Linear Momentum of Quasistatic Electromagnetic Fields*

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(Received 2 March 1966)

It is shown that the net mechanical impulse one must exert to change one static system of currents and charges slowly into another such system is independent of the method used, and is given by the change in $\sum q_i \mathbf{A}^T(\mathbf{x}_i,t)$, where $\mathbf{A}^T(\mathbf{x}_i,t)$ is the transverse vector potential, evaluated at the position $\mathbf{x}_i$ of the charge q_i. The linear momentum stored in a quasistatic electromagnetic field is thus given by $\sum q_i \mathbf{A}^T(\mathbf{x}_i,t)$. These considerations show that it is possible to give the transverse vector potential $\mathbf{A}^T$ of a static system a physical meaning. Namely, that $\mathbf{A}^T(\mathbf{x})$ is the net mechanical impulse one must exert to bring a unit point charge slowly from infinity to the point $\mathbf{x}$.

INTRODUCTION

IT has long been realized that in order to conserve energy and momentum, it is necessary to endow electromagnetic (EM) radiation with energy and momentum. In quasistatic fields, that is, EM fields in which there is no radiation, the situation is not so satisfactory. Although the concept of energy in quasistatic electric and magnetic fields is discussed at length in most standard texts on EM theory, it is not commonly appreciated that it is necessary to assume that quasistatic EM fields also possess linear momentum. This fact is mentioned by Feynman,[1] who introduces the angular momentum of a quasistatic EM field to resolve a paradox, and by Cullwick,[2] who discusses the idea of action and reaction in a system interacting via EM forces. The following remarks are based on the ideas discussed in their books.

I. AN EXAMPLE

In view of the unfamiliarity of the concept of linear momentum in a quasistatic EM field, it is useful to begin the discussion by considering a specific situation, adapted from Cullwick. This serves to introduce most of the ideas which later are treated in more generality. A system consists of an infinitely long circular solenoid, sitting at rest and initially carrying no current, and a point charge q also sitting at rest outside the solenoid a distance r from the axis of the solenoid (Fig. 1). Initially, there is no linear momentum. The current in and hence the magnetic flux Φ through the solenoid is now slowly increased. According to Faraday's law an electric field $\mathbf{E}$ is generated, which, at the position of the charge, has the magnitude

$$\mathbf{E} = (\dot{\Phi}/2\pi r)\mathbf{i}\,. \tag{1}$$

The charge is acted upon by a force

$$\mathbf{F}_{\mathrm{EM}} = q\mathbf{E} = (q\dot{\Phi}/2\pi r)\mathbf{i} \tag{2}$$

and tends to move. Suppose, however, that one exerts on the charge a mechanical force equal and opposite to that exerted by the electric field

$$\mathbf{F}_{\mathrm{ME}} = -\mathbf{F}_{\mathrm{EM}} = -(q\dot{\Phi}/2\pi r)\mathbf{i}\,. \tag{3}$$

The charge thus remains at rest. The solenoid feels no net force so it too remains at rest. When the flux through the solenoid reaches some value Φ, the current and hence the flux is held constant. In the final state the solenoid is at rest and no net force acts on it. The charge is at rest and no force acts on it. There is again no mechanical linear momentum. However, to reach this state from the initial one, one has had to supply to

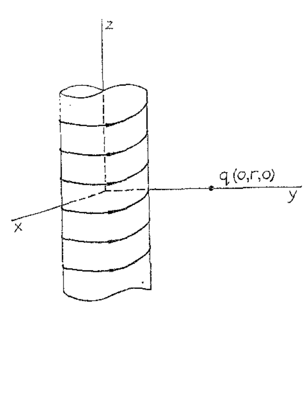

FIG. 1. The charge and solenoid.

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* This research was supported by NRC Grant A-2207.

[1] R. P. Feyman, *The Feyman Lectures on Physics* (Addison–Wesley Publishing Co., Inc., Reading, Mass., 1964), Vol. II, Secs. 17.4, 27.6.

[2] E. G. Cullwick, *Electromagnetism and Relativity* (Longmans, Green and Co. Ltd., London, 1959), 2nd ed., Chap. 17.

the system an impulse

$$\mathbf{I}_{\rm ME}=\int dt\mathbf{F}_{\rm ME}=-(q\Phi/2\pi r)\mathbf{i}\,. \qquad (4)$$

Thus, the linear momentum of the system has changed. Since the mechanical linear momentum is the same before and after, the only place to store this momentum, equal to the right-hand side of Eq. (4), is in the static EM field which exists once the flux is established in the solenoid.

