Electromagnetic energy changes due to charges moving through constant, or zero, magnetic field

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Abstract: The paper examines the interaction between a charge q moving at constant velocity through a region in which the magnetic flux density may be zero, but the $\boldsymbol{A}$ vector is not. The simplest example of the source of $\boldsymbol{A}$ is a closed toroid carrying constant current. The motion of q induces an EMF in the toroid, causing an exchange of energy with the current source, although the toroid produces no external magnetic flux or induced electric field. It is shown that the apparent anomaly of a transfer of energy without a force on q is resolved by the changes of internal energy and by the definition of what is meant by the term 'force'. The interaction depends on the vector $\boldsymbol{A}$, and illustrates the consequences of a change of electrokinetic momentum, a concept, due to Maxwell, whose practical application has been explored in previous papers. Its predictions are, necessarily, consistent with those of field theory, but assigning the momentum and stored energy to the charges, instead of the field, clarifies the operating principle, defines the parameters in terms of inductance, and provides an equivalent circuit, as in other electrical machines, whereas the alternative field concept of flux linkage is difficult to apply to a charge q moving through no flux. The Aharonov-Bohm effect, which is commonly stated to be inconsistent with classical electromagnetism, provides an example of the momentum change in terms of the quantum-mechanical phase change due to the vector $\boldsymbol{A}$.

1 Introduction

The practical advantages of Maxwell's 'dynamical' interpretation of electromagnetism have been argued elsewhere [1], and have been shown to be a part of an electromagnetic theory in which the electric potential ϕ and the vector $\boldsymbol{A}$ replace the field vectors as the quantities of primary physical significance [2]. $\boldsymbol{A}$ is defined as the electrokinetic momentum per unit charge [3], rather than as the magnetic vector potential. Electrical and magnetic energies become the potential and kinetic energies of the charges, and are mutual in the sense that the energy allocated to any charge group depends on the proximity, or otherwise, of others. Momentum is likewise a mutual property. The action of a transformer depends on changes of mutual momentum with time, whereas that of a DC machine depends on changes with position.

One of the simplest examples of such a change is the movement of a charge q along the axis of a toroidal coil (Fig. 1). The progressive increase in the $\boldsymbol{A}$ vector due to q induces an EMF in the coil, in which the electron stream is the electromagnetic equivalent of a flywheel, exchanging momentum with q. Expressed in field terms, the moving charge is a source of magnetic flux, and the EMF is due to the rate-of-change of flux linkage with the toroid as q approaches. Thus the device forms an electrical generator, capable of delivering current and power into an electrical load. It offers possibilities for MHD applications and the like, and variants have been examined for such purposes [4].

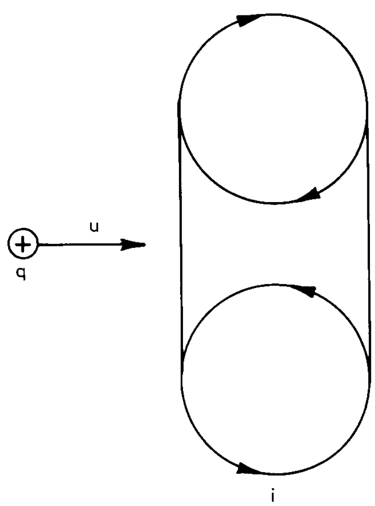

Fig. 1 *Mutual momentum exchange*
Toroid shown in cross-section

The objective of the paper is to clarify the nature of the interaction, which provides one of the most fundamental examples of an exchange of mutual momentum, and was studied for this reason. The device shown in Fig. 1 is unusual in several respects. Its operation is magnetic in the conventional sense, so that adding an iron core to the toroid, for example, will increase the voltage and power generation capability. But there is no magnetic, or $\boldsymbol{u} \times \boldsymbol{B}$, force on q, since the magnetic flux density $\boldsymbol{B}$ due to an ideal toroid is limited to the region enclosed by the winding, and is zero everywhere outside (except for the

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single-turn effect, which can be ignored if the number of turns is sufficiently large). Even if q were subjected to a magnetic field, the nature of the $\boldsymbol{u} \times \boldsymbol{B}$ force prohibits any exchange of energy.

Any change in the current i in the toroid causes a force on q because, in field theory, a change in flux in the interior region causes an electric field outside by 'transformer action'. But this is not a necessary part of the interaction, since an external current source may be used to keep i constant without affecting the power generation capability. The toroid then produces neither a magnetic field, nor an induced electric field, at any exterior point, although it is a source of $\boldsymbol{A}$. It is for this reason that the Aharonov-Bohm effect [5], which describes the same interaction in terms of the motion of a modulated electron beam around a small coil or magnetised whisker (Fig. 2), is widely regarded as anomalous [5–7]. The

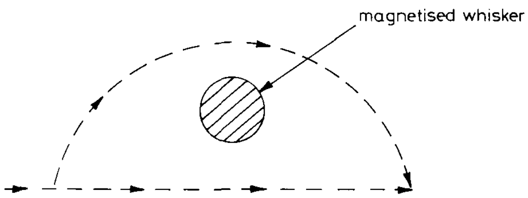

Fig. 2 *Aharonov-Bohm effect*

interference pattern formed by splitting the beam, and then recombining it, is observed to depend on the magnetisation in the whisker, in according with the Schrödinger equation, although the electron packets pass through no $\boldsymbol{B}$ field.

Forces due to changes in i are utilised in many applications, including betatrons and other forms of particle accelerator. Simple devices of this nature have been described as paradoxical [8] for precisely the opposite reason, namely that there appears to be no reaction on the coil to the angular acceleration forces in the external charges [9]. The toroid is usually replaced by a short coil, giving a flux density $\boldsymbol{B}$ outside it, which combines with the electric flux density $\boldsymbol{D}$ due to q, to give a field momentum, $\boldsymbol{D} \times \boldsymbol{B}$, whose changes with time account for the reaction, and measurements made with such a device have been given as direct experimental evidence of forces acting on empty space [10]. As has been shown in Reference 1, the electrodynamical description of the interaction clarifies it, and shows that the field momentum argument can be misleading in that there is a (delayed) reaction on the coil. The present paper extends this work to include the effects of the motion of q.

The operation of an energy convertor in the manner illustrated in Fig. 1 requires the separation of q from an opposite charge $-q$, and the process of separation and recombination is obviously important, partly because of the role of the 'electrostatic' forces, and also because the two charges must be recombined as a part of the operating cycle. Bringing up a charge $-q$ to meet q reverses the original action and gives zero net energy exchange, which is also the result of allowing q to continue through the coil to a remote point before recombining it with $-q$, since then the net change in flux linkage is zero. Thus the current must be varied at some stage in the operating cycle of any continuously acting generator, and in practical devices [4] the changes in current and in position will normally occur simultaneously. Magnetrons, klystrons and the like provide examples in which the operating frequency makes the wavelength comparable with the device dimensions, and the current in the toroidal coil is replaced by those on the surface of a resonant cavity, but the coupling principles remain the same, and the geometry similar. The superposition of the effects of the changes in current (i.e. of charge acceleration) with those due to charge velocity then tend to obscure the latter, and one of the purposes of the paper is to separate them. It is assumed, for simplicity, that q moves at a sufficiently low velocity to neglect retardation and also effects which are explicitly relativistic, although special relativity theory cannot be ignored since it accounts for the forces which are conventionally regarded as 'magnetic' (Section 9).

2 Dynamical theory

The approach to electromagnetics set out in References 1 and 2 is characterised by the assumption that the electromagnetic stored energy, power and momentum are properties of the charges, instead of the field, giving energy densities $\rho\phi/2$ and $\boldsymbol{J} \cdot \boldsymbol{A}/2$, in place of $\boldsymbol{E} \cdot \boldsymbol{D}/2$ and $\boldsymbol{H} \cdot \boldsymbol{B}/2$, where ρ is the charge density, $\boldsymbol{J}$ is the current density, and ϕ is the electric potential. $\boldsymbol{D} \times \boldsymbol{B}$ is replaced by the canonical momentum density $\rho\boldsymbol{A}$, which defines $\boldsymbol{A}$ as the momentum per unit charge, instead of the magnetic vector potential, and the term 'potential' will be reserved for ϕ, commonly referred to as the 'electrostatic potential', although the name can be misleading since the ϕ interaction is not confined to statics. In general, $\rho\phi/2$ is the potential, or separation, energy of the charges, and the term was used in this sense by Maxwell [3] to distinguish it from the kinetic energy, $\boldsymbol{J} \cdot \boldsymbol{A}/2$, of the 'electrical fluid', or moving charges, whose momentum defines $\boldsymbol{A}$. Maxwell developed his 'general equations of the electromagnetic field' by applying this 'dynamical theory' [3, 11], and his name provides, perhaps, the most descriptive way of distinguishing it from field theory*. The change in definitions means that the field vector $\boldsymbol{E}$ and $\boldsymbol{B}$, given by

$$\boldsymbol{E}_\phi = -\mathbf{grad}\ \phi$$

$$\boldsymbol{E}_A = -\partial \boldsymbol{A}/\partial t$$

combining to

$$\boldsymbol{E} = \boldsymbol{E}_\phi + \boldsymbol{E}_A \qquad (1a)$$

and

$$\boldsymbol{B} = \mathbf{curl}\ \boldsymbol{A} \qquad (1b)$$

are merely convenient labels for the differentials of ϕ and $\boldsymbol{A}$, and become auxiliary functions, thus reversing the customary roles of the field and potential quantities in field theory. The difference in approach is illustrated by Maxwell's use of eqn. $1b$ as the first of the 'general equations' of the *Treatise* [3], written with $\boldsymbol{B}$ on the left-hand side, not the right, $\boldsymbol{A}$ denoting 'the electromagnetic momentum' (art.604).

The observable behaviour of the charges is given by the force, of density

$$\boldsymbol{f} = \rho[\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}] \qquad (2)$$

or

$$\boldsymbol{f} = \rho[-\mathbf{grad}\ \phi - \partial \boldsymbol{A}/\partial t + \boldsymbol{u} \times (\mathbf{curl}\ \boldsymbol{A})] \qquad (3)$$

* What is meant by the 'dynamical theory' here is described in References 1 and 2, and differs in detail from Maxwell's; for example, he included the displacement current, $\partial \boldsymbol{D}/\partial t$, as a component of $\boldsymbol{J}$ in vacuo. Maxwell's treatment is examined in a paper in preparation entitled 'Maxwell's equations and the Maxwell force'.

and is the same whether expressed in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$, or $\boldsymbol{A}$ and ϕ. Eqn. 2 is conventionally referred to as the Lorentz force, but eqn. 3, with $\boldsymbol{B}$ in place of **curl** $\boldsymbol{A}$, was given by Maxwell [3] as the second of his 'general equations' (putting it in place of one of those universally referred to as the 'Maxwell equations'), and the separation into three components is fundamental to the definition of ϕ and $\boldsymbol{A}$, and thus to the dynamical theory. Field theory, on the other hand, is based on the vector $\boldsymbol{E}$ (like $\boldsymbol{B}$) as the underlying physical concept, defining a single condition of space which accounts for the force on a stationary charge, so that any components of $\boldsymbol{E}$ are mathematical fictions having no physical significance. Hence the common view that the 'back EMF' in a transformer, represented by the $\partial A/\partial t$ term in eqn. 3, and the 'applied voltage', obtained by integrating **grad** ϕ, are artificial concepts, and, some suggest, should be avoided. The argument is based on the assumption that the only meaningful velocity is that of the charge on which the force $\boldsymbol{f}$ acts, whereas the dynamical theory attaches equal importance to the velocities of the source charges, expressed in terms of the vector $\boldsymbol{A}$. That is, it distinguishes the interactions due to the excess ('static') charges on the transformer winding surface from those due to the conduction electrons in the conductor interior (together with the electron spins in any ferromagnetic material) whose motion causes the current. The symbol ϕ is needed, in place of V, to make the essential distinction between these two different sources of voltage, one of which is conservative whilst the other is not.

