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Transverse Electromagnetic Waves with $\vec{E} \parallel \vec{B}$

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It is shown that a general class of transverse electromagnetic waves with $\vec{E} \parallel \vec{B}$ exists. These waves possess magnetic helicity. In the case of plasma, both a high-frequency branch with $\omega^2 = \omega_p{}^2 + k^2c^2$ and a low-frequency branch with $\omega \approx 0$ are allowed. The zero-frequency branch corresponds to the force-free magnetic field $\nabla \times \vec{B} = k\vec{B}$. These waves also exist in magnetized plasmas over a wide frequency range.

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It is generally believed that in transverse electromagnetic waves electric field $\vec{E}$ and magnetic field $\vec{B}$ are always perpendicular to each other. In this Letter we show that, however, a general class of transverse electromagnetic waves with $\vec{E} \parallel \vec{B}$ exists. We show how to obtain these waves in general and give examples in vacuum and plasmas. All these waves carry magnetic helicity. In a cold collisionless plasma, the magnetostatic mode[1-3] of this class becomes the more familiar force-free field $\nabla \times \vec{B} = k\vec{B}$.

We consider transverse electromagnetic waves in a uniform medium. These transverse waves can be described by

$$\vec{B} = \nabla \times \vec{A}, \tag{1}$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \tag{2}$$

in which the vector potential $\vec{A}$ satisfies $\nabla \cdot \vec{A} = 0$ and the wave equation

$$\nabla \times \nabla \times \vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{j}. \tag{3}$$

Here

$$\vec{j} = \overleftrightarrow{\sigma} \cdot \vec{E},$$

where $\overleftrightarrow{\sigma}$ is the conductivity tensor operator of the medium under consideration. After Fourier analysis in time, we have

$$\nabla \times \nabla \times \vec{A} - (\omega^2/c^2)\overleftrightarrow{K}(\omega) \cdot \vec{A} = 0 \tag{4}$$

with the dielectric tensor

$$\overleftrightarrow{K}(\omega) = \overleftrightarrow{I} - 4\pi\overleftrightarrow{\sigma}(\omega)/i\omega.$$

For simplicity, we consider only cases where $\overleftrightarrow{K}(\omega)$ is independent of wavelength.

We first look at the vacuum case. In vacuum $\sigma = 0$ and Eq. (4) becomes

$$(\nabla^2 + k^2)\vec{A}_k = 0 \tag{5}$$

with $\omega^2 = k^2c^2$. This wave equation allows the well-known linear polarized plane waves with $\vec{E} \perp \vec{B}$.[4] For every solution of Eq. (5), it is straightforward to show that

$$\vec{F}_k = \vec{A}_k + k^{-1}\nabla \times \vec{A}_k \tag{6}$$

satisfies not only Eq. (5) but also

$$\nabla \times \vec{F}_k = k\vec{F}_k. \tag{7}$$

For those vector potentials $\vec{A}$ satisfying Eq. (7), the electric field $\vec{E}$ and magnetic field $\vec{B}$ are parallel to each other and both are perpendicular to the vector $\vec{k}$. Therefore, for every plane wave solution, a wave solution with $\vec{E} \parallel \vec{B}$ can be constructed. An example is

$$\vec{k} = k(0,0,1),$$

$$\vec{A} = A(\sin kz, \cos kz, 0)\cos\omega t,$$

$$\vec{E} = (\omega A/c)(\sin kz, \cos kz, 0)\sin\omega t,$$

and

$$\vec{B} = kA(\sin kz, \cos kz, 0)\cos\omega t.$$

This solution corresponds to two circularly polarized waves[4] propagating opposite to each other in such a way that their Poynting vectors are cancelled out. It is interesting to note that this vacuum wave possesses magnetic helicity[5] $\int \vec{A}\cdot\vec{B}\,dV$. The time-averaged magnetic helicity density is related to the energy density ϵ by

$$\langle \vec{A}\cdot\vec{B}\rangle = \frac{2\pi}{k}\left\langle\left(\frac{E^2}{4\pi} + \frac{B^2}{4\pi}\right)\right\rangle = \frac{2\pi}{k}\epsilon.$$

Therefore, a single helical photon with energy $\hbar\omega$ carries a magnetic helicity of hc.

