

Fluids in electric and magnetic fields: Pressure variation and stability

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Received June 5, 1981

The variation of the hydrostatic pressure with position in a dielectric or magnetic fluid acted upon by electric or magnetic forces may be somewhat delicate. For instance, the dominant role played by the striction forces in fluids having positive susceptibility is a physical phenomenon whose importance does not seem to be sufficiently well-known. Closely related to the pressure balance is the thermodynamic stability problem. In this paper two examples are analysed in detail in order to elucidate the pressure variation and thermodynamic stability for dielectric, paramagnetic, and diamagnetic liquids.

La variation de la pression hydrostatique avec la position, dans un fluide diélectrique ou magnétique soumis à des forces électriques ou magnétiques peut donner lieu à des problèmes assez délicats. Par exemple, le rôle dominant joué par les forces de striction dans les fluides ayant une susceptibilité positive est un phénomène physique dont l'importance ne semble pas suffisamment bien connue. Le problème de la stabilité thermodynamique est étroitement relié à l'équilibre de pression. Dans cet article, on analyse deux exemples en détail, dans le but de clarifier la variation de pression et la stabilité thermodynamique pour les liquides diélectriques, paramagnétiques et diamagnétiques.

[Traduit par le journal]

Can. J. Phys., 60, 449 (1982)

I. Introduction

Consider a homogeneous dielectric fluid acted upon by electric forces in an electrostatic field. To find the electric force density f^{el} one can make use of Helmholtz' variational principle, equating the variation of the free energy of the field, $\mathcal{F} = \frac{1}{2} \int \mathbf{E} \cdot \mathbf{D} \, dV$ under reversible, isothermal conditions to the mechanical work being done by external forces in displacing the matter. As the external force density must everywhere be equal and opposite to f^{el} , one has for the variation $\delta\mathcal{F} = - \int f^{el} \cdot s \, dV$, where s is the arbitrary displacement within the medium. For an uncharged medium this procedure leads to the expression

$$[1] \quad f^{el} = -\frac{1}{2} E^2 \nabla \epsilon + \frac{1}{2} \nabla [E^2 \rho (\partial \epsilon / \partial \rho)_T]$$

where ρ is the mass density and ϵ the permittivity (we use SI units). This expression is called the Helmholtz force. The detailed derivation of it can be found in various places, for instance in Panofsky and Phillips (1), Landau and Lifshitz (2), and Brevik (3). The last term in [1] is the electrostriction term. In magneto-statics, the paramagnetic and diamagnetic cases are handled analogously, the only difference in the final expression for the force being that the magnetic field \mathbf{H} replaces the electric field \mathbf{E} and the permeability μ replaces the permittivity ϵ :

$$[2] \quad f^{magn} = -\frac{1}{2} H^2 \nabla \mu + \frac{1}{2} \nabla [H^2 \rho (\partial \mu / \partial \rho)_T]$$

In writing these expressions we have assumed linear

constitutive relations, $\mathbf{D} = \epsilon \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$.

In the derivation of these expressions for forces acting on dielectric fluids it is characteristic that only the *first order* variation of the free energy had to be taken into account. In order to understand the mechanics of dielectric fluids there is, however, one important additional factor that also ought to be included, namely the thermodynamic *stability* of the system. This brings the *second order* variation of the free energy into the problem. The physical system is, under ordinary circumstances, stable with respect to a small external disturbance. This means that the amount of mechanical work necessary to bring the system from its equilibrium state to some neighbouring state is generally positive. Expressed in terms of the free energy, we may say that a dielectric fluid, when acted upon by a field, generally adjusts itself such that its free energy becomes a minimum. (We shall always assume that the temperature is kept constant.) The condition that the free energy be an *extremum* leads to the force density expressions, [1] and [2] above, whereas the additional condition that the extremum be a *minimum* leads to the thermodynamic *inequalities*. We shall not derive the inequalities here; the reader is referred to Sect. 18 in Landau and Lifshitz (2). The resulting inequalities are

$$[3] \quad (\partial \zeta / \partial \rho)_{D,T} > 0$$

$$[4] \quad (\partial E / \partial D)_{\zeta,T} > 0$$

where ζ is the chemical potential per unit mass.

