12, 187 (1964).
[18]J. B. Mehl and W. Zimmerman, Jr., Phys. Rev. Letters 14, 815 (1965).
[19]K. R. Atkins, *Liquid Helium* (Cambridge U. P., New York, 1959).

PHYSICAL REVIEW A VOLUME 2, NUMBER 4 OCTOBER 1970

Runaway Solutions: Remarks on the Asymptotic Theory of Radiation Damping*

William L. Burke†
Lick Observatory, Board of Studies in Astronomy and Astrophysics, University of California, Santa Cruz, California 95060
(Received 13 April 1970)

This paper examines the "runaway solutions" (exponentially diverging solutions) which often arise in slow-motion approximations such as classical electron theory. Detailed study of a typical runaway shows that it is a spurious solution attributable to faulty approximation technique rather than a physical result connected with, for example, the point-electron assumption. Methods for consistently using the slow-motion approximation while avoiding these spurious solutions are given.

In this paper we will discuss some general features that arise in systems where one has an oscillator coupled to a field (continuum) capable of transmitting waves with a finite speed. One expects the coupling both to lower the frequency of the oscillator due to an increased effective inertia and also to cause the amplitude of the oscillator to decay due to energy lost in outgoing radiation. For systems where the oscillator is much smaller than a wavelength of the resulting radiation, these effects are computable by perturbation theory. However, along with the physically reasonable solutions, one also finds unexpected solutions characterized by an exponential growth which is obviously physically impossible. These impossible solutions grow on a time scale so short that the initial mathematical assumptions of the perturbation scheme are violated. Hence, the occurrence of such solutions must be due to faulty perturbation technique. The problem is not a routine singular perturbation problem, but incorporates special features coming from delay terms implicit in the equations. The delay expresses the dependence of the motion on its past history and is due to the finite propagation speed of the medium to which the oscillator is coupled. Runaway solutions seem to be a common feature of systems incorporating a delay mechanism. Such spurious solutions have arisen in many places, most notably in classical electromagnetism,[1,2] where a great deal of effort has been expended in trying to interpret the results physically. Here we shall discuss this phenomenon in detail, using a simple mechanical system as a concrete example. For such a system, explanations involving infinite negative self-energies or noncausal behavior over short times are clearly inappropriate. Instead, we will resolve the difficulties by a careful use of ideas from singular perturbation theory.

The system that we will use as an example consists of a spherical elastic shell which oscillates radially, radiates acoustic waves, and suffers radiation damping.

Let $\tilde{r}$, $\tilde{t}$ equal the dimensional radius and time coordinates, l equal the equilibrium radius of the sphere, $Z(\tilde{t})$ equal the radial position of the sphere at time $\tilde{t}$, a_0 equal the speed of sound, and $\tilde{\phi}$ equal the velocity potential. If $\tilde{\phi}$ satisfies

$$\nabla_{\tilde{r}}^2 \tilde{\phi} - \frac{1}{a_0^2} \frac{\partial^2 \tilde{\phi}}{\partial \tilde{t}^2} = 0, \tag{1}$$

then we can arrive at a solution of the linearized fluid equations by taking a velocity given by

$$\vec{v} = \vec{\nabla} \tilde{\phi} \tag{2}$$

and a pressure given by

$$p - p_0 = \rho_0 \frac{\partial \tilde{\phi}}{\partial t}, \tag{3}$$

where p_0 and ρ_0 are the static pressure and density of the medium. The boundary conditions are the expression

$$\frac{\partial \tilde{\phi}}{\partial \tilde{r}} [Z(t)] = \frac{dZ}{d\tilde{t}} \tag{4}$$

and the restriction to outgoing waves for large $\tilde{r}$.

We can introduce the mechanical parameters of the elastic shell by means of an "effective mass" m and "effective elasticity" k such that the equation of motion for the shell becomes

$$m\frac{d^2\tilde{Z}}{d\tilde{t}^2}+k[Z(t)-l]=4\pi\rho_0 Z^2\frac{\partial\tilde{\phi}}{\partial\tilde{t}}[Z(t)]. \tag{5}$$

We will now make several approximations. First, we will assume small displacements. We formalize this by introducing η, the ratio of a typical displacement to the radius of the sphere, and a dimensionless amplitude variable of order unity,

$$\zeta\equiv(Z-l)/\eta l. \tag{6}$$

We will assume that η is much smaller than any power of ϵ of interest, where ϵ is the small parameter that will occur in the slow-motion approximation. So severe a restriction on the size of η is unnecessary, but will serve to keep the example uncluttered. Some restriction on the size of η is necessary to justify our use of linearized acoustics.

