r^4} + \frac{eE}{4\pi r^2}(\cos^2\psi + 1)\right]\frac{r\dot\theta}{c} \\ & + \frac{1}{2}\left[\left(\frac{e}{4\pi r^2}\right)^2\left(3\cos^2\psi - \frac{1}{2}\right) + \frac{eE}{4\pi r^2}\left(\frac{3}{4}\cos^2\psi + \frac{3}{4}\right)\cos\psi\right]\frac{r^2\dot\theta^2}{c^2} \\ & + \frac{1}{2}\left[\frac{e^2}{12\pi^2 r^4}\cos\psi + \frac{eE}{3\pi r^2}\right]\frac{r^2\ddot\theta}{c^2} \end{aligned} \tag{100}$$

$$T_4^1 = -\frac{eE}{4\pi r^2}\frac{r^2\ddot\theta}{2c^2}\sin^2\psi \tag{101}$$

$$T_4^2 = -\left[\frac{e}{4\pi r^2} + E\cos\psi\right]\frac{e}{4\pi r^3}\frac{r^2\ddot\theta}{2c^2}\sin\psi \tag{102}$$

The calculation of the force on the charged particle located at the origin of R is to be based on

$$T^i_{k;i} = -f_k \tag{103}$$

where the semicolon denotes the covariant derivative and f_k is the four-force density on the charged matter. One might try to define the four-force on the particle by means of

$$F_k = \lim_{\delta V\to 0}\int f_k\, dV \tag{104}$$

where δV is an element of three-volume surrounding the particle. This leads to difficulties as one cannot integrate a vector in covariant fashion in a noninertial frame of reference. There is no such problem with the integration of invariants, however. Suitable invariants can be constructed by forming the inner product of T^i_k with some vector and taking the divergence of the resulting expression:

$$(T^i_k v^k)_{;i} = T^i_{k;i}v^k + T^i_k v^k_{;i} \tag{105}$$

By integration over a suitable region of space–time we find

$$\int (T^i_k v^k)_{;i}\, d\Sigma = -\int (f_k v^k)\, d\Sigma + \int (T^i_k v^k_{;i})\, d\Sigma \tag{106}$$

where $d\Sigma$ represents an element of four-volume. As vectors v^k, the four-velocity (21) and four-acceleration (22) of a fixed point of R can be used. The use of the definition of the covariant derivative and the symmetry properties of T^{ik} leads to

$$T^i_k v^k_{;i} = T_m{}^i\frac{\partial v^m}{\partial x^i} + \frac{1}{2}\frac{\partial g_{ik}}{\partial x^m}v^m T^{ik} \tag{107}$$

If the velocity vector u^k and the metric (11) are used in (107) then

$$T^i_k u^k_{;i} = T_1{}^4\dot\theta \tag{108}$$

whereas the acceleration vector a^k gives

$$T^i_k a^k_{;i} = \frac{(T_4{}^4 - T_1{}^1)\dot\theta^2 - T_1{}^4\ddot\theta}{(1 + x\dot\theta/c)^2} \tag{109}$$

The divergence term in (106) is to be converted to a surface integral:

$$\int (T^i_k v^k)_{;i}\, d\Sigma = \int (T^i_k v^k)\, dV_i \tag{110}$$

where dV_i is the three-dimensional surface element given by

$$dV_i = \frac{1}{3!}\epsilon_{ikmn}dV^{kmn} \tag{111}$$

The quantity ϵ_{ikmn} is the ϵ-tensor defined by

$$\epsilon_{ijkm} = \sqrt{-g}\,\delta_{ijkm}, \quad \epsilon^{ijkm} = \frac{1}{\sqrt{-g}}\,\delta^{ijkm} \tag{112}$$

where δ_{ijkm} and δ^{ijkm} are the permutation symbols. The tensor dV^{kmn} is constructed from infinitesimal four-vectors lying in the three-dimensional surface of integration:

$$dV^{kmn} = \begin{vmatrix} a^k & b^k & c^k \\ a^m & b^m & c^m \\ a^n & b^n & c^n \end{vmatrix} \tag{113}$$

The element of four-volume is an invariant and can be expressed

$$d\Sigma = \sqrt{-g}\, dx^1\, dx^2\, dx^3\, dx^4 = (1 + x\dot{\theta}/c)\, dV\, dx^4 \tag{114}$$

where dV is the ordinary three-dimensional volume element.

