large displacement regions, where the velocity is approaching constancy because of air resistance, can probably be interpreted interestingly, but we did not attempt that.

In order to find test mass accelerations, the spot accelerations were scaled down by the magnification factor of 84.4. This quantity can be calculated from the lengths of the lever arms listed in Table I, along with a factor of 2 introduced by the mirror. The differences between left and right test mass accelerations are of the order of 10^{-6} cm/s^2, corresponding to angular displacements from equilibrium position of about $\frac{1}{3}$ rad. This is based on the calculated torque of 2.1×10^{-3} dyn cm/rad using the parameters in Table I and the tabulated value for the bulk modulus of drawn tungsten.

Geometric asymmetries in the centering of the dumbbell in the housing are canceled out in first order by left–right averaging. The next-order effect is only some 1% for a zero-location error corresponding to some 25 cm in spot location, even in the closest gravitating position, so we can neglect centering error throughout. The data here were taken before we realized the effectiveness of the cancellation, and so includes the errors in our direct measurements of the distances r. Instead, only the distances between the pseudospheres in their left-run and right-run positions need be measured, reducing the errors in line 4 of Table II effectively to zero. In that case, it would be worthwhile to measure the accelerations more accurately.

The final left–right averages have been corrected for several small effects, as shown in Table II. (None of the following corrections are made by students in the 2-h lab.) The averaging cancels out all the torsion except that from the small difference in the regions of definition of the parabolas. In all four runs in Table II, the effective operating regions for left and right runs were within 3 cm of each other on the scale; the small differences in acceleration are shown as corrections in Table II. Other corrections include both the gravitational torque on the rod and its moment of inertia, the nonspherical geometry of each test mass, and the countertorque exerted by the far pseudosphere on each test mass. The contribution from the rod, as deduced from the parameters of Table I, is less than one might think because the parts of the rod that contribute most to its moment of inertia are also those that are attracted most effectively by the pseudospheres.

To get n in the $1/r^n$ distance dependence of the acceleration a, the quantity $r^n a$ was plotted versus r. The best horizontal fit and the error in the slope were estimated by eye. The quite satisfactory results for G (including the iron–uranium comparison) and for n are given in Table II.

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[1]H. Cavendish, "Experiments to Determine the Density of the Earth," Philos. Trans. R. Soc. London, Part II, 469–526 (1798). This article is reprinted in *The Scientific Papers of the Honourable Henry Cavendish, F. R. S., Vol. II (Chemical and Dynamical)* (Cambridge U. P., London, 1921), pp. 249–286.

Angular momentum of a charge–monopole pair

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The angular momentum of a static electric charge–magnetic monopole is, in SI units, $\mathbf{L} = (eg/4\pi)\,\hat{\mathbf{a}}$. Here, e and g are their respective strengths and $\hat{\mathbf{a}}$ is a unit vector along the line from the electric charge to the magnetic pole. A simple derivation is presented of this well-known result. The derivation makes use of vector (and tensor) analysis; no actual integrations involving the electromagnetic field are involved.

Figure 1 shows a static "charge–monopole" pair. The pair consists of an electric charge of strength e located at $\mathbf{r} = -\mathbf{a}/2$ and a magnetic pole of strength g located at $\mathbf{r} = \mathbf{a}/2$. Thomson[1,2] stated that the angular momentum $\mathbf{L}$ stored in the associated electromagnetic field is independent of the separation distance $a = |\mathbf{a}|$ between the electric charge and the magnetic pole and that $\mathbf{L}$ is given by the simple expression

$$\mathbf{L} = (eg/4\pi)\hat{\mathbf{a}}\,. \tag{1}$$

Here, $\hat{\mathbf{a}} = \mathbf{a}/a$ is a unit vector in the direction of $\mathbf{a}$ (i.e., from the electric charge to the magnetic pole). [We have converted Thomson's original expression, $\mathbf{L} = (eg/c)\hat{\mathbf{a}}$, from Gaussian units to SI.[3]] Dirac[4] has shown that the product of the two strengths eg is quantized so as to make the angular momentum quantized in integer multiples of $\hbar/2$. The purpose of this article is to present a simple method of obtaining Eq. (1). The method employs only vector (and tensor) analysis; no actual integrations involving the electromagnetic fields are necessary. This is intended to complement the treatment by Adawi,[5] who has demonstrated three different methods (including the use of spherical polar and prolate spheroidal coordinates) to evaluate the necessary integrals.