We introduce a dimensionless parameter κ to measure the coupling between the shell and the medium:

$$\kappa\equiv 4\pi\rho_0 l^3/m. \tag{7}$$

Here we will assume only that κ is not large.

The approproximation of particular interest involves the assumption that the size of the radiating system l is much smaller than the wavelength of the resulting acoustical radiation, λ. If the coupling between the radiation field and the shell is not extremely large, then the oscillations will take place on a time scale $1/\omega$, where

$$\omega=(k/m)^{1/2}. \tag{8}$$

This leads to radiation whose wavelength is of order λ:

$$\lambda\equiv a_0/\omega. \tag{9}$$

We now assume that

$$\epsilon\equiv l/\lambda\ll 1. \tag{10}$$

Let us transform to dimensionless variables. The length scale appropriate for the acoustical radiation is λ; therefore, a suitable dimensionless radial coordinate is

$$r\equiv\tilde{r}/\lambda. \tag{11}$$

The time scale for the acoustical radiation is $1/\omega$; therefore, we take

$$t\equiv\omega\tilde{t}=a_0\tilde{t}/\lambda. \tag{12}$$

Using these variables, we write the wave equation [Eq. (1)]

$$\nabla_r^2\phi-\partial_t^2\phi=0. \tag{13}$$

Near the body, the appropriate length scale is l; the radial coordinate is

$$R\equiv\tilde{r}/l. \tag{14}$$

We define a dimensionless velocity potential

$$\phi\equiv\tilde{\phi}/\epsilon\eta a_0 l. \tag{15}$$

The boundary condition [Eq. (4)] becomes

$$\left(\frac{\partial\phi}{\partial R}\right)_{(R=1)}=\frac{d\zeta}{dt}, \tag{16}$$

where the smallness of η has been used to transfer the boundary condition to the equilibrium position of the shell. The equation of motion of the shell becomes

$$\frac{d^2\zeta}{dt^2}+\zeta=\kappa\left(\frac{\partial\phi}{\partial t}\right)_{(R=1)}. \tag{17}$$

There is another time scale l/a representing the time required to communicate around the body. This time scale will be discussed later.

To get a uniformly valid expansion of the velocity potential ϕ for small ϵ, we will have to match an expansion valid where $R=O(1)$, from which we can compute the pressure on the sphere, with an expansion valid where $r=O(1)$, to which we can apply our outgoing-wave boundary condition. [3]

Let us assume an inner expansion for ϕ of the form

$$\phi\sim A(R,t)+\epsilon B(R,t)+\cdots. \tag{18}$$

The terms satisfy the equations

$$\nabla_R^2 A=0,\quad \nabla_R^2 B=0,\ldots, \tag{19}$$

with boundary conditions

$$\frac{\partial A}{\partial R}(1,t)=\frac{d\zeta}{dt},\quad \frac{\partial B}{\partial R}(1,t)=0. \tag{20}$$

These have solutions

$$A(R,t)=-\frac{d\zeta}{dt}\frac{1}{R}+\alpha(t), \tag{21}$$

$$B(R,t)=\beta(t), \tag{22}$$

and the functions α and β are to be determined by matching this expansion with the appropriate outgoing-wave solution of the outer expansion.

Let us assume an outer expansion for ϕ of the form

$$\phi\sim a(r,t)+\epsilon b(r,t)+\cdots. \tag{23}$$

All of the terms satisfy equations of the form

$$\partial_r^2(ra)-\partial_t^2(ra)=0. \tag{24}$$

The general outgoing-wave solution is

$$a(r,t)=W(t-r)/r. \tag{25}$$

The functions W, α, β, and b can now be determined by matching. Expanding Eq. (25) for small r, we have

$$a(r,t) \to W(t)/r - W'(t) + \cdots , \tag{26}$$

and matching this with Eqs. (21) and (22) gives us

$$W(t) = -\frac{d\zeta}{dt} , \tag{27}$$

$$\alpha(t) = 0, \tag{28}$$

$$\beta(t) = \frac{d^2\zeta}{dt^2} , \tag{29}$$

$$b(r,t) = 0. \tag{30}$$

The expansion for ϕ valid at $\tilde{r} = l$ is therefore

$$\phi \sim -\frac{d\zeta}{dt} + \epsilon\frac{d^2\zeta}{dt^2} + O(\epsilon^2) . \tag{31}$$