The invariant surface integral in (110) is most conveniently expressed in polar coordinates. The acceleration vector (22) expressed in polar coordinates is

$$a'^k = \left(\frac{\dot{\theta}c\cos\psi}{1 + x\dot{\theta}/c},\ -\frac{\dot{\theta}c\sin\psi}{r(1 + x\dot{\theta}/c)},\ 0,\ 0\right) \tag{115}$$

The use of the velocity vector (21) in (106) gives

$$\int T'^{\,i}_4 \frac{c}{1 + x\dot{\theta}/c}\, dV_i' = -\int f_4 c\, dx^4 dV + \int T_1{}^4\dot{\theta}\, d\Sigma \tag{116}$$

whereas the acceleration vector (115) gives

$$\begin{aligned} &\int \left(T'^{\,i}_1 \cos\psi - T'^{\,i}_2 \frac{1}{r}\sin\psi\right) \frac{\dot{\theta}c}{1 + x\dot{\theta}/c}\, dV_i' \\ &\quad = -\int f_1\dot{\theta}c\, dV dx^4 + \int [(T_4{}^4 - T_1{}^1)\dot{\theta}^2 + T_1{}^4\ddot{\theta}]\, \frac{dV dx^4}{1 + x\dot{\theta}/c} \end{aligned} \tag{117}$$

where use is made of (108), (109), and (110). In (116) and (117) primed quantities are in polar coordinates, unprimed in rectangular coordinates. The terms involving the force are to be interpreted as

$$\lim_{\delta V \to 0} \int f_4 c\, dV dx^4 = c \int F_4\, dx^4 \tag{118}$$

$$\lim_{\delta V \to 0} \int f_1\dot{\theta}c\, dV dx^4 = c \int F_1\dot{\theta}\, dx^4 \tag{119}$$

where F_1 and F_4 are components of the four-force acting on the particle. This agrees with the result given by Moller (*4*). (See equation 87, p. 306 of Moller's book. Note that $d\mathring{V}^0 = dV$ since the matter under consideration is at rest.)

As a four-dimensional region of integration, let us take a thin tube of radius r (Fig. 4) surrounding the world line of the particle, which is the x^4-axis. The vectors a^i, b^i, c^i in (113) can be chosen as

$$\begin{aligned} a^i &= (0, d\psi, 0, 0) \\ b^i &= (0, 0, d\phi, 0) \\ c^i &= (0, 0, 0, dx^4) \end{aligned} \tag{120}$$

From (111) the only nonzero component of dV_i is found to be

$$dV_1 = \sqrt{-g}\, d\psi\, d\phi\, dx^4 = (1 + x\dot{\theta}/c)r^2 \sin\psi\, d\psi\, d\phi\, dx^4 \tag{121}$$

By use of Eq. (101) we see that the surface integral appearing in (116) is proportional to r^2 and hence is zero in the limit $r \to 0$. The volume integral in (116) also vanishes giving the result that the fourth component of the four-force is zero in the permanent rest frame of the particle as it should be. The component T'^{1}_{4} is the radial component of Poynting's vector and, as is well known, a calculation of the type indicated in Eq. (116) performed in the instantaneous (inertial) rest frame of the particle leads to the conventional energy radiation rate which, however, cannot be properly called the fourth component of the radiation

