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A Theoretical Energy Paradox in the Lorentz-Dirac-Wheeler-Feynman-Rohrlich Electrodynamics.

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Summary. — In this article we deduce the Lorentz-Dirac-Wheeler-Feynman-Rohrlich equation of motion of a charged particle by considering a special case of the Sommerfeld, Markoff, Bohm-Weinstein and Shih-Prastein-Erber equations of motion. We can prove or classical examples that 1) there is conservation of energy in the Lorentz-Dirac-Wheeler-Feynman-Rohrlich electrodynamics for the « good solution » of these theories and time-dependent (one-dimensional) forces; 2) there appears a theoretical « paradox » for the constant homogeneous magnetic field. Here the total radiation is bigger than the initial kinetic energy! Here there is no conservation of the total observable energies.

1. – Introduction.

The Sommerfeld electrodynamics are very well known ([1-3]). Mostly one considers a general equation of motion

$$m_0 \boldsymbol{a} = \boldsymbol{F}^{\text{ex}} + (\boldsymbol{F}^1 - m_{\text{el}} \boldsymbol{a}) \,, \tag{1.1}$$

where m_0 is the mechanical mass, m_{el} the electromagnetic mass, $\boldsymbol{F}^{\text{ex}}$ the external force and $(\boldsymbol{F}^1 - m_{\text{el}}\boldsymbol{a})$ the total contribution of the « self-interaction force ».

(1) M. WEINSTEIN: Thesis, Princeton University (1953).
(2) S. M. PRASTEIN: Thesis, Illinois Institute of Technology (1965).
(3) H.-C. SHIH: Thesis, Illinois Institute of Technology (1962).

One has in the limit $v/c \ll 1$

$$m_{el} = \frac{4e^2}{3c^2\pi}\int_0^\infty |g(k_0, k)|^2 \, dk \,, \tag{1.2}$$

$$\boldsymbol{F}^1 = \frac{4e^2}{3c^2\pi}\int_0^\infty \int_0^\infty \cos ckt' |g(k_0, k)|^2 \dot{\boldsymbol{a}}(t-t') \, dk \, dt' \,. \tag{1.3}$$

The experimental mass is $m = m_0 + m_{el}$.

The « form-function » (or « cut-off ») is related to the charge density by

$$g(k_0, k) = \frac{1}{e}\int d^3\boldsymbol{r} \exp[+i\boldsymbol{kr}]\varrho(\boldsymbol{r}, k_0^{-1}) \,, \tag{1.4}$$

where k_0^{-1} is a unit length of the order of the classical electron radius. We analyse here the limit of a extended particle:

$$\varrho(\boldsymbol{r}, k_0^{-1}) = e\delta(x)\delta(y)\delta(z) = e\delta(\boldsymbol{r}) \,. \tag{1.5}$$

In this case

$$g(k_0, k) = 1 \,, \tag{1.6}$$

$$m_{el} = \frac{4e^2}{3c^2\pi}\int_0^\infty dk \,, \tag{1.7}$$

$$m_0 = m - \frac{4e^2}{3c^2\pi}\int_0^\infty dk \,, \tag{1.8}$$

$$\boldsymbol{F}^1 = \frac{2e^2}{3c^3}\dot{\boldsymbol{a}}(t) \,. \tag{1.9}$$

One thus obtains

$$(m_0 + m_{el})\dot{\boldsymbol{a}} = m\dot{\boldsymbol{a}} = \boldsymbol{F}^{ex} + \frac{2e^2}{3c^3}\dot{\boldsymbol{a}}(t) \,. \tag{1.10}$$

This is what is usually called the Lorentz-Dirac equation of motion.

2. – The Dirac-Plass solution of the equation of motion [4-8].

One has for linear motions ($F^{ex} = 0$)

$$ma = eE(t) + \frac{2e^2}{3c^3}\dot{a}(t) \,. \tag{2.1}$$

Using Laplace transforms one can prove easily that the most general solution of the equation is

$$a(t) = \exp[bt]\left[a(0) - \frac{b}{m}\int_0^t \exp[-bt']eE(t')\, dt'\right], \tag{2.2}$$

where

$$\frac{1}{b} = \frac{2e^2}{3mc^3} \,. \tag{2.3}$$

In physical problems the value of $a(0)$ is determined from the real appropriate initial conditions.

One can take, for instance, a physical situation where

$$a(0) = 0 \tag{2.4}$$

and then the solution

$$a(t) = -\frac{b}{m}\exp[bt]\int_0^t \exp[-bt']eE(t')\, dt' \tag{2.5}$$

is unstable for most of the external forces.