In deriving the thermodynamic inequalities an arbitrary relationship was assumed between \mathbf{E} and \mathbf{D} , the relationship need not be linear. In the specific case of a

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linear medium, [4] becomes trivially satisfied as the permittivity ϵ is always positive. This inequality thus gives no new information in this case. The remaining inequality [3] is, however, also of importance when the medium is linear. In the following we shall assume a linear medium (as we already did in writing [1] and [2]). The pressure distribution throughout the fluid is thus described by [1] (or [2]), whereas the thermodynamic stability of the system is described by [3].

It may feel natural for the reader now to ask: were not the formulas above after all developed long ago? What then is the motivation for dealing with the thermodynamics of electromagnetic media once more? The answer is that these issues are much more delicate and subject to more controversies than one might be inclined to think at first, without detailed background in this field of research. One important point is the *correctness* of Helmholtz' force expression. Within the scientific community no general consensus has in fact in the past existed on the correctness of Helmholtz' force. This expression has had to exist side by side with other rivalling expressions, the most significant of which is the Kelvin (or Einstein-Laub) expression $(\mathbf{P} \cdot \nabla)\mathbf{E} = -\frac{1}{2}E^2\nabla\epsilon + \frac{1}{2}\nabla(\mathbf{E} \cdot \mathbf{P})$. What has gradually become clear during recent years is, however, that the Helmholtz expression stands out as the best alternative for explaining experimental results in a straightforward way. We may in this context refer to our review on the electromagnetic energy-momentum tensor in material media (4), where various proposals for the electromagnetic force were examined and compared with observations. Especially the Hakim-Higham experiment (5), along with the Goetz-Zahn experiment (6, 7), show convincingly good agreement with the Helmholtz theory. It is interesting to notice also that Lai *et al.* (8) recently seem to have been able to derive the Helmholtz force from first principles, using statistical mechanics.

Our second motivation for the present treatment is to stress the importance of the *electrostriction* (or *magnetostriction*) term, as following from the Helmholtz force, for the distribution of hydrostatic pressure throughout a dielectric liquid. For instance, in electrostatics it is just the electrostriction force that furnishes the compressive pressure needed to hold a column of liquid together as a whole as it rises between two charged vertical condenser plates partly immersed in the liquid. These issues in fact seem to be insufficiently known. Thus, in the standard textbook of Panofsky and Phillips (1) one will find that the pressure distribution is given incorrectly.

Third, we wish to stress a thermodynamic stability property, namely the central role played by the chemical potential ζ . It is quite instructive to carry out in detail the analysis of how the value of ζ generally adjusts itself so as to resist an external disturbance. We are

not aware that the stabilizing role of ζ , although simple, has ever been discussed in detail in the standard treatises on electrodynamics. In particular, it is worthwhile to notice the behaviour of a *diamagnetic* medium, $\mu < \mu_0$. In this case the direction of each force term in [2] is opposite to the direction in a paramagnetic medium. Nevertheless, the thermodynamic inequality [3] is general enough to describe the stability of the system in this special case also.

It may be worthwhile to notice in passing that the features discussed here can be of interest not only in classical electromagnetism but also in different fields of physics, such as in the theory of the Casimir (i.e., zero-point) force acting on dielectric balls and in the modern bag-model theory in quantum chromodynamics (cf. for instance, the paper by Milton (9) on the dielectric Casimir effect and the related treatment by the author (10) when electrostriction is included).

We shall now consider two examples which, although relatively simple, are general enough to serve our demonstration purposes. The reader may consult also the treatment by Lahoz (11).

II. Example 1

This example is very simple; in the electrostatic version it is a parallel-plate condenser completely immersed in a dielectric liquid. The distribution of pressure in the liquid is described by [1]. Now the first term to the right in [1] is usually negligible. In the experiment of Hakim and Higham (5) the maximum average electric field between the electrodes was about 5×10^7 V/m; even in such a strong field the change in ϵ was found to be very small. The significant term in [1] is thus the electrostriction term. The effect of this force is to draw liquid into the field region between the plates and compress it, thus increasing the density and pressure of the liquid. Assuming for simplicity a nonpolar liquid, so that ϵ according to the Clausius-Mossotti relation $(\epsilon - \epsilon_0)/(\epsilon + 2\epsilon_0) = \text{const.}$ ρ is a function of ρ but not of the temperature T , we obtain for the excess pressure in the region of the field