Substituting this into the equation of motion [Eq. (17)] we find an equation of motion

$$(1+\kappa)\frac{d^2\zeta}{dt^2} + \zeta - \epsilon\kappa\frac{d^3\zeta}{dt^3} + O(\epsilon^2) = 0, \tag{32}$$

the result of routine singular-perturbation techniques.

The difficulty can now be seen by looking at the modes of this equation of motion. There are two of the form

$$\zeta = \exp(\pm i\nu t - \tfrac{1}{2}\epsilon\kappa\nu^4 t), \tag{33}$$

where

$$\nu^2 \equiv 1/(1+\kappa), \tag{34}$$

which show both the modification of the original modes and also a radiation damping of order ϵ. In addition, there is a third mode, given approximately by

$$\zeta \approx e^{+t/\kappa\epsilon} , \tag{35}$$

which diverges at $t \to +\infty$. When such a divergent solution appeared in the theory of classical electromagnetism applied to the "point" electron, it was called a "runaway solution," and prompted some remarkable attempts to circumvent its appearance, attempts including even abandoning causality. [2] In the present problem, the runaway mode is clearly impossible. A clue to its formal occurrence is given by the fact that its growth time violates the original perturbation assumption.

To investigate the runaway solution further, we now solve Eqs. (13), (16), and (17) without assuming that $\epsilon \ll 1$.

The general outgoing-wave solution of (13) can be written

$$\phi = W(t-r)/r. \tag{36}$$

The boundary condition [Eq. (16)] gives us the requirement

$$-\epsilon W'(t-\epsilon) - W(t-\epsilon) = \epsilon\frac{d\zeta}{dt} . \tag{37}$$

The motion of the oscillator is thus described by the third-order system

$$\zeta'' + \zeta - \kappa\Phi' = 0, \tag{38}$$

$$\zeta' + \epsilon\Phi' + \Phi = 0, \tag{39}$$

$$\Phi(t) \equiv W(t-\epsilon)/\epsilon . \tag{40}$$

The characteristic equation satisfied by the frequency of an exponential solution

$$\zeta = Ae^{pt}, \quad \Phi = Be^{pt} \tag{41}$$

is

$$(1+\epsilon p)(p^2+1) + \kappa p^2 = 0. \tag{42}$$

This equation gives us two oscillatory damped modes with decay time $t \sim 1/\epsilon$; one sees that Eq. (33) gives a correct approximation to these modes. In addition there is an overdamped mode

$$p \approx -(1+\kappa)/\epsilon, \tag{43}$$

with decay time $t \sim \epsilon$.

Now Eq. (42) contains no trace of the runaway solution. To see how the runaway solution comes in, let us write the exact equation of motion explicitly. We can solve Eq. (39) for Φ:

$$\Phi = ce^{-t/\epsilon} - \epsilon^{-1}\int_{-\infty}^{t} e^{(s-t)/\epsilon}\zeta'(s)\,ds. \tag{44}$$

This leads to an exact (in ϵ) equation of motion:

$$\frac{d^2\zeta}{dt^2} + \zeta + \frac{\kappa}{\epsilon}\int_{-\infty}^{t} e^{(s-t)/\epsilon}\zeta'(s)\,ds = 0. \tag{45}$$

We see that a delaylike term [4] has appeared explicitly. The characteristic equation corresponding to this equation of motion is

$$p^2 + 1 + \kappa p^2/(1+\epsilon p) = 0. \tag{46}$$

By integrating Eq. (45) by parts repeatedly, we can generate a sequence of approximate equations of arbitarily high order. Our previous approximation [Eq. (32)] is recovered by truncating this sequence to $O(\epsilon)$. In terms of the algebraic equation (46), this procedure is just the expansion of $(1+\epsilon p)^{-1}$ in powers of ϵ. Such an expansion will generate algebraic equations of arbitrarily high degree. These equations have only two roots which are $O(1)$; the rest are $O(1/\epsilon)$, and give rise to more and more runaway solutions as more and more terms in the expansion are kept. These spurious solutions arise as our expansion tries to

describe a genuine fast mode [Eq. (43)], which violates the assumptions underlying the approximation scheme (i.e., that time derivatives are of order unity).