Therefore Dirac imposes the boundary condition

$$a(\infty) = 0 \,. \tag{2.6}$$

It is found to be that the value of the initial acceleration which satisfies this

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(4) H. A. Lorentz: *Theory of Electrons*, 2nd ed. (New York, 1952).
(5) P. A. M. Dirac: *Proc. Roy. Soc.*, A **167**, 148 (1938).
(6) J. A. Wheeler and R. P. Feynman: *Rev. Mod. Phys.*, **17**, 157 (1945).
(7) G. Plass: *Revs. Mod. Phys.*, **33**, 37 (1961).
(8) F. Rohrlich: *Classical Charged Particles* (Cambridge, 1965).

boundary condition is

$$a(0) = \frac{b}{m}\int_0^\infty \exp[-bt']eE(t')\,dt' . \tag{2.7}$$

The « regularized » solution is now for the choice of $a(0)$

$$a(t) = \frac{b}{m}\exp[bt]\int_t^\infty \exp[-bt']eE(t')\,dt' . \tag{2.8}$$

This is the « good solution » of the Dirac theory.

3. – Example number 1.

One chooses

$$eE(t) = eE\exp[-|at|]u(t) . \tag{3.1}$$

Thus the « good solution » is

$$a(t)\begin{cases} \dfrac{eEb}{m(b+|a|)}\exp[bt]u(-t) & \text{if } t<0 , \\ \dfrac{eEb}{m(b+|a|)}\exp[-|a|t]u(t) & \text{if } t>0 . \end{cases} \tag{3.2}$$

One imposes again $v(-\infty)=0$. According to ROHRLICH this choice does not affect the problem in any way (see ref. (8) p. 178). Thus

$$v(t)\begin{cases} \dfrac{eE}{m(b+|a|)}\exp[bt]u(-t) & \text{if } t<0 , \\ \dfrac{eEb}{m(b+|a|)|a|}\exp[-|a|t]u(t)+C_1 & \text{if } t>0 . \end{cases} \tag{3.3}$$

One has

$$a(0^-) = a(0) = a(0^+) . \tag{3.4}$$

Therefore one must choose C_1 in such a way that

$$v(0^-) = v(0) = v(0^+) . \tag{3.5}$$

This implies

$$C_1 = \frac{eE}{m(b+|a|)}\left(1+\frac{b}{|a|}\right) . \tag{3.6}$$

One can see easily that

$$a(-\infty) = a(+\infty) = 0 . \tag{3.7}$$

Thus one has

$$\int_{-\infty}^{+\infty} dE_{\text{Schott}} = 0 . \tag{3.8}$$

So we have to evaluate only T, E_{kin} and W_{rad}.

One obtains easily that

$$T = \int_{-\infty}^{+\infty} eE\,dr = \frac{e^2E^2}{2m(b+|a|)|a|}\left(2+\frac{b}{|a|}\right) \geqslant 0 \tag{3.9}$$

and that

$$W_{\text{rad}} = \frac{m}{b}\int_{-\infty}^{+\infty} a^2\,dt = \frac{T}{(2+b/|a|)} \geqslant 0 . \tag{3.10}$$

Finally

$$E_{\text{kin}} = \int_{-\infty}^{+\infty} dE_{\text{kin}} = \frac{T}{(2+b/|a|)}\left(1+\frac{b}{|a|}\right) \geqslant 0 . \tag{3.11}$$

One has in this case conservation of the total observable energies

$$T = W_{\text{rad}} + E_{\text{kin}} . \tag{3.12}$$

There is no variation in near-field energy nor internal energy.

4. – Example number 2.

The constant homogeneous magnetic field (treated according the Plass solution of the Dirac equation).

The motion is determined by the system

$$(4.1)\qquad \begin{cases} a_x - \omega v_x - \dfrac{1}{b}\,\dot a_x = 0\,, \\ a_y + \omega v_y - \dfrac{1}{b}\,\dot a_y = 0\,, \end{cases}$$

where

$$(4.2)\qquad \omega = \frac{eH}{mc}\,.$$

The particle is injected with constant velocity into the magnetic field at $t = 0$.

