# Correspondence

## ELECTROMAGNETIC ENERGY AND POWER IN TERMS OF CHARGES AND POTENTIALS INSTEAD OF FIELDS

*Indexing terms: Electromagnetic theory, Mathematical techniques, Magnetic fields*

In paper 6471A (*IEE Proc. A*, 1989, **136**, (2), pp. 55–65) Dr Carpenter presented an alternative method of calculating the energy associated with an electromagnetic field. Instead of using the classical expression for the energy density given by

$$u_f = \tfrac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}) \tag{A}$$

Dr Carpenter assumes the expression

$$u_c = \tfrac{1}{2}(\rho\phi + \boldsymbol{j} \cdot \boldsymbol{A}) \tag{B}$$

where $\phi$ and $\boldsymbol{A}$ are the scalar and vector potentials respectively. The purpose of this note is to point out that $\int u_f \, dV$ and $\int u_c \, dV$ are not equal, in general, although they are identical when the fields are not changing with time. In general the electric field is given by

$$\boldsymbol{E} = -\operatorname{grad} \phi - \frac{\partial \boldsymbol{A}}{\partial t} \tag{C}$$

and it is easily shown, using Maxwell's equations that

$$u_f = u_c - \tfrac{1}{2}\operatorname{div}(\phi\boldsymbol{D} + \boldsymbol{H} \times \boldsymbol{A}) + \frac{1}{2}\left(\boldsymbol{A} \cdot \frac{\partial \boldsymbol{D}}{\partial t} - \boldsymbol{D} \cdot \frac{\partial \boldsymbol{A}}{\partial t}\right) \tag{D}$$

which, when integrated over all space, yields

$$\int u_f \, dV = \int u_c \, dV - \frac{1}{2}\int_S (\phi\boldsymbol{D} + \boldsymbol{H} \times \boldsymbol{A}) \cdot d\boldsymbol{S} + \frac{1}{2}\int \left(\boldsymbol{A} \cdot \frac{\partial \boldsymbol{D}}{\partial t} - \boldsymbol{D} \cdot \frac{\partial \boldsymbol{A}}{\partial t}\right) dV \tag{E}$$

where $S$ is a closed surface situated at an infinite distance from the charges and currents. Gauss's theorem has been employed to obtain this form.

For static fields the surface integral is zero because, at large distances, the potentials vary as $1/r$, and the fields vary as $1/r^2$. For time-varying fields we can state that the surface integral is still zero. This is because the fields and potentials propagate with a finite velocity and so would take an infinite time to reach the surface $S$. Therefore eqn. E reduces to

$$\int u_f \, dV = \int u_c \, dV + I \tag{F}$$

where

$$I = \frac{1}{2}\int \left(\boldsymbol{A} \cdot \frac{\partial \boldsymbol{D}}{\partial t} - \boldsymbol{D} \cdot \frac{\partial \boldsymbol{A}}{\partial t}\right) dV \tag{G}$$

It will be remembered that different choices of the vector $\boldsymbol{A}$ are possible, consistent with the definition (or partial definition) $\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}$. The different possibilities correspond to different choices for div $\boldsymbol{A}$ and this is referred to as the choice of gauge. The left-hand side of eqn. F is not dependent on the particular gauge which is used because it depends on the electromagnetic field and not explicitly on the scalar and vector potentials. On the other hand each integral on the right-hand side does depend on the choice of gauge; their sum is of course invariant under a gauge transformation. In general the potentials may be transformed as follows:

$$\boldsymbol{A}_1 = \boldsymbol{A} + \operatorname{grad} \Lambda(x, y, z, t) \tag{H}$$

and

$$\phi_1 = \phi - \frac{\partial \Lambda(x, y, z, t)}{\partial t} \tag{I}$$

where $\Lambda$ is an arbitrary gauge function.

