Energy balance in coherent electromagnetic radiation

R Coïsson

Dipartimento di Fisica, Università di Parma, 43100 Parma, Italy

Received 15 June 1993, in final form 6 September 1993

Abstract. Bunched charges, as in the 'free electron laser', radiate more energy than unbunched ones. For a better understanding of how the forces between particles determine the conservation of energy, we take the simple model of two charges within a wavelength of a sinusodoidal wave, and show that the relative phase of the particle's motion with respect to the wave is modified by the force between the two particles, and this explains the extra work done by the wave. The phase shift is proportional to the emitted field and depends on the retardation (particle distance divided by speed of light), and turns out to be independent of distance.

Résumé. Quand un faisceau de particules chargées est modulé, comme dans le cas du 'laser à electrons libres', il émet plus de radiation qu'un faisceau non modulé. Pour une meilleure compréhension de comment les forces entre les charges determinent la conservation de l'énergie, nous prenons un simple modèle: deux charges qui se trouvent sous l'action d'une onde harmonique et à une distance entre elles inférieure à sa longuer d'onde, et nous montrons que la phase entre les charges et l'onde est modifiée par la force qui s'exerce entre les particules, et cela explique le travail en plus fournit par l'oude. Ce déphasage dépend de la force et du temps que la lumière emploie pour couvrir leur distance, et résulte indépendent de cette distance.

1. Introduction

When a charged particle is subject to an electromagnetic (EM) field and is then accelerated, it emits radiation. In particular, we call this 'synchrotron radiation' for ultrarelativistic particles in a magnetic field, and Compton (or Thompson) scattering for a particle in an electromagnetic wave.

The radiation from a single particle can be calculated from Liénard's formula (which is the Green's function of Maxwell's equations) [1].

For many electrons at random positions, the resulting radiation is an incoherent superposition of fields, and the total intensity is proportional to N, the number of particles. But if the beam is modulated (the electrons are bunched) on a spatial scale comparable to the emitted wavelengths 'coherent' radiation is emitted with power proportional to N^2 in addition to the 'incoherent' one, proportional to N [2]. In particular we have this in free electron lasers (FEL) [3], and in the harmonic generation by the FEL or optical klystron [4].

As the modulated beam emits more radiation than an unmodulated one, it must lose more energy. But how is this produced? The descriptions of the interchange of energy between the electron beam and radiation in the FEL do not tell us what is the force doing the extra work, as they assume the conservation of energy. In particular, it is not evident how, while the increase in emitted power is independent of the distance between the two particles (as long as it is $\ll \lambda$), the force between them decreases with the distance: how can this force produce a work independent of distance?

This discussion is also relevant to the speed-up of emission of radiation when many oscillators are excited simultaneously by a short pulse and are within a coherence volume of the emitted radiation, as in the excitation of Møssbauer radiation by synchrotron radiation [5,6].

In order to analyse in detail the mechanism by which the various forces cause this extra work, we take a simple example: Thompson scattering of monochromatic radiation on two electrons spatially separated by a distance that is small with respect to the wavelength.

For a better physical understanding of the procedure, we begin with a rough, naïve description (simple enough to show the effect of retardation and phase shifts, but not quantitative), followed by a more formal and general description, using the scheme taken from the analysis of the classical model of the electron.

2. Radiation from one electron

Let us first recall the energy balance and radiation pressure (momentum balance) in Thompson scattering from one electron.

The electron, of charge e and mass m, is in a plane monochromatic wave propagating in the z direction, with electric field along y:

$$\boldsymbol{E} = E_0 \hat{y} \sin(\omega t) \tag{1}$$

$$\boldsymbol{B} = E_0 \hat{x} \sin(\omega t). \tag{2}$$

The amplitude of the motion is $y_0 = eE_0\lambda^2/4\pi^2 mc^2$. We assume $y_0 \ll \lambda$ (which also implies non-relativistic motion).

We also assume $\hbar\omega \ll mc^2$.

This is the classical scheme for the description of Thompson (low-energy Compton) scattering and, using a Lorentz transformation and the Weiszäcker–Williams approximation, it can be used for an electron in an undulator (as in a FEL).