The difference is illustrated most clearly by writing the equations for ϕ and $\boldsymbol{A}$ in integral form (Reference 3, p. 257 or Reference 12),

$$\phi = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho}{r}\, dv \tag{4}$$

and

$$\boldsymbol{A} = \frac{1}{4\pi\varepsilon_0 c^2} \int \frac{\boldsymbol{J}}{r}\, dv \tag{5}$$

although any of the usual methods can be used for practical calculations. Here ρ is the net, or excess, charge density, and $\boldsymbol{J}$ is the current density in any volume element dv (taking the time-retarded values when propagation effects are significant, so that div $\boldsymbol{A}$ satisfies the Lorentz gauge [2]). Thus, in some applications (such as antennas), the dynamical and field theories differ only in interpretation, and they will always give identical results, provided that field theory is used in a self-consistent way. But, as has been shown elsewhere [1], this requirement is not always met, as in the treatment of the betatron 'paradox', because one consequence of assigning properties to the field tends to be a lack of adequate attention to the sources. The dynamical theory, in replacing eqn. 2 by eqn. 3, draws attention to both the two groups of charge which are involved in every electromagnetic interaction, and it is these which form the elements of the 'connected system' whose properties Maxwell examined in terms of Lagrangian dynamics, and expressed in terms of the momentum vector $\boldsymbol{A}$.

The energy convertor (Fig. 1) provides an illustration of the practical difference between the two theories. Although the flux density $\boldsymbol{B}$ (i.e. **curl** $\boldsymbol{A}$) is zero everywhere outside the coil, $\boldsymbol{A}$ is not (Fig. 3). The current i in the wire sets up a momentum vector $\boldsymbol{A}_i$, which forms closed lines parallel to the current in regions close to the wires, and here takes its maximum value, in accordance with eqn. 5. Thus $\boldsymbol{A}_i$ is not confined to the interior, but also encircles the coil outside, as we see from the vector potential interpretation, in which the $\boldsymbol{B}$ lines inside are

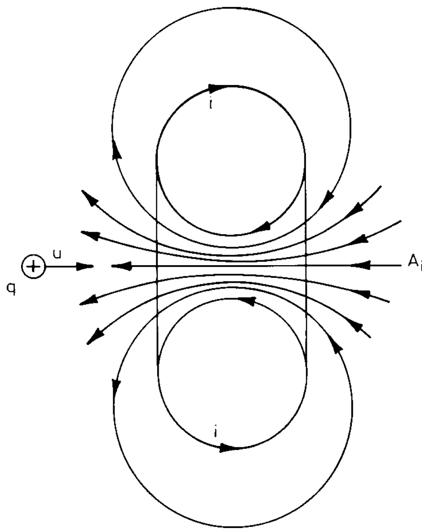

Fig. 3 *Momentum vector due to i*

the source of $\boldsymbol{A}_i$ outside, giving an electric field, $-\partial \boldsymbol{A}_i/\partial t$ (eqn. 1), as in a transformer, when i changes in time. Since i depends on velocity, any charges of density ρ outside the coil acquire a mutual momentum

$$\boldsymbol{G}_{12} = \int \rho \boldsymbol{A}_i\, dv \tag{6a}$$

due to the force exerted on them by changes in current, and it is these forces which cause the EMF in the secondary winding of a transformer in which the toroid is the primary. When the charges form a group with a total charge q, as in Fig. 1, it is convenient to abbreviate eqn. 6a to

$$G_{12} = q\boldsymbol{A}_i \tag{6b}$$

for simplicity and clarity, although the charge group q need not be small. (Here the symbol q is used to distinguish between total charge and charge density.)

The interaction is obviously reciprocal, since $\boldsymbol{A}_i$ is due to the movement of the conduction electrons, and the motion of q at velocity $\boldsymbol{u}$ likewise generates an $\boldsymbol{A}_q$ vector given by eqn. 5. Comparing the integrand with eqn. 4 shows that the result can be expressed in the form

$$\boldsymbol{A}_q = \boldsymbol{u}\phi_q/c^2 \tag{7}$$

where ϕ_q denotes the potential due to q, giving a simple and clear picture of the $\boldsymbol{A}_q$ 'field' due to q (and, in general, a very succinct, useful and general statement of the relationship between electrodynamics and electrostatics [13]). The EMF in the coil, due to the rate-of-change of $\boldsymbol{A}_q$, is the same as is given by the change in magnetic flux, since integrating eqn. 1b defines flux linkage

$$\Phi = \oint \boldsymbol{A} \cdot \boldsymbol{dl} \tag{8}$$

as the integral of the $\boldsymbol{A}$ vector around any turn of the coil. But the underlying explanation is very different. The forces on the conduction electrons, which are, in field terms, caused by the changes in $\boldsymbol{B}_q$ (due to q), are attributed instead to the mutual kinetic properties of the two sets of charge. Both represent action-at-a-distance effects

[14], one due to the fluxes in empty space, and the other due to the source charges. The EMF provides a direct illustration of the meaning of the vector $\boldsymbol{A}$, which is defined in momentum terms by the force which it causes when it changes in time.

The momentum interpretation shows at once the nature of the difficulties underlying both the field and momentum descriptions. The charge q is subjected to a change in $\boldsymbol{A}_i$, and thus to a force of density

$$f_q = \rho(-\mathbf{grad}\ \phi - \partial \boldsymbol{A}_i/\partial t) \tag{9}$$

in the reference frame R_q in which q is stationary, where

$$\partial \boldsymbol{A}_i/\partial t = (\partial \boldsymbol{A}_i/\partial z)(dz/dt) \tag{10}$$

is due to the movement of q in the direction z, and thus to the relative motion of the coil in the opposite direction. But the $\partial \boldsymbol{A}_i/\partial t$ term is not the only force on q in eqn. 9, and the point which is fundamental to the operation is whether the interaction is wholly 'magnetic', in the sense that it is confined to the $\boldsymbol{E}_A$ part of $\boldsymbol{E}$ in eqn. 3, or whether it also depends on $\boldsymbol{E}_\phi$. In the dynamical theory [2] the energy-flow vector is $\boldsymbol{J}\phi$, showing that ϕ is essential to all electrical generators (although disguised in the Poynting vector $\boldsymbol{E} \times \boldsymbol{H}$ by the contributions of both ϕ and $\boldsymbol{A}$ to $\boldsymbol{E}$), and this raises the question of the role of the 'electrostatic' forces in the description of a 'magnetic' device.

3 Electrokinetic momentum

The property of mutual momentum, and the forces which it causes, can be illustrated most directly by a system in which the toroidal coil is replaced by one or more insulated discs carrying charges around their rims (Fig. 4),

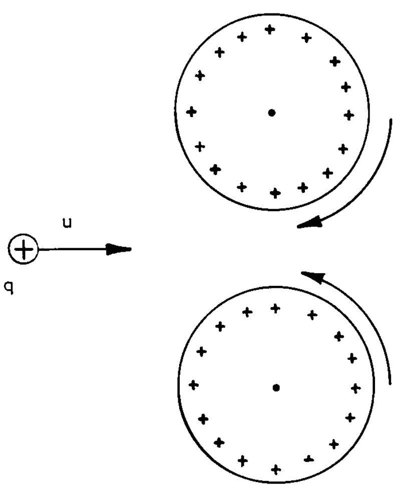

Fig. 4 *Spinning-disc equivalent*

each spinning in its own plane about its axis. The configuration is similar to that considered by Feynman *et al.* [9] as an example of the betatron-type force, caused by an acceleration of the discs, but here the forces on the discs are of direct interest, together with the reaction on q when the discs are spun at constant speed. By fixing the charges to the material the $\partial \boldsymbol{A}_q/\partial t$ forces due to q can be resisted by 'mechanical' forces*, in place of the $\mathbf{grad}\ \phi$ restraints which are typical of conductors, and the EMF tending to move the conduction electrons and crystal lattice charges in opposite directions is replaced by a net torque. That is, the discs replace electrokinetic, or differential, momentum by 'mechanical' momentum by removing the lattice charges. The electric forces due to ϕ_q are, by definition, conservative, so that the torque caused by the approach of q is accounted for by the momentum force of density

$$f_d = -\rho_d\, \partial \boldsymbol{A}_q/\partial t \tag{11}$$

on the disc charges, of density ρ_d. The torque depends on

$$\oint f_d \cdot d\boldsymbol{l} = -\rho_d \frac{d}{dt} \oint \boldsymbol{A}_q \cdot d\boldsymbol{l} \tag{12}$$

in accordance with the change in flux linkage (eqn. 8). If the mechanical constraints were removed, the discs would accelerate so as to conserve momentum,

$$\oint (m\boldsymbol{u}_d + \rho_d \boldsymbol{A}_d + \rho_d \boldsymbol{A}_q) \cdot d\boldsymbol{l} = 0 \tag{13}$$

where m is the relevant mass density, concentrated in the rim, $\boldsymbol{u}_d$ is the local velocity, and $\boldsymbol{A}_d$ is due to the charges on the discs. When m is negligible $\boldsymbol{A}_d$ opposes $\boldsymbol{A}_q$, in accordance with Lenz's law, which is a statement of momentum conservation.

The EMF consists of the sum of the peripheral components of the forces on the elements δq_2 of the charges on the discs, due to the elements δq_1 of q. The mutual momentum force on δq_2 is given by

$$\delta f_2 = -\delta q_2\, \partial \delta \boldsymbol{A}_{21}/\partial t$$

where $\delta \boldsymbol{A}_{21}$ is the $\boldsymbol{A}$ due to δq_1 'seen' by δq_2. Substituting from eqn. 7,

$$\delta \boldsymbol{A}_{21} = (\boldsymbol{u}_1 - \boldsymbol{u}_2)\delta\phi_{21}/c^2$$

if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are the linear velocities of the two elements, and $\delta\phi_{21}$ is the mutual potential. Combining this with the reciprocity condition

$$\delta q_1 \delta\phi_{12} = \delta q_2 \delta\phi_{21} \tag{14}$$

shows that the reaction is symmetrical, and that the kinetic, like the electrostatic, forces are equal but opposite

$$\delta f_1 + \delta f_2 = 0 \tag{15}$$

where

$$\boldsymbol{\delta f}_1 = -\delta q_1 \partial \delta \boldsymbol{A}_{12}/\partial t$$

However, the reference frames in which $\boldsymbol{\delta f}_1$ and $\boldsymbol{\delta f}_2$ are observed are those of the two different charges, and are not the same, so that the simplicity of electrostatic interaction is, inevitably, lost. Although each charge experiences a force due to the $\delta \boldsymbol{A}$ vector generated by the other, it sets up no $\boldsymbol{A}$ vector itself in the reference in which it is stationary and 'sees' no contribution from $\boldsymbol{A}$ to the force on the other charge. Moreover, although eqn. 15 leaves no net linear force, it leaves a net torque on the system comprising the two charges δq_1 and δq_2 since the forces δf_1 and δf_2 are in the two (opposite) directions of the $\boldsymbol{\delta A}$ vectors. If the charges are moving in straight lines, both $\delta \boldsymbol{A}$ and $\partial \delta \boldsymbol{A}/\partial t$ are in the direction of motion of the source charge, not along the line joining the two. Since $\boldsymbol{A}$ depends on the reference, the electromagnetic effects of motion are clearly not limited to the momentum forces defined by eqn. 11, but the $\mathbf{grad}\ \phi$ terms contribute; i.e. they also are electrodynamic. The same point emerges in eqn. 2 from the asymmetry of the $\boldsymbol{u} \times \boldsymbol{B}$ term, which acts

* Using the term 'mechanical' in the pragmatic sense. It raises the underlying question of the nature of the 'nonelectromagnetic' forces acting on electric charge.

at right-angles to the velocity $\boldsymbol{u}$, so that it cannot satisfy the action-reaction principle, and disappears in the reference in which the source is stationary*. Thus the effect of motion is not confined to the $\boldsymbol{u} \times \boldsymbol{B}$ term, but includes $\boldsymbol{E}$ (which is not the same as the ϕ term in eqn. 3).