It might, of course, turn out that the net impulse one must apply to the system to reach the final state depends not only on the state but on the method by which the state was reached. If this were the case, then one would not only need to know the currents and charges in the final state but at all times in the past. It is shown later that the net impulse one must apply is indeed independent of the method used, provided that the system is set up sufficiently slowly. For the moment, a consideration of the following alternate method for reaching the final state will serve to make this plausible.[3]

Suppose one initially places the charge a great distance from the solenoid, say on the y axis, then slowly increases the current in the solenoid to its final value, and then slowly brings the charge up to its final position along the y axis. In this case negligible force acts on either the charge or the solenoid as the current is increased in the solenoid. Further, no force acts on the charge as it is brought up, since it always moves through a field-free region. However, the moving charge generates a magnetic field. This magnetic field acts on the current flowing in the solenoid, giving a net EM force on the solenoid in the positive x direction. To prevent the solenoid from accelerating, one must counterbalance this force by a mechanical force on the solenoid in the negative x direction. A direct computation shows that the total impulse one must give the system to reach the final state is again given by Eq. (4).

II. GENERALIZATIONS

It remains to generalize these ideas. For this purpose, consider a system consisting of stationary current-carrying loops of wire, the current density at a point $\mathbf{x}$ being $\mathbf{J}(\mathbf{x})$, and stationary charges, the charge density at a point $\mathbf{x}$ being $\rho(\mathbf{x})$. The individual loops and charges are acted on by EM forces, which one assumes to be counterbalanced by mechanical means. The net mechanical force one must apply to the system as a whole to keep it from accelerating is zero. This system can be changed in various ways. First, one can slowly change the current flowing through each of the loops. Then, according to Faraday's law, an electric field $\mathbf{E}^T$ is induced, where

$$\mathbf{E}^T(\mathbf{x},t)=-\dot{\mathbf{A}}^T(\mathbf{x},t). \qquad (5)$$

$\mathbf{A}^T(\mathbf{x},t)$ is the transverse vector potential[4], which, for sufficiently slowly varying currents, is given by

$$\mathbf{A}^T(\mathbf{x},t)=\frac{\mu_0}{4\pi}\int d^3\mathbf{x}'\frac{\mathbf{J}^T(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\,. \qquad (6)$$

This electric field acts on the charges, exerting on them a net force

$$\mathbf{F}_{\rm EM}=\int d^3\mathbf{x}\rho(\mathbf{x})\mathbf{E}^T(\mathbf{x},t)$$
$$=-(d/dt)\int d^3\mathbf{x}\rho(\mathbf{x})\mathbf{A}^T(\mathbf{x},t)\,. \qquad (7)$$

To keep the charges at rest, one must counterbalance this force by mechanical forces, the net mechanical force one must apply to the charges being

$$\mathbf{F}_{\rm ME}=-\mathbf{F}_{\rm EM}=d/dt\int d^3\mathbf{x}\rho(\mathbf{x})\mathbf{A}^T(\mathbf{x},t). \qquad (8)$$

The forces which the individual current loops exert on one another change as the currents change. If the loops are to remain stationary, the mechanical forces applied to the loops must also change. However, the net mechanical force one must apply to all the loops remains zero. Thus, the net impulse one must apply to the system during the time in which the currents and hence the vector potential change from some initial to some final value is simply

$$\mathbf{I}_{\rm ME}=\int dt\mathbf{F}_{\rm ME}=\Delta\int d^3\mathbf{x}\rho(\mathbf{x})\mathbf{A}^T(\mathbf{x},t)\,. \qquad (9)$$

[3] In this paper the word "state" refers to a particular configuration of currents and charges. Then, in the general, although not in the quasistatic, case the net impulse required to reach some final state *will* depend not only on the state but on the history of the system. In practice one avoids this by generalizing the concept of "state" to include electric and magnetic fields.

[4] The transverse vector potential $\mathbf{A}^T$ is the solution to the equations $\nabla\times\mathbf{A}^T=\mathbf{B}$, $\nabla\cdot\mathbf{A}^T=0$, which vanishes at infinity. $\mathbf{A}^T$ is invariant under gauge transformations.