$$[5] \quad \Delta p = \frac{1}{2}E^2\rho \, d\epsilon/d\rho \\ = E^2(\epsilon - \epsilon_0)(\epsilon + 2\epsilon_0)/(6\epsilon_0)$$

which is always positive since $\epsilon > \epsilon_0$. The physical meaning of our pressure symbol p ought to be emphasized: it is the *hydrostatic* (or elastic) pressure which, ignoring changes in entropy, is a function only of the mass density ρ . Thus p is a mechanical quantity, not directly associated with the electric field: if we imagine that the electric forces were at each position replaced by external mechanical forces of the same magnitude and direction, the resulting pressure throughout the liquid would be exactly the same as p . (The necessity of giv-

ing this explicit definition of what is meant by the pressure in a dielectric fluid has been stressed also by R. Peierls (personal communication). What is here called p is just the same as what in Landau and Lifshitz' book is called p_0 . See ref. 2, Sect. 15.)

The excess pressure, as given by [5], was measured in the Hakim-Higham experiment to within $\pm 5\%$ (4, 5).

It is convenient to introduce the symbol f^{tot} , defined as the total force density acting on the liquid. Thus $f^{\text{tot}} = -\nabla p + f^{\text{el}}$, and at equilibrium we must necessarily have $f^{\text{tot}} = 0$. As ∇p can at constant temperature be replaced by $\rho \nabla \zeta_0$, in accordance with the thermodynamic identity for the chemical potential $\zeta_0 = \zeta_0(\rho, T)$ in the absence of a field, $\rho d\zeta_0 = dp - S_0 dT$, we can write, in accordance with ref. 2, Sect. 15,

$$[6] \quad f^{\text{tot}} = -\rho \nabla \zeta_0 - \frac{1}{2} E^2 \nabla \epsilon + \frac{1}{2} \nabla [E^2 \rho (\partial \epsilon / \partial \rho)] \\ = -\rho \nabla \zeta$$

where ζ is the chemical potential in the presence of the field. The equilibrium condition at constant temperature may be expressed as

$$[7] \quad \zeta = \zeta_0(\rho, T) - \frac{1}{2} E^2 (\partial \epsilon / \partial \rho)_T = \text{constant}$$

If $\rho = \text{constant}$, as we may assume in the present example, we may transform [7] into the form

$$[8] \quad p(\rho, T) - \frac{1}{2} E^2 \rho (\partial \epsilon / \partial \rho)_T = \text{constant}$$

which is in accordance with [1]. The constant in [8] is the pressure outside the condenser.

Consider now the stability problem. We may imagine that the system is brought from the equilibrium state to a neighbouring state by a compression of the liquid within the condenser, so that ρ increases slightly. The physical conditions in the example are such that it is convenient to let the voltages on the plates be constant during the compression. This corresponds to constant E rather than constant D . We may replace [3] by the inequality

$$[9] \quad (\partial \zeta / \partial \rho)_{E, T} > 0$$

For a linear medium, the transition from [3] to [9] is trivial. (In the general nonlinear case, we can also easily derive the inequality [9] on the basis of the assumed positiveness of the quantity $(\partial D / \partial E)_\rho = \partial(D, \rho) / \partial(E, \rho)$. Namely, the following expression becomes then positive:

$$[10] \quad \left(\frac{\partial D}{\partial E} \right)_\rho \left(\frac{\partial \zeta}{\partial \rho} \right)_D = \frac{\partial(D, \rho)}{\partial(E, \rho)} \frac{\partial(D, \zeta)}{\partial(D, \rho)} \\ = \frac{\partial(E, \zeta)}{\partial(E, \rho)} \frac{\partial(D, \zeta)}{\partial(E, \zeta)} \\ = \left(\frac{\partial \zeta}{\partial \rho} \right)_E \left(\frac{\partial D}{\partial E} \right)_\zeta > 0$$

from which [9] follows, when [4] is taken into account.)

