Elliptic vortices of electromagnetic wave fields

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We demonstrate the existence of elliptic vortices of electromagnetic scalar wave fields. The corresponding intensity profiles are formed by propagation-invariant confocal elliptic rings. We have found that copropagation of this kind of vortex occurs without interaction. The results presented here also apply for physical systems described by the (2 + 1)-dimensional Schrödinger equation. © 2001 Optical Society of America

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Electromagnetic vortices occur for some solutions of linear and nonlinear wave equations. Depending on their particular features, they may be referred to as wave dislocations or disclinations, field defects, singular fields, helical waves, rotating waves, and dark beams.[1–6] Potential applications of electromagnetic vortices as optical waveguides,[2] in manipulations of microparticles,[7] and in the detection of weak optical signals[8] have motivated theoretical and experimental research in recent years.

For the linear case, high-order Bessel beams are the fundamental vortex solutions of the three-dimensional Helmholtz wave equation, whereas Laguerre–Gaussian beams are their counterparts in the paraxial approximation. For the nonlinear case, vortices can occur as solutions of the nonlinear Schrödinger equation that represents vortex solitons.[2] These kinds of solution can also be found for the Gross–Pitaevskii equation, which is used in the studies of Bose–Einstein condensates,[3] and for the Ginzburg–Landau equation, which occurs in solid-state physics, superfluidity, superconductivity, and the spatiotemporal dynamics of laser resonators.[4,5]

The propagation and interaction of optical vortices have been studied in linear and nonlinear media. In both cases it has been observed that vortices can interact through the intensity and the phase gradient between them and hence be displaced from their original transverse positions.[9–13]

In this Letter we demonstrate the existence of electromagnetic elliptic vortices that can occur as fundamental solutions of the Helmholtz and Schrödinger wave equations. We show that high-order elliptic vortices are composed of a number of unitary vortices that do not interact on propagation, even in a nonlinear medium. In this case the overall phase of the field appears to rotate about a thin lamina as it propagates.

Propagation of scalar electromagnetic fields in linear media is described by the Helmholtz equation $\nabla^2 E + k^2 E = 0$, where, for wave fields of angular frequency ω traveling with speed v in a transparent medium, the wave number is $k = \omega / v$. The most elemental vortex of the Helmholtz equation (HE) and the (2 + 1)-dimensional Schrödinger equation (SE) is found for fields with radial symmetry that carry an azimuthal phase factor of the form $\exp(\pm i m \varphi)$. The integer m defines the winding number or topological charge; and its sign, the direction of rotation.

We investigate only the scalar HE because the SE can be treated in the same way. In the transverse plane, elliptic coordinates with foci at $(\pm h, 0)$ are defined by the transformation $x + iy = h \cosh(\xi + i\eta)$, where the eccentric radial and angular curvilinear coordinates vary in the range $0 \le \xi < \infty$ and $0 \le \eta < 2\pi$, respectively. In these coordinates, the three-dimensional HE separates into a longitudinal part that has a solution with dependence $\exp(\pm i k_z z)$ and a transverse part that further separates into the eccentric radial (modified) Mathieu equation and the eccentric angular (ordinary) Mathieu equation.[14] These are, respectively,

$$\frac{\partial^2 R(\xi)}{\partial \xi^2} - (a - 2q \cosh 2\xi) R(\xi) = 0\,, \tag{1}$$

$$\frac{\partial^2 \Phi(\eta)}{\partial \eta^2} + (a - 2q \cos 2\eta) \Phi(\eta) = 0\,, \tag{2}$$

where the eigenvalue a emerges as a separation constant. Parameter $q = k_t^2 h^2 / 4$ carries information about transverse frequency k_t and the elliptic coordinate system through h. The transverse and longitudinal wave-vector components satisfy $k^2 = k_t^2 + k_z^2$.

The solutions to Eqs. (1) and (2) are known as Mathieu functions.[14–18] The values of a for which angular equation (2) has periodic solutions are denoted a_m $(m = 0, 1, 2 \ldots)$ for the even solutions $\mathrm{ce}_m(\eta; q)$ and b_m $(m = 1, 2, \ldots)$ for the odd solutions $\mathrm{se}_m(\eta; q)$. For a given order m (i.e., a_m and b_m), radial equation (1) has four associated solutions: two even solutions, $\mathrm{Je}_m(\xi; q)$ and $\mathrm{Ne}_m(\xi; q)$, and (when $m \neq 0$) two odd solutions, $\mathrm{Jo}_m(\xi; q)$ and $\mathrm{No}_m(\xi; q)$. The