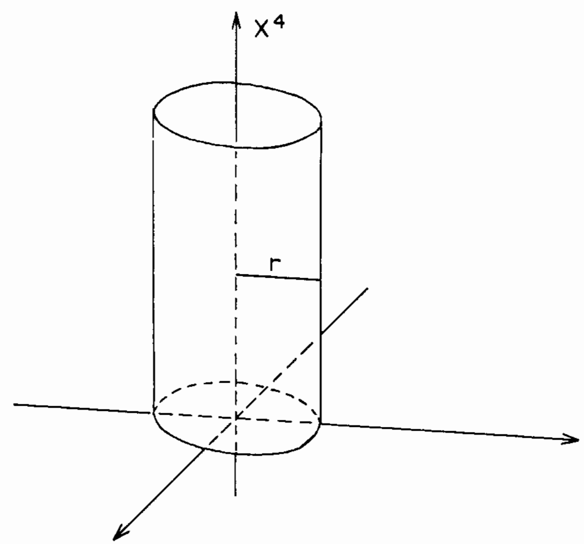

FIG. 4. Four-dimensional region of integration in the accelerated frame

four-force. This is because the general form of the four-force on a particle which conserves proper mass is of the form

$$F^k = \left(\frac{\mathbf{F}}{\sqrt{1-(v/c)^2}}, \frac{\mathbf{F}\cdot\mathbf{v}}{c\sqrt{1-(v/c)^2}}\right) \tag{122}$$

where $\mathbf{F}$ is the ordinary three-force.

In the calculation of the surface integrals appearing in (117), only terms which do not vanish as $r \to 0$ need be retained. The important terms in the integrand are

$$\begin{aligned} r^2(T'^1_1 \cos\psi - T'^2_1 r \sin\psi) \\ = -\frac{eE}{4\pi} + \frac{1}{2}\left(\frac{e}{4\pi}\right)^2\left[-\frac{\cos\psi}{r^2} + \frac{\dot\theta}{rc}(1+\cos^2\psi)\right. \\ \left. - \frac{\dot\theta^2}{c^2}\left(\cos^2\psi + \frac{3}{2}\right)\cos\psi - \frac{4\ddot\theta}{3c^2}\right] \end{aligned} \tag{123}$$

where use has been made of

$$T'^1_2 = g_{22}g^{11}T'^2_1 = r^2 T'^2_1 \tag{124}$$

The evaluation of the surface integrals gives

$$\begin{aligned} \int\left(T'^i_1 \cos\psi - T'^i_2 \frac{1}{r}\sin\psi\right)\frac{\dot\theta c}{1 + x\dot\theta/c}\,dV'_i \\ = \int\left(-eE + \frac{e^2\dot\theta}{6\pi rc} - \frac{e^2\ddot\theta}{6\pi c^2}\right)\dot\theta c\,dx^4 \end{aligned} \tag{125}$$

In evaluating the volume integrals appearing in (117) it must be remembered that this part of the expression is in rectangular coordinates. The rectangular components of the energy momentum tensor can be obtained from the polar form (Eqs. (95) through (102) represent the energy-momentum tensor in polar coordinates) by means of the usual tensor transformation:

$$T_1^1 = (\cos^2\psi - \sin^2\psi)T'^1_1 - 2r\sin\psi\cos\psi T'^2_1 \tag{126}$$

$$T_1^4 = \cos\psi\, T'^4_1 - \frac{1}{r}\sin\psi\, T'^4_2 \tag{127}$$

The important terms are

$$T_1^4 = -\left(\frac{e}{4\pi r^2}\right)^2 \frac{r^2\ddot\theta}{2c^2}\sin^2\psi \tag{128}$$

$$T_4^4 - T_1^1 = -\sin^2\psi\left(\frac{e}{4\pi r^2}\right)^2 + \frac{3e^2\dot\theta\sin^2\psi\cos\psi}{16\pi^2 r^3 c} \tag{129}$$