The inequality [9] shows that during the compression, which is performed at constant values of E and T , the chemical potential ζ is increasing monotonically with ρ . Thus ζ becomes greater inside than outside the condenser, and the reaction force f^{tot} from the system will according to [6] act *outwards*, against the applied external compressive force. This is in accordance with what we should expect, as the total free energy is at minimum at equilibrium and the supplied mechanical work during the compression therefore positive. Corresponding results are obtained if we let the external forces be directed outwards instead of inwards: the density ρ becomes smaller between the plates than outside, the chemical potential ζ also becomes smaller, and so the reaction force f^{tot} acts inwards in a direction opposite to the applied force. The performed work is positive as before. The stability condition generally implies that the chemical potential between the plates adjusts itself at the right level to produce the appropriate compensating force.

The electrostatic case considered so far is the most familiar case. If we now proceed to the magnetostatic field, we are confronted with phenomena which are qualitatively different. This is due to the fact that the permeability μ can, on the basis of thermodynamic arguments, be both greater or less than μ_0 . The paramagnetic case with $\mu > \mu_0$ is in our context equivalent to the electrostatic case and shall not be further considered. What is of interest here is the diamagnetic case, $\mu < \mu_0$, as the factor $(\partial \mu / \partial \rho)_T$ in the magnetostriction term then becomes negative. This is easily seen explicitly when it is assumed that the liquid is satisfying the relation analogous to the Clausius-Mossotti relation; cf. the analogue to [5] where the right-hand side becomes negative when $\mu < \mu_0$. This means that the magnetostriction force acts *outwards*, tending to expel liquid from the field region (we assume for definiteness that the homogeneous magnetic field is produced by an electromagnet with plane pole surfaces immersed in the liquid). At first sight one might think that the situation is unstable. However, this is not so, if it can be assumed that the pressure in the field-free region outside the magnet is positive, as it always is in practice. One may describe the behaviour of the system as follows: the purpose of the magnetostriction forces is to push out so much of the liquid that the hydrostatic chemical potential $\zeta_0(\rho, T)$, which is at constant temperature a monotonically decreasing function for decreasing values of ρ , becomes sufficiently low in the field region to permit a constant value for the total chemical potential ζ to be established throughout the liquid:

$$[11] \quad \zeta = \zeta_0(\rho, T) - \frac{1}{2} H^2 (\partial \mu / \partial \rho)_T = \text{constant}$$

Let us finally consider the stability of the diamagnetic system. As before, the stability is governed by the thermodynamic inequality expressing that the total free energy be a minimum at equilibrium. Assuming constant magnetizing current in the coils, i.e., constant H , the inequality analogous to the inequality [9] is

$$[12] \quad (\partial\zeta/\partial\rho)_{H,T} > 0$$

Thus, assuming for instance an isothermal compression of the liquid in the field region, the chemical potential ζ will be increasing in that region, and the total reaction force f^{tot} will according to [6] be acting outwards, against the external compression force. The performed mechanical work during the compression is positive, as it should be. If we consider an expansion instead of a compression of the liquid, we obtain similar consistent results.

III. Example 2

Our second example is more complicated in that two new features have to be taken into account; first, the gravitational field plays a role for the pressure; secondly, the liquid has a free surface on which electric or magnetic forces act. The situation in electrostatics is the well-known one where a parallel-plate condenser is partly immersed in a dielectric liquid, cf., Fig. 1. Above the liquid we assume atmospheric pressure. When a strong electric field is formed between the plates, the liquid can be observed to rise slightly in the interior region. We shall make use of this situation to discuss some salient points in the force-pressure balance.

First, in order to avoid unnecessary complications we shall assume that the distance d between the plates is so great that surface tension plays no important role. This will be the case if $d \gg (2\alpha/\rho g)^{1/2}$, where α is the surface tension coefficient, see Landau and Lifshitz (12). The free surface between the plates will then essentially remain as a plane surface after it has risen, and become curved only in the immediate vicinity of the walls.

