The Ampère and Biot–Savart force laws

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Abstract. The known equivalence between the Ampère and Biot–Savart force laws, for closed circuits carrying an electric current, is here extended to the case of the force on a part of a circuit and due to the action of the other part of the same circuit. Our theorem invalidates some criticism made to the Biot–Savart law and the experimental results favouring Ampère's law. A recent experiment is in agreement with the here proved theorem.

Riassunto. La nota equivalenza fra le leggi di Ampère e di Biot–Savart per circuiti chiusi percorsi da corrente elettrica, viene qui estesa al caso della forza su una parte di un circuito e dovuta all'azione delle altre parti dello stesso circuito. Questo teorema invalida la critica fatta alla legge di Biot–Savart ed anche i risultati sperimentali che sembrerebbero favorire la legge di Ampère. Un recente esperimento è in accordo col teorema qui dimostrato.

1. Introduction

Since the formulation of Ampère's law and that of Biot–Savart for the interaction force between current elements, several articles on the subject have characterized an increasingly interesting theoretical controversy on the fundamentals of electromagnetism. The salient aspect of the controversy refers to the experimental discrimination of the two laws, and this is related to several measurements of the interaction force: some were performed more than a decade ago (Graneau 1982, Pappas 1983), while the last is quite recent (Cavalleri *et al* 1995).

Ampère's law (Ampère 1823) for the force one current element $i\,\mathrm{d}\boldsymbol{s}$ exerts on another $i'\,\mathrm{d}\boldsymbol{s}'$ is given by

$$\boldsymbol{A}_{ds',ds} = -\boldsymbol{r}ii'[2\,\mathrm{d}\boldsymbol{s}\cdot\mathrm{d}\boldsymbol{s}'/r^3 - 3(\mathrm{d}\boldsymbol{s}\cdot\boldsymbol{r})(\mathrm{d}\boldsymbol{s}'\cdot\boldsymbol{r})/r^5] \qquad (1)$$

where $\boldsymbol{r}$ is the vector in the direction from the point P$'$ at $\mathrm{d}\boldsymbol{s}'$ to the point P at $\mathrm{d}\boldsymbol{s}$.

The law of Biot and Savart can be expressed in the following form (Grassmann 1845): the field at P$'$, caused by $i\,\mathrm{d}\boldsymbol{s}$ at P, is $i(\mathrm{d}\boldsymbol{s}\times\boldsymbol{r})/r^3$ and the force on $i'\,\mathrm{d}\boldsymbol{s}'$ at P$'$ is

$$\boldsymbol{B}_{ds',ds} = ii'\,\mathrm{d}\boldsymbol{s}\times(\mathrm{d}\boldsymbol{s}\times\boldsymbol{r})/r^3. \qquad (2)$$

The crucial point of the controversy is that Newton's third law is obeyed with Ampère's law, because $\boldsymbol{A}_{ds',ds} = -\boldsymbol{A}_{ds,ds'}$ while, with force law (2), the action and reaction principle is violated because $\boldsymbol{B}_{ds',ds}$ is not in general equal to $-\boldsymbol{B}_{ds,ds'}$. This circumstance has led some authors (Moon and Spencer 1954, Aspden 1969) to develop and propose alternative electrodynamical theories based on law (1) in order to preserve the action and reaction principle. Actually the violation implied in the Biot–Savart law regards the forces only and is striking if one thinks in terms of steady-state situations. However, an isolated current element is made of positive and negative electrical charges in relative motion and the charges that appear at both ends of the element produce a time varying electric field of a dipole kind. The first cardinal equation of dynamics is saved if we consider also the electromagnetic momentum and its time variation. In closed circuits carrying constant currents the variation of the electromagnetic momentum vanishes and we are left with the macroscopic forces on the circuits which, in this case, satisfy Newton's third law (Cavalleri *et al* 1988, Yoneda 1994).

Moreover, since law (2) forms the basis for expressing the Lorentz force in a covariant form, the development of special relativity has made the Biot–Savart law the familiar law accepted by the scientific community.

Nevertheless, Wesley (1983) has argued that the Biot–Savart law is untenable, because, when applied to a single current loop, it gives a net resultant force which is not zero. In this case, he reasons, the current loop, which is an isolated system with no external forces applied, could be set in motion thanks to the internal action of the current on itself.