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nomenclature that we have adopted here ensures a direct correspondence to the Bessel circular cylindric functions J_m and N_m when $h \to 0$, namely, Je_m, $\mathrm{Jo}_m \to J_m$ and Ne_m, $\mathrm{No}_m \to N_m$. In the mathematical literature, these function pairs are normally written Ce_m, Se_m and Fe_m, Ge_m, respectively.[15–18]

A suitable linear combination of products of the angular and the radial solutions can be used to represent elliptic traveling waves of the HE in elliptic coordinates in the same way as in circular coordinates.[6,19] To find appropriate values of the coefficients, we use the solution of the three-dimensional HE expressed by a reduced form of the Whittaker integral,[16] namely,

$$E(\mathbf{r}) = \exp(ik_z z)\int_0^{2\pi} A(\varphi) \times \exp[ik_t(x\cos\varphi + y\sin\varphi)]\mathrm{d}\varphi\,, \tag{3}$$

where $A(\varphi)$ is the angular spectrum of field $E(\mathbf{r})$. Inasmuch as the field intensity that is due to Eq. (3) is independent of propagation coordinate z, it represents propagation-invariant electromagnetic fields. For instance, in circular coordinates, where $x + iy = r\exp(i\theta)$, setting $A(\varphi) = \exp[im(\varphi - \theta)]$ results in $E(r,\theta,z) = J_m(k_t r)\exp(im\theta + ik_z z)$, which, for optical fields, is identified as the propagation-invariant mth-order Bessel beam.[20]

For the elliptic case, the eccentric angular solutions of Eq. (2) are $\mathrm{ce}_m(\eta;q)$ and $\mathrm{se}_m(\eta;q)$, which we use to construct the complex function $\mathrm{ee}_m(\varphi;q) = \mathrm{ce}_m(\varphi;q) + i\,\mathrm{se}_m(\varphi;q)$. Setting $A(\varphi) = \mathrm{ee}_m(\varphi;q)$, using the corresponding coordinate transformation in Eq. (3), and evaluating the integral,[17] we obtain the final form for the electromagnetic field:

$$E(\xi,\eta,z;q) = [A_m(q)\mathrm{Je}_m(\xi;q)\mathrm{ce}_m(\eta;q) + iB_m(q)\mathrm{Jo}_m(\xi;q)\mathrm{se}_m(\eta;q)]\exp(ik_z z)\,, \tag{4}$$

which includes the coefficients $A_m(q)$ and $B_m(q)$ mentioned above. We have found that, for $m \geq 2$ and $q \lesssim m^2/2 - 1$, numerical evaluation of these constants[17,18] yields values that are such that $A_m(q) \simeq B_m(q)$ and hence can be factored out.

The physics of Eq. (4) becomes clear once the field is displayed graphically. In Fig. 1, the transverse intensity patterns and corresponding phase at $z = 0$ are shown for $m = 1, 4, 7$. The field intensity has an elliptic ringed structure, and its phase rotates following an elliptic trajectory. For the total topological charge $m = 1$, the phase is formed by a single elliptic vortex. For $m \geq 2$, the phase is formed by m in-line vortices, each with unitary topological charge such that the total charge (along a closed trajectory enclosing all the vortices) is m. The branch cuts lie upon confocal hyperbolas, implying that, on propagation, a point in the phase front travels along an elliptic helix of constant ξ. Observe that the phase gradient along the line joining adjacent vortices is zero.

From Eq. (4) it is clear that, unlike vortices of the circular cylindric HE, elliptic vortices also depend on the eccentric radial variable. The principal branch points of the elliptic vortices are found where the real and imaginary components of the field, and the eccentric radial coordinate ξ, are zero. This occurs at $x_\alpha = h\cos(\eta_\alpha)$, where $\alpha = 1,\ldots,m$ and η_α are the zeros of $\mathrm{ce}_m(\eta;q)$ in the interval $[0, \pi]$. Whereas in first order the phase rotates about the longitudinal axis, for higher order it apparently rotates about a strip in the x–z plane defined by points x_1 and x_m, the width of which is always smaller than $2h$.

The field in Eq. (4) is a fundamental solution of the HE and defines the mth-order Mathieu propagation-invariant electromagnetic field. It constitutes the mth member of a two-dimensional Fourier expansion in elliptical coordinates. The corresponding Fourier coefficients can be obtained by use of the two-dimensional orthogonality theorem for Mathieu functions.[15,21] These functions form a complete basis for wave fields described by the HE (or wave functions for the SE). The $m = 0$ solution has been studied theoretically in the optical context[22] and was realized experimentally in the research reported in Ref. 23. Although as it stands Eq. (4) represents a field of infinite width, it must be stressed than any experimental realization of a propagation-invariant field will inevitably be of finite lateral extent; the beam in the laboratory will therefore propagate without change of shape within the Fresnel region defined by the size of the physical aperture and its associated conical geometry.[19] Outside this region, diffraction effects modify the dynamics of the vortices.