When discussing the force balance in the liquid we shall first avoid introducing the chemical potential and instead apply the force expression in [1] directly. Both terms in this expression are of importance here. The term $-\frac{1}{2}E^2\nabla\epsilon$ acting in the boundary region at the free surface gives, upon integration over the boundary, the surface force density $\frac{1}{2}E^2(\epsilon - \epsilon_0)$. This is the force which is responsible for the elevation of the liquid. The liquid is thus lifted up at the free surface. Equating the surface force to the gravitational pressure ρgh , we obtain for the equilibrium height h

$$[13] \quad h = (\epsilon - \epsilon_0)E^2/(2\rho g)$$

Next consider the electrostriction term in [1]. Its role

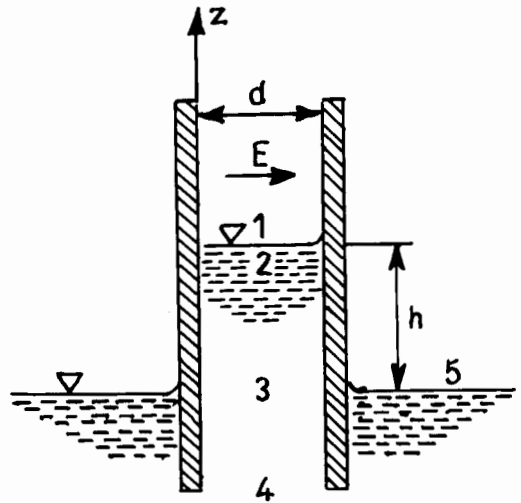


FIG. 1. Two charged condenser plates partly immersed in a dielectric liquid.

is very important for the stability of the system, as it gives rise to a uniform pressure $\frac{1}{2}E^2\rho(\partial\epsilon/\partial\rho)_T$ in the liquid. At the free surface the electrostriction force acts downwards, and it is *stronger* than the surface force $\frac{1}{2}E^2(\epsilon - \epsilon_0)$ acting upwards. This is easily seen explicitly if the liquid satisfies the Clausius-Mossotti relation: the difference between the pressure p_2 in the liquid just below the free surface and the air pressure p_1 just above the surface is

$$[14] \quad p_2 - p_1 = \frac{1}{2}E^2[\rho d\epsilon/d\rho - (\epsilon - \epsilon_0)] \\ = E^2(\epsilon - \epsilon_0)^2/(6\epsilon_0)$$

cf. Fig. 1. This expression is always positive. Further, if we assume that the liquid instead is polar and satisfies a more complicated relation such as the Onsager relation, we will also see explicitly that $\rho(\partial\epsilon/\partial\rho)_T > (\epsilon - \epsilon_0)$, for instance from the data compiled by Lahoz (ref. 11, p. 87). It is important to realize that the condition for the liquid to be lifted up at the free surface is that the compression force is always stronger than the elevating force; this is an immediate consequence of the fact that a liquid is unable to withstand tension. The main role played by the electrostriction in the force balance is that it is responsible for the necessary stabilizing force. On the other hand, it ought to be pointed out for completeness why the electrostriction force, being the gradient of a scalar function, cannot contribute at all to the elevation height of the liquid: the z -component of the total electrostriction force can be written as a surface integral which has to vanish at both limits; at the upper integration surface (above the free surface) because $\rho = 0$ there, at the lower integration surface (far beneath the plates) because $E = 0$ in that region.

Let us now write down the expression for the hydrostatic pressure $p(z)$ in the liquid column as a function of the height z . Below the free surface the gravitational pressure $\rho g(h-z)$ has to be included in the description, and we obtain

$$[15] \quad p(z) - p_1 = \frac{1}{2}E^2\rho(\partial\epsilon/\partial\rho)_T - \frac{1}{2}E^2(\epsilon - \epsilon_0) + \rho g(h-z)$$

This expression, which gives a pressure which varies linearly with z , will be valid down to the transition region between the lower edges of the plates where the electrostriction force $\frac{1}{2}\nabla[E^2\rho(\partial\epsilon/\partial\rho)_T]$, which now is acting upwards, gradually diminishes the electrostriction pressure. As in Brevik (4), we give in Fig. 2 a schematic illustration of the variation in pressure along a path from points 1 to 5 in Fig. 1. When there is air above the liquid, the difference between the air pressures p_1 and p_5 is negligible. The given values for the pressure variation in the figure correspond to the following example: the liquid is benzene, for which $\epsilon_r = 2.28$ (at 20°C), $\rho = 880 \text{ kg/m}^3$, and the field is taken equal to $E = 3 \times 10^6 \text{ V/m}$. The pressure in the main part of the liquid column is then calculated from [15], where the Clausius–Mossotti relation is used in the electrostriction term and the value $h = 5.9 \text{ mm}$ is inserted as a result calculated separately from [13].