Furthermore, Graneau and Pappas have published some accounts of experiments which favour the Ampère force and disprove the Biot–Savart force. These

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experiments are related to the original test performed by Ampère himself, who tried to show the existence of the longitudinal forces on current elements. These longitudinal forces are a peculiarity intrinsic to force law (1). Force law (2) indicates that only perpendicular forces exist.

The existence of longitudinal forces was claimed previously even by Robertson in his interpretation of a simple experiment performed by Hering in 1923. Robertson (1945) suggests that Hering's experiment, which does not account for the law of maximum flux, can be explained only by Ampère's theory.

Our intention in this paper is to dispute the theoretical and experimental considerations against the Biot–Savart law considered above.

2. Limits of validity of the action and reaction principle for the Biot–Savart law

Although it is not generally explicitly stated, both laws (1) and (2) imply that the force on $i'\,\mathrm{d}\boldsymbol{s}'$ is the only force acting on this current element when only the two current elements $i\,\mathrm{d}\boldsymbol{s}'$ and $i\,\mathrm{d}\boldsymbol{s}$ are considered. Therefore, it is taken for granted that the force due to the interaction of $i'\,\mathrm{d}\boldsymbol{s}'$ on itself is just zero.

However, for Biot–Savart, even assuming that an infinitesimal current element does not act on itself, the net force

$$\boldsymbol{B}_{C,C} = i^2 \int_C \int_C \mathrm{d}\boldsymbol{s}' \times (\mathrm{d}\boldsymbol{s} \times \boldsymbol{r})/r^3 \qquad (3)$$

due to the action of a finite part C of a current loop on itself is generally different from zero. For Ampère's force, instead, one can show that

$$\boldsymbol{A}_{C,C} = -i^2 \int_C \int_C \boldsymbol{r}[2\,\mathrm{d}\boldsymbol{s}\cdot\mathrm{d}\boldsymbol{s}'/r^3 - 3(\mathrm{d}\boldsymbol{s}\cdot\boldsymbol{r})(\mathrm{d}\boldsymbol{s}\cdot\boldsymbol{r})/r^5] = 0. \qquad (4)$$

In fact, when integrating over C the elements $\mathrm{d}\boldsymbol{s}'$ interchanges position with $\mathrm{d}\boldsymbol{s}$, and $\boldsymbol{r}$ changes sign so that to every term of the integral will correspond an equal and opposite term.

The argument of Wesley (1983) against the Biot–Savart law goes as follows. By expanding the vector product $\mathrm{d}\boldsymbol{s}' \times (\mathrm{d}\boldsymbol{s} \times \boldsymbol{r}) = (\mathrm{d}\boldsymbol{s}'\cdot\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{s} - (\mathrm{d}\boldsymbol{s}'\cdot\mathrm{d}\boldsymbol{s})\boldsymbol{r}$, and using vector identities, the net force on both current elements may be written as

$$\boldsymbol{B}_{ds',ds} + \boldsymbol{B}_{ds,ds'} = i^2\boldsymbol{r} \times (\mathrm{d}\boldsymbol{s} \times \mathrm{d}\boldsymbol{s}')/r^3. \qquad (5)$$

Then Wesley integrates over the single closed loop ℓ, composed of the two parts $C + C'$, obtaining

$$\boldsymbol{B}_W = i^2\left[\int_{C'}\int_C \mathrm{d}\boldsymbol{s}' \times (\mathrm{d}\boldsymbol{s} \times \boldsymbol{r})/r^3 - \int_{C'}\int_C \mathrm{d}\boldsymbol{s} \times (\mathrm{d}\boldsymbol{s}' \times \boldsymbol{r})/r^3\right] = i^2 \int_C \int_{C'} \boldsymbol{r} \times (\mathrm{d}\boldsymbol{s} \times \mathrm{d}\boldsymbol{s}')/r^3 \qquad (6)$$

which is different from zero. However, we notice that in the first integral of (6) Wesley has neglected the action of C' on itself, namely $\int_{C'}\int_{C'}$, and in the second, the action of C on itself, $\int_{C'}\int_C$, is also missing. If these two contributions are added we obtain for the net force due to the action of a circuit ℓ on itself

$$\boldsymbol{B} = i^2 \oint\oint [\mathrm{d}\boldsymbol{s}' \times (\mathrm{d}\boldsymbol{s} \times \boldsymbol{r})/r^3].$$