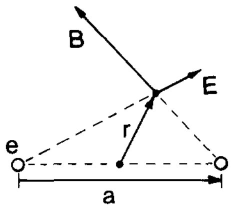

Fig. 1. An electric charge of strength e and a magnetic pole of strength g form a "charge–monopole" pair. The vector from the electric charge to the magnetic pole is $\mathbf{a}$. The origin is chosen midway between the electric charge and the magnetic pole.

The angular momentum stored in the electromagnetic field of the charge–monopole pair is given by the volume integral

$$\mathbf{L} = \epsilon_0 \int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})d\tau . \qquad (2)$$

Because the total linear momentum stored in this field is zero, the angular momentum $\mathbf{L}$ is independent of the particular choice of origin. The electric and magnetic fields for this situation are[3]:

$$\mathbf{E} = (e/4\pi\epsilon_0)[(\mathbf{r}+\mathbf{a}/2)/(|\mathbf{r}+\mathbf{a}/2|^3)] , \qquad (3)$$

$$\mathbf{B} = (g/4\pi)[(\mathbf{r}-\mathbf{a}/2)/(|\mathbf{r}-\mathbf{a}/2|^3)] . \qquad (4)$$

The electric field $\mathbf{E}$ is derivable from a scalar potential ϕ,

$$\mathbf{E} = -\nabla\phi , \qquad (5)$$

where

$$\phi = e/4\pi\epsilon_0|\mathbf{r}+\mathbf{a}/2| . \qquad (6)$$

The term $\mathbf{E}\times\mathbf{B}$ can be written

$$\mathbf{E}\times\mathbf{B} = -\nabla\phi\times\mathbf{B} = -\nabla\times(\phi\mathbf{B}) \qquad (7)$$

because $\nabla\times\mathbf{B} = 0$. Letting

$$\mathbf{W} = \phi\mathbf{B} , \qquad (8)$$

the following identity is readily verified:

$$\begin{aligned}\mathbf{r}\times(\mathbf{E}\times\mathbf{B}) &= \mathbf{r}\times(-\nabla\times\mathbf{W}) \\ &= -\nabla(\mathbf{r}\cdot\mathbf{W}) + \nabla\cdot(\mathbf{r}\mathbf{W}) \\ &\quad - 2\nabla\cdot(\mathbf{W}\mathbf{r}) + 2\mathbf{r}\nabla\cdot\mathbf{W} .\end{aligned} \qquad (9)$$

When integrated over all space, the first three terms on the right-hand side of Eq. (9) can be transformed into surface integrals over the surface of a "large" sphere centered at the origin. Although the argument is somewhat subtle due to the fact that $\mathbf{W}$ goes to zero only as rapidly as $1/r^3$ for large r, these surface integrals can be shown to vanish (see Appendix). As for the last term in Eq. (9),

$$\begin{aligned}2\mathbf{r}\nabla\cdot\mathbf{W} = 2\mathbf{r}\nabla\cdot(\phi\mathbf{B}) &= 2\mathbf{r}(\nabla\phi)\cdot\mathbf{B} + 2\mathbf{r}\phi\nabla\cdot\mathbf{B} \\ &= -2\mathbf{r}\mathbf{E}\cdot\mathbf{B} + 2\mathbf{r}\phi[g\,\delta(\mathbf{r}-\mathbf{a}/2)] .\end{aligned} \qquad (10)$$