The Lorentz–Abraham equation of motion is

$$m\ddot{\boldsymbol{r}} = e\boldsymbol{E} + e\boldsymbol{v} \wedge \boldsymbol{B} + m\tau_0 \dot{\ddot{\boldsymbol{r}}} \tag{3}$$

where

$$\tau_0 = \frac{2r_0}{3c} = \frac{e^2}{6\pi\epsilon_0 mc^3} \tag{4}$$

(r_0 is the 'classical electron radius') and the radiation reaction term τ_0 is a small perturbation; then we can write for the y motion to first order:

$$m\ddot{y} = eE + m\tau_0 \dot{\ddot{y}} \tag{5}$$

where $\dot{\ddot{y}}$ is taken from the solution of equation (5) without radiation reaction ($\tau_0 = 0$) : $\ddot{y}(t) = eE(t)/m$. The steady state solution of equation (5) gives:

$$\dot{y} = \frac{eE_0}{m\omega}(-\cos\omega t + \omega\tau_0 \sin\omega t). \tag{6}$$

The average radiated power is:

$$\bar{P} = m\tau_0 \bar{\ddot{y}}^2 = \frac{m\tau_0 e^2 E_0^2}{m^2}\overline{\sin^2(\omega t)} = \tfrac{1}{3}\kappa E_0^2 \tag{7}$$

with $\kappa = 4\pi\epsilon_0 c r_0^2$, which, divided by the wave intensity $\frac{1}{2}\epsilon_0 c E_0^2$, gives the Thompson cross-section: $\sigma = 8\pi r_0^2/3$.

$\bar{P}$ is emitted symmetrically forward and backwards, and the work is done by the wave, then the corresponding momentum per unit time $\bar{P}/c$ is a force pushing the particle in the forward $\hat{z}$ direction.

P must correspond to a work per unit time W done by the field on the electron, and P/c to an average longitudinal force (radiation pressure):

$$W = eE\dot{y} = \frac{e^2 E_0^2}{m\omega}\sin\omega t(-\cos\omega t + \omega\tau_0 \sin\omega t) \tag{8}$$

$$\bar{W} = e^2 E_0^2 r_0/3mc = \tfrac{1}{3}\kappa E_0^2 \tag{9}$$

$$F = eB\dot{y} = \frac{e^2 E_0^2}{mc\omega}\sin\omega t(-\cos\omega t + \omega\tau_0 \sin\omega t) \tag{10}$$

$$\bar{F} = e^2 E_0^2 r_0/3mc^2 = \bar{W}/c. \tag{11}$$

The radiation reaction force is very small, but produces a phase shift between $F = eE$ and $\ddot{y}$, so that F and $\dot{y}$ are no longer in quadrature.

3. Radiation from two electrons (naïve description)

Let us now take two electrons, at positions $x = -r/2$ and $x = r/2$, with oscillation amplitude $y_0 \ll r \ll \lambda$. Now the emitted power is four times larger than before, as we can understand by the fact that the emitted power is proportional to e^2.

Neglecting the Coulomb term, each particle emits a field which, at the position of the other, is

$$E_r(t) = -\frac{r_0}{r}E\left(t - \frac{r}{c}\right). \tag{12}$$

The equation of motion of each electron is as before, plus the force due to the radiation field of the other particle:

$$m\ddot{y} = eE - e\frac{r_0}{r}E\left(t - \frac{r}{c}\right) + m\tau_0 \dot{\ddot{y}}. \tag{13}$$

Approximating:

$$E\left(t - \frac{r}{c}\right) \simeq E_0 \sin\left(\omega t - \frac{\omega r}{c}\right) \simeq E_0\left(\sin\omega t - \frac{\omega r}{c}\cos\omega t\right) \tag{14}$$

we get, as in equation (6),

$$\dot{y} = -\frac{eE_0}{m\omega}\left(1 - \frac{r_0}{r}\right)\cos\omega t + \frac{er_0 E_0}{mc}\left(1 + \frac{2}{3}\right)\sin\omega t. \tag{15}$$

The work per unit time and the radiation pressure on each particle are now:

$$\bar{W} = e\overline{E\dot{y}} = \frac{5e^2 E_0^2 r_0}{6mc} = \frac{5}{6}\kappa E_0^2 \tag{16}$$

$$\bar{F} = e\overline{B\dot{y}} = \bar{W}/c. \tag{17}$$

If the segment connecting two particles is not perpendicular to the acceleration, but forms an angle θ with it, equation (12) and then the result equation (16) are multiplied by $\cos\theta$.

We see that, qualitatively, things operate in the right sense: the phase shift produced by the small interparticle force, which is proportional to $1/r$, is proportional to the retardation, which is proportional to r, so the work done is independent of r and is added to that of the radiation reaction.

But the work has increased by a factor 2.5, not 2. And in the case where $\boldsymbol{r}$ is parallel to $\dot{\boldsymbol{v}}$ the field produced by one particle on the other is zero, so we do not understand how the extra work is done.

4. Radiation from two electrons: a general description

The reason is that we have neglected the Coulomb field; although it decreases as $1/r^2$, if the approximation is pushed to second order, a term proportional to the square of the retardation, multiplied by it, still gives a term independent of r.