In general the integral $I$ of eqn. F *is not zero* and eqn. B cannot be employed in time-varying situations. This has been clearly stated by Feynman, Leighton and Sands [A]. In some cases, however, the integral $I$ does vanish, e.g. for a transverse electromagnetic wave propagating along a transmission line with a suitable choice of gauge. In this case the following potentials may be used (see Allen [B]),

$$\phi(x, y, z, t) = \theta(y, z)F(x - ct) \tag{J}$$

and

$$\boldsymbol{A} = \boldsymbol{i}\phi(x, y, z, t)/c \tag{K}$$

where $c$ is the velocity of propagation. These potentials satisfy the Lorentz condition

$$\operatorname{div} \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{L}$$

We may note in passing that this condition represents a class of potentials and does not specify $\phi$ and $\boldsymbol{A}$ [C]. The electric field is perpendicular to $\boldsymbol{A}$, as may be readily verified using eqn. C, and the integral $I$ vanishes.

Consider now the same physical situation, but with the gauge transformed according, to eqns. H and I. The integral $I$ is now replaced by $I^1$ where

$$I^1 = \frac{1}{2}\int \left(\boldsymbol{A}^1 \cdot \frac{\partial \boldsymbol{D}}{\partial t} - \boldsymbol{D} \cdot \frac{\partial \boldsymbol{A}^1}{\partial t}\right) dV \tag{M}$$

so that

$$I^1 = \frac{1}{2}\int \left(\operatorname{grad} \Lambda \cdot \frac{\partial \boldsymbol{D}}{\partial t} - \boldsymbol{D} \cdot \frac{\partial}{\partial t}\operatorname{grad} \Lambda\right) dV \tag{N}$$

This integral is not zero, in general, because $\Lambda(x, y, z, t)$ is an arbitrary function of position and time. Thus the integral $I$, which was zero in the original gauge, is replaced by an integral $I^1$ which is not zero in the new gauge.

If we choose

$$\Lambda = \lambda(y, z)F(x - ct) \tag{O}$$

it can be verified that $I^1$ becomes zero. With this choice of $\Lambda$ we have

$$\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = F(x - ct)\left\{\frac{\partial^2 \lambda}{\partial u^2} + \frac{\partial^2 \lambda}{\partial z^2}\right\} \tag{P}$$

The right-hand side of eqn. P is not necessarily zero, so that the gauge is not necessarily Lorentzian [C]. It will become so if the Laplacian of $\lambda$ vanishes. This is the expression in the braces on the right-hand side.

The main purpose of this note, however, is not to discuss the choice of gauge. The Lorentz gauge is the natural choice in most cases. The main purpose is simply to state that $\int u_f \, dV$ and $\int u_c \, dV$ are not equal in general. The validity of the latter expresssion must therefore be carefully checked in any given application of Dr Carpenter's theory.

M. UEHARA
J.E. ALLEN

*14th January 1991*

*Department of Engineering Science*
*University of Oxford*
*Parks Road*
*Oxford OX1 3PJ*
*UK*

#### References

A FEYNMAN, R.P., LEIGHTON, R.B., and SANDS, M.: 'The Feynman lectures on Physics-Vol II'. 2nd edn. (Addison-Wesley, 1966) section 15–6
B ALLEN, J.E.: 'Digital-pulse approach to electromagnetism' (correspondence), *IEE Proc. A*, 1989, **136**, p. 113
C JACKSON, J.D.: 'Classical electrodynamics' (Wiley, 1962) p. 181

**8442A**

Dr Uehara and Dr Allen draw attention to a fundamental point in arguing the need for the integral $I$ in eqn. G to vanish, but give no evidence that it does not, in the Lorentz gauge. Their demonstration that it may not do so in other gauges is helpful, and supports other reasons (which I give in Section 6 of the paper) for choosing the Lorentz gauge, together with the difficulties in interpretation which otherwise follow. In considering their expressions it is pertinent to observe that what is usually known as the Slepian version of the Poynting vector

$$\boldsymbol{J}\phi + \phi \partial \boldsymbol{D}/\partial t + \boldsymbol{H} \times \partial \boldsymbol{A}/\partial t$$

provides the terms which are necessary to convey the energy from the conductors out into the field, when it is assumed to be stored with density $(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B})/2$, so that they transfer the energy between these sites and the alternative $(\rho\phi + \boldsymbol{J} \cdot \boldsymbol{A})/2$ sites. Thus the investigation becomes a matter of showing whether or not the net energy transfer is zero, and the underlying question is that of the validity of different interpretations of the Poynting vector.