When integrated over all space, the first term on the right-hand side of Eq. (10) vanishes because the integrand is an odd function of $\mathbf{r}$ for our choice of origin. Because of the δ function, the integral of Eq. (10) over all space becomes

$$\begin{aligned}\int 2\mathbf{r}\nabla\cdot\mathbf{W}\,d\tau &= 2g\mathbf{r}\phi|_{\mathbf{r}=\mathbf{a}/2} \\ &= 2g(\mathbf{a}/2)(e/4\pi\epsilon_0 a) = (eg/4\pi\epsilon_0)\hat{\mathbf{a}}.\end{aligned} \qquad (11)$$

Finally, combining Eqs. (2) and (9)–(11) leads to the desired result,

$$\mathbf{L} = (eg/4\pi)\hat{\mathbf{a}} . \qquad (12)$$

After this article was submitted, a similar treatment of this problem was found in Portis' text.[6] Portis attacks the problem using two scalar potentials ($\mathbf{E} = -\nabla\phi$ and $\mathbf{B} = -\nabla\bar{\phi}$).

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APPENDIX

We will discuss one of the four terms on the right-hand side of Eq. (9), the others are treated in a similar manner. Consider the term $\nabla\cdot(\mathbf{r}\mathbf{W})$. The volume integral of this can be converted into a surface integral over a "large" sphere centered at the origin,

$$\int \nabla\cdot(\mathbf{r}\mathbf{W})d\tau = \int \hat{\mathbf{r}}\cdot(\mathbf{r}\mathbf{W})dS . \qquad (A1)$$

Here, $\hat{\mathbf{r}} = \mathbf{r}/r$ is a unit vector in the $\mathbf{r}$ direction and dS is a differential element of surface area on this large sphere. For large r, the leading term of $\hat{\mathbf{r}}\cdot\mathbf{r}\mathbf{W}$ (arising from the $1/r^3$ contribution of $\mathbf{W} = \phi\mathbf{B}$) is

$$\begin{aligned}\hat{\mathbf{r}}\cdot\mathbf{r}\mathbf{W} &\simeq \hat{\mathbf{r}}\cdot\mathbf{r}(e/4\pi\epsilon_0 r)(g\mathbf{r}/4\pi r^3) , \\ \hat{\mathbf{r}}\cdot\mathbf{r}\mathbf{W} &\simeq (eg/16\pi^2\epsilon_0)\hat{\mathbf{r}}/r^2 .\end{aligned} \qquad (A2)$$

Using $dS = r^2\,d\Omega$ ($d\Omega$ = element of solid angle) the integral (A1) becomes

$$\int \nabla\cdot(\mathbf{r}\mathbf{W})\,d\tau \simeq (eg/16\pi^2\epsilon_0)\int \hat{\mathbf{r}}\,d\Omega , \qquad (A3)$$

which vanishes because of the vector nature of $\hat{\mathbf{r}}$.

[1] J. J. Thomson, *Elements of the Mathematical Theory of Electricity and Magnetism* (Cambridge U. P., Cambridge, 1909), 4th ed., p. 532.

[2] J. J. Thomson, *Recollections and Reflections* (Macmillan, New York, 1937), p. 370.

[3] We view e and g as sources of $\mathbf{D}$ and $\mathbf{B}$, respectively. The pertinent Maxwell equations are $\nabla\cdot\mathbf{D} = \rho_{\text{electric}} = e\,\delta(\mathbf{r}+\mathbf{a}/2)$ and $\nabla\cdot\mathbf{B} = \rho_{\text{magnetic}} = g\,\delta(\mathbf{r}-\mathbf{a}/2)$.

[4] P. A. M. Dirac, Proc. R. Soc. London Ser. A **133**, 60 (1931).

[5] I. Adawi, Am. J. Phys. **44**, 762 (1976).

[6] A. M. Portis, *Electromagnetic Fields: Sources and Media* (Wiley, New York, 1978), pp. 405–408.