For point charges q_i located at the points $\mathbf{x}_i$, this becomes

$$\mathbf{I}_{ME}=\Delta\sum q_i\mathbf{A}^T(\mathbf{x}_i,t)\ . \qquad (10)$$

A second way in which the system can be changed is to move the charges slowly from their original positions to some final position. It suffices to consider the effects of moving a single point charge q. As the charge moves, the electrostatic force which it feels varies. The electrostatic force which it exerts on the other charges also changes. However, the sum of the electrostatic forces acting on all the charges remains zero. In addition to the electrostatic force, the moving charge is acted on by the magnetic field of the loops, the force which the loops exert on the charge being

$$\mathbf{F}_{EM}(\text{charges})=q\mathbf{v}\times\mathbf{B}(\text{loops})$$
$$=\frac{\mu_0 q}{4\pi}\int d^3\mathbf{x}\frac{\mathbf{v}\times(\mathbf{r}\times\mathbf{J})}{\mathbf{r}^3}, \qquad (11)$$

where $\mathbf{r}=\mathbf{x}-\mathbf{x}'$ is the radius vector from the instantaneous position of the charge $\mathbf{x}'$ to the point $\mathbf{x}$, and $\mathbf{v}=\dot{\mathbf{x}}'$ is the instantaneous velocity of the charge. Since the charge is moving, it generates a magnetic field

$$\mathbf{B}(\text{charge})=(\mu_0 q/4\pi)(\mathbf{v}\times\mathbf{r}/r^3). \qquad (12)$$

This magnetic field exerts a force on the current carrying loops, the net force which it exerts being

$$\mathbf{F}_{EM}(\text{loops})=\int d^3\mathbf{x}\mathbf{J}\times\mathbf{B}(\text{charge})$$
$$=\frac{\mu_0 q}{4\pi}\int d^3\mathbf{x}\frac{\mathbf{J}\times(\mathbf{v}\times\mathbf{r})}{r^3}. \qquad (13)$$

This force is in addition to the forces which the loops exert on one another, and which sum to zero. If one is to prevent the loops and charges from accelerating, each of these forces must be counterbalanced by mechanical means. The net mechanical force one must apply to the system is

$$\mathbf{F}_{ME}(\text{system})$$
$$=-\mathbf{F}_{EM}(\text{charges})-\mathbf{F}_{EM}(\text{loops})$$
$$=-\frac{\mu_0 q}{4\pi}\int d^3\mathbf{x}\frac{\mathbf{v}\times(\mathbf{r}\times\mathbf{J})+\mathbf{J}\times(\mathbf{v}\times\mathbf{r})}{r^3}$$
$$=\frac{\mu_0 q}{4\pi}\left[\int d^3\mathbf{x}\frac{\mathbf{J}\mathbf{v}\cdot\mathbf{r}}{r^3}-\int d^3\mathbf{x}\frac{\mathbf{v}\mathbf{J}\cdot\mathbf{r}}{r^3}\right]. \qquad (14)$$

The second of these two integrals is zero. The first can be rewritten to give

$$\mathbf{F}_{ME}(\text{system})=(d/dt)(q\mathbf{A}^T), \qquad (15)$$

where $\mathbf{A}^T$ is the transverse vector potential due to the loops evaluated at the instantaneous position of the moving charge. The net impulse one must give the system as the charge is moved from its initial to its final position is

$$\mathbf{I}_{ME}=\int dt\mathbf{F}_{ME}=\Delta(q\mathbf{A}^T)\ . \qquad (16)$$

By combining these two processes, one in which the current density is changed, and one in which the charge density is changed, one can slowly change any static system of currents and charges into any other such system. In so doing, one must give the system a net impulse, equal to the change in $\sum q\mathbf{A}^T$. Since the change in mechanical linear momentum is zero, this impulse must equal the change in linear momentum stored in the EM field. Thus, the linear momentum of a static EM field is given by $\sum q\mathbf{A}^T$.

Although the above derivation clarifies the various forces acting on various parts of the system which contribute to the net force acting on the system as a whole, it does not make obvious the fact that the impulse depends only on the initial and final states, and not on the method by which one passes from the initial to the final state. To rectify this, a somewhat more sophisticated, although closely related, approach to the problem follows. Again, consider a system containing currents, the current density being $\mathbf{J}(\mathbf{x},t)$, and charges, the charge density being $\rho(\mathbf{x},t)$. For the moment, the currents and charges can vary arbitrarily with time. These currents and charges generate electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, which exert forces back on the currents and charges which generated them. The net force which the fields exert is given by

$$\mathbf{F}_{EM}=\int d^3\mathbf{x}(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B})\ . \qquad (17)$$