We could analyze the runaway solutions in the above problem by comparing the approximation with the exact result.[5] Suppose now that we deal with a more complicated but basically similar problem. The analytical work will then have to rely only on perturbation methods.

In some cases, we will be faced with an initial-value problem. Typical initial values to be prescribed for our model problem are position ζ, velocity ζ', and pressure Φ'. The highly damped mode will be needed to satisfy general initial conditions, and we will need to consider a boundary layer in time of thickness l/a. The problem would thus be singular both in space *and* in time. Motions are not slow in the initial boundary layer, and the detailed description (resonances, etc.) will present formidable difficulties.

In other cases, and these are more common, we will be given a system excited by a slowly varying external force, or will be interested only in the behavior of the system after some unspecified initial excitation. In these cases very little ($\sim\epsilon^2$) of the fast mode is excited and we deal essentially with outer solutions. A routine approach would be to insert an expansion for $\zeta(t)$:

$$\zeta(t)\sim\zeta_0(t)+\epsilon\zeta_1(t)+\cdots \tag{47}$$

into Eq. (32), to find

$$(1+\kappa)\zeta_0''+\zeta_0=0, \tag{48}$$

$$(1+\kappa)\zeta_1''+\zeta_1=\kappa\zeta_0'''. \tag{49}$$

These equations have no runaway solutions, and simple two-variable methods[6] allow one to remove the secular terms and compute the damping. An alternative method, which also uses the fact that we deal with outer solutions, is the following.

Consider the more general equation resulting from a nonlinear oscillator coupled to the medium

$$\frac{d^2\zeta}{dt^2}+f(\zeta)-\epsilon\kappa\frac{d^3\zeta}{dt^3}=O(\epsilon^2). \tag{50}$$

Differentiating (50) and using the result to eliminate the third derivative [which is $O(1)$ for outer solutions] gives us the equation

$$\frac{d^2\zeta}{dt^2}+f(\zeta)+\epsilon\kappa f'(\zeta)\frac{d\zeta}{dt}=O(\epsilon^2). \tag{51}$$

In the linear case, we could simply throw away the unstable mode. However, when numerical integration is necessary, the instability of Eq. (50) will cause difficulties which are eliminated by using Eq. (51) instead. This equation still contains the damping information but is not unstable.

For example, the equation of motion for a classical electron derived from a slow-motion (i.e., point-electron) assumption is

$$m\ddot{Z}+V'(Z)-\tfrac{2}{3}(e^2\dddot{Z}/c^3)=F(t), \tag{52}$$

where $V(Z)$ is a potential and $F(t)$ is an applied force. This equation clearly suffers from the presence of spurious runaway solutions. If the function $F(t)$ varies only on a time scale long compared to the time $\tau\equiv e^2/mc^3$, we can use the above trick to derive a new equation:

$$m\ddot{Z}+\tfrac{2}{3}\tau V''(Z)\dot{Z}+V'(Z)=F(t+\tfrac{2}{3}\tau)+O(\tau^2). \tag{53}$$

This equation does not contain any spurious solutions. Any attempt to use the advance term $F(t+\frac{2}{3}\tau)$ to violate causality requires changing $F(t)$ on the time scale τ and violates the assumptions under which Eq. (52) describes the system. An exact equation, corresponding to Eq. (45), suitable for such short time scales requires a detailed description of the charge distribution.

There is a general mathematical feature of the present problem which occurs in many important physical problems. Consider Eq. (45): The integral relates the force to the time history of the motion. It is formally equivalent to an infinite sum of derivatives. This is typical of delay equations. If, for instance, the term $(t-\epsilon)$ had been present, it would have been formally equivalent to

$$\zeta(t-\epsilon)=e^{-\epsilon d/dt}\zeta(t)=\sum_{k=0}^{\infty}\frac{(-\epsilon)^k}{k!}\frac{d^k\zeta}{dt^k}. \tag{54}$$

Use of any partial sum may lead to difficulties when $d/dt=O(1/\epsilon)$. Formally, Eq. (45) may be regarded as a differential equation of infinite order. In the singular perturbation, we lose infinitely many derivatives in the outer equation. For an inner equation, no finite number of them will do. In many radiation problems and also in statistical mechanics, integrals over finite domains occur. Even if these domains are $O(\epsilon)$, use of a cutoff series of derivatives may lead to difficulties when we deal with inner solutions or when we deal with outer solutions but fail to utilize this fact fully.