We can also use the solutions of Eq. (7) to find the $\vec{E}\parallel\vec{B}$ waves in other media. In the case of unmagnetized plasma, the dielectric tensor $\overleftrightarrow{K}$ becomes diagonal, and from Eq. (4) we obtain the dispersion relation[1,6]

$$k^2c^2/\omega^2 = 1 - \omega_p^{\,2}/\omega(\omega + i\nu),$$

where ω_p is the plasma frequency and ν is the collision frequency. This dispersion relation gives both a high-frequency branch[1,6]

$$\omega = \pm(\omega_p^{\,2} + k^2c^2)^{1/2} - \frac{1}{2}\frac{i\nu}{1 + k^2c^2/\omega_p^{\,2}},$$

and a low-frequency branch[1,2]

$$\omega = -\frac{i\nu}{1 + \omega_p^{\,2}/k^2c^2}.$$

The high-freqency mode is very similar to the vacuum modes. The low-frequency mode, in which conducting current dominates over displacement current, has no counterpart in vacuum. It is easily verified that the magnetic helicity of these waves decays at the same rate as the wave energy.

In the low-frequency mode, a small electric field proportional to ν exists to give the necessary current $\vec{j}$ parallel to $\vec{B}$. In the limit, $\nu \to 0$, both the electric field $\vec{E}$ and resistivity vanish, and the low-frequency mode becomes the force-free field $\nabla\times\vec{B} = k\vec{B}$.[7-9] These force-free fields have been used to describe plasma discharges and turbulences in fusion researches.[8-10]

These purely transverse $\vec{E}\parallel\vec{B}$ waves can also propagate in plasmas in a uniform external field $\vec{B}_0$. With $\vec{k}$ parallel to $\vec{B}_0$, the dispersion relation for a cold plasma is given by

$$k^2c^2/\omega^2 = R \tag{8}$$

and

$$k^2c^2/\omega^2 = L, \tag{9}$$

where R and L are the dielectric constants for right-hand and left-hand polarization, respectively.[6] Equation (8) covers electron cyclotron waves,[6] whistler waves,[6] and fast waves. Equation (9) includes ion cyclotron waves.[6] In the low-frequency limit ($\omega \ll \Omega_i$, where Ω_i is the ion cyclotron frequency in $\vec{B}_0$), both R and L approach the value $\omega_{pi}^{\,2}/\Omega_i^{\,2}$ and the waves become helical shear Alfvén waves.[6] The plasma fluid velocity $\vec{v} = c(\vec{E}\times\vec{B}_0)/B_0^{\,2}$ in these waves is perpendicular to the wave magnetic field $\vec{B}$ in contrast to the case of ordinary shear Alfvén wave in which $\vec{v}\parallel\vec{B}$. This class of helical shear waves has been studied from magnetohydrodynamics equations by Murata.[11]

In conclusion, we have shown that a general class of transverse electromagnetic waves with $\vec{E}\parallel\vec{B}$ exists. The familiar force-free field $\nabla\times\vec{B} = k\vec{B}$ is a member of this family.

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Comment on "Transverse Electromagnetic Waves with E∥B"

Chu and Ohkawa[1] (CO) have proposed that a class of transverse electromagnetic (TEM) waves with **E**∥**B** exists. This paper has provoked critical reaction[2–6] and a rebuttal.[7] Most of this discussion appears to be due to the failure of CO[1] to define their terminology carefully and explain their assumptions. Consequently, the respondents assumed instinctively that all TEM waves propagate and attempted to prove that **E**∥**B** waves could not propagate and hence, could not exist. The example of CO[1] implied that their proposed class consisted of only standing waves and Chu[7] was explicit in his rebuttal. This Comment derives the general conditions under which TEM standing waves with **E**∥**B** exist and remedies these deficiencies.

It is useful to classify TEM wave solutions of Maxwell's equations according to whether their Poynting vector $\mathbf{S}(\mathbf{r},t)=0$ or $\neq 0$. The former are **E**∥**B** TEM standing waves and the latter **E**⊥**B** TEM traveling or standing waves.[8] CO[1] made implicit use of the Coulomb gauge in their derivation. The Coulomb gauge introduces the constraints $\nabla\cdot\mathbf{A}=0$ and $\Phi(\mathbf{r},t)=C$. This gauge, which is consistent with the Lorentz gauge required to obtain independent wave equations for **A** and Φ, is used in the following analysis. However, the fields calculated are independent of the gauge.