There is no contribution from the $T_1^4\ddot\theta$ term in (117). The remaining term can be expressed

$$\begin{aligned} \int (T_4^4 - T_1^1)\dot\theta^2\left(1 + \frac{x\dot\theta}{c}\right)^{-1} dV dx^4 = \int \frac{3e^2\dot\theta^3 \sin^3\psi\cos\psi}{8\pi rc}\,dr\,d\psi\,dx^4 \\ - \int \sin^3\psi\left(\frac{e\dot\theta}{4\pi r^2}\right)^2\left(1 - \frac{r\dot\theta\cos\psi}{c}\right)2\pi r^2\,d\psi\,dr\,dx^4 \end{aligned} \tag{130}$$

where the integral is over the volume of the tube of radius r surrounding the x^4-axis (Fig. 4). Evaluation of the integrals over ψ yields

$$\int (T_4^4 - T_1^1)\dot\theta^2 \frac{1}{1 + x\dot\theta/c}\,dV dx^4 = -\frac{4}{3}\iint_0^r \dot\theta^2 \frac{1}{2}\left(\frac{e}{4\pi r^2}\right)^2 4\pi r^2\,dr\,dx^4 \tag{131}$$

The use of (125), (119), and (131) in (117) yields the result

$$\int\left(-eE + \frac{e^2\dot\theta}{6\pi rc} - \frac{e^2\ddot\theta}{6\pi c^2} + F_1 + \frac{4\dot\theta}{3c}\int_0^r \frac{1}{2}\left(\frac{e}{4\pi r^2}\right)^2 4\pi r^2\,dr\right)\dot\theta c\,dx^4 = 0 \tag{132}$$

The term proportional to r^{-1} can be expressed

$$\frac{e^2\dot\theta}{6\pi rc} = \frac{4\dot\theta}{3c}\int_r^\infty \frac{1}{2}\left(\frac{e}{4\pi r^2}\right)^2 4\pi r^2\,dr \tag{133}$$

in which form it can be formally combined with the divergent integral in (132). We write the resulting integral as

$$m_e = \frac{1}{c^2}\int_0^\infty \frac{1}{2}\left(\frac{e}{4\pi r^2}\right)^2 4\pi r^2\,dr \tag{134}$$

where m_e is interpreted as the electromagnetic mass of the electron. Since the length of the tube is arbitrary, we set the integrand in (132) equal to zero with the result

$$\frac{e^2\ddot\theta}{6\pi c^2} + eE = F_1 + \frac{4}{3}m_e(\dot\theta c) \tag{135}$$

The divergent term is formally independent of the surface over which the flow of energy and momentum is computed. If the calculation is performed in an inertial frame, only the term proportional to r^{-1} is obtained. See for instance Dirac's (*1*) original calculation. The extra term that makes this possible, specifically, the integral from 0 to r in (132), comes from terms that appear in $T^i_{k;i}$ in the noninertial frame that do not appear in an inertial frame. To see this, note that in general

$$T^i_{k;i} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}(\sqrt{-g}\,T_k^i) - \frac{1}{2}\frac{\partial g_{rs}}{\partial x^k}T^{rs} \tag{136}$$

For the particular metric under consideration, the case $k = 1$ gives

$$\frac{\partial T_1{}^i}{\partial x^i} + \frac{(T_1{}^1 - T_4{}^4)\dot{\theta}/c}{1 + x\dot{\theta}/c} = -f_1 \tag{137}$$

The integral (134) is not over the coordinates of the accelerated frame; it is a formal way of expressing the divergent terms and shows that the electromagnetic mass of the electron is to be computed in the accelerated frame in the same way as in an inertial frame. Covariance problems that arise from the introduction of an electron radius are thus eliminated.

The force F_1 is interpreted as

$$F_1 = m_{\text{mech}} \left[\frac{du_1}{dt} - \frac{1}{2} \frac{\partial g_{rs}}{\partial x} u^r u^s \right] \tag{138}$$

where m_{mech} is the mass of nonelectromagnetic origin of the electron, and $u_k = (0, 0, 0, -c)$ is the four-velocity of the electron relative to R. For the particular metric under consideration, (138) gives

$$F_1 = m_{\text{mech}}(\dot{\theta} c) \tag{139}$$

(This is really another way of deriving Eq. (14).) If in (135) the masses m^e and m_{mech} are lumped together into the empirically determined rest mass m of the electron the result is the equation of motion

$$\frac{e^2\ddot{\theta}}{6\pi c^2} + eE = m\dot{\theta}c \tag{140}$$

Dirac's equation is

$$m\frac{dU^i}{dt} = \frac{e}{c} F^{ij} U_j + \frac{e^2}{6\pi c^3}\left[\ddot{U}^i - \frac{U^i}{c^2}(\dot{U}^j \dot{U}_j) \right] \tag{141}$$

where U^i is the four-velocity of the electron relative to I. That (140) is identical to (141) for one-dimensional motion is found by substituting the four-velocity (4) into (141) and remembering that the only nonvanishing component of the external field is $F^{14} = -E$.