This expression is now rewritten in a more convenient form. Consider first the force exerted on the charges by the electric field

$$\mathbf{F}_E=\int d^3\mathbf{x}\rho\mathbf{E}\ . \qquad (18)$$

The electric field $\mathbf{E}$ can be written as the sum of a longitudinal field $\mathbf{E}^L(\nabla\times\mathbf{E}^L=0)$ and a transverse field, $\mathbf{E}^T(\nabla\cdot\mathbf{E}^T=0)$

$$\mathbf{E}=\mathbf{E}^L+\mathbf{E}^T. \tag{19}$$

Explicit expressions for $\mathbf{E}^L$ and $\mathbf{E}^T$ are

$$\mathbf{E}^L=(1/4\pi\epsilon_0)\int d^3\mathbf{x}'(\rho\mathbf{r}/r^3) \tag{20}$$

and

$$\mathbf{E}^T=-\dot{\mathbf{A}}^T. \tag{21}$$

$\mathbf{A}^T$ is the transverse vector potential. If the expression (19), together with (20) and (21), is substituted into the formula for $\mathbf{F}_\mathrm{E}$, Eq. (18), one obtains

$$\mathbf{F}_\mathrm{E}=\frac{1}{4\pi\epsilon_0}\int\int d^3\mathbf{x}d^3\mathbf{x}'\frac{\rho(\mathbf{x},t)\rho(\mathbf{x}',t)\mathbf{r}}{r^3}-\int d^3\mathbf{x}\rho\dot{\mathbf{A}}^T. \tag{22}$$

The first term vanishes. One is left with

$$\mathbf{F}_\mathrm{E}=-\int d^3\mathbf{x}\rho\dot{\mathbf{A}}^T. \tag{23}$$

Now consider the force exerted on the currents by the magnetic field

$$\mathbf{F}_\mathrm{M}=\int d^3\mathbf{x}\mathbf{J}\times\mathbf{B}=\int d^3\mathbf{x}\mathbf{J}\times(\nabla\times\mathbf{A}^T). \tag{24}$$

By using the vector identity

$$\mathbf{U}\times(\nabla\times\mathbf{V})+\mathbf{V}\times(\nabla\times\mathbf{U})=\mathbf{U}\nabla\cdot\mathbf{V}+\mathbf{V}\nabla\cdot\mathbf{U}+\nabla\cdot(\mathbf{I}\mathbf{U}\cdot\mathbf{V}-\mathbf{U}\mathbf{V}-\mathbf{V}\mathbf{U}) \tag{25}$$

and Gauss's theorem one can write $\mathbf{F}_\mathrm{M}$ in the form

$$\mathbf{F}_\mathrm{M}=-\int d^3\mathbf{x}\mathbf{A}^T\times(\nabla\times\mathbf{J}^T)+\int d^3\mathbf{x}\mathbf{A}^T\nabla\cdot\mathbf{J}+\oint ds(\mathbf{A}^T\cdot\mathbf{J}\mathbf{n}-\mathbf{n}\cdot\mathbf{A}^T\mathbf{J}-\mathbf{n}\cdot\mathbf{J}\mathbf{A}^T). \tag{26}$$

The surface integral vanishes. By using the equation of continuity to rewrite the second term, one finds

$$\mathbf{F}_\mathrm{M}=-\int d^3\mathbf{x}\dot{\rho}\mathbf{A}^T-\int d^3\mathbf{x}\mathbf{A}^T\times(\nabla\times\mathbf{J}). \tag{27}$$

These expressions for the electric and magnetic forces, Eqs. (23) and (27), can now be combined to give the net force exerted by the EM field on the currents and charges which generated it

$$\mathbf{F}_\mathrm{EM}=-\frac{d}{dt}\int d^3\mathbf{x}\rho\mathbf{A}^T-\int d^3\mathbf{x}\mathbf{A}^T\times(\nabla\times\mathbf{J}). \tag{28}$$