The devices shown in Figs. 1 and 4 differ from conventional 'magnetic' machines in two essentials. Whereas the energy exchange is usually between closed circuits, the motion of the charge q around a closed path would make the net output energy zero, so that, whereas we may ignore ϕ in calculating the EMF in a closed winding, we may not do so in examining the force on q. Also, since q is not paired, its magnetic field $\boldsymbol{B}$ depends on the reference from which it is observed, and, although the operation must be self consistent in any reference, care is needed in specifying which one. Likewise, as in mechanics, the momentum must necessarily depend on the reference frame used to define the velocity, but with one essential difference. As is shown explicitly by eqn. 6, the momentum density $q\boldsymbol{A}$ acquired by a small charge q is a mutual property (it is merely another way of describing inductance) which depends on the velocity of the source of $\boldsymbol{A}$, not that of q. The force which is caused by a moving charge is nonconservative, and the difference between the ϕ and $\boldsymbol{A}$ components is not that the former is 'electrostatic', but that it is the conservative part of a force interaction which is relativistic in origin (as is shown directly by eqn. 7). Hence the ϕ term is not sufficient, and it is, of course, this which makes electrical machines, and the associated electromagnetic energy transfer, possible. But likewise neither is $\boldsymbol{A}$ (or $\boldsymbol{B}$) sufficient to describe the force on an isolated charge q.

Eqn. 7 shows that the spinning-disc model is wholly impractical as an energy convertor, since the dynamic forces on the discs, due to q, are too small to observe. Replacing them by a conducting wire, moving at half the drift velocity, introduces two sets of charges moving in opposite directions, and the symmetrical nature of this system gives $\boldsymbol{A}$ without any net ϕ, but only if the charges are constrained to a uniform charge density, which is not the property of a conductor. It is the vast amount of charge in the conduction energy band, compared with what is possible on a disc, which makes 'electromotive' force more important than the 'mechanical' torque, and gives the interaction which is illustrated in Fig. 1 a practical value.

4 Electrokinetic energy

A more practical form of 'flywheel' is obtained by short-circuiting a superconducting winding. The opposite forces on the conduction electron stream and lattice charges produce electrokinetic momentum due to the differential velocity, in place of the net, or 'mechanical', momentum, and the density of the charges produces a large $\boldsymbol{A}_i$ vector, due to the resulting current i, replacing $\boldsymbol{A}_d$ in eqn. 13. The $m\boldsymbol{u}$ term, due to the self-mass of the electrons, is now assumed to be negligible.

The electrokinetic energy, unlike the momentum, depends on the velocities of the charges which carry it, as well as those of the source of $\boldsymbol{A}$. The electrokinetic energy density (Reference 3, art. 634, or Reference 12) is $\boldsymbol{J} \cdot \boldsymbol{A}/2$, where the current density is

$$\boldsymbol{J} = \rho \boldsymbol{u}$$

due to charges of density ρ moving at velocity $\boldsymbol{u}$, and the total kinetic energy of the system becomes

$$U_A = \frac{1}{2} m_q u^2 + \frac{1}{2} \int_q \rho \boldsymbol{u} \cdot (\boldsymbol{A}_q + \boldsymbol{A}_i)\, dv + \frac{1}{2} \oint_{coil} \boldsymbol{J} \cdot (\boldsymbol{A}_q + \boldsymbol{A}_i)\, dv \tag{16}$$

in the inertial reference R_{coil} in which the coil is stationary. Here m_q is the mass of q, and the other two terms are integrated over q and over the circuit, respectively. The mutual components $\rho \boldsymbol{u} \cdot \boldsymbol{A}_i$ and $\boldsymbol{J} \cdot \boldsymbol{A}_q$ are negative when the current is induced, flowing in the direction shown in Figs. 1 and 3, and both the momentum and the energy are conserved.

The relationship between the two is illustrated by writing the first mutual term in eqn. 16 in the form

$$(U_A)_{12} = \boldsymbol{u} \cdot \boldsymbol{G}_{12}/2$$

(eqn. 6) assuming uniform velocity $\boldsymbol{u}$. Substituting from eqn. 5 shows that the two mutual terms are equal, and are given by

$$(U_A)_{12} = (U_A)_{21} = \frac{1}{8\pi\varepsilon_0 c^2} \int_q \oint_{coil} \frac{1}{r} \rho \boldsymbol{u} \cdot \boldsymbol{J}\, dv_q\, dv_c$$

where r is the distance between the volume elements dv_q and dv_c of q and of the coil, respectively. This simplies to

$$(U_A)_{12} = (U_A)_{21} = \frac{1}{8\pi\varepsilon_0 c^2} q i_c \oint_{coil} \frac{1}{r} \boldsymbol{u} \cdot \boldsymbol{dl}$$

when q is sufficiently small in volume, and the coil consists of line elements $\boldsymbol{dl}$ all carrying the same current i_c. In practice, q is likely to be cylindrical in shape, of length $\boldsymbol{\delta}_q$, so that

$$q\boldsymbol{u} = i_q \boldsymbol{\delta}_q$$

or this forms an equivalent. The energy can then be written in the form

$$U_A = [m_q u^2 + i_q^2 L_{11} + i_q i_c L_{12} + i_c i_q L_{21} + i_c^2 L_{22}]/2 \tag{17}$$

and the concept of inductance is not limited to closed circuits, as is implied by the customary field definition in terms of flux linkage (eqn. 8). Eqn. 17 shows that the operation of the device, like that of other machines, depends on the rate-of-change of inductance, and that the change is reciprocal, even though q moves through no magnetic field due to i_c.

Suppose i is initially zero when q is remote. The equilibrium, in the axial direction, of the conduction electrons, of density ρ_-, in the wire is given by

$$f_- = \rho_-[-\mathbf{grad}\ \phi - \partial(\boldsymbol{A}_q + \boldsymbol{A}_i)/\partial t] = 0 \tag{18}$$

since the $\boldsymbol{u} \times \mathbf{curl}\ \boldsymbol{A}$ term in eqn. 3 is zero in the direction of motion. The local mismatch between $\boldsymbol{A}_q$ and $\boldsymbol{A}_i$ produces nonzero div $\boldsymbol{J}$, and generates ϕ, but this makes no net contribution in the closed circuit (although the same term is important in considering the force on q). The momentum balance makes the second integral in eqn. 16 zero, and likewise the corresponding mutual and self

* Note also that, in field theory, the problem is complicated by the rotation. The electric field $\boldsymbol{E}$ experienced at any point on the discs, moving at local velocity $\boldsymbol{u}$, is $\boldsymbol{u} \times \boldsymbol{B}$, and has nonzero divergence so that it does not satisfy the Maxwell field equations. This is characteristic of any rotating (noninertial) reference frame, and it is, perhaps, unfortunate that most devices of interest to the machines engineer rotate. Hence the extensive literature on the electromagnetic mysteries of homopolar machines, whose operation depends on the nonzero divergence of $\boldsymbol{u} \times \boldsymbol{B}$.

energies in eqn. 17,

$$i_c i_q L_{21}/2 + i_c^2 L_{22}/2 = 0$$

so that the component of mutual energy assigned to the coil, and likewise the component assigned to q, goes negative. Suppose that the system is now returned to its original energy state by switching the current off in a sufficiently short time to allow no significant change in the position of q. The work which has to be expended on q to keep it moving at uniform velocity is given by

$$\int \boldsymbol{f}_q \cdot \boldsymbol{u}\, dt = \int q\boldsymbol{u} \cdot \partial \boldsymbol{A}_i/\partial t\, dt$$

$$= \int i_q L_{12}\, di_c/dt\, dt$$

and integrates to $i_q i_c L_{12}$; i.e. twice the amount of mutual energy which was extracted previously in the coil. The other half must have been extracted from q during the first part of the cycle, requiring a force, but due only to the change in i_c, not to the change in position.

This is shown most simply by again imposing the condition that $\boldsymbol{u}$, and hence i_q, are constant. If the required force $\boldsymbol{f}_q$, now a restraint, were given by the total differential of $\boldsymbol{A}_i$, it would absorb an energy

$$\int \boldsymbol{f}_q \cdot \boldsymbol{u}\, dt = q\boldsymbol{u} \cdot \boldsymbol{A}_i = i_q i_c L_{12}$$

where i_c and L_{12} are the final values, and this is twice the amount needed for the energy balance. The total differential includes the two components

$$q\boldsymbol{u} \cdot d\boldsymbol{A}_i/dt = i_q[L_{12}\, di_c/dt + i_c\, dL_{12}/dt]$$

and i_c varies linearly with L_{12}, since the coil voltage (eqn. 18) takes the form

$$L_{22}\, di_c/dt + i_q\, dL_{12}/dt = 0$$

Thus the di_c/dt part of dA_i/dt at q satisfies the energy balance requirements, and the dL_{12}/dt part as given by eqn. 10 makes no contribution, showing the importance of the **grad** ϕ term in eqn. 9.

The energy can (necessarily) be accounted for in magnetic field terms, although field theory tends to obscure the various interactions between the charges. The kinetic energy is replaced by the magnetic energy density $\boldsymbol{H} \cdot \boldsymbol{B}/2$, giving

$$U_A = \int (\boldsymbol{H}_q + \boldsymbol{H}_i) \cdot (\boldsymbol{B}_q + \boldsymbol{B}_i)/2\, dv \qquad (19)$$

where the subscripts indicate the sources, and the integral extends over all space (introducing problems of retardation). The mutual energy

$$\int (\boldsymbol{H}_q \cdot \boldsymbol{B}_i + \boldsymbol{H}_i \cdot \boldsymbol{B}_q)/2\, dv = \int \boldsymbol{H}_i \cdot \boldsymbol{B}_q\, dv$$

is confined to the region inside the toroid, and the two components which the dynamical theory separates between the two sources are overlaid on one another within the same space. The result is the same, when retardation effects are neglected, as follows at once from the usual proof of the equivalence between the field and kinetic energy forms (e.g. Reference 12, p. 124), which allows independent sources of $\boldsymbol{H}$ and $\boldsymbol{B}$. Thus there is no difference in the two approaches when the fields are separated out into the various 'overlaid' components, but terms which are physically separate in eqn. 16 now occupy the same space, and the superposition conflicts with the conventional view that it is only the resultant $\boldsymbol{E}$ and $\boldsymbol{B}$ vectors, defined by eqn. 2, which are physically significant. The different inductances in eqn. 17 are generally regarded as separate entities when applied to closed circuits, and this is one of the causes of the conceptual divide between the field and circuit descriptions of any given device. The dynamical theory, by removing the energy from the 'field' (i.e. the $\boldsymbol{A}$ map) and assigning it to the charges, provides a 'circuit-theory' view expressed in terms of the interactions between the various 'circuit' parts, including any isolated charges, and it is this which brings to the fore the key concept of mutual energy. Although very familiar in circuit terms, the idea of separation into four parts, two of which combine to give a total mutual energy which omits the usual factor 1/2, appears artificial when applied to the field, and is seldom considered in treatments of energy in texts on 'electromagnetic theory'. In consequence, although all the $\boldsymbol{J} \cdot \boldsymbol{A}/2$ energies described as 'kinetic' can equally be interpreted in field terms as 'magnetic' (provided that retardation effects are negligible), the operation of the device (like others) tends to be confused, rather than clarified, by combining the energies of all the component interactions, and redistributing them in empty space. But making the comparison shows that the 'kinetic' forces due to $\boldsymbol{A}$ and 'magnetic' forces due to $\boldsymbol{B}$ are not the same.