In place of a tube r = const. surrounding the x^4-axis, it is also possible to use a tube δ = const. (Fig. 5). This surface has the property that radiation measured on it over a time interval $t_P' - t_P$ was emitted over an equal time interval $t_Q' - t_Q$. This follows from $\delta = t_P' - t_Q' = t_P - t_Q$ = const. Such a surface is no longer parallel to the x^4-axis and the surface elements (111) become more complicated. A calculation done in this manner leads to the same equation of motion demonstrating that at least for these two cases the flow of energy and momentum is independent of the shape of the surface used in its computation.

It is possible to write down Eqs. (116) and (117) for advanced fields as well as retarded. If this is done and one-half the difference between (117) and the

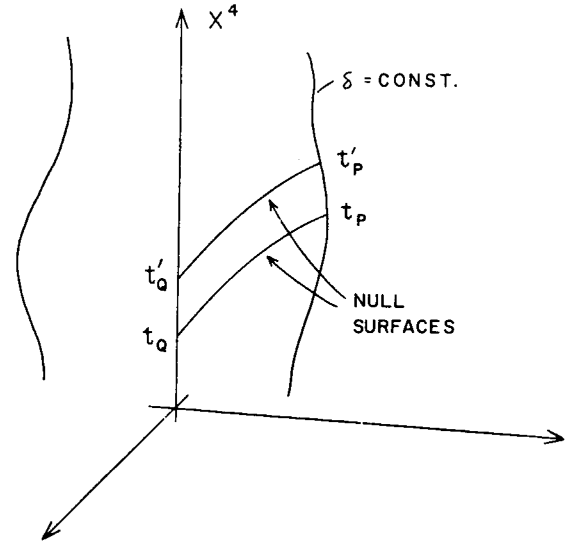

FIG. 5. Four-dimensional region of integration defined by δ = const.

corresponding equation for advanced fields is used to calculate the force, one finds that all divergent terms cancel out. In other words, calculation of the flow of energy and momentum is done by means of the vector

$$\frac{c}{1 + x\dot{\theta}/c} \frac{1}{2} (T_j{}^i(\text{ret})\, dV_i - T_j{}^i(\text{adv})\, dV_i) \tag{142}$$

The same equation of motion results, and there seems little point in introducing advanced fields.

VI. INTERPRETATION OF THE EQUATIONS OF MOTION

It is convenient to introduce into the equation of motion the characteristic time

$$t_0 = \frac{e^2}{6\pi mc^3} = 6.25 \times 10^{-24} \text{ sec} \tag{143}$$

which is approximately the time required for a light signal to traverse the electron. Equation (140) can then be expressed

$$\ddot{\theta} - \dot{\theta}/t_0 = -eE(t)/(mct_0) \tag{144}$$

The above equation can easily be integrated regarding E as a function of t:

$$\dot{\theta}e^{-t/t_0} - \dot{\theta}(0) = -\frac{e}{mct_0}\int_0^t E(t)e^{-t/t_0}\, dt \tag{145}$$

Rohrlich (*5*) has suggested the use of the asymptotic condition

$$\lim_{t\to\infty} e^{-t/t_0}\dot{U}^i(t) = 0 \tag{146}$$

It is convenient here to apply the condition to the gravitational field strength measured in the rest frame of the electron:

$$\lim_{t\to\infty} e^{-t/t_0}\dot{\theta}(t) = 0 \tag{147}$$

Application of (147) to (145) leads to

$$\dot{\theta}(t) = \frac{e}{mct_0} e^{t/t_0} \int_t^\infty E(t)e^{-t/t_0}\, dt \tag{148}$$

In practice, E is not given as a function of t but rather of X. Equation (48) allows E to be re-expressed in terms of t.