The present example was discussed also in Sect. 6-7 in the well known treatise of Panofsky and Phillips (1). These authors let the symbol p mean the mechanical or hydrostatic pressure in the liquid, just as we do. It ought to be pointed out, however, that the pressure distribution as given in their Fig. 6-8 by comparison with our Fig. 2 is not correct.

We next consider the chemical potential ζ in the presence of the gravitational field. We recall that [6] and [7], which were derived when disregarding the gravitational field, were sufficiently general to hold also when ρ varied with position. Thus [6] will be applicable even across the boundary region at the free surface where ρ changes abruptly. The inclusion of the gravitational field merely means that a term $g(z-h)$ has to be added to the chemical potential, due to the hydrostatic pressure. The condition for mechanical equilibrium, as before, is that $\zeta = \text{constant}$ throughout the field. The constant must be equal to p_1/ρ_1 , where subscript 1 refers to the atmosphere above the liquid. The expression for ζ replacing [7] becomes thus

$$[16] \quad \zeta = \zeta_0(\rho, T) - \frac{1}{2}E^2(\partial\epsilon/\partial\rho)_T + g(z-h) = p_1/\rho_1$$

This expression holds within the column of liquid, $z \leq h$.

Note that the effect of the elevating force term $-\frac{1}{2}E^2\nabla\epsilon$ is not shown explicitly in [16]. The reason for

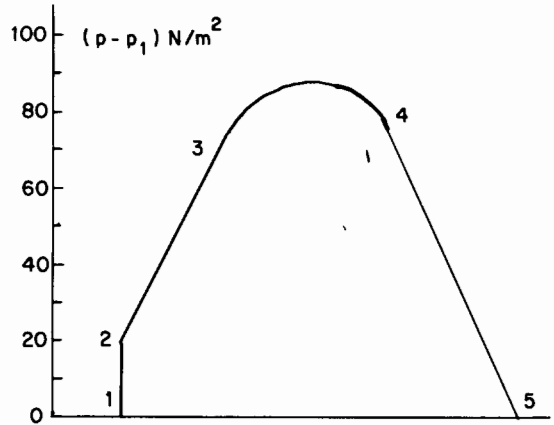


FIG. 2. The hydrostatic pressure variation from point 1 to point 5 in Fig. 1. The variation is linear between points 2 and 3.

this is that the effect of the term has been hidden in the dependence of ϵ on ρ : $\nabla\epsilon = (\partial\epsilon/\partial\rho)_T\nabla\rho$ under isothermal conditions. Some care should here be exerted so that one does not on the basis of [16] conclude that the elevating force is without significance for the chemical potential. For instance, the density ρ appearing in $\zeta_0(\rho, T)$ is the result of the action of three different external forces: gravitational force, electrostriction force, and surface elevating force. A related point worth noticing is that we cannot on the basis of [16] directly infer an equation for the pressure, like we did when passing from [7] to [8]. The last mentioned passage required constant density. When discussing the pressure, it is most convenient to use [1] directly, such as we did above.

Considerations of thermodynamic stability do not require special attention in this example. The stability of the system is governed by [9], as before, when the voltages on the condenser plates are held constant during imaginary compressions or expansions of the liquid. The chemical potential ζ adjusts itself appropriately so as to counteract the external forces, in accordance with [6].

We finally come to the diamagnetic case, $\mu < \mu_0$. The experimental arrangement is imagined to be analogous to the electrostatic one, for instance in the form of an electromagnet partly immersed in the diamagnetic liquid such that the magnetic field is horizontal. The force density is given by the expression in [2], augmented by the gravitational force. A characteristic of the diamagnetic case is that the free surface of the liquid becomes *depressed* in the field. This is so because the surface force density $\frac{1}{2}H^2(\mu - \mu_0)$, arising from the first term in [2], is negative and thus acts downwards. In direct analogy to [13] we can write for the displacement of the free surface:

$$[17] \quad h = (\mu - \mu_0) H^2 / (2\rho g)$$

which is less than zero, corresponding to a depression. The magnetostriction force in this case acts *upwards* at the free surface and *downwards* in the inhomogeneous field region between the lower ends of the magnet. This is easily seen explicitly if we assume the magnetostatic analogue to the Clausius–Mossotti relation, cf. [5]. As before, the magnetostriction effect cannot have any influence on the displacement h of the free surface.