5 Induced charge

Eqn. 17 shows that, in the reference R_{coil} in which the coil is stationary, the interaction appears similar to that in any 'magnetic' machine, but it is different when 'seen' from q. In the reference R_q in which q is stationary, $\boldsymbol{A}_q$ disappears, along with the corresponding magnetic field $\boldsymbol{B}_q$, so that there appears to be no induced EMF, and no change in mutual inductance. This reflects the fundamental difference between a conventional machine, in which all the sources of $\boldsymbol{A}$ (i.e. of momentum) are due to the differential motion of the charges within the conductors, and a device whose operation depends on the separation of q from $-q$.

The effect of the separation is to induce a charge of density ρ_s on the coil, opposite in sign to q, and ρ_s is the most obvious source of a **grad** ϕ term in the force on q. Cullwick [15, 16], in drawing attention to some of the unusual features of the device shown in Fig. 1, pointed out that $\boldsymbol{B}_q$ depends on the reference, so that, as argued by Howe [17], the self-consistency of the field description requires that the induced EMF in R_q must be accounted for by the motion of ρ_s, and the electrostatic part of the interaction cannot be isolated from the electrokinetic part. A demonstration, in field terms, of the equivalence of the q and ρ_s sources is difficult [18], and it is, perhaps, unfortunate that a long discussion of this point distracted attention from its underlying implications, and from more important aspects of the operating principle.

The proof of the equivalence in charge-potential terms follows at once if the coil is open-circuited, so that $\partial A_i/\partial t$ can be ignored, and we assume that **grad** ϕ_s, due to ρ_s, is equal but opposite to **grad** ϕ_q, due to q, at all points on the coil. That is,

$$\phi_q + \phi_s = 0 \qquad (20)$$

if the coil is the potential reference. Assuming that all the ρ_s charges move at the same velocity $-\boldsymbol{u}$ gives

$$\boldsymbol{A}_s = -\boldsymbol{u}\phi_s/c^2 \qquad (21)$$

in R_q (from eqn. 7), and substituting from eqn. 20 shows that A_s, in R_q, takes the same value as A_q, in R_{coil}, at all points on the wire (neglecting the small change in ϕ between R_{coil} and R_q). Thus the equivalence applies not only to the EMF due to the line integrals of A_q and A_s, but to the distribution of the $\partial A/\partial t$ force around the coil. Since the operation of the device depends on the EMF, and this is essentially a relativistic modification of ϕ_q (as is shown by eqn. 7), the independence of reference is correspondingly important, and the result illustrates the practical advantages of eqn. 7 as a means of describing kinetic, or 'magnetic', effects.

Although the potential reference is not arbitrary [2], it may be chosen to simplify the point here, since it is only the derivatives which produce the forces. The coil potential is, of course, zero if it is earthed sufficiently effectively (i.e. through a low-inductance connection), and the charge density, ρ_s, on it then integrates to $-q$; more generally some of $-q$ is located elsewhere, but this does not affect the proof (other than in the need to examine the potential reference). Eqn. 20 depends on the assumption that the open-circuit removes the momentum terms, and that the velocity $-\boldsymbol{u}$ is uniform, which ignores the effect of the current due to the changes in ρ_s, causing an electric, or differential, effect, but this is the same in both references, and can be ignored in making the comparison between them. More obvious, perhaps, is the neglect of the A_s term (i.e. A_q in eqn. 18) in the equilibrium condition (eqn. 20), obtained by equating the potential gradients. Eqn. 21 shows that the force due to $\partial A_s/\partial t$ is small, compared with **grad** ϕ_s, and is a part of the relativistic changes in ϕ which have also been ignored. It is not the relative magnitude of A_s which is important, but its nonconservative property acting on the large amount of charge in the conduction energy band of the wire.

Eqn. 16 is replaced by

$$U_A = \frac{1}{2} m_c u^2 + \frac{1}{2}\int_{coil} \rho_s(-\boldsymbol{u}) \cdot (\boldsymbol{A}_s + \boldsymbol{A}_i)\, dv + \frac{1}{2}\int_{coil} \boldsymbol{J} \cdot (\boldsymbol{A}_s + \boldsymbol{A}_i)\, dv \tag{22}$$

where m_c denotes the mass of the coil. The electromagnetic, like the 'mechanical', part of the kinetic energy of q is transferred to the coil in the change from R_{coil} to R_q, but retains the same sign. Since A_s is equal to A_q, the mutual term in the second integral is unchanged, and so also is the other half of the mutual energy, whilst the momentum reciprocity condition derived from eqn. 14 shows that the change from ρA_q to $\rho_s A_s$ transfers the same self energy from q to ρ_s (i.e. $-q$).

Thus the choice of reference does not affect the electrokinetic energy, but it locates both halves on the coil, and this changes the way in which the interaction is described. 'Viewed' from R_q it is the motion of the 'static' charges ρ_s on the surface of the coil, at velocity $-\boldsymbol{u}$, which induces the EMF, causing the conduction electron 'flywheel' to rotate, and imparting $i^2L_{22}/2$ energy to it. Expressed in field terms, the magnetic flux due to the relative movement of the two parts of the system is generated by the coil, which thus acquires inductance components due to its net motion, in addition to its more familiar self inductance, due to A_i. These fluxes and inductances provide the kinetic coupling, due to A_s and A_i, between the moving ρ_s charges on the surface of the wire and the conduction electrons, defining the $\boldsymbol{J}$ vector, inside it.

If the superconductor is replaced by a resistive wire of variable section, the EMF induced at any given instant may be matched to the resistance, so that i remains uniform and constant, $\partial A_i/\partial t$ is zero, and the energy is dissipated instead of stored. Then the only force on q is that due to ϕ_s, i.e. is the same as when the coil is stationary, and any additional $\partial A_i/\partial t$ component, due to changes in i, represent the usual 'transformer action' (Section 4), as in a static device. The net force on the coil is likewise limited to ϕ_q, acting on ρ_s, so that the R_q view of the resistance-matched constant-i operation shows that the net forces on both q, and on the coil, are the same as when the coil is stationary, if retardation delays are ignored.

6 Components of $\boldsymbol{u} \times \boldsymbol{B}$

The R_q reference shows the importance of the potential energy due to ϕ_q and ϕ_s (i.e. of the $\boldsymbol{E}$ term in eqn. 2). But it does not explain the zero $\boldsymbol{u} \times \boldsymbol{B}_i$ component in eqn. 2, due to the current, i, giving zero magnetic force on q in R_{coil}, whereas the $\partial A_i/\partial t$ term in eqn. 10 is not zero. The difference requires another source of **grad** ϕ, as is obvious from the vector expansion

$$\nabla(\boldsymbol{u} \cdot \boldsymbol{A}) = \boldsymbol{u} \times \nabla \times \boldsymbol{A} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{A} + \boldsymbol{A} \times \nabla \times \boldsymbol{u} + (\boldsymbol{A} \cdot \nabla)\boldsymbol{u}$$

showing that

$$\boldsymbol{u} \times \boldsymbol{B} = \boldsymbol{u} \times \nabla \times \boldsymbol{A} = \nabla(\boldsymbol{u} \cdot \boldsymbol{A}) - (\boldsymbol{u} \cdot \nabla)\boldsymbol{A} \tag{23}$$

if ∇ operates only on $\boldsymbol{A}$, not $\boldsymbol{u}$. Since the first term is a scalar, it can be written

$$\phi' = -\boldsymbol{u} \cdot \boldsymbol{A} \tag{24}$$

(complementing eqn. 7), and the $\boldsymbol{u} \times \boldsymbol{B}$, like the $\boldsymbol{E}$, part of eqn. 2 can be separated into components expressed in terms of ϕ and $\boldsymbol{A}$

$$\boldsymbol{f} = \rho[-\mathbf{grad}\,\phi - \partial \boldsymbol{A}/\partial t - \mathbf{grad}\,\phi' - (\boldsymbol{u} \cdot \nabla)\boldsymbol{A}] \tag{25}$$

The four components can be reduced back to two by adding ϕ' to ϕ, and observing that the total change in $\boldsymbol{A}$ is

$$d\boldsymbol{A}/dt = \partial \boldsymbol{A}/\partial t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{A} \tag{26}$$

which is the more general form of eqn. 10. Then

$$\boldsymbol{f} = \rho[-\mathbf{grad}\,(\phi' + \phi) - d\boldsymbol{A}/dt] \tag{27}$$

in accordance with eqn. 3 when the potential, and $d\boldsymbol{A}/dt$, terms are those 'seen' by the moving charge. The transformation is Galilean, so that the force is that predicted from the reference relative to which the charge is moving, rather than that experienced by the charge. Thus eqn. 3 can be reduced to two terms, but the condensation is not necessarily helpful.

The significance of the components is shown by the diagram of $\boldsymbol{u} \times \boldsymbol{B}$ (Fig. 5), that is of the vector

$$\boldsymbol{E}_i' = \boldsymbol{u} \times \boldsymbol{B}_i$$

representing the electric field 'seen' by q, as a consequence of its motion past the magnetic field $\boldsymbol{B}_i$ due to the current i in the toroid. Here the prime is used to distinguish the reference frame R_q in which q is stationary. Since $\boldsymbol{B}_i$ is confined to the interior of the coil, so also is $\boldsymbol{E}_i'$, and the electric field lines terminate on the wires. Thus the sources of $\boldsymbol{E}_i'$ consists of

$$\mathbf{curl}\,\boldsymbol{E}_i' = -\partial \boldsymbol{B}_i/\partial t$$

generating the component of E_i' given by

$$-(\partial A_i/\partial t)' = -(u \cdot \nabla)A_i$$

together with a scalar component, $-\nabla\phi_i'$, given by eqn. 24, using the suffix i to emphasise the source. Comparing

$$\nabla^2\phi = -\rho/\varepsilon_0$$

with

$$\nabla^2 A = -J/\varepsilon_0 c^2$$

(or, more generally, their retarded equivalents) shows that the sources of ϕ_i' are

$$\rho_i' = -u \cdot J/c^2 \quad (28)$$

Hence the conduction electron density in an uncharged current-carrying wire is not equal but opposite to the lattice charge density, in the reference R_q moving relative to the wire at velocity u.

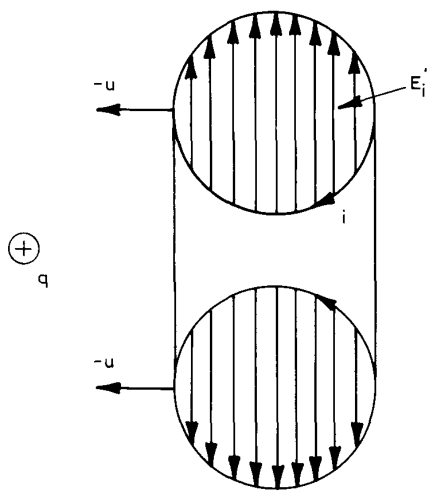

Fig. 5 *Electric field E_i', 'seen' by q, in R_q*

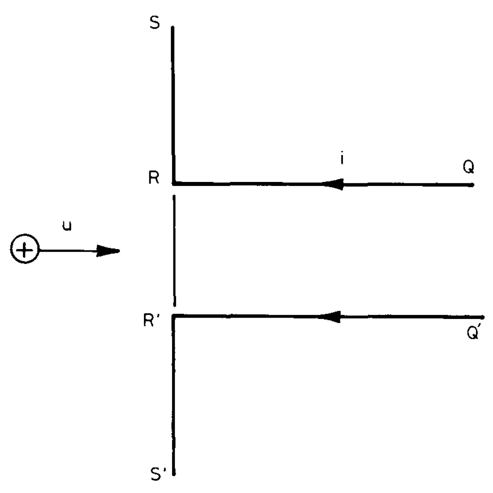

Fig. 6 *Rectangular-section coil*

If the coil is rectangular in shape (Fig. 6), the ρ_i' sources are inside the wires, such as RQ, parallel to u, whereas the sources of the component of $(u \cdot \nabla)A_i$ which contributes to E_i' are confined to wires normal to u, such as RS. These terminate E_i' tangentially, and, as in the division of the E vector in eqn. 2, the two terms in eqn. 23 represent the effects of different sets of charge. When used to calculate the 'mechanical' forces, due to the motion of the conduction electrons, $\nabla(u \cdot A)$ gives the interaction between long, current-carrying wires which are parallel to each other, whilst $(u \cdot \nabla)A$ describes that at a right-angled bend, the two cancelling out when applied to the axial components of force on parallel currents*. When applied to E_i', they also cancel outside the toroid (where **curl** A_i is zero), although the relevant components of A_i are at right-angles; but in the interior the contributions add, in the direction of E_i'. They cancel everywhere, in the direction u, but separate conservative from nonconservative components, and adding the conservative part makes the result in R_q the same as in R_{coil}.