We consider the solution of (148) for the special case

$$\begin{aligned} E(t) &= 0 & t &< 0 \\ E(t) &= \text{const.} & 0 &< t < t_1 \\ E(t) &= 0 & t_1 &< t \end{aligned} \tag{149}$$

We assume that the electron is initially moving in the positive x-direction and that the field further accelerates it in the same direction. The solutions, along with the expressions for $\ddot{\theta}$, are

$$\dot{\theta}(t) = \frac{eE}{mc} e^{t/t_0}[1 - e^{-t_1/t_0}] \qquad t < 0 \tag{150}$$

$$\dot{\theta}(t) = \frac{eE}{mc} [1 - e^{-(t_1-t)/t_0}] \qquad 0 < t < t_1 \tag{151}$$

$$\dot{\theta}(t) = 0 \qquad t_1 < t \tag{152}$$

$$\ddot{\theta}(t) = \frac{eE}{mct_0} e^{t/t_0}[1 - e^{-t_1/t_0}] \qquad t < 0 \tag{153}$$

$$\ddot{\theta}(t) = -\frac{eE}{mct_0} e^{-(t_1-t)/t_0} \qquad 0 < t < t_1 \tag{154}$$

$$\ddot{\theta}(t) = 0 \qquad t_1 < t \tag{155}$$

These functions are shown in Fig. 6. The phenomenon of pre-acceleration is clearly shown. For further discussion of this point, see Rohrlich (*5*) and Plass (*6*).

Whereas hyperbolic motion does not, strictly speaking, actually exist, we see from (150), (151), and (152) that if $t_1 \gg t_0$, i.e., if the electron spends a longer

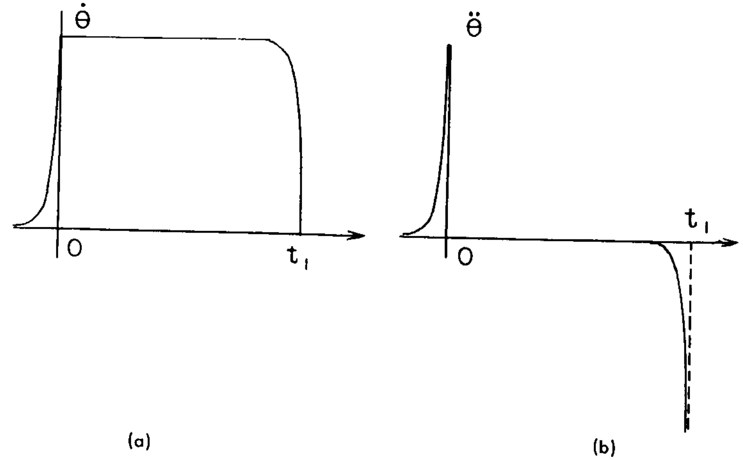

FIG. 6. (a) The function $\dot{\theta}$. (b) The function $\ddot{\theta}$

time in the field than it takes a light signal to cross its diameter, the motion will be very accurately hyperbolic in the region of the field:

$$\dot{\theta} = eE/(mc) \tag{156}$$

Another point to be noted is that Rohrlich's condition (146) and the condition (147) are not equivalent. Rohrlich's condition gives

$$e^{-t/t_0}\dot{U}^1 = e^{-t/t_0}\dot{\theta}c \cosh \theta \to \tfrac{1}{2}\dot{\theta}ce^{-t/t_0+teE/(mc)} \tag{157}$$

where the hyperbolic approximation $\theta = eEt/(mc)$ has been employed. Thus (146) follows from (157) provided that

$$\frac{1}{t_0} - \frac{eE}{(mc)} > 0 \tag{158}$$

We can write

$$\frac{eEt_0}{(mc)} = \frac{gt_0}{c} = \frac{gr_0}{c^2} \tag{159}$$

where r_0 is the "electron radius." It was necessary to assume in the derivation of (140) that $gr_0/c^2 \ll 1$. Hence (140) is assured.

