Origin of "Hidden Momentum Forces" on Magnets*

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Shockley and James have noted that an electrically neutral magnet whose moment is changing with time exerts a force on an electric charge of negligible velocity at a large distance from it, but that there is no obvious corresponding back-action of the charge on the magnet, although required by general considerations of the conservation of momentum. In the present paper, it is shown that the back-force is a consequence of the relativity corrections to the motion of the particles composing the magnet. The proof is given generally in terms of the relativistic theorem on the motion of the "center of energy" and explicitly in terms of a Lagrangian for a system of particles obtained many years ago by Darwin. The effect of spin is examined and found not to affect the action-reaction balance. In the Appendix, the properties of the center of energy are utilized to show how the Darwin Lagrangian should be modified when there are nonelectric classical forces acting between the particles of the magnet.

1. INTRODUCTION

In a recent paper, Shockley and James[1] proposed the following interesting thought experiment: Consider a charged particle at rest in the time-independent field of a magnet. The magnetization of the magnet then is allowed to change. (An especially simple model is a magnet which consists of two oppositely charged disks of equal mass, rotating in opposite directions about a common axis. The magnetization is changed by bringing the two disks into contact and allowing friction to bring them to rest.) The changing magnetic field induces an electric field which exerts a force on the particle. It is straightforward to show that if the particle is far from the magnet, this force is given by the expression

$$\mathbf{F}=\frac{e}{c}\left(\frac{\mathbf{r}}{r^3}\times\frac{d\mathbf{M}}{dt}\right), \tag{1}$$

where e is the charge of the particle, $\mathbf{r}$ is the vector from the magnet to the particle, and $\mathbf{M}$ is the magnetization of the magnet. However, there is no obvious back-force of equal magnitude on the magnet.

Thus we are confronted with a striking paradox. It appears that if we put the entire experimental apparatus (magnet, particle, and a mechanical timing device to turn off the magnet at a preassigned time) in a sealed box, such a box would remain at rest until the timing device went off, shortly after which time the box would begin moving. (We can imagine the interior of the box to be coated with a thick layer of putty, so that eventually both the test particle and the magnet find themselves stuck to the walls of the box, and the whole system moves as a unit.) In brief, we would have a true recoilless rifle.

However, the analysis of the preceding paragraph is wrong. The source of the error is this: Despite the presence of only one factor of $1/c$ in Eq. (1), the effect it describes is in fact of order $1/c^2$, because the magnetization, when written in terms of the motions of charge carriers, is of order $1/c$.

Thus, in calculating what happens in the thought experiment, other effects of the same order (such as retardation effects, relativistic variation of mass with velocity, etc.) must be taken into account. In other words, the Stark-effect problem of the particles moving in the sensibly uniform electric field exerted on it by the distant test charge must be treated by relativistic rather than Newtonian mechanics. When this is done (as we shall show), the paradox disappears; the magnet feels a force equal and opposite to that which it exerts upon the particle, and the box referred to above does not move. This is what one expects if the total momentum is to remain zero, as Shockley and James emphasize, but it is instructive to trace out explicitly the mechanism by which this balance of momentum is secured.

Our discussion will proceed from the general to the particular, and from the exact to the approximate. In Sec. 2, we show that for any field theory (classical or quantum) described by a local, Lorentz-invariant Lagrangian, the box cannot move. Thus, in particular, in electrodynamics, it cannot move. Likewise, in any theory of magnetic media derivable from such a Lagrangian, it cannot move.

However, such a result does not give much insight into the actual balance of forces in the thought experiment. Therefore, in Sec. 3, we pass from the full theory to a theory which is correct only if one neglects terms of order $1/c^3$. To this order, we can eliminate the radiation modes from the theory and describe the interaction of charged particles in pure action-at-a-distance terms. The Lagrangian for such a description is the famous one first derived by Darwin[2] in 1920. We explicitly show that, according to this Lagrangian, the box cannot move, and discuss the balance of forces in some detail.

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[1] W. Shockley and R. P. James, Phys. Rev. Letters **18**, 876 (1967). The need of a back-force the negative of (1) to secure balance of action and reaction was also noted in a slightly earlier paper by O. Costa de Beauregard, Phys. Letters **24A**, 177 (1967).

[2] C. G. Darwin, Phil. Mag. **39**, 537 (1920).