The change in ϕ is reflected in the potential energy (PE), given by

$$U_\phi = \frac{1}{2}\int \rho\phi \, dv \quad (29)$$

The charges of density ρ, in q, acquire additional PE, in R_q, of density

$$\rho\phi_i'/2 = \rho(-u \cdot A_i)/2 \quad (30)$$

from eqn. 24, and the corresponding PE of the ρ_i' charges in the ϕ_q 'field' of q, is, from eqns. 28 and 7

$$\rho_i'\phi_q/2 = -u \cdot J\phi_q/2c^2$$
$$= -J \cdot A_q/2 \quad (31)$$

where A_q is interchangeable with A_s. Thus the transfer of kinetic energy from q to ρ_s, as a consequence in the change in reference, is accompanied by equal, but opposite, changes in potential energy, so that the 'potential', like the 'kinetic', energy depends on velocity, and the essential distinction between them is in the nonconservative part of the associated forces. The distinction is, however, self consistent in any given reference.

7 Operating cycle

Eqns. 25 and 27 illustrate the way in which the change in reference affects the nature of the forces and the extent to which the description of the device in terms of the A vector, or inductance, is incomplete. In consequence the dynamical and field theories (necessarily) agree in predicting the same force on q when it is moving as when it is stationary, assuming that the output energy is absorbed locally (Section 8). But this does not prevent the energy exchange, due to the motion of q, and, as shown in Section 4, the inductance coefficients are sufficient and self consistent in describing the exchange, when 'viewed' from any given reference. But they are derived from A, not from the concept of flux; indeed the A vector can be defined and interpreted as the current times the inductance per unit length between circuit elements†

$$A_{12} \cdot \delta l_1 = i_2 \delta L_{12} \quad (32)$$

where the source element need not be small. The increase (or reduction) in mutual inductance, as q approaches the coil, is due to the changes in A_i, and A_q, and the field

* Showing that it is this separation of terms which underlies the distinction between Ampère's law of force between current elements and that deduced in the customary way from the $u \times B$ term in eqn. 2. The former gives equal but opposite action and reaction, whereas the latter does not.

† A measures the 'inductivity' of the conductor. The dynamical theory extends the local conductor property of resistivity to all of the circuit parameters, including capacitance, impedance and radiation resistance.

interpretation fails not only because of the difficulty in defining flux linkage, but because q 'sees' no change in the flux produced by the coil.

It is helpful, in comparing the different points of view and the effect of constant-current operation, to examine an operating cycle which includes a constant-current phase. A period during which the current does change has to be included, to provide for the recombination of q with the opposite charge $-q$, without nullifying the energy transfer, and the complete cycle might consist of four parts. During the first, q is separated from $-q$ and set in motion, acquiring a kinetic energy

$$U_q = m_q u^2/2 + i_q^2 L_{11}/2$$

in R_{coil} (eqn. 17). The toroid is energised, when remote from q, requiring an electrical input

$$U_c = i_c^2 L_{22}/2$$

from an external source. During the second part, q approaches the toroid at uniform velocity, whilst i_c is kept constant, requiring a further external input to the coil

$$U_c = \int\oint i_c(\partial A_q/\partial t) \cdot \boldsymbol{dl}\, dt = i_c i_q L_{21}$$

where L_{21} is the final value. U_c is positive if i_c is in the direction defining positive L_{21} (i.e. opposite to that shown in the diagrams). The current is then switched off, during the third part of the cycle, and the fourth part consists of the recombination of q and $-q$, while the coil is open-circuited. The original 'electrostatic' separation energy, which is included in the **grad** ϕ forces on q (and on ρ_s) throughout, is then recovered, and these capacitive energy and force terms can be ignored for the present purpose. They require a supplementary energy balance, which is considered in Section 8.

It is convenient to assume, as in Section 4, that the current switch-off is sufficiently rapid to make the change in mutual inductance negligible during stage 3, and that a force, $\boldsymbol{f}_q$, keeps the velocity $\boldsymbol{u}$ constant. The work done by $\boldsymbol{f}_q$ is

$$U_q = \int \boldsymbol{f}_q \cdot \boldsymbol{u}\, dt = \int q\boldsymbol{u} \cdot d\boldsymbol{A}_i = i_q i_c L_{12}$$

as before, and the energy obtained from the coil

$$U_c = \int\oint i_c \boldsymbol{A}_i \cdot \boldsymbol{dl}\, dt = i_c^2 L_{22}/2$$

since $\boldsymbol{A}_q$ is constant. The kinetic energy which is then recovered from q, during stage 4,

$$U_q = m_q u^2/2 + i_q^2 L_{11}/2$$

is that originally supplied during stage 1.

The force $\boldsymbol{f}_q$ is introduced to simplify the calculation by separating the self and mutual energy terms, but is not needed in practice since stages 3 and 4 could be combined by choosing the current i_c so that the $\partial A_i \partial t$ force, due to 'transformer action', brings q to rest. The practical details of this part of the cycle do not affect the kinetic energy balance, which allows no force on q during stage 2, when both the velocity $\boldsymbol{u}$ and the current i_c are constant. More specifically, it allows no change in the force which is exerted on q when stationary. This is because the electrical input U_c, if positive, increases the internal (mutual) energy by supplying a current in the direction which makes L_{12} positive. The characteristic feature of the device, operating as a generator, is that the direction of the induced current drives L_{12} negative, giving a lower internal energy state at the end of the stage than at the beginning, even though the initial mutual energy is zero. Field theory accounts for the absence of a force f_q in the same way; that is the $\boldsymbol{H}_i \cdot \boldsymbol{B}_q$ component goes negative, reducing the total $\boldsymbol{H} \cdot \boldsymbol{B}/2$ energy in the region inside the toroid.

The negative energy is subsequently recovered in restoring the system to its original energy state, and in this respect the operation is no different from that of any device during a period in which the net internal energy diminishes. It illustrates the physical significance of the negative sign of $\boldsymbol{J} \cdot \boldsymbol{A}/2$, whenever the vectors $\boldsymbol{J}$ and $\boldsymbol{A}$ have opposite direction, which is characteristic of the mutual energy terms in all transformers and 'magnetic' machines, and gives a useful insight into their operation. Eqn. 17 shows that the inductive effect of q on the toroid is the same as that of another closed winding, and it is helpful to make the comparison with an equivalent two-winding system, in which the two currents are kept constant whilst one coil is brought into the vicinity of the other. The mechanical work on moving coil 1 is given by

$$\int f_1 \cdot u_1\, dt = \int i_1 i_2\, \partial L_{12}/\partial x\, u\, dt = i_1 i_2 L_{12}$$

and the electrical work obtained by integrating $i_1\, dA_2$ is equal but opposite, since i_1 is constant. Thus there is no net expenditure of energy on the moving coil, just as none is required on q, and the stationary coil requires an electrical work which accounts for the whole of the mutual energy $i_1 i_2 M$.

The comparison illustrates an essential difference between the self-energy and mutual-energy terms (eqns. 16 and 17), in that the latter are not, in general, injected into, or recovered from, the components to which they are assigned. 'Self energies' analyse, in turn, into the sum of mutual subenergies, since all electromagnetic energy is mutual, so that the density $\boldsymbol{J} \cdot \boldsymbol{A}/2$, like $\boldsymbol{H} \cdot \boldsymbol{B}/2$, is merely a convenient way of making the system energy account balance, and may not be interpreted literally. The observable energy expenditure on the charges, and thus the stored energy density, is given [2] by the integral of $\boldsymbol{J} \cdot \boldsymbol{dA}$, which has arbitrary values whenever $\boldsymbol{J}$ and $\boldsymbol{A}$ are varied independently, and it is only the total system energy which can be accounted for by assuming a stored energy density $\boldsymbol{J} \cdot \boldsymbol{A}/2$. The energy density provides a local energy balance in systems driven in such a way that $\boldsymbol{A}$ varies linearly with $\boldsymbol{J}$, but not when driven in other ways, even though the system remains linear in the sense that it contains no ferromagnetic materials. The energy transfer by remote action is a consequence of the action-at-a-distance of the observable forces, i.e. those which act on the charges*. This point is fundamental to the operation of the device with constant i_c, since this transfers to the coil the energy which is assigned to q in the third term of eqn. 17, and accounts for energy generation without a force on q.

The constant-current operating mode is not essentially different from that described in Section 4, in which the short-circuited coil stores the exchange energy in self-inductance, and this may be supplied subsequently to an electrical load. The details of the actual cycle are a matter

* Field theory 'explains' the transfer of energy, but only if we take an equally literal view of the mechanism transferring the force, i.e. of the physical existence of Maxwell stresses in empty space [2].

of practical convenience, subject only to the requirement that the current is switched to zero when the charges are recombined, which means that a relatively large amount of inductive energy must be injected into the coil and recovered, giving the device the equivalent of a poor power factor. The mutual and self energies are obviously closely linked, and both depend on the cross-sectional area of the coil, so that their ratio imposes severe limitations on the operation at low speed. It does not, however, make the exchange impractical, as is shown by its many applications (including magnetrons).

8 Energy transfer to external source

In field theory there is a tendency to ignore ϕ as wholly 'electrostatic', but ϕ_i' in R_q shows that care is needed in interpretation, and the ϕ in R_{coil} clearly cannot be the same as when q is stationary, since the axial equilibrium of the conduction electrons in the coil (eqn. 18) must depend on the current. This adds energy terms, and requires further examination. The potential which is observed in R_{coil} separates into three parts,

$$\phi = \phi_q + \phi_s + \phi_e \tag{33}$$

where ϕ_q is due to q, and ϕ_s is the opposite 'static' component defined by the condition that ϕ_q and ϕ_s sum to zero at all points on the coil (eqn. 20), leaving the ϕ_e to balance the forces due to the total A (eqn. 18) in a resistanceless conductor. The transfer of energy around the coil requires a $\boldsymbol{J}\phi$ vector [2] due to the net ϕ_e, which depends on the shape of the conductors, as well as their resistivity, but is controlled by the potential difference between the coil terminals. That is, by the external source providing the 'applied voltage', due to ϕ_e, which opposes the 'back EMF', or induced voltage, due to A. A transformer provides a familiar example. ϕ_e causes the electric stress in the insulation and determines the rate of energy transfer along the conductors. It also contributes to the force on q.