In order to show this, let us write the electric field emitted at a distance $\boldsymbol{r}$ from a charge e to first order in v/c:

$$4\pi\epsilon_0 \boldsymbol{E} = \frac{e}{r'^2}\frac{\hat{r}' - \boldsymbol{v}'/c}{1 - 3\hat{r}'\cdot\boldsymbol{v}'/c} + \frac{e}{cr'}\frac{(\hat{r}' \wedge ((\hat{r}' - \boldsymbol{v}'/c)\wedge\dot{\boldsymbol{v}}'))}{1 - 3\hat{r}'\cdot\boldsymbol{v}'/c} \tag{18}$$

(where primed quantities refer to the time $t' = t - r'/c$) and calculate the Coulomb field alone, on a particle momentarily at rest, to first order in $\ddot{v}$ and in $\dot{v}$. In this case we can write the primed quantities as series expansions of $t - t' = r'/c \simeq r/c$:

$$\begin{aligned}
\boldsymbol{v} &= -\frac{r}{c}\dot{\boldsymbol{v}} + \frac{r^2}{2c^2}\ddot{\boldsymbol{v}}\\
\boldsymbol{r} &= \boldsymbol{r} + \frac{r^2}{2c^2}\dot{\boldsymbol{v}} - \frac{r^3}{6c^3}\ddot{\boldsymbol{v}}\\
r'^2 &= r^2 + \frac{r^3}{c^2}(\hat{r}\cdot\dot{\boldsymbol{v}}) - \frac{r^4}{3c^3}(\hat{r}\cdot\ddot{\boldsymbol{v}})\\
\hat{r}' &= \hat{r} + \frac{r}{2c^2}(\dot{\boldsymbol{v}} - (\hat{r}\cdot\dot{\boldsymbol{v}})\hat{r}) + \frac{r^2}{6c^3}((\hat{r}\cdot\ddot{\boldsymbol{v}})\hat{r} - \ddot{\boldsymbol{v}})
\end{aligned} \tag{19}$$

and substituting in the first term of equation (18) and taking for the moment $\dot{\boldsymbol{v}} = 0$, we find a term

$$4\pi\epsilon_0\boldsymbol{E}_1 = \frac{e\hat{r}}{r^2} + \frac{e}{c^3}\left((\hat{r}\cdot\ddot{\boldsymbol{v}})\hat{r} - \frac{1}{3}\ddot{\boldsymbol{v}}\right) \tag{20}$$

(the first term is the Coulomb field calculated at time t, i.e. neglecting the retardation: this, when summed over the two electrons, is zero). In the case where $\hat{r}$ is parallel to $\ddot{v}$, we get a 'force' equal to $2\ddot{v}/3c^3$ as expected (the second term in equation (18) vanishes as the double vector product is zero).

If we do the same algebra on the second term of equation (18), we find (still for $\dot{\boldsymbol{v}} = 0$):

$$4\pi\epsilon_0\boldsymbol{E}_2 = \frac{e}{c^3}(\ddot{\boldsymbol{v}} - (\hat{r}\cdot\ddot{\boldsymbol{v}})\hat{r}). \tag{21}$$

Equation (21), inserted in equation (13) instead of the retardation term (second term in the RHS of equation (14)) gives the same result as equation (16) (with the factor $\cos\theta$); in particular a value $\frac{3}{2}$ too large for $\theta = \pi/2$ and equal to zero for $\theta = 0$.

This means that the correction is indeed due to the retardation of the Coulomb field (equation (20)).

Adding equation (20) and (21) the two $(\hat{r}\cdot\ddot{\boldsymbol{v}})\hat{r}$ terms cancel each other and we get an 'equivalent' force (between the two electrons) equal to $m\tau_0\ddot{\boldsymbol{v}}$, independent of θ, equal to the so-called 'self-force' (last term in equation (3)) and therefore able to double the work and pressure on each particle.

It is important to remember that by 'equivalent force' we mean that instead of a retarded force we can consider an instantaneous one plus an 'equivalent' force which corrects for the retardation.

5. An additional remark

What we have described here is part of the procedure used to calculate the reaction force in the classical electron model [7–10]. We can now make an additional remark: if we keep also the terms of equation (19) linear in the acceleration $\dot{\boldsymbol{v}}$ when substituting in equation (18), we find, summing over the two electrons, the reaction term proportional to acceleration:

$$4\pi\epsilon_0\boldsymbol{E} = -\frac{e^2}{c^2 r}(\dot{\boldsymbol{v}} + (\hat{r}\cdot\dot{\boldsymbol{v}})\hat{r}) \tag{22}$$

which is interpreted as an additional inertial mass due to the interaction between the two particles. This is just equal to the interaction energy divided by c^2, as shown in [11] when $\hat{r}$ is perpendicular to $\dot{\boldsymbol{v}}$. When they are parallel, the result is twice as large (a similar remark is made in [12]). Physically, this can be interpreted in the following way: if particle 1 is at $z = 0$ and 2 at $z = r$, with acceleration in the positive z direction, and both are positive, 1 does positive work on 2, and 2 negative on 1. Therefore an energy Fv is transferred forward per unit time on a distance r. This means a momentum $Fvr/c = e^2v/cr$, and as they are accelerated, a momentum change per unit time (i.e. a force) e^2a/cr. This corresponds to a (slightly) non-equal acceleration of the charges because they repel each other and they are not linked together. If they were, the tension of the link would create a backward-travelling 'hidden' momentum which would compensate that part (see the 'Poincaré stresses' and the Trouton–Noble experiment [13–16]).