We also discuss the effects of electron spin, which are also important in order $1/c^2$.

It is well known that in order to construct a classical theory of extended bodies, nonelectromagnetic cohesive forces must be introduced. Therefore, for logical completeness, in the Appendix we show how to construct Lagrangians for such forces that are like Darwin's in that they are "relativistic" up to and including effects of order $1/c^2$. We show that such forces do not affect our arguments.

We wish to stress that the intent of this paper is didactic; we want to show that the conventional principles of electrodynamics are sufficient to resolve the apparent paradox. None of the general theorems used in our discussion is new[3]; we give their proofs here for the sake of clarity and completeness.

2. GENERAL ARGUMENTS IN A FULLY RELATIVISTIC THEORY

It is known[4] that in any theory described by a local, Lorentz-invariant Lagrangian, it is possible to find a second-rank tensor (the stress-energy-momentum tensor) that is symmetric and conserved[5]:

$$T^{\mu\nu}=T^{\nu\mu}, \tag{2a}$$

$$\partial T^{\mu\nu}/\partial x_\mu=0. \tag{2b}$$

A closed system is one that can be enclosed in a surface upon which all components of $T^{\mu\nu}$ can be neglected. The box containing a magnet, a charge, and a timing device, discussed in Sec. 1, is obviously a closed system in this sense.

In the usual way, we define

$$E=\int T^{00}d^3x, \tag{3a}$$

$$P^i=c^{-1}\int T^{0i}d^3x, \tag{3b}$$

and

$$X^i=E^{-1}\int T^{00}x^i d^3x, \tag{3c}$$

where the integrals run over the volume occupied by the closed system. E is the energy of the system, $\mathbf{P}$ is the momentum, and $\mathbf{X}$ is the position of the center of energy, the relativistic generalization of the center of mass. The following laws follow directly from Eqs. (2) and (3):

$$dE/dt=0, \tag{4a}$$

$$d\mathbf{P}/dt=0, \tag{4b}$$

and

$$d\mathbf{X}/dt=c^2\mathbf{P}/E. \tag{4c}$$

These are, respectively, the law of conservation of energy, the law of conservation of momentum, and the law of steady motion of the center of energy.

We shall now use these equations to analyze the apparent paradox discussed in Sec. 1. Before the timing device fires, T^{00} is independent of time; hence $\mathbf{X}$ is independent of time. Thus, by Eq. (4c), $\mathbf{P}$ is zero. Since $\mathbf{P}$ is conserved, it remains zero even after the timing device is fired. Hence $d\mathbf{X}/dt$ remains zero, and the box does not move when it absorbs the momentum both of the test charge and the recoiling magnet. This completes the argument.

Note that to construct this argument, we had to use (4c) as well as (4b). This may puzzle the reader who, misled by elementary mechanics texts, thinks of the nonrelativistic law of motion for the c.m. as a trivial consequence of the conservation of momentum. However, although the derivation is indeed trivial, it depends critically upon the identification of p with mv: This, in turn, is a consequence of Galilean invariance. Thus the nonrelativistic law depends as much upon Galilean relativity as its relativistic counterpart does upon Einsteinian relativity.[6]

3. DETAILED THEORY TO ORDER $1/c^2$ INCLUSIVE

Darwin Lagrangian

In 1920, Darwin observed that, to order $1/c^2$ inclusive, the radiation modes could be eliminated from electrodynamics, and the theory written completely in terms of instantaneous (but velocity-dependent) action at a distance. Shockley and James prefaced their title with the words "try simplest case." The motto for the present section of our paper might be said to be "try simplest microscopic case—back to Darwin."

The Lagrangian Darwin found is

$$L=\sum_a \tfrac{1}{2}m_a v_a^2+\sum_a \tfrac{1}{8}c^{-2}m_a v_a^4 -\tfrac{1}{2}\sum_{a\neq b} e_a e_b r_{ab}^{-1}+\tfrac{1}{4}\sum_{a\neq b} c^{-2}e_a e_b r_{ab}^{-1} \times[\mathbf{v}_a\cdot\mathbf{v}_b+(\mathbf{v}_a\cdot\mathbf{r}_{ab})(\mathbf{v}_b\cdot\mathbf{r}_{ab})r_{ab}^{-2}], \tag{5}$$

[3] The results of the Appendix are a possible exception. However, the law of motion of the center of energy, Eq. (4c), is derived in almost every text on relativity, and the center of energy for the Darwin Lagrangian, Eq. (9), was known at least as early as 1948, since it occurs as a problem in a text of that date [L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1948), p. 184]. We do not know whether it was mentioned anywhere in the literature in the almost three decades between the publication of Darwin's paper and that of this text.