The elliptic patterns of Fig. 1 were obtained under the assumption that the coefficients $A_m(q)$ and $B_m(q)$ are practically equal and can be neglected. However, for a given order m the numerical values of these coefficients depend on q, and there exists a critical value, q_c, for which $A_m(q) \neq B_m(q)$ if $q > q_c$. A typical pattern for this situation is shown in Fig. 2 for a fourth-order

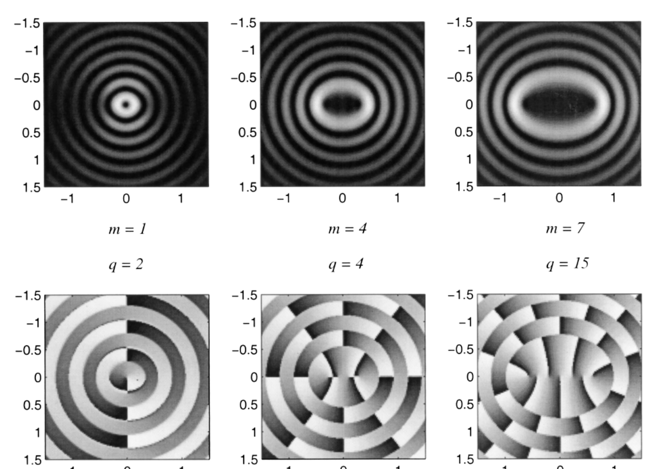

Fig. 1. Intensities and elliptic phases of higher-order Mathieu beams. The ellipticity of the phase is best observed in the central region of the ringed pattern.

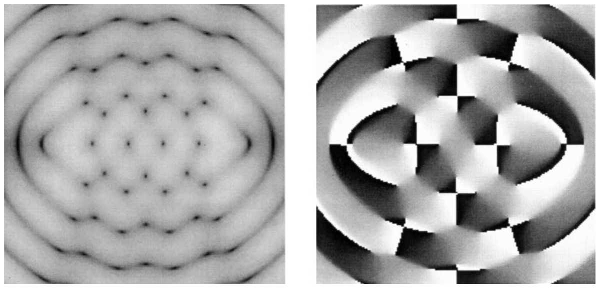

Fig. 2. Breakup of a fourth-order elliptic vortex as a result of excessive stretching of the elliptic ring pattern by an increase of the value of interfocal distance h of the elliptic coordinates such that now the coefficients $A_m(q)$ and $B_m(q)$ in Eq. (4) are not equal, as is the case in Fig. 1. Observe that new vortices are created and that some of them rotate in the opposite direction to the original.

pattern with $q = 20$. Observe that a number of vortices have been created and are arranged in a crystallike pattern. We remark that this intensity pattern is propagation invariant, a possible interpretation of which is that, for a fixed wavelength, we can stretch an elliptic pattern of order m by increasing the value of h and consequently of q; doing so results in breaking the elliptic pattern, and a new vortex structure is formed. The phase picture shows that some of the newly created vortices rotate in the opposite direction to the original, so the total topological charge m must remain unaltered. This breaking of the pattern shows the relevance of the coefficients in the description of elliptic waves of the HE.

The fact that Eq. (4) represents a propagation-invariant field implies that (for $m \geq 2$) the constituent vortices remain in the same transverse position. This behavior contrasts with that reported in Refs. 9–13, where the copropagating vortices shifted as a result of the amplitude and phase gradients of the diffracted background field. The difference arises because the field distributions discussed in this Letter are natural modes of the wave equation and possess the structure necessary for invariant propagation within the limits mentioned above. We performed several simulations to confirm that propagation-invariant behavior does indeed occur, even in the presence of Kerr-type nonlinearity of both focusing and defocusing types.

One can apply the wave field presented here to investigate elliptic optical tweezers and atom traps as well to study the transfer of angular momentum to microparticles or atoms.[7,24] It may be enlightening to analyze the effects of an induced elliptic rotating phase and its angular momentum on Bose–Èinstein condensates[3] or on the formation of quantum mirages in elliptic corrals.[25]

In conclusion, we have demonstrated the existence of electromagnetic elliptic vortices of the Helmholtz and Schrödinger wave equations. We have introduced the function $\mathrm{ee}_m(\varphi; q)$ that is the required angular spectrum for their creation and obtained the appropriate coefficients to represent elliptic traveling waves. Our results show that electromagnetic elliptic vortices comprise a number of in-line vortices. Because the vortices presented here belong to the family of propagation-invariant optical fields, they do not interact, even in nonlinear media. We have also shown that there exist situations for which the elliptic pattern is broken and new vortices are formed and arranged in a crystallike structure. Field distributions of this type can be produced in the laboratory by use of holographic techniques.[20]

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