6. Conclusions

We have analysed the mechanism by which the force between two electrons causes a work corresponding to the increase in radiated power in coherent radiation. We have seen that the role of the interparticle force is to produce a phase shift between the external force and the acceleration, which increases the work done by the force accelerating the charges; and the increase of the delay due to the interparticle distance compensates the decrease in field with the distance, so that the work is independent of the distance.

This result can be used for a clearer understanding of the increased loss of energy and momentum from a beam of electrons, interacting, for example, with a magnetic field, when it emits 'coherent' radiation because it is modulated, as in a FEL and 'optical klystron' oscillator and harmonic generation schemes, and the 'speed up' of the decay of coherently excited Mössbauer media.

Acknowledgments

The author thanks the graduate students in physics of the University of Parma for many useful discussions, in particular D Cassi and G Oriani. This work has also been stimulated by discussions on the FEL with B Kincaid and on Mössbauer–Bragg scattering with S Ruby.

References

[1] Jackson J D 1962 *Classical Electrodynamics* (New York: Wiley)
[2] Coïsson R 1986 *Proc. SPIE* **582** 20–23
[3] Luchini B and Motz H 1990 *Undulators and Free Electron Lasers* (Oxford: Clarendon)
[4] Coïsson R 1983 *Nucl. Instrum Meth.* **208** 185–8
[5] Ruby S 1974 *J. Physique* **C6** 209
[6] van Bürck U 1986 *Hyperf. Interactions* **27** 219–30
[7] Heitler W 1954 *The Quantum Theory of Radiation*, (Oxford: Clarendon)
[8] Panofsky W K H and Philips M 1955 *Classical Electricity and Magnetism* (Reading, MA: Addison-Wesley)
[9] Griffiths D 1981 *Introduction to Electrodynamics* (Englewood Cliffs, NJ: Prentice-Hall) (the results of [7,8,9] are the same when integrated over the sphere, but the intermediate results, which are relevant for us with two charges, apparently are not).
[10] Oppenheimer J R 1970 *Lectures on Electrodynamics* (New York: Gordon and Breach)
[11] Boyer T H 1978 *Am. J. Phys.* **46** 383
[12] Griffiths D J and Szeto E W 1978 *Am. J. Phys* **46** 244–8
Griffiths D J and Owen R E 1983 *Am. J. Phys.* **51** 1120–6
[13] We remark that it has been shown that the whole question of Poincaré stresses, 'hidden momentum', etc, can be made to disappear 'by definition' (and stability and covariance apparently decoupled) by a different, covariant, definition of energy–momentum density of the electromagnetic field, as done by Mandel and Kwal [17] and Rohrlich [18]. We agree with Jackson ([1] section 17.6) that it is a matter of taste if one wants to define a covariant energy–momentum density, but we think that with the 'traditional' approach we get some interesting physical insights. (In any case, as the integrated energy–momentum of a closed system is a 4-vector [19], Poincaré's stresses are not an 'ad hoc' assumption, as *any* force which keeps the electron together also compensates the 4/3 factor in the electromagnetic mass, momentum and kinetic energy in the classical electron model.) For discussions of the various viewpoints, see
Boyer T H 1982 *Phys. Rev.* D **25** 3246
Rohrlich F 1982 *Phys. Rev.* D **25** 3251
Tangherlini F R 1963 *Am. J. Phys.* **31** 285 and reference [18]
[14] Poincaré H 1906 *Rend. Circ. Mat. Palermo* **21** 129
[15] Becker R 1964 *Electromagnetic Fields and Interactions* (New York: Dover) pp 273 and 397–8
[16] Feynman R, Leighton R and Sands M 1964 Lectures on Physics vol II (Reading, MA: Addison-Wesley) ch 27
[17] Mandel H 1926 *Z. Phys* **39** 40
Kwal B 1949 *J. Phys. R. Radium* **10** 103
[18] Rohrlich R 1960 *Am. J. Phys.* **28** 639; 1965 *Classical Charged Paticles* (Reading, MA: Addison-Wesley)
[19] Møller C 1952 *The Theory of Relativity* (London: Oxford University Press) VI-63 (p 166)