Expanding the vector product, and considering that $\oint[\mathrm{d}\boldsymbol{s}'\cdot\boldsymbol{r})/r^3] = \oint[\mathrm{d}\boldsymbol{s}'\cdot\boldsymbol{\nabla}(1/r)] = 0$, the above expression yields

$$\boldsymbol{B} = i^2 \oiint [(\mathrm{d}\boldsymbol{s}'\cdot\mathrm{d}\boldsymbol{s})\boldsymbol{r})/r^3] = 0.$$

In fact, as for Ampère's force in (4), the above integral can be thought of as being composed of a series of equal and opposite contributions when $\mathrm{d}\boldsymbol{s}'$ is interchanged with $\mathrm{d}\boldsymbol{s}$ in the process of integration.

We complete our considerations on the Biot–Savart law by showing that even the torques on two current loops are equal and opposite. Let $\boldsymbol{s} = \mathrm{OP}$, and $\boldsymbol{s}' = \mathrm{OP}'$, be the distance from the current elements $i\,\mathrm{d}\boldsymbol{s}$ and $i'\,\mathrm{d}\boldsymbol{s}'$ respectively from the pole O of the torque, with $\boldsymbol{s}' - \boldsymbol{s} = \boldsymbol{r}$. The torque on the loop ℓ' due to the action of the loop ℓ is

$$\begin{aligned} M' &= \oint_{l'} \boldsymbol{s}' \times [i'\,\mathrm{d}\boldsymbol{s}' \times \oint_l i(\mathrm{d}\boldsymbol{s} \times \boldsymbol{r})/r^3] \\ &= ii' \oiint \boldsymbol{s}' \times [(\mathrm{d}\boldsymbol{s}'\cdot\boldsymbol{r})\,\mathrm{d}\boldsymbol{s} - (\mathrm{d}\boldsymbol{s}'\cdot\mathrm{d}\boldsymbol{s})(\boldsymbol{s}' - \boldsymbol{s})]/r^3 \\ &= ii' \oiint [-\boldsymbol{s}' \times \mathrm{d}\boldsymbol{s}(\mathrm{d}\boldsymbol{s}'\cdot\boldsymbol{\nabla}'(1/r) \\ &\quad + \boldsymbol{s}' \times \boldsymbol{s}(\mathrm{d}\boldsymbol{s}'\cdot\mathrm{d}\boldsymbol{s})/r^3]. \end{aligned} \qquad (7)$$

The first term of (7) can be developed as

$$\oint_{l'} \boldsymbol{s}' \times \mathrm{d}\boldsymbol{s}\,\mathrm{d}(1/r) = \oint_{l'} \mathrm{d}(\boldsymbol{s}'/r) \times \mathrm{d}\boldsymbol{s} - \oint_{l'} (1/r)\,\mathrm{d}\boldsymbol{s}' \times \mathrm{d}\boldsymbol{s} = 0 - \oint_{l'} \mathrm{d}\boldsymbol{s}' \times \mathrm{d}\boldsymbol{s}/r.$$

With the help of the above expression (7) yields

$$M' = -ii' \oiint [\mathrm{d}\boldsymbol{s}' \times \mathrm{d}\boldsymbol{s}/r + \boldsymbol{s}' \times \boldsymbol{s}(\mathrm{d}\boldsymbol{s}'\cdot\mathrm{d}\boldsymbol{s})/r^3],$$

which is antisymmetric in $\boldsymbol{s}$ and $\boldsymbol{s}'$, implying that $M' = -M$.

For a single loop, the result $M' = 0$ follows from considerations analogous to those made previously about the action of a loop on itself.