Although this puts the matter into a different perspective, it does not directly answer the objection that I am assuming an expression for energy density which is often specifically rejected as a general alternative to the customary $(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B})/2$. Moreover the paper has been criticised as lacking adequate proof, and my own views have changed in some respects since it was written. In particular, I have come to the conclusion that the two total energies are not always the same, even in the Lorentz gauge. The method has been further explored in a paper which is currently under consideration [D], and this is immediately relevant, but it does not address the question directly, and I am glad of the opportunity to examine the stored energy equivalence more explicitly.

The foundation of the $\phi$, $A$ formulation is the equilibrium condition

$$\rho(-\nabla\phi - \partial \boldsymbol{A}/\partial t + \boldsymbol{u} \times \nabla \times \boldsymbol{A}) + \boldsymbol{f} = 0$$

or

$$\rho(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) + \boldsymbol{f} = 0 \qquad \text{(Q)}$$

giving the nonelectromagnetic force of density $\boldsymbol{f}$ on charges of density $\rho$. $\boldsymbol{f}$ remains the same whether the 'field' is interpreted as the $(\phi, A)$ 4-vector, or the $(\boldsymbol{E}, \boldsymbol{B})$ tensor obtained by differentiating $\phi$ and $\boldsymbol{A}$, so that both methods of description must necessarily give the same observations, in all possible experiments, since what we observe is the behaviour of charges*. Maxwell's bellringing metaphor [E] aptly summarises the point. Any electromagnetic system is equivalent to a belfry whose mechanism is accessible to the bellringer below via a set of ropes emerging through holes in the ceiling. Each can be used, either separately or in combination, to exchange force, energy and momentum with the mechanism, and our task is to devise models by observing the response of the ropes to the various inputs applied by the bellringer. Different models will have associated with them different physical concepts, but these are no more than convenient mnemonics, and, provided that the forces comply with eqn. Q, the only other requirement is that the concepts are interpreted in a way which is internally self-consistent.

One concept is stored energy. If any model correctly predicts all the forces, then it must necessarily include the correct amount for stored energy, since this is no more than a way of accounting for the bellringer's observation that he (or she) gets back energy expended on the ropes. Thus we have only to ensure that the energy density is calculated from the force $\boldsymbol{f}$, in eqn. Q, which is the more general form of eqn. 14$a$ of the paper. The $J^2/\sigma$ term in eqn. 18$a$ is replaced by $\boldsymbol{f} \cdot \boldsymbol{u}$, representing the force multiplied by the velocity of the bellringer's rope. Some of the energy thus expended is partly accounted for by the $w_c = \boldsymbol{J}\phi$ power that we (essentially arbitrarily) attribute to any moving charges, leaving the difference, $\phi\partial\rho/\partial t + \boldsymbol{J} \cdot \partial \mathbf{A}/\partial t$. This gives the rate-of-change of $(\rho\phi + \boldsymbol{J} \cdot \boldsymbol{A})/2$ if the local $\phi$ and $\boldsymbol{A}$ vary linearly with the local $\rho$ and $\boldsymbol{J}$, and we define the local energy density accordingly. If the variation is not linear, then an energy transfer, or radiation, across the empty spaces, is required to match the relevant terms in eqn. 18$a$ with the change in stored energy density. Radiation is, of course, very familiar in our observation of antenna behaviour, but is not limited to communication frequencies, as is often suggested by $\boldsymbol{E}$, $\boldsymbol{B}$ theory. All interactions between charges in which we expend energy on one, by moving it, and then recover the same energy by moving another, illustrate the radiation consequence of the remote action of the electromagnetic forces. The essential purpose of a transformer, for example, is a controlled transfer of energy across the empty space between the windings, on its way from the supply to the load.