Consider a toroidal coil of rectangular cross-section (Fig. 6) whose opening RR' is sufficiently small compared with the rectangle dimensions as to make the effects of the other corners negligible as q approaches R. Keeping the current i_c constant removes the A_i term from eqn. 18, and the equilibrium of the conduction electrons, in the axial direction, requires

$$f_- = \rho_-(-\mathbf{grad}\ \phi_e - \partial A_q(\partial t) = 0 \tag{34}$$

in R_{coil}, assuming a resistanceless wire. The $\partial A_q/\partial t$ force on the electrons is concentrated in the vicinity of R, and is confined to the wires, such as RQ, parallel to $\boldsymbol{u}$, so that charge accumulates, and produces the ϕ_e needed to transfer energy to the rest of the turn, and to the external load. ϕ_e provides the electrical equivalent of the 'mechanical' restraints on the rotating discs in Fig. 4, and $\boldsymbol{J}\phi_e$ provides the alternative to the 'mechanical' power.

Consider first the equilibrium forces on the $-q$, or ρ_s, charge with which q is paired. If the system consisting of the two components shown in Fig. 1 is to be closed, the coil must be 'earthed', so that ρ_s integrates to $-q$, and, as q approaches, the induced charge is transferred progressively to the nearer surface of the toroid. In the absence of resistance the transfer cannot be achieved in a stable way, since the axial electron equilibrium condition in the earth lead,

$$f_- = \rho_-(-\mathbf{grad}\ \phi - \partial A/\partial t) = 0$$

leads to sustained oscillations between potential and kinetic energy, as when a charged capacitor is connected in parallel with one which is discharged (Reference 2, p. 61), giving a net loss of $\rho\phi/2$ energy because of the addition $\boldsymbol{J}\cdot A/2$ term. The only way of achieving equilibrium is by some form of viscous damping, for example by including resistance in the earthing lead, and this absorbs the separation energy which is lost as q approaches ρ_s. In general, a second energy convertor is needed in the earthing lead in addition to that connected between the terminals, and the assumption made in Section 7 that all the potential energy of the system is recovered requires that the second converter is reversible.

The separation energy due to ϕ_e likewise has to be injected and recovered. The force

$$\boldsymbol{F}_q = -q\nabla\phi_e \tag{35}$$

on q, together with the corresponding force density

$$f_e = -\rho_e\nabla\phi_q$$

on the sources, ρ_e, of ϕ_e on the coil, produces an additional potential energy, $q\phi_e/2$, and $\rho_e\phi_q/2$, which is sufficiently small to ignore in most two-winding devices. It here requires a supply energy balance, which depends on the current i_c, but it is additional to the kinetic energy exchange, not a part of it, since, although ρ_e and ϕ_e control the transfer of energy, the control power is not directly related to the amount transferred.

This is shown by considering, for example, a short-circuited coil in which the conductivity σ is distributed so as to keep i temporarily constant, by matching $\partial A_q/\partial t$ to the resistive volt-drop J/σ. The mutual-energy exchange U_c (Section 7) is then converted locally into heat, and ϕ_e is eliminated. In general, the magnitude of ϕ_e, together with the corresponding force on q, depends on the extent of the redistribution of energy within the circuit, rather than on the amount transferred, and the same capacitive energy appears as an additional effect in field theory (although easily overlooked when concentrating attention on the fields, instead of the charge equilibrium conditions). It supplements the 'magnetic' component of the interaction, but does not form a part of it. The role of the ϕ_e term is evident in high-frequency applications, such as cavity resonators, in which a resistanceless conductor obviously cannot be assumed to be an equipotential. Its relative importance depends directly on the frequency, geometry and resistivity.

9 Lorentz transformation

The ϕ_e term shows that the force on q depends on the current in the coil, but it is the components of $\boldsymbol{u} \times \boldsymbol{B}$, in R_q, which are more directly involved in the energy exchange, and these illustrate the practical implications of the relativistic nature of the 'magnetic', or 'kinetic', forces. Separating the third term in eqn. 3 into its components answers the question whether or not $\boldsymbol{u} \times \nabla \times \boldsymbol{A}$ is a 'momentum force', by distinguishing between $\partial A/\partial t$ and ϕ', but this depends, in turn, on the definition of ϕ by its conservative property, and 'momentum' as the nonconservative term.

The transformation can be made self-consistent only by stating it in relativistic, instead of Galilean, form, i.e. as a Lorentz transformation [19, 20] from one inertial reference, R, to another, R', moving at velocity $\boldsymbol{u}$,

$$\phi' = \gamma(\phi - \boldsymbol{u}\cdot A) \tag{36}$$

where

$$\gamma = 1/\sqrt{(1 - u^2/c^2)} \tag{37}$$

The component of $\boldsymbol{A}'$ in the direction $\boldsymbol{u}$ is

$$A'_u = \gamma(A_u - u\phi/c^2) \tag{38}$$

and the component in the transverse direction is the same in R' as in R. Eqn. 7 follows from eqn. 38 by substituting $\boldsymbol{A}' = 0$, and eqn. 24 is an approximate form of eqn. 36, likewise taking the exact form

$$\phi = \boldsymbol{u} \cdot \boldsymbol{A} \tag{39}$$

if $\phi' = 0$. Thus a current-carrying wire whose surface is an equipotential in the reference, now R', in which the crystal lattice is stationary, acquires a different value of ϕ in any other reference.

The interchange between ϕ and $\boldsymbol{A}$, and the associated energy change, is comparable to that between the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields, although with the proviso that the field components are not the same as those due to ϕ and $\boldsymbol{A}$. The distinction between ϕ, as a 'static' effect, and either $\boldsymbol{A}$, or $\boldsymbol{u} \times \boldsymbol{B}$, as 'kinetic' or 'dynamic', is essentially arbitrary, not only because it depends on the reference used to define velocity, and requires the motion of two sets of interacting charges, but because the 'static' charge density on a conductor is not invariant (despite the invariance of charge). The underlying interaction is between two moving charges, causing a force on one, say q_1, which is given by the $\boldsymbol{E}$ field of the source charge, q_2, in the reference in which q_1 is stationary, and is directed along the radius vector from the retarded position of q_2 [19, 20]. This is in contrast with $\boldsymbol{u} \times \boldsymbol{B}$, directed in the plane normal to $\boldsymbol{u}$, and $\partial \boldsymbol{A}/\partial t$, which is in the direction of motion if the charges are moving parallel to each other. The concept of momentum (like that of flux) describes a component of the relativistic change in the potential, ϕ, due to a stationary charge, and necessarily depends on the choice of reference. It is defined by the distinction between conservative and nonconservative components, where the definition of 'conservative' is, in turn, affected by retardation, and it is only the Lorentz gauge which retards both ϕ and $\boldsymbol{A}$, in accordance with the fundamental postulate of special relativity. Because ϕ, due to q_2, is retarded, its local gradient is not in the direction of $\boldsymbol{E}$, i.e. of the radius vector. In general, neither the $\boldsymbol{u} \times \boldsymbol{B}$ nor the $\partial \boldsymbol{A}/\partial t$ term is sufficient to account for the 'dynamic' force on a moving charge.

10 Equivalent circuit

Eqn. 17 shows that the $\partial \boldsymbol{A}/\partial t$ terms can be represented by an equivalent circuit (Fig. 7), whose graphical symbolism

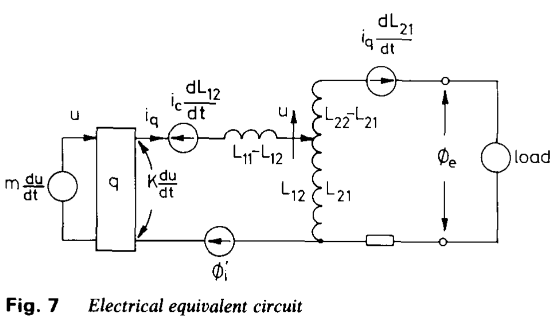

Fig. 7 *Electrical equivalent circuit*
$K = m\delta_q/q$

provides a useful summary of the operating principles, as in other forms of energy convertor. In the coil the changes in momentum produce electrical voltages given by

$$v_2 = L_{22}\, di_c/dt + L_{21}\, di_q/dt + i_q\, dL_{21}/dt \tag{40}$$

in which the two 'transformer' terms are represented in the usual way by inductance elements, and the last requires an additional generator representing the effect of the relative movement between the parts, giving a voltage which depends on i_q. These together represent the coil term in eqn. 16. They combine with the resistive volt drop to account for the terminal voltage ϕ_e.

The charge q can likewise be replaced by an equivalent winding whose inductances are defined by equating energies, so that the corresponding terms in eqns. 16 and 17 are equivalent to the voltages

$$v_1 = L_{11}\, di_q/dt + L_{12}\, di_c/dt + i_c\, dL_{12}/dt \tag{41}$$

representing the various momentum forces on q. An additional circuit element is required to represent changes in $m\boldsymbol{u}$, caused by changes in i_c, as in a betratron, and another is required to balance the $i_c\, dL_{12}/dt$ effect of changes in position. The additional component is the ϕ'_i, or $\nabla(\boldsymbol{u} \cdot \boldsymbol{A}_i)$, generator, which combines with $i_c\, dL_{12}/dt$, or $(\boldsymbol{u} \cdot \nabla)\boldsymbol{A}_i$ term to produce zero net force on q. Thus, when 'viewed' from the coil (i.e. from R_{coil}), the two generators on the q side combine, and can be removed, corresponding to the $\boldsymbol{u} \times \boldsymbol{B}$ description, but they are needed to separate the momentum and conservative force components.

When i_c is kept constant, the approach of q, at constant velocity, gives zero voltage across all the inductance elements, and the $i_q\, dL_{21}/dt$ generator exchanges $i_c i_q L_{21}$ energy with the external source. On the q side the two generators describe opposite changes in internal energy, summing to zero, i.e. they represent an energy exchange, in which neither component is accessible, separately, to external observation. During a sufficiently rapid current change, on the other hand, all three generators can be ignored, and the remaining elements exchange energy with the inductors, which store the relevant $i_1 i_2 L_{12}/2$ components, accounting jointly for the total mutual energy $i_c i_q L_{21}$. Thus the internal energy is accessible, in electrical form, via both the inductors and the generators, and the latter represent the $\partial \boldsymbol{A}/\partial t$ forces due to changes in position, although not the mechanical energy. If q were to be replaced by a second winding, giving a symmetrical equivalent circuit, and both carry constant current, the movement of one requires a mechanical work which balances the electrical input to it, and the net change in mutual energy is provided by the electrical input to the fixed winding, as in Fig. 7. The equivalent circuit then represents electrical voltage and energy components in both windings, but no such distinction is possible when representing q, other than in units. The force on q due to ϕ_e is excluded, and requires a separate equivalent circuit in which the elements are capacitors.

Fig. 7 illustrates the close relationship between the dynamical and circuit descriptions. The former analyses the lumped parameters into the components contributed by each part of the circuit, in contrast with field theory, which concentrates attention on the different parts of the field, and their contributions to the flux linkages. The practical importance of the separation of the EMF into components is shown by the dependence of the kinetic (or magnetic) stored energy density $\boldsymbol{J} \cdot \boldsymbol{A}/2$ on $\boldsymbol{A}$, whereas the power flow $\boldsymbol{J}\phi$ is accounted for by ϕ. The operation of the device depends on the interaction between these terms.

The circuit also shows the implications of the high leakage reactance, and the consequences of increasing the momentum, per unit current, by filling the interior of the toroid with iron. This increases L_{12} in the ratio of the permeability μ, and thus the corresponding motion generator voltage, per unit current, but it also increases L_{22}, and the stored energy, per unit current, introduces a dL_{11}/dt term, and imposes other limitations. Fig. 7 gives the two-port form, in which the inductance $L_{11} - L_{12}$ is usually negative, and the turns ratio is not shown explicitly. Both can be corrected by introducing an ideal transformer, but this contributes little of importance, other than in defining what is meant by 'turns ratio' in a device in which one of the elements is an isolated group of moving charge.