Let us write down the expression for the pressure difference ($p_2 - p_1$), where p_2 refers to a point just below the free surface and p_1 to a point just above it:

$$[18] \quad p_2 - p_1 = \frac{1}{2} H^2 [\rho d\mu/d\rho - (\mu - \mu_0)] \\ = H^2 (\mu - \mu_0)^2 / (6\mu_0)$$

This is a quadratic expression, positive even when $\mu < \mu_0$. The positiveness of ($p_2 - p_1$) is actually what we should expect on physical grounds, for a negative pressure difference would simply mean that p_2 was negative in the special case of a vacuum above the liquid ($p_1 = 0$), and that would be meaningless. The formalism thus leads to consistent results. We see from [18] that the reason why ($p_2 - p_1$) is positive is that the downwards directed surface force $\frac{1}{2} H^2 (\mu - \mu_0)$ is stronger than the upwards directed magnetostriction force $\frac{1}{2} H^2 d\mu/d\rho$. Thus, not only are the *directions* of the two force terms in [2] opposite to what we are accustomed to from the more familiar case of positive susceptibilities, but also the relative *strengths* of the two terms have been interchanged.

On the whole, one ought to exert some care in distinguishing material-dependent properties of a physical system from the underlying general principles. We have seen above that the dominating influence from the striction forces on the variation of the local hydrostatic pressure in a liquid is no general principle but only a material-dependent property, dependent on the sign of the susceptibility. In order to distinguish the underlying general principles we shall usually have to go to thermodynamics. The thermodynamic principle of main importance here, being independent of whether the fluid in question has positive or negative susceptibility, is that the free energy at constant temperature tends towards a minimum at equilibrium. In other words, the work required to bring the system out of the equilibrium state is always positive.

IV. Conclusion

Our discussion has throughout been based upon the Helmholtz force, as given in [1] and [2] for the electrostatic and magnetostatic cases respectively. The correctness and straightforward applicability of the Helmholtz force in experimental situations is an im-

portant point that should be stressed. Putting the Helmholtz force equal to the gradient of a pressure p at equilibrium implies that p is the *hydrostatic* pressure, at the same value of fluid density. That is, the role played by the electric or magnetic field is essentially to act as an external agency, compressing the fluid. If the field were removed, and there were instead some other device producing the same local force density at each point, the pressure would everywhere be the same as p .

The fact that the Helmholtz force is derived from a thermodynamic variational principle can be considered as just one example of the surprisingly great power of thermodynamics. Similarly the stability of the system, as expressed basically by the inequalities [3] and [4], is also a thermodynamic result. As such it is strong enough, as we have seen, to cover also the diamagnetic case, despite the fact that the direction of each term in the force expression [2] is now reversed relative to the case of a paramagnetic fluid. The essential point is thus not the sign of the susceptibility, or the direction of the field force, but rather that the work required to bring the system out from equilibrium is always positive.

It is important to be aware of the essential role played by the electrostriction (or magnetostriction) force for the distribution of the hydrostatic pressure in a fluid. We dealt with the electrostriction force in both examples above. As regards example 1, we have noticed the good agreement between the Hakim–Higham experiment (5) and the Helmholtz theory. As regards example 2, we have taken the opportunity to point out the incorrect pressure variation given in Panofsky and Phillips' book (1). While recognizing the importance of striction forces for the variation of pressure throughout the fluid, it should, however, be borne in mind that these forces are peculiar in that they are expressible as gradients of scalar functions. Therefore, they are in principle unable to contribute to the total, integrated force on a test body. For this reason striction forces are frequently omitted in the formalism; for instance, this is the case both for Minkowski's and Abraham's energy–momentum tensors. When only overall forces or torques on a test body are desired, this omission is fully legitimate.

In this paper we have considered only dielectric, paramagnetic, and diamagnetic fluids; ferromagnetic media have been left out.

Finally we mention that the relation between Helmholtz' and Kelvin's force expressions has recently been the subject of a discussion between Gingras (13) and Lahoz (14). The reader may consult these references for further information.

V. Acknowledgement

I wish to thank Professor P. C. Hemmer for a useful discussion on the manuscript.

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