The most general TEM solution of the vector wave equation for **A** obtained from Maxwell's equations is[9] $\mathbf{A}(\mathbf{r},t)=\mathbf{A}_+(\eta)+\mathbf{A}_-(\zeta)$, where $\eta\equiv\mathbf{k}\cdot\mathbf{r}-\omega t$ and $\zeta\equiv\mathbf{k}\cdot\mathbf{r}+\omega t$. Then $\mathbf{B}(\mathbf{r},t)=\nabla\times\mathbf{A}=\mathbf{k}\times(\mathbf{A}'_++\mathbf{A}'_-)$, where $\mathbf{A}'_+\equiv[d\mathbf{A}_+(\eta)/d\eta]$ and $\mathbf{A}'_-\equiv[dA_-(\zeta)/d\zeta]$, and so **B** is transverse since $\mathbf{k}\perp\mathbf{B}$. $\nabla\cdot\mathbf{A}=\mathbf{k}\cdot(\mathbf{A}'_++\mathbf{A}'_-)=0$, so that $\mathbf{B}\neq 0$ if $\mathbf{A}'_+\neq-\mathbf{A}'_-$. If $\nabla\Phi=0$, then $\mathbf{E}(\mathbf{r},t)=-\partial\mathbf{A}/\partial t=\omega(\mathbf{A}'_+-\mathbf{A}'_-)$, and so $\mathbf{E}\parallel(\mathbf{A}'_+-\mathbf{A}'_-)$. **E** is transverse if $\mathbf{k}\cdot(\mathbf{A}'_+-\mathbf{A}'_-)=0$. It can be shown that the Poynting vector $\mathbf{S}(\mathbf{r},t)\propto\mathbf{E}\times\mathbf{B}=\mathbf{k}[(\mathbf{A}'_++\mathbf{A}'_-)\cdot(\mathbf{A}'_+-\mathbf{A}'_-)]$ for TEM waves. $\mathbf{S}(\mathbf{r},t)=0$ if $(\mathbf{A}'_++\mathbf{A}'_-)\cdot(\mathbf{A}'_+-\mathbf{A}'_-)=0$, which means that $(\mathbf{A}'_++\mathbf{A}'_-)\perp(\mathbf{A}'_+-\mathbf{A}'_-)$. It follows that $|\mathbf{A}'_+|=|\mathbf{A}'_-|$. If $\mathbf{A}'_+\nparallel\mathbf{A}'_-$, then $\mathbf{E}\neq 0$, $\mathbf{B}\neq 0$, and $\mathbf{E}\cdot\mathbf{B}\neq 0$. Moreover, $\mathbf{k}\cdot(\mathbf{A}'_++\mathbf{A}'_-)=0$ and $\mathbf{k}\cdot(\mathbf{A}'_+-\mathbf{A}'_-)=0$ for these TEM waves. Consequently, TEM standing waves exist with **E**∥**B** and $\mathbf{S}(\mathbf{r},t)=0$. The example of CO[1] satisfies these conditions.

A similar analysis for TEM traveling or standing waves with **E**⊥**B** and $\mathbf{S}(\mathbf{r},t)\neq 0$ yields the following results: $\mathbf{E}\cdot\mathbf{B}=\omega\mathbf{k}\cdot[(\mathbf{A}'_++\mathbf{A}'_-)\times(\mathbf{A}'_+-\mathbf{A}'_-)]=0$ if either $(\mathbf{A}'_++\mathbf{A}'_-)\parallel(\mathbf{A}'_+-\mathbf{A}'_-)$ or $(\mathbf{A}'_++\mathbf{A}'_-)\times(\mathbf{A}'_+-\mathbf{A}'_-)\perp\mathbf{k}$, and $\mathbf{S}(\mathbf{r},t)\neq 0$ if $(\mathbf{A}'_++\mathbf{A}'_-)\not\perp(\mathbf{A}'_+-\mathbf{A}'_-)$.

The approach taken by CO[1] of defining another vector potential $\mathbf{F}_k(\mathbf{r})=\mathbf{A}_k(\mathbf{r})+k^{-1}\nabla\times\mathbf{A}_k(\mathbf{r})$, which leads to $\nabla\times\mathbf{F}_k(\mathbf{r})=k\mathbf{F}_k(\mathbf{r})$ if $\nabla\cdot\mathbf{F}_k(\mathbf{r})=\nabla\cdot\mathbf{A}_k(\mathbf{r})=0$, is insufficient to define those TEM standing waves with **E**∥**B** and $\mathbf{S}(\mathbf{r},t)=0$, unless $\mathbf{A}_k(\mathbf{r})$ is constrained to be real since all standing waves must satisfy $\mathbf{A}(\mathbf{r},t)=\mathrm{Re}[\exp(-i\omega t)]\times\mathrm{Re}[\mathbf{A}_k(\mathbf{r})]$. Otherwise, it can be shown that the vector potential