According PLASS the « good solution » of this equation is then

$$(4.3)\qquad \begin{cases} v_x = +\, v_0 \cos\beta t \exp[-|\alpha| t]\,, \\ v_y = -\, v_0 \sin\beta t \exp[-|\alpha| t]\,, \end{cases}$$

where

$$(4.4)\qquad \begin{cases} |\alpha| = \alpha = \tfrac{1}{2} b\{[\tfrac{1}{2} + \tfrac{1}{2}(1 + 16 b^{-2}\omega^2)^{\frac{1}{2}}]^{\frac{1}{2}} - 1\}\,, \\ \beta = \tfrac{1}{2} b[-\tfrac{1}{2} + \tfrac{1}{2}(1 + 16 b^{-2}\omega^2)^{\frac{1}{2}}]^{\frac{1}{2}}\,. \end{cases}$$

There is no total external work ($T = 0$). There is as well no variation of near-field energy nor variation of internal energy.

Here the observable energies are thus the kinetic energy and the radiation energy W_{rad}. One can never « see » the Schott energy of the velocity (r^{-2}) fields. The initial kinetic energy is

$$(4.5)\qquad \frac{mv_0^2}{2} = \frac{mv(0^-)^2}{2} = \frac{mv(0)^2}{2} = \frac{mv(0^+)^2}{2}\,.$$

This kinetic energy should be transformed at last ($\lim t \to +\infty$) into the total radiation energy W_{rad} (because everybody accepts that the Schott energy of a particle without acceleration ($t = 0^-$) and also of a particle at rest ($t = +\infty$) is zero). We should have at $t = \infty$ that $|\Delta E_{\text{kin}}| = |\Delta W_{\text{rad}}|$. We can see that the total gain in radiation energy is

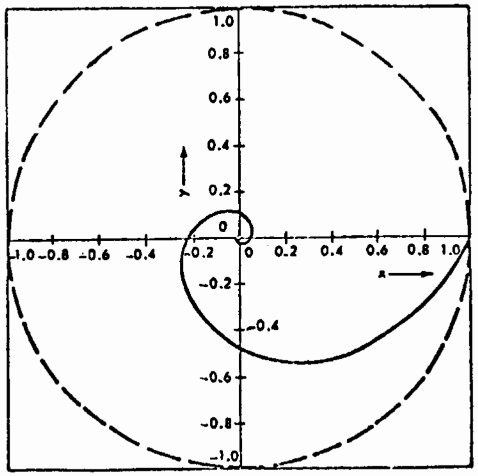

Fig. 1. – Motion of a radiating charged body in a uniform magnetic field for the particular case when $\omega/b = 0.968$. ——— With force of radiative reaction; – – – without radiative reaction.

$$(4.6)\qquad W_{\text{rad}} = \frac{2e^2}{3c^3}\int_{0^-}^{\infty} a^2\,\mathrm{d}t = \left[\frac{\alpha^2 + \beta^2}{\alpha b}\right]\frac{mv_0^2}{2}\,.$$

One takes now the explicit example of PLASS ($\omega/b = 0.968$). Then W_{rad} (see (4.4)) is given by

$$(4.7)\qquad W_{\text{rad}} = 1.57\,\frac{mv_0^2}{2}\,.$$

So the particle radiates 57 percent energy more than its initial kinetic energy! This is a « paradox ». This « creation of energy » can only be compensated by a total negative Schott energy for the particle at rest ($\lim t \to +\infty$), but this cannot be accepted because then two different point particles at rest could have a different total Schott energy.

The Schott energy for a particle at rest must be zero. So there is « something wrong » with this « good solution » of the Dirac theory ([9]).

According to this example there is no conservation of the observable energies. The deviation could lead to measurable effects in the ultra-relativistic region.

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([9]) W. J. M. CLOETENS: Thesis, Brussels University (1967).

RIASSUNTO (*)

In questo articolo si ricava l'equazione del moto di una particella carica data da Lorentz-Dirac-Wheeler-Feynman-Rohrlich, considerando un caso speciale dell'equazione di moto di Sommerfeld, Markoff, Bohm-Weinstein e Shih-Prastein-Erber. Si dimostra con esempi classici che 1) vi è conservazione dell'energia nell'elettrodinamica di Lorentz-Dirac-Wheeler-Feynman-Rohrlich per la « soluzione buona » di queste teorie e forze dipendenti dal tempo (unidimensionali), 2) compare un « paradosso » teorico per il campo magnetico costante omogeneo. Qui la radiazione totale è maggiore dell'energia cinetica iniziale! Qui non c'è conservazione delle energie totali osservabili.

(*) *Traduzione a cura della Redazione.*