No approximations have so far been made. If one now makes the assumption that ρ and $\mathbf{J}$ change slowly with time, further simplification occurs. In this quasistatic case the vector potential $\mathbf{A}^T$ is given by Eq. (6) to a first approximation. Substituting this expression for $\mathbf{A}^T$ into the second term in Eq. (28), one finds that the resulting integral vanishes. The corrections to Eq. (6) for $\mathbf{A}^T$ involve the second and higher time derivatives of $\mathbf{J}^T$. Thus, if the currents change sufficiently slowly with time, the second term in Eq. (28) can be made as small as one wishes in comparision to the first term in Eq. (28), which involves only the first time derivative. Thus, in the quasistatic case one can write

$$\mathbf{F}_\mathrm{EM}=-(d/dt)\int d^3\mathbf{x}\rho\mathbf{A}^T, \tag{29}$$

where now $\mathbf{A}^T$ is related to the current density by Eq. (6). The net mechanical force one must exert on the system to prevent it from accelerating is the negative of $\mathbf{F}_\mathrm{EM}$

$$\mathbf{F}_\mathrm{ME}=d/dt\int d^3\mathbf{x}\rho\mathbf{A}^T. \tag{30}$$

Since $\mathbf{F}_\mathrm{ME}$ is expressed as a total time derivative, the net impulse one must exert on the system to pass slowly from some initial to some final state is independent of the method used, and is given by

$$\mathbf{I}_\mathrm{ME}=\Delta\int d^3\mathbf{x}\rho\mathbf{A}^T. \tag{31}$$

Since the net mechanical linear momentum of the system remains unchanged, the change in the linear momentum of the EM field is given by the right-hand side of Eq. (31) and the field linear momentum itself by

$$\mathbf{P}_\mathrm{EM}=\int d^3\mathbf{x}\rho\mathbf{A}^T. \tag{32}$$

Note that $\mathbf{P}_\mathrm{EM}$ depends solely on the currents and charges in the state, and not on the method by which the state was reached, provided that the state was reached sufficiently slowly.

These considerations show that the transverse vector potential of a static EM field can be measured, although not by a process carried out at a single point, as in the case of the field strengths. To be specific, the transverse vector potential $\mathbf{A}^T(\mathbf{x})$ is the net impulse one must exert in slowly moving a unit point charge from infinity to the point $\mathbf{x}$. Alternatively, $\mathbf{A}^T(\mathbf{x})$ is the linear momentum acquired by the EM field when a unit point charge is brought from infinity to the point $\mathbf{x}$. This interpretation of $\mathbf{A}^T(\mathbf{x})$ is completely analogous to that given the electrostatic potential $\phi(\mathbf{x})$, which, if ϕ vanishes at infinity, is the work required to move a unit point charge from infinity to the point $\mathbf{x}$.

ACKNOWLEDGMENTS

The author would like to thank Dr. G. Stabilini and Dr. M. H. Jericho for helpful discussions.

APPENDIX

It is of interest to compare the expression (32) for the momentum stored in a static EM field to the expression for the momentum stored in an arbitrary EM field, namely

$$\mathbf{P}_{\mathrm{EM}}=\int d^3\mathbf{x}\epsilon_0\mathbf{E}\times\mathbf{B}. \qquad \text{(A1)}$$

It is relatively easy to show that the two expressions are equivalent in the appropriate limit. To see this, one writes the electric field in Eq. (A1) as the sum of a longitudinal and a transverse part, $\mathbf{E}^L$ and $\mathbf{E}^T$, obtaining

$$\mathbf{P}_{\mathrm{EM}}=\int d^3\mathbf{x}\epsilon_0\mathbf{E}^L\times\mathbf{B}+\int d^3\mathbf{x}\epsilon_0\mathbf{E}^T\times\mathbf{B}\,. \qquad \text{(A2)}$$

Using the vector identity (25), one can write the first term in Eq. (A2) in the form

$$\int d^3\mathbf{x}\mathbf{A}^T\nabla\cdot(\epsilon_0\mathbf{E}^L)$$
$$+\oint ds(\mathbf{A}^T\cdot\mathbf{E}^L\mathbf{n}-\mathbf{n}\cdot\mathbf{A}^T\mathbf{E}^L-\mathbf{n}\cdot\mathbf{E}^L\mathbf{A}^T)\,. \qquad \text{(A3)}$$

The surface integral vanishes, since at large distances from the source $\mathbf{A}^T$ falls off at least as fast as r^{-1}, and $\mathbf{E}^L$ falls off at least as fast as r^{-2}. One finds that, in general, the momentum stored in an EM field is given by

$$\mathbf{P}_{\mathrm{EM}}=\int d^3\mathbf{x}\rho\mathbf{A}^T+\int d^3\mathbf{x}\epsilon_0\mathbf{E}^T\times\mathbf{B}\,. \qquad \text{(A4)}$$

In the quasistatic limit, the second term in Eq. (A4), which involves $\mathbf{E}^T=-\dot{\mathbf{A}}^T$, is negligibly small in comparison to the first term. One then recovers Eq. (32).