$$\mathbf{A}(\mathbf{r},t)=\mathbf{A}_0[a\cos(\mathbf{k}\cdot\mathbf{r}+\omega t)+b\sin(\mathbf{k}\cdot\mathbf{r}+\omega t)]$$
$$=\mathrm{Re}[\exp(-i\omega t)\mathbf{A}_k(\mathbf{r})],$$

where $\mathbf{A}_k(\mathbf{r})=\mathbf{A}_0\exp[-i(\mathbf{k}\cdot\mathbf{r}-\delta)]$ and $\delta=\tan^{-1}(b/a)$, can be used to obtain a derived vector potential $\mathbf{F}_k(\mathbf{r})$ for which **E**⊥**B** and $\mathbf{S}(\mathbf{r},t)\neq 0$.

It has been shown that a class of TEM waves with **E**∥**B** exists that can be derived from a vector potential $\mathbf{A}(\mathbf{r},t)=\mathbf{A}_+(\eta)+\mathbf{A}_-(\zeta)$, satisfying $\mathbf{k}\cdot[d\mathbf{A}_+(\eta)/d\eta]=0$ and $\mathbf{k}\cdot[d\mathbf{A}_-(\zeta)/d\zeta]=0$, and a scalar potential $\Phi=C$, if $|d\mathbf{A}_+(\eta)/d\eta|=|d\mathbf{A}_-(\zeta)/d\zeta|$ and $d\mathbf{A}_+(\eta)/d\eta\parallel d\mathbf{A}_-(\zeta)/d\zeta$, where $\eta\equiv\mathbf{k}\cdot\mathbf{r}-\omega t$ and $\zeta\equiv\mathbf{k}\cdot\mathbf{r}+\omega t$. These are the most general conditions for TEM waves with **E**∥**B** to exist. Those **E**∥**B** solutions obtained by the condition given by CO[1] can be obtained by use of the above formalism. These waves do not propagate since $\mathbf{S}(\mathbf{r},t)=0$, and should be described as TEM standing waves with **E**∥**B** to distinguish them from those classical TEM traveling and standing waves with **E**⊥**B** and $\mathbf{S}(\mathbf{r},t)\neq 0$.

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Chu and Ohkawa Respond: In their Comment on our paper,[1] Zaghloul, Volk, and Buchmaster[2] demonstrate once again the existence of $\mathbf{E}\parallel\mathbf{B}$ waves and give a general condition under which transverse electromagnetic waves with $\mathbf{E}\parallel\mathbf{B}$ exist. They point out correctly that $\mathbf{E}\parallel\mathbf{B}$ exists only for standing transverse electromagnetic waves and our recipe is not sufficient.

In addition to the $\mathbf{E}\parallel\mathbf{B}$ waves discussed in Ref. 1, there is a different family of $\mathbf{E}\parallel\mathbf{B}$ standing waves. They are of the form

$$\mathbf{A}(\mathbf{r},t)=f(\mathbf{r})\mathbf{G}(t).$$

It can be shown that if $\mathbf{G}(t)$ is a rotating vector about the direction of variation $\nabla f(\mathbf{r})$,

$$d\mathbf{G}/dt \propto \nabla f(\mathbf{r})\times\mathbf{G},$$

$\mathbf{E}$ and $\mathbf{B}$ are parallel to each other. An example[3] is a standing wave in vacuum with $\omega=kc$ and

$$\mathbf{A}(\mathbf{r},t)=\cos kZ(\hat{\mathbf{x}}\sin\omega t+\hat{\mathbf{y}}\cos\omega t),$$

$$\mathbf{E}(\mathbf{r},t)=-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}=-k\cos kZ(\hat{\mathbf{x}}\cos\omega t-\hat{\mathbf{y}}\sin\omega t),$$

$$\mathbf{B}(\mathbf{r},t)=\nabla\times\mathbf{A}=k\sin kZ(\hat{\mathbf{x}}\cos\omega t-\hat{\mathbf{y}}\sin\omega t).$$

Both of the two classes of transverse electromagnetic $\mathbf{E}\parallel\mathbf{B}$ waves are standing waves, and the Poynting vector $\mathbf{S}=0$ everywhere and can be obtained from the method presented in the preceding Comment.[2]

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