I emphasise this point to show that the assumption that the energy density is $(\rho\phi + \boldsymbol{J} \cdot \boldsymbol{A})/2$, and power $\boldsymbol{J}\phi$, requires a corresponding definition of radiation to make the model self-consistent, and also to show that we can then readily investigate the stored energy equivalence. If the total $\phi$ and $\boldsymbol{A}$ radiation obtained in this way sums to zero over the system as a whole, then the integral of the $(\rho\phi + \boldsymbol{J} \cdot \boldsymbol{A})/2$ term must necessarily, give the net stored energy required to account for the 'mechanical' work $\boldsymbol{f} \cdot \boldsymbol{u}$, and the 'electrical' power $\boldsymbol{J}\phi$. When we assume the Lorentz gauge, the radiation terms in $\phi$ and $\boldsymbol{A}$ separate, and it is then a simple matter to show that, if there is no retardation, the requirement that the net radiation is zero

* This is subject to the difficult requirement that we can satisfactorily define what we mean by a 'nonelectromagnetic' force, on a charge such as an electron [D], and also that we can distinguish between electromagnetic and 'nonelectromagnetic' mass [F]. However the definitions are irrelevant for the present purpose provided that they are the same in both theories.

is merely a restatement of the reciprocity conditions for $\rho\phi$ and $\boldsymbol{J} \cdot \boldsymbol{A}$. I give the details of the $\phi$ calculation in Reference D. If we use other gauges, then the radiation does not separate into independent $\phi$ and $\boldsymbol{A}$ components, and the simplicity of the calculation disappears. I have not explored the consequences, but the example given in Section 6 of the paper shows why I am entirely happy to accept the suggestion that the $(\rho\phi + \boldsymbol{J} \cdot \boldsymbol{A})/2$ density may not then integrate to give the stored energy of the system.

Thus the energy densities are equivalent at all frequencies, in the Lorentz gauge, if the system† as a whole does not radiate, or, alternatively, if there is no significant retardation in the interactions between the system parts (since this is required for reciprocity); one of the advantages of the $\phi$, $\boldsymbol{A}$ description is that it shows directly the equivalence between these two statements. However, the radiation terms do not, in general, sum to zero, and when we come to consider the behaviour of a system which radiates a net energy a difficulty arises in comparing the two models.

When viewed in $\phi$, $\boldsymbol{A}$ terms, the characteristic feature of radiated energy is that it is not recoverable by the system that is radiating it (assuming an open universe), so that it is dissipated and not stored. When viewed in $\boldsymbol{E}$, $\boldsymbol{B}$ terms, on the other hand, the system includes the field around the charges, and the energy remains recoverable if the boundary surface is sufficiently remote. In terms of the belfry metaphor, the ropes no longer provide the only access, because the function of the bells is to produce an acoustic energy field, and we may add other transducers to include reception as well as transmission. This suggests that the equivalence no longer applies, or, at best, suggests the need for care in the way in which we apply it. Since the forces are still correctly predicted, both within the radiating system, and also in any other system within the radiation field, all observable effects necessarily remain the same in both descriptions, and the point under investigation is no more than a matter of interpretation. It seems to me extremely important to recognise this very clearly, or we may otherwise lose sight of what actually matters to the engineer.