[4] F. J. Belinfante, Physica 7, 305 (1940).

[5] In this section, we adopt the usual conventions: Greek indices run from 0 to 3, Latin indices from 1 to 3, x^0 is ct, and repeated indices are summed.

[6] Indeed, the correspondence is even closer. In Hamiltonian mechanics, for a system of Newtonian particles interaction through potentials, $M\mathbf{X}$ is the infinitesimal generator of Galilean transformations, just as the Hamiltonian is the infinitesimal generator of time translation. Likewise, for relativistic theories, $E\mathbf{X}$ is the generator of Lorentz transformations. When looked at in this way, Eq. (4c) is simply the statement that the commutator of an infinitesimal time translation and an infinitesimal Lorentz transformation is an infinitesimal space translation.

where the indices a, b label the particles, m is mass, e is charge (in unrationalized Gaussian units), $\mathbf{v}$ is velocity, and $\mathbf{r}_{ab}$ is the vector from particle b to particle a. The first two terms in Eq. (5) are simply the usual relativistic Lagrangian for a free particle, expanded to order $1/c^2$; the third term is the usual Coulomb interaction; the last term represents the combined effect of magnetic interactions and retarded Coulomb interactions. Equation (5) leads to the following equations of motion:

$$\mathbf{p}_a=\partial L/\partial \mathbf{v}_a=m_a\mathbf{v}_a(1+\tfrac{1}{2}c^{-2}v_a^2)+\textstyle\sum_b \tfrac{1}{2}c^{-2}e_ae_br_{ab}^{-1}[\mathbf{v}_b+\mathbf{r}_{ab}(\mathbf{v}_b\cdot\mathbf{r}_{ab})r_{ab}^{-2}] \quad (6)$$

and

$$d\mathbf{p}_a/dt=\partial L/\partial \mathbf{r}_a=\textstyle\sum_b e_ae_b\mathbf{r}_{ab}r_{ab}^{-3}-\sum_b \tfrac{1}{2}c^{-2}e_ae_br_{ab}^{-3}[\mathbf{v}_a\cdot\mathbf{v}_b+3(\mathbf{v}_a\cdot\mathbf{r}_{ab})(\mathbf{v}_b\cdot\mathbf{r}_{ab})r_{ab}^{-2}]\mathbf{r}_{ab}+\sum_b \tfrac{1}{2}c^{-2}e_ae_br_{ab}^{-3}[\mathbf{v}_a(\mathbf{v}_b\cdot\mathbf{r}_{ab})+\mathbf{v}_b(\mathbf{v}_a\cdot\mathbf{r}_{ab})]. \quad (7)$$

(Here and elsewhere, in summations over b the "self" terms $b=a$ are, of course, to be excluded. In summations of the type $\sum_{b\neq a}$, a given pair of particles is involved twice, viz., through b, a and a, b. By $\partial L/\partial \mathbf{v}_a$ we mean a vector whose x component is $\partial L/\partial \dot{x}_a$, etc.)

Since the Darwin Lagrangian is translationally invariant, the momentum, defined in the canonical way as

$$\mathbf{P}=\textstyle\sum_a \mathbf{p}_a, \quad (8)$$

is a constant of the motion. There is no canonical definition of the center of energy; however, it is easy enough to guess the proper definition:

$$c^{-2}E\mathbf{X}=\textstyle\sum_a m_a(1+\tfrac{1}{2}c^{-2}\mathbf{v}_a^2)\mathbf{r}_a+\sum_{a\neq b}\tfrac{1}{2}c^{-2}e_ae_br_{ab}^{-1}\mathbf{r}_a. \quad (9)$$