11 Electromagnetic mass

The separation of $\boldsymbol{E}$, in eqn. 2, into two components (eqn. 3) separates the electromagnetic momentum from the forces which change it, and identifies a property which can also be expressed in terms of mass. Eqn. 17 shows that the momentum of q, on its own, is

$$\boldsymbol{G}_q = m\boldsymbol{u} + m_{11}\boldsymbol{u} \tag{42}$$

where

$$m_{11} = (q/\delta_q)^2 L_{11} \tag{43}$$

is its electromagnetic mass. By substituting from eqn. 7 in the relevant term in eqn. 16,

$$m_{11}u^2/2 = \frac{1}{2}\int_q \rho\boldsymbol{u}\cdot\boldsymbol{A}_q\,dv = (\boldsymbol{u}\cdot\boldsymbol{u}/2)\int_q (\rho\phi_q/c^2)\,dv \tag{44}$$

showing that m_{11} is the mass of the energy which is required to separate the charge from its opposite counterpart (the total separation energy requiring integration over both charge groups). Although m_{11} is normally too small to be significant compared with m, showing that L_{11} also is then negligible, this may not be so for the other inductances, and their corresponding masses. Here m represents the 'ordinary' mass of the charges, but, if these are electrons, m can, in turn, be explained in the same way [9, 19, 21, 22], i.e. as the mass of its electromagnetic energy, usually expressed in field terms*.

The mutual momentum density $\boldsymbol{g}_{12}$ of charges of density ρ_1, in a 'field' $\boldsymbol{A}_2$, is $\rho_1\boldsymbol{A}_2$, so that, if the velocity of ρ_1 is $\boldsymbol{u}_1$, the mutual energy assigned to a charge group q is

$$\int_q (\boldsymbol{J}_1\cdot\boldsymbol{A}_2/2)\,dv = \int_q (\rho_1\boldsymbol{u}_1\cdot\boldsymbol{A}_2/2)\,dv = \int_q (\boldsymbol{g}_{12}\cdot\boldsymbol{u}_1/2)\,dv$$

If the source is also a small group, moving at uniform velocity $\boldsymbol{u}_2$, the kinetic energy density takes the form

$$\boldsymbol{J}_1\cdot\boldsymbol{A}_2/2 = (\text{mutual mass density})\boldsymbol{u}_1\cdot\boldsymbol{u}_2/2 \tag{45}$$

* The effects of retardation have to be taken into account in making the comparison.

where the mass density is given by

$$(\rho_1\phi_2 + \rho_2\phi_1)/2c^2$$

from eqn. 7, together with the reciprocity principle. A source consisting of a uniform wire carrying a current

$$i_c = q_- u_-$$

due to electrons of density q_-, per unit length of wire, is characterised by the velocity u_- of the electrons. The kinetic interaction between q and each section of the wire can be interpreted as a contribution to the mutual mass, summed in accordance with the local scalar product of the vectors $\boldsymbol{u}$ and $\boldsymbol{u}_-$. Assuming that both are uniform in magnitude defines a net mutual mass m_{12}, so that the mutual-energy term (eqn. 17)

$$i_q i_c L_{12}/2 = uu_- m_{12}/2$$

giving

$$m_{12} = q_- qL_{12}/\delta_q \tag{46}$$

The reciprocity evident in eqns. 17 and 46 shows that the same mutual mass is assigned to the current.

The mutual momentum assigned to q can be summed around the circuit and written in vector form

$$\boldsymbol{G}_{12} = q\boldsymbol{A}_i = m_{12}\boldsymbol{u}_-$$

since the inductance is given by

$$L_{12} = \boldsymbol{A}_i\cdot\boldsymbol{\delta}_q/i_c \quad\text{or}\quad \boldsymbol{A}_i = q_-\boldsymbol{u}_-L_{12}/\delta_q \tag{47}$$

where $\boldsymbol{A}_i$ defines the direction of the equivalent velocity vector $\boldsymbol{u}_-$. Thus the momentum of q (eqn. 42) is changed by the presence of the coil to

$$\boldsymbol{G}_q = m\boldsymbol{u} + m_{11}\boldsymbol{u} + m_{12}\boldsymbol{u}_- \tag{48}$$

giving a momentum force

$$\boldsymbol{F}_q = m\,d\boldsymbol{u}/dt + m_{11}\,d\boldsymbol{u}/dt + m_{12}\,d\boldsymbol{u}_-/dt + \boldsymbol{u}_-\,dm_{12}/dt \tag{49}$$

with magnitude

$$F_q = m\,du/dt + (q/\delta_q)(L_{11}\,di_q/dt + L_{12}\,di_c/dt + i_c\,dL_{12}/dt) \tag{50}$$

by substitution from eqns. 43 and 46. This compares directly with the corresponding expression (eqns. 40 and 41) for EMF, which gives the line integral of the electric field, and thus the total force per unit line charge density.

Although the change in terms does not add anything new to the result, it provides a different perception of it, and the concept of the electromagnetic mass of individual electrons becomes an example of a wider view of the kinetic interactions between all moving charges as an inertial effect. Two electrons moving in the same direction acquire more mass than when they are separated, for the same reasons as each acquires its own, if we adopt an electromagnetic model of the electron. Viewed in this way, the inertial property of the moving electron stream in a coil, which 'explains' the inductance, is a necessary consequence of the retardation of the 'electrostatic' interaction between the electrons, as can be shown directly by considering the electromagnetic effects of a pulse in parallel wires [13]. The interaction between q and the coil follows.

The description in terms of momentum and mass helps to clarify the difference between the dynamical and field theory concepts. Whereas the Lorentz force equilibrium is

$$F_L = q(E + u \times B) = m\, du/dt \qquad (51)$$

of a charge q with 'nonelectromagnetic' mass m, defining force in terms of momentum, F_q, in eqn. 49, takes the m_{12} component of $u \times B$ from one side of the equation to the other. Thus, although replacing momentum by mass merely restates what was implied at the outset by using the vector A (i.e. the inductance), it helps in showing that 'force' is not an unambiguous concept when its origins are electromagnetic. Whether or not there is a force on q is, in the end, a matter of definition.

12 Aharonov-Bohm effect

As has been pointed out by Feĭnberg [23], an electron, q, moving past a magnetised whisker (Fig. 2), produces an H_q field which changes the magnetisation energy in the whisker by an amount depending on which way q goes round it. The change in magnetisation replaces the load energy, in Fig. 7, and shows that classical field theory predicts an interaction between the two, even though the whisker produces no B field at q. The point is often ignored, because of the tendency of the classical approach to concentrate attention on the fields, at the expense of the sources, although the effect on the magnetisation shows that any apparent discrepancy is not with quantum mechanics, as usually stated, but is internal to field theory. Replacing B by the A 'field' shows the physical significance of A as a measure of the mutual kinetic energy, and provides an answer to the question posed by Imry and Webb [6] and others about the extent to which the Aharonov-Bohm effect can be scaled up.

The apparatus used in various experimental investigations (see, for example, Reference 7) includes an earthed conducting screen around the whisker, suggesting that there is no essential difference between this and the coil shown in Fig. 1, except that ϕ_e is zero. The quantum-mechanical phase (which can, of course, be observed only up to 2π), provides a measure of the mutual energy between q and the whisker [23], and is equivalent to F_q (eqn. 49), instead of the Lorentz force F_L (eqn. 51), as is to be expected since the Hamiltonian implies a definition of force in terms of momentum. q acquires a mutual mass due to the coupling of its motion with the electron spins in the whisker, as is shown directly by Maxwell's dynamical treatment of classical theory, which can be rephrased in conventional field terms by the equivalence [7]

$$qA_i = \int D_q \times B_i\, dv \qquad (52)$$

where the suffixes denote the sources, and thus a momentum exchange since D_q and B_i are components, not total fields. However, the absence of B_i at q illustrates the way in which A provides a much more direct interpretation of momentum than $D \times B$, as is reflected in the Schrödinger equation, but is equally true of classical theory. To quote Feynman *et al.* (Reference 9, pp. 15–14) 'E and B are slowly disappearing from the modern expression of physical laws; they are being replaced by A and ϕ'. Perhaps the most remarkable aspects of the change is the neglect by all except a few writers such as Macdonald [24] of Maxwell's own use of the 'dynamical' ideas as the basis for the classical theory.

Difficulties in understanding how q can 'sense' the presence of flux at points remote from it [25], or on the existence of a long-range quantum state [26], are evidence of the conceptual problems associated with the B and D vectors, and are not peculiar to the Aharonov-Bohm device. Integrating the equation

$$\mathbf{curl}\, E = -\partial B/\partial t$$

shows that flux has a long-range or action-at-a-distance effect in all magnetic interactions; moreover one which is instantaneous, not retarded [14]. Remarks by many authors on the rather mysterious nature of the canonical momentum density

$$g = mu + \rho A$$

likewise show the essential artificial nature of the difficulties which are caused by interpreting electromagnetic momentum as $D \times B$, so that related quantum effects appearing in other applications, such as the behaviour of Josephson junctions in circuits linked by flux, and of SQUID devices, are also usually presented as examples of a failure of classical theory. The 'trapping' of flux by superconductors becomes rather less mysterious when viewed in ϕA terms, since the removal of extraneous forces from the conduction charges necessarily implies the conservation of their momentum (as is recognised in the London theory [17]), and this is quantised as a direct consequence of the quantum nature of the Cooper pairs which carry it.

13 Force on circuit

Portis [7], in examining the Aharonov-Bohm effect, argues that, although there may be no force on q, there is one on the coil, because the $J \times B_q$ interaction with the current does not integrate to zero. This point, together with the proof, which is given by demonstrating the equivalence between the total $J \times B_q$ force and the rate-of-change of mutual momentum given by the right-hand side of eqn. 52, is of interest for several reasons, including the obvious implications of an apparent lack of balance between the action and the consequent reaction. It amplifies the point made previously [1] about the use of one component of the $D \times B$ momentum to obtain the net force on the system, when the current in the same device changes in time. The force which is then exerted on q is accounted for by the rate-of-change of mutual momentum, giving an apparent reaction on empty space, instead of on the coil, when expressed in field terms. Replacing the right-hand side of eqn. 52 by the left-hand side helps to clarify the physical significance of the mutual term.

Clearly the $J \times B$ forces on the coil cannot balance out, because B_q is everywhere at right-angles to the cross-section (which may be a rectangle consisting of parallel conductors of equal length), and the field is greater in the regions close to q than in those further away. But the lack of a $J \times B$ term at q does not mean there is no force, and likewise $J \times B$ is not the only component of force on the circuit. The interaction depends on the separation energy of q, so that the E term in eqn. 2 cannot be ignored, nor can it be assumed to be identical to the static field in the absence of i; i.e. motional effects are not limited to $u \times B$, as is evident when considering the interactions between two moving charges.

It is also evident by changing the reference frame from R_{coil} to R_q, as was pointed out in Section 5, since B_q then disappears, and with it the force density $J \times B_q$ on the winding. It is replaced by a $J \times B$ term due to the B vector in R_q, which is accounted for by the movement of the $-q$, or ρ_s, charges on the surface of the coil. That is, the field which appears in the absence of any current in the coil. The $\boldsymbol{J} \times \boldsymbol{B}$ term now represents a force on the current within the wire due to a charge on its surface, and can only be one component of a total force which must be zero; otherwise the coil would propel itself along. It can be argued that this does happen, because the mutual component $\boldsymbol{D}_s \times \boldsymbol{B}_i$ is changing in time, where $\boldsymbol{D}_s$ is due to ρ_s, but, as when considering the 'transformer' effect of a change in current [1], the other components of $\boldsymbol{D} \times \boldsymbol{B}$ have to be taken into account. As observed from R_{coil}, the net $\boldsymbol{D}$ vector is zero everywhere inside the coil, if ϕ_e is zero in eqn. 33 (as it is if the resistance is suitably distributed), and there is no net $\boldsymbol{D} \times \boldsymbol{B}_i$. The rate-of-change of one component of the momentum certainly indicates a component of force, as we see at once by substituting the term on the right-hand side of eqn. 52 by that on the left, but it does not follow that this is the net force, and is observable; i.e. that it changes the 'mechanical' momentum $m\boldsymbol{u}$ due to the 'ordinary' mass m. There will be a change in the total $\boldsymbol{D} \times \boldsymbol{B}$ when integrated over all space, indicating a force on the system as a whole, but this merely confirms the lack of an instantaneous balance between the observable forces on q, and on the coil, due to the finite velocity of propagation, and is of little practical interest, even at high frequencies.