The equation of motion (140) can be expressed in the form

$$eEv = \frac{d}{dT}(mc^2 \cosh \theta) - \frac{e^2\ddot{\theta}}{6\pi c^2} v \tag{160}$$

where use is made of Eq. (1) and $dt = dT/\cosh \theta$. Equation (160) is an expression of conservation of energy. Let us consider a case where the external field is confined to a limited region of space such as that given by (149). If (160) is integrated between any two limits, the result is

$$\int eE\, dX = \Delta(\text{K.E.}) - \int \frac{e^2\ddot{\theta}}{6\pi c^2} v\, dT \qquad (161)$$

where use is made of $vdT = dX$. If both limits lie outside the region of the field E, then $\dot{\theta} = 0$ at both limits of integration, and the term representing radiation loss can be integrated by parts to give

$$-\int \ddot{\theta} v\, dT = -\int \ddot{\theta} \sinh \theta\, dt = +\int \dot{\theta}^2 \cosh \theta\, dt = +\int \dot{\theta}^2\, dT \qquad (162)$$

The above result shows that the over-all energy loss can be accounted for either by the conventional radiation rate $e^2\dot{\theta}^2/(6\pi c)$ or the radiation four-force $e^2\ddot{\theta}/(6\pi c^2)$. Since we are considering a case where the electron always moves the same distance X in the driving field, it emerges with less kinetic energy than it would have if radiation were absent. Still, the motion in the region of the field is accurately hyperbolic—i.e., the same as if radiation were neglected. The important thing is the inclusion of the points where $\ddot{\theta}$ comes into play, i.e., where the electron enters and leaves the field.

VII. CONCLUSION

Much attention has been given recently to the interpretation and solution of the equations of motion (141). See, for example, the papers by Rohrlich (*5*) and Plass (*6*). The present paper serves to put the basic equation on firmer ground by demonstrating that it can be obtained in a straightforward manner without recourse to advanced fields. The calculation is based on the standard form of the energy-momentum tensor (94), no modifications such as was done by Pryce (*8*) being found necessary. It is found that the divergent term is formally independent of any "electron radius."

By obtaining exact expressions for the electromagnetic field in the rest frame of a charged particle in hyperbolic motion, we have shown that there is no radiation present in the accelerated frame. Thus the presence of radiation in the inertial frame, but none in the accelerated frame, does not contradict the principle of equivalence as was suggested by Bondi and Gold (*7*). It is possible to transform the radiation away when other than Lorentz transformations are used. This fact has also been noted by Rohrlich (*9*).

The problem of hyperbolic motion is best approached as a limiting case where a uniform driving field is defined over a limited region of space which is then allowed to become large compared to the electron. The Dirac equation gives rise to a sensible conservation of energy equation. The total energy loss can be accounted for either by the radiation force which comes into play as the electron enters and leaves the field or by the conventional radiation rate.

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REFERENCES

1. P. A. M. DIRAC, *Proc. Roy. Soc.* **A167,** 148 (1938).
2. J. A. WHEELER AND R. P. FEYNMAN, *Revs. Modern Phys.* **17,** 157 (1945).
3. T. FULTON AND F. ROHRLICH, *Ann. Phys.* (*NY*) **9,** 499 (1960).
4. C. MOLLER, "The Theory of Relativity." Clarendon Press, Oxford, 1955.
5. F. ROHRLICH, *Ann. Phys.* (*NY*) **13,** 93 (1961).
6. G. N. PLASS, *Revs. Modern Phys.* **33,** 37 (1961).
7. H. BONDI AND T. GOLD, *Proc. Roy. Soc.* **A229,** 416 (1955).
8. M. H. L. PRYCE, *Proc. Roy. Soc.* **A168,** 389 (1938).
9. F. RORLICH, *Nuovo cimento.* **21,** 811 (1961).