The first term is the center of energy for a free particle; the last term is the interaction energy of a pair of particles, times their mean position, divided by c^2. To show that this is the correct guess, we must verify the law of motion for the center of energy, Eq. (4c).[7] The calculation is trivial, since, to the order to which we are working, the only terms for which we need to explicitly invoke the equations of motion are already of order $1/c^2$; therefore we can use the nonrelativistic equations in an electrostatic field to eliminate $d\mathbf{v}_a/dt$. We find

$$\frac{d}{dt}c^{-2}E\mathbf{X}=\textstyle\sum_a m_a(1+\tfrac{1}{2}c^{-2}v_a^2)\mathbf{v}_a+\sum_{a\neq b}c^{-2}e_ae_br_{ab}^{-3}(\mathbf{r}_{ab}\cdot\mathbf{v}_a)\mathbf{r}_a+\tfrac{1}{2}\sum_{a\neq b}c^{-2}e_ae_b\{r_{ab}^{-3}[-\mathbf{r}_{ab}\cdot(\mathbf{v}_a-\mathbf{v}_b)]\mathbf{r}_a+r_{ab}^{-1}\mathbf{v}_a\}+O(c^{-4}). \quad (10)$$

[7] The result (9) can, as is to be expected, also be obtained from evaluation of the integral (3c) for a system of particles. The first part of (9) arises from the center of the relativistic kinetic energy (to $1/c^2$ inclusive), a δ function for each particle. The second part of (9) is the center of the mutual electrostatic energy $E^2/8\pi$ of a system of particles. (The "cross" but not the infinite "self" terms are included in E^2.) Magnetic energy in the system is of higher order in $1/c$ and can be omitted.

If we change the summation indices in the double sums in such a way that all the velocities which appear carry the index b, this equation becomes

$$c^{-2}d(E\mathbf{X})/dt=\textstyle\sum_a m_a(1+\tfrac{1}{2}c^{-2}v_a^2)\mathbf{v}_a+\sum_{a\neq b}\tfrac{1}{2}c^{-2}e_ae_br_{ab}^{-1}\times[\mathbf{v}_b+\mathbf{r}_{ab}(\mathbf{v}_b\cdot\mathbf{r}_{ab})r_{ab}^{-2}]+O(c^{-4})=\mathbf{P}+O(c^{-4}), \quad (11)$$

which is the desired result.

Thus, just as one would expect, the law of motion of the center of energy is valid in the Darwin theory to that order in $1/c^2$ to which the Darwin theory itself is valid. Therefore all the general arguments of Sec. 2 go through unaltered; the box described in Sec. 1 still cannot be set permanently in motion.

Verification of Balance of Forces with Darwin Lagrangian

Unfortunately, the equations which we have obtained are singularly resistant to a simple physical interpretation in terms of particles exchanging forces; the expression for the momentum of any given particle, Eq. (6), involves the coordinates of all the other particles in the world. Likewise, the expression for the center of energy, Eq. (9), involves not just a weighted sum over particle position, but also a sum over pairs of positions. However, for the particular case of interest (a test particle in the field of a magnet), these expressions simplify considerably, and a simple interpretation is possible.

In this case, since the magnet is electrically neutral and devoid of a permanent electric moment, the particle-magnet cross terms in Eq. (9) may be neglected. (Of course, the electric field of the particle induces a small polarization in the magnet; but for large separations this effect is negligible compared with those in which we are interested.) Thus Eq. (9) becomes

$$c^{-2}E\mathbf{X}=m(1+\tfrac{1}{2}c^{-2}v^2)\mathbf{r}+c^{-2}E_m\mathbf{X}_m, \quad (12)$$

where the unsubscripted variables are those of the test particle and the variables bearing a subscript m are those of the magnet. (The latter are calculated by doing an appropriate sum over the particles which compose the magnet.) Also, since the particle is initially at rest, and since the force which the magnet exerts on the particle is of order $1/c^2$, to the order of interest we may neglect the v^2/c^2 corrections to the energy of the particle. Likewise, we may write

$$c^{-2}E_m=\textstyle\sum_a m_a(1+\tfrac{1}{2}c^{-2}v_a^2)+\tfrac{1}{2}\sum_{a\neq b}c^{-2}e_ae_br_{ab}^{-1}. \quad (13)$$

The first term is a constant; the remainder would also be a constant, by the conservation of energy, were it not for the interaction of the particle and the magnet. However, as we have argued, this is itself an effect of order $1/c^2$; therefore, in the order to which we are working, we may neglect the variation of E_m and set $c^{-2}E_m$ equal to a constant M, the rest mass of the magnet.