The usual example of a simple dipole helps to amplify, and clarify, the point. Suppose that the dipole consists of two spheres, carrying charge but no current, linked by a straight wire carrying current but no charge, so that the spheres are the sources of $\phi$ and the link the source of $\boldsymbol{A}$. The energy radiation can be obtained by calculating $\phi$ and $\boldsymbol{A}$, which are then customarily differentiated to obtain $\boldsymbol{E}$ and $\boldsymbol{B}$, followed by the integration of $\boldsymbol{E} \times \boldsymbol{H}$ over a closed surface drawn around the dipole. We may alternatively use the 'EMF' method, in which $\boldsymbol{E}$ is reintegrated along the dipole, and this calculation can be interpreted more directly in $\phi$, $\boldsymbol{A}$ terms, without any reference to $\boldsymbol{E}$. The potential difference between the two spheres acquires a component in phase quadrature to the local charge, as a direct consequence of the time delay, so that the capacitance acquires a resistive component (not as an analogy, but as a literal description of the nature of the forces). So also does the inductance of the link, as given by $\boldsymbol{A}$, which is the inductance per unit length, multiplied by the current. The capacitive contribution to the resistance is negative, and the inductive contribution positive, showing an energy exchange as one important aspect, but the point that is immediately relevant is the net loss of energy due to the retardation. Because any observer, either inside or outside the system, sees the spheres at two different time phases, the dipole acquires a net charge, together with a corresponding change in the link current. These are the sources of the 'far-field', giving the net energy radiation, or dissipation, as distinct from the 'near-field', whose energy is recoverable, and thus is 'stored', at all operating frequencies.

The details of the calculation, part of which has been given in Reference D, are not needed to illustrate the sense in which the energy storage terms can no longer be the same, and why this does not matter to the engineer. We may observe that, even in the $\boldsymbol{E}$, $\boldsymbol{B}$ view, the surface used to integrate $\boldsymbol{E} \times \boldsymbol{H}$ defines, by its contents, a system which is dissipating energy, if we maintain a steady sinusoidal excitation, and wait long enough. This shows the underlying point, and also the consequences of neglecting the surface term in Drs Uehara and Allen's eqn. E. The resistive components of the capacitance and inductance assume a steadystate sinusoidal supply, whereas the energy which is carried by the wavefront, in the $\boldsymbol{E}$, $\boldsymbol{B}$ view, represents a perpetual transient, described by the $\phi$, $\boldsymbol{A}$ conditions when the dipole was originally energised. There is no disagreement in what we observe, and the $\boldsymbol{E} \times \boldsymbol{H}$ calculation gives exactly the same results as the 'EMF', or $\phi$, $\boldsymbol{A}$ method, when we allow time for steady sinusoidal conditions everywhere inside the integration surface. If we examine the instant at which the wavefront goes through the surface, we find that the $\boldsymbol{E}$, $\boldsymbol{B}$ view of the energy changes from storage to dissipation, so that the source of any misunderstanding is not in the comparison between the $\phi$, $\boldsymbol{A}$ and $\boldsymbol{E}$, $\boldsymbol{B}$ models, but is internal to the latter.

I cannot find anything in Reference A to show that the integral $I$ may not be zero, other than the unsupported assertion, common to a great many texts, that $(\rho\phi + \boldsymbol{J} \cdot \boldsymbol{A})/2$ is valid only under rather ill-defined conditions. On the other hand Professor Feynman and his co-authors remark in Section 15.5 that $\boldsymbol{A}$ and $\phi$ are replacing $\boldsymbol{E}$ and $\boldsymbol{B}$ in the modern expression of physical laws, and if this is to have any meaning in macroscopic electromagnetism, it implies the $\phi$, $\boldsymbol{A}$ view of the energy density. Clearly neither of the models is to be taken literally. They suggest that stored energy is a system property that cannot be distributed in any observable sense (except where there is energy conversion, as in polarisable materials).

C.J. CARPENTER *7th May 1991*

*Visiting Fellow*
*Department of Electrical and Electronic Engineering*
*University Walk*
*Bristol BS8 1TR*

† This assumes that the 'system' consists of all the parts of devices such as capacitors and transformers.

### References

D CARPENTER, C.J.: 'Electromagnetic theory without electric flux', under consideration for IEE Proc. A (paper SMT 887)

E MAXWELL, J.C.: 'Scientific papers' (Cambridge University Press, 1890, reprinted Dover, 1965), p. 783

F CARPENTER, C.J.: 'Electromagnetic energy changes due to charges moving through constant, or zero, magnetic field', *IEE Proc. A*, 1991, 138, (1), pp. 55–70, section 11