I should remark that the approach used here is quite different from that of those, e.g., Rohrlich[2] or Plass,[7] who attempt to use the extra integration constant to select a solution which does not, in fact, run away. Such an approach is justified in electromagnetism only if the equation arrived at by Dirac is considered exact despite the difficulties of the limit process (which suffers always from spurious solutions) used to derive it. Such an approach is of no use in resolving the runaway

difficulties that arise when the slow-motion approximation is applied to other physical problems.

The ideas presented here arose during a study[8] of gravitational radiation damping, where the problem of runaway solutions had to be solved before work could safely proceed. I would like to thank Professor P. A. Lagerstrom for help with this problem.

---

*Lick Observatory Bulletin No. 608

†Also member of the Board of Studies in Physics, U. of Calif. Santa Cruz, Calif.

[1] P. A. M. Dirac, Proc. Roy. Soc. (London) A167, 148 (1938).

[2] F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965).

[3] This is a routine problem in matching. See, for example, J. Cole, *Perturbation Methods in Applied Mathematics* (Blaisdell, Waltham, Mass., 1968).

[4] For a discussion of delay equations, see T. L. Saaty, *Modern Nonlinear Equations* (McGraw-Hill, New York, 1967), Chap. 5.

[5] D. J. Kaup, Phys. Rev. 152, 1130 (1966) has also discussed the role of higher terms in removing runaway solutions for the case of the classical electron. Here we are interested in addition with reconciling the approximate results with the correct results to improve the approximation scheme.

[6] See J. Cole, Ref. 3.

[7] G. N. Plass, Rev. Mod. Phys. 33, 37 (1961).

[8] W. L. Burke (unpublished).

PHYSICAL REVIEW A VOLUME 2, NUMBER 4 OCTOBER 1970

Thermodynamic Properties of hcp He^4

Guenter Ahlers
Bell Telephone Laboratories, Murray Hill, New Jersey 07974
(Received 13 March 1970)

Precision heat-capacity measurements at constant volumes for hcp He^4 are presented for molar volumes between 13.7 and 20.8 cm^3 and for temperatures between 1.3 °K and the melting temperature. These data are used to calculate the deviations of the equation of state for hcp He^4 from the Grüneisen equation of state. It is found that the reduced Debye temperature Θ/Θ_0 is *not* a volume-independent function of the reduced temperature T/Θ_0 and that the Grüneisen parameter γ is both volume and temperature dependent. It is observed that γ and Θ/Θ_0 at a given volume and T/Θ_0 are the same for hcp He^4 and hcp He^3, and it is suggested that the observed volume and temperature dependence of γ and Θ/Θ_0 are typical for close-packed van der Waals solids in general. The temperature-dependent contributions to other thermodynamic functions are given as well.

I. INTRODUCTION

In this paper detailed and precise constant-volume heat-capacity (C_v) measurements for hcp He^4 are reported. These data are sufficiently extensive to yield complete information about the temperature-dependent contributions to the equation of state for molar volumes greater than about 13 cm^3. The work was undertaken because a careful comparison of the thermal properties of hcp He^4 with available results[1] for hcp He^3 is expected to reveal any nonclassical isotope effect on the thermodynamic properties of the solids. If such nonclassical effects exist, they would be more noticeable in helium than in other simple solids, because the relative contribution of the zero-point energy to the total internal energy is larger here. If no such effects are observable, or if their nature is simple, it is not unreasonable to use solid helium as a model substance for the prediction of the properties of other close-packed solids whose binding is by van der Waals forces. There has been great temptation in the past[2] to look upon solid helium as such a model substance because its thermal properties can be studied readily over a large volume range. For most other solids, such an investigation would be very difficult because very large pressures would be required.

If solid helium is to be looked upon as a model solid, then it is of interest to examine the thermal properties of He^4 in detail and to compare them to simple equations of state such as the one proposed by Grüneisen.[3] If the isotope effect in helium is found to be essentially classical, it might be hoped that any observed deviations from the