3. Experimental equivalence of Ampère and Biot–Savart laws

Lyness (1961–62) has shown that the only force between current loops, measurable experimentally, is the force on a current element C' due to the action of a current loop ℓ, i.e. $\boldsymbol{B}_{C',\ell}$ or $\boldsymbol{A}_{C',\ell}$. Here C' can represent a part, or the whole, of a closed circuit ℓ'. Moreover, he has shown that, for two different circuits ℓ and ℓ', no discrimination is possible between the Ampère and Biot–Savart laws, because

$$\boldsymbol{B}_{C',\ell} = \boldsymbol{A}_{C',\ell}. \tag{8}$$

Let us ideally divide the loop into two parts, C and C'. Graneau and Pappas performed experiments to measure the force, due to part C of ℓ, acting on part C' of the same loop. They claim that discrimination is possible between $\boldsymbol{B}_{C',C}$ and $\boldsymbol{A}_{C',C}$, and that they have found evidence of the longitudinal forces predicted by Ampère. Furthermore, Pappas states that (8) is not valid when C' is part of ℓ, probably because of the presumed divergences in (1) and (2) when $\mathrm{d}\boldsymbol{s} \equiv \mathrm{d}\boldsymbol{s}'$. However, if the action of a current element on itself is not zero, both laws (1) and (2) are wrong even when they are applied to two different loops, because both neglect the self-interaction of the current elements of each loop. Thus, we may assume that (8) has general validity. Since, from (4), $\boldsymbol{A}_{C',C'} = 0$, we can write $\boldsymbol{A}_{C',C} = \boldsymbol{A}_{C',C} + \boldsymbol{A}_{C',C'} = \boldsymbol{A}_{C',\ell'}$. Furthermore, for the Biot–Savart force, we have to take into account the action of C' on itself, $\boldsymbol{B}_{C',C'}$, which cannot be separated, experimentally, from the action of C on C'. Finally, we obtain, by means of (8)

$$\boldsymbol{A}_{C',C} = \boldsymbol{A}_{C',\ell} = \boldsymbol{B}_{C',C} + \boldsymbol{B}_{C',C'} = \boldsymbol{B}_{C',\ell}.$$

From the above result, discrimination is not possible even for the interaction between parts of the same circuits. Our result invalidates the claim of the experiments of Graneau and Pappas, and implies a complete experimental equivalence between the Ampère and Biot–Savart laws.

The other evidence for the existence of longitudinal forces, claimed by Robertson, refers to Hering's experiment described in figure 1. A current flows in a circuit ABCD. Part of the side BC is the wire W, pivoted at P, the other end of W being free to move on the mercury in a curved trough M. Contrary to the law of maximum flux, the wire W moves inwards. According to Ampère's law this fact can be explained because of longitudinal repulsion between the current in the mercury and that in the free end of the wire. However, this experiment can be also explained by the Biot–Savart law. In fact, the current element BB′ along the mercury trough generates a magnetic field about the end of the wire B′, with the result that the magnetic force acting on the wire is inwards.

4. Conclusions

We have shown that the force and torque calculated according to the Biot–Savart law do not violate the action and reaction principle when the law is applied to closed circuits. Furthermore, even when considering only a part of the circuit, it is not possible to discriminate

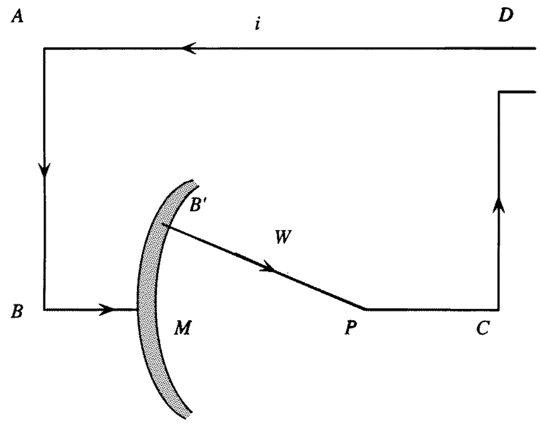

Figure 1. An electric current flows in the circuit ABCD. The horizontal wire W is pivoted at P, the other end of W being free to move over mercury in the curved trough M. Starting from this position, the segment W is pulled inward by the magnetic force.

the two basic laws of Ampère and Biot–Savart, which are to be considered experimentally equivalent.

Finally, because one could object that any theory can be disproved by experiments, the Graneau–Pappas experiment has been repeated by Cavalleri *et al* and is described in another paper. The result disproves the Graneau–Pappas experiments and confirms the standard theory emphasized in this paper.

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