Although the field momentum can be separated into components by eqn. 52, with the same physical implications as the $q\boldsymbol{A}$ alternatives, this is unlikely to be helpful even to those familiar with the practical application of the $\boldsymbol{D} \times \boldsymbol{B}$ concept, and it will not be further pursued here. The more important point is that, although there is a $\boldsymbol{J} \times \boldsymbol{B}$ force on the coil, and this is directly related to the mutual momentum, the assumption that either proves that there is a net force on the coil, or on the system as a whole [7], is clearly unwarranted.

14 Force on q due to curl A

The condition that $\boldsymbol{B}_i$ (or **curl** A_i) is zero at q simplifies the investigation of the energy change, but is not necessary to it. Any coil replacing the toroid in Fig. 1, for example, will share a mutual kinetic energy with q, given by $\boldsymbol{J} \cdot A/2$, or $\rho\boldsymbol{u} \cdot A/2$, as will any two groups of charge when the velocity $\boldsymbol{u}$ of one through the A 'field' of the other has a component in the direction of A. The energy changes if the $\boldsymbol{u}$ component of A is not constant, and this principle is unaffected by the curl of A. The change does not, of itself, produce a force on q, in the sense that it does not alter the 'mechanical' momentum $m\boldsymbol{u}$, whereas **curl** A does. The $\boldsymbol{u} \times$ **curl** A force, on the other hand, has the well-known property of producing no change in energy. In general, the total interaction separates into two parts, one an energy change without a force, and the other a force without an energy change.

The $\boldsymbol{u} \times \nabla \times A$ forces constrain moving charges into a spiral motion, which can be interpreted in the conventional way as evidence of the $\boldsymbol{B}$ vector (Fig. 8*a*), but is equally evidence of A, since the charges conform to the momentum pattern of the source (Fig. 8*b*). Devices such as storage rings, for example, are designed to produce an A vector in the form of closed circles, thus forcing the charges into similar paths, parallel to the local sources of A in eqn. 5, and momentum conservation suggests that the direction of motion of positive charges is opposite to that of A (in accordance with the result obtained from the field by applying the two crossproduct rules). The resulting mutual energy is a familiar property of any betatron device, or transformer, in which the charges are accelerated by changing the source current. If A does not vary with position, the mutual energy does not change during the orbit, and, even if it does, the closing of the orbit, or circuit, as in a second transformer winding, makes the net energy change zero when the source

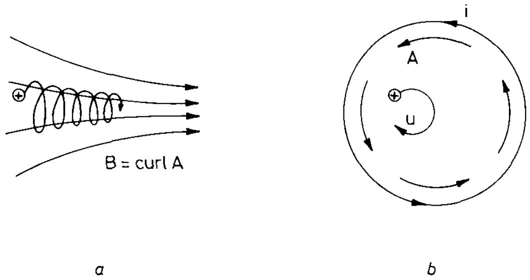

Fig. 8 *Forces due to* **curl** *A*
a Interaction with $\boldsymbol{B}$
b Interaction with A

current is constant. The $(\boldsymbol{u} \cdot \nabla)A$ term in eqn. 26 then has little practical interest. More generally, any isolated charge moving through the earth's A 'field', for example, will produce a change of flux linkage with the source current i and inject energy into it, if the component of A_i, due to i, in the direction of motion is not constant. The reciprocity principle shows the A_i provides a convenient alternative to A_q as a measure of this exchange. The effect tends to change the net source current, and is additional to that of any 'electrostatic' component, such as ϕ_s and ϕ_e (eqn. 33).

The amount of energy involved is commonly too small to be significant and it integrates to zero when q completes a closed path or is recombined without a change in the source of A. Nevertheless, its neglect conceals an interaction which may be useful. It becomes most evident when **curl** A is zero, as in Fig. 1, or in other examples, such as waveguides, in which the H-mode (or A-mode) gives nonzero A outside the waveguide, where $\boldsymbol{B}$ is zero, showing that a moving charge can couple with the mode. More generally, the energy changes due to $\boldsymbol{u} \cdot A$ appear in combination with the forces due to $\boldsymbol{u} \times \nabla \times A$, requiring a more complete statement of the dynamical interaction than is possible here, but it may be helpful to make a brief comparison between the momentum and field description.

Since the magnitude of $\boldsymbol{u}$ is constant, the centrifugal force, normal to $\boldsymbol{u}$, can be expressed either as $m(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ (cf. eqn. 23) or in the form

$$f_{cent} = m(\boldsymbol{u} \times \mathbf{curl}\ \boldsymbol{u})/2$$

Comparison with eqn. 3 shows that the 'magnetic' equilibrium condition is

$$m\ \mathbf{curl}\ \boldsymbol{u} + 2q\ \mathbf{curl}\ A = 0 \quad (53)$$

for a freely moving particle of 'nonelectromagnetic' mass m, and charge q, neglecting the electrical self-momentum qA_q. Integrating, and equating integrands,

$$m\boldsymbol{u} + 2qA = \text{constant} \quad (54)$$

if the 'field' is uniform. These are alternatives to the more usual scalar relationship

$$r = mu/qB$$

for the radius of the path. They give the sign of $\boldsymbol{u}$ and the direction of the axis of rotation by inspection, and show the linear relationship between the velocity and the amount of source current which is needed to keep the orbit geometry unchanged.

The factor 2 is the same as appears in superconductor theory, in which the concept of momentum qA plays an important role [27]. Here m is the mass of the electron, and $2q$ is the charge of the Cooper pair (e.g. Reference 9, vol. III). In general, the canonical momentum

$$G = m\boldsymbol{u} + qA$$

is not conserved for reasons which depend on the choice of reference frame. When viewed from one, such as R_{coil}, relative to which q is in motion, the $\boldsymbol{u} \times \nabla \times \boldsymbol{A}$ term appears as a force which is additional to the momentum term, $\partial A/\partial t$, so that, in this reference, q is not in equilibrium under the action of $\nabla\phi$ and momentum forces only. As 'seen' from R_q, there is no $\boldsymbol{u} \times \boldsymbol{B}$ term, and the $\nabla\phi$ and momentum forces are sufficient, but both contribute to the circular motion of q, since, as the direction of $\boldsymbol{u}$ changes, so also do the parts of the current source which are selected by the $\boldsymbol{u} \cdot \boldsymbol{J}$ source of ϕ' (eqn. 28). R_q is not an inertial reference if it is defined so that q is continually stationary in it, but choosing it so that q is momentarily stationary shows that the $\nabla\phi'$ and $(\boldsymbol{u} \cdot \nabla)A$ terms make equal contributions to the total force. This illustrates the magnitude of the $\nabla\phi'$ term, as a part of the familiar $\boldsymbol{u} \times \boldsymbol{B}$ force. The equality of components occurs repeatedly in different guises, and is a result of the invariance of the scalar product of the $(\boldsymbol{J}, \rho)$ and (A, ϕ) 4-vectors, i.e. of the corresponding Lagrangian, or energy difference.

15 Conclusions

The study has extended previous work on electrokinetic momentum to the interaction between a charge q and a closed winding, carrying a constant current i, due to the motion of q past the winding. It has been shown that the kinetic, or 'magnetic', coupling between them causes an exchange of energy with the current source, at a rate which depends on the A_i vector due to i, whilst the magnetic field $\boldsymbol{B}_i$, due to i, at q, is not directly relevant and may be zero. In this respect the exchange is essentially the same as the 'transformer' action due to a change in the current i, which defines the momentum, and depends on the local A value at q, not the local $\boldsymbol{B}$. Both effects illustrate the physical significance of the (retarded) A, manifest either as an interaction due to charge acceleration, as in a transformer, or as a kinetic effect due to changes in position. Both suggest that the common view of the A vector as a mathematical fiction can be misleading, since it provides a direct measure of the kinetic interaction between the two sources of A.

The change in A_i experienced by q, due to its motion, provides a convenient reciprocal measure of the effect of q on the coil; that is, of the rate-of-change of flux linkage, given by A_q, due to q. The essential role of the vector A_q is to separate the nonconservative part of the electromagnetic effect of q from the contribution due to the potential ϕ_q, and thus select the part of the interaction which is important in a closed circuit. The induced EMF gives no net linear force on a uniformly-charged circuit, and likewise none on q, in accordance with field theory, which gives no $\boldsymbol{u} \times \boldsymbol{B}_i$ force due to i, if $\boldsymbol{B}_i$ is zero. Thus the energy exchange requires no conversion of momentum from qA to 'mechanical' ($m\boldsymbol{u}$) form.

The interaction depends on the electromagnetic effect of the charge velocity $\boldsymbol{u}$ in a form which was given explicitly by Maxwell, in his 'general equations', although absent from what are now referred to as the 'Maxwell equations'. The $\boldsymbol{E}$ term in the Lorentz force (eqn. 2) separates into two components, and this separation is essential to the concept of charge momentum qA. The $\boldsymbol{u} \times \boldsymbol{B}$ term likewise separates into parts, corresponding to ϕ and A terms, when 'seen' from the reference R_q in which q is stationary, and the corresponding forces are equal but opposite when $\boldsymbol{B}$ is zero. This gives two different views of the interaction. Whereas in R_q the momentum and $\nabla\phi$ forces are sufficient, any reference in which q is in motion requires an additional $\boldsymbol{u} \times \boldsymbol{B}$ term. Although this depends on the differential of A, it is not a momentum force, although it has a momentum component.

Expressing the charge momentum in terms of A shows that this is merely another view of inductance, or 'inductivity', links electromagnetic theory directly to circuit theory, and provides an equivalent circuit describing the operation. The energy conversion within the winding is at the expense of the internal energy of the system, which may be driven negative, so that the exchange demands no kinetic force on q when the current i is kept constant. This does not, however, imply the absence of a force, since the interaction is due to the 'electrostatic' effect of q, which means that the circuit is not uniformly charged, and the ϕ terms are correspondingly important. These include the ϕ_i' component of $\boldsymbol{u} \times \boldsymbol{B}$, in the reference R_q, and the ϕ_e term which is needed to transfer energy from the coil to the current source. The $\boldsymbol{J}\phi$ equivalent of the Poynting vector shows the importance of ϕ in any transfer of energy by the moving charges, and the device also illustrates the transfer by remote action, since the energy input to one component may be recovered from another.

The net change in the kinetic energy, $\boldsymbol{J} \cdot A/2$, or $q\boldsymbol{u} \cdot A/2$, is zero when q completes a closed path, or is recombined with the opposite charge, whilst the source current is kept constant, so that the exchange can be ignored in most practical applications, but not in all. It is illustrated by the Aharonov-Bohm, and related quantum-mechanical, effects, which are usually regarded as inconsistent with classical electromagnetic theory because they depend on mutual momentum and energy. The momentum concept, and the associated electromagnetic mass, shows that there are alternative definitions of what is meant by the force on a charge, although there can be no essential difference between field theory and Maxwell's 'dynamical' theory, since they share the same ϕ and A, and predict the same observable forces. However, the field description tends to obscure the interaction by combining all the components, and concentrating attention on the total field vectors, at the expense of the source charges and the mutual terms. It also tends to conceal the role of the momentum, which is fundamental to all electromagnetic charge interactions, regardless of frequency.

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