Thus we obtain

$$c^{-2}E\mathbf{X}=m\mathbf{r}+M\mathbf{X}_m. \qquad (14)$$

Hence, by the law of motion of the center of energy, Eq. (11), we have

$$md^2\mathbf{r}/dt^2=-Md^2\mathbf{X}_m/dt^2. \qquad (15)$$

The force on the test particle is precisely balanced by an equal and opposite force on the magnet. (In the preceding sentence, we have used the word "force" in its most naive sense: "Force" is the right-hand side of Newton's second law.[8]) If the particles composing the magnet are moving with velocities slow enough that the relativistic mass M is essentially the classical mass, then Eq. (15) tells us that the c.m. of the magnet recoils with exactly the same velocity as one would obtain from Newton's third law and the fact that the test charge moves in conformity with the force given in Eq. (1). However, as we shall show, this balance is secured only because of the relativistic corrections to the motion of the particles of the magnet in the presence of the electric field of the distant test particle of small velocity.

We shall now explicitly calculate these forces. We shall begin with the force on the test particle. Under the approximations discussed above, the equations of motion, (6) and (7) become

$$d\mathbf{p}/dt=0 \qquad (16)$$

and

$$\mathbf{p}=m\mathbf{v}+\tfrac{1}{2}c^{-2}e\sum\nolimits_{\text{magnet}}|\mathbf{r}-\mathbf{r}_b|^{-1}e_b \times[\mathbf{v}_b+|r-r_b|^{-2}(\mathbf{r}-\mathbf{r}_b)(\mathbf{v}_b\cdot(\mathbf{r}-\mathbf{r}_b))], \qquad (17)$$

where we have inserted the explicit definition of the difference vector. If the particle is far from the magnet, we can choose the center of the magnet to be the origin of our coordinates and expand in inverse powers of r. Retaining only the first two terms, we find

$$\mathbf{p}=m\mathbf{v}+\tfrac{1}{2}c^{-2}e\sum r^{-1}e_b\{\mathbf{v}_b+r^{-2}(\mathbf{r}\cdot\mathbf{r}_b)\mathbf{v}_b+r^{-2}(\mathbf{r}\cdot\mathbf{v}_b) \times\mathbf{r}[1+3r^{-2}(\mathbf{r}\cdot\mathbf{r})]-r^{-2}\mathbf{r}_b(\mathbf{v}_b\cdot\mathbf{r})-r^{-2}\mathbf{r}(\mathbf{v}_b\cdot\mathbf{r}_b)\}. \qquad (18)$$

Now to zeroth order in inverse powers of c, a real magnet has charge density zero. This is also true of the idealized magnet discussed in the Introduction if the separation of the two disks is small. In particular,

$$\sum e_b\mathbf{v}_b=\frac{d}{dt}\sum e_b\mathbf{r}_b=0 \qquad (19a)$$

and

$$\sum e_b\mathbf{v}_b\cdot\mathbf{r}_b=\tfrac{1}{2}\frac{d}{dt}\sum e_b r_b^2=0,$$
$$\sum e_b(\mathbf{r}\cdot\mathbf{r}_b)(\mathbf{r}\cdot\mathbf{v}_b)=0. \qquad (19b)$$

[8] If we had used "force" in the sense of Lagrangian mechanics (the time rate of change of a canonical momentum), Eq. (16) would tell us that the "force" on the test particle vanishes. We shall return to this point later.

Thus Eq. (18) becomes

$$\mathbf{p}=m\mathbf{v}-\tfrac{1}{2}c^{-2}er^{-3}\sum e_b\mathbf{r}\times(\mathbf{r}_b\times\mathbf{v}_b). \qquad (20)$$

If we insert the standard definition of the magnetization

$$\mathbf{M}=\tfrac{1}{2}c^{-1}\sum\nolimits_b e_b\mathbf{r}_b\times\mathbf{v}_b, \qquad (21)$$

we obtain

$$\mathbf{p}=m\mathbf{v}-er^{-3}c^{-1}\mathbf{r}\times\mathbf{M}, \qquad (22)$$

and Eq. (16) becomes

$$md\mathbf{v}/dt=er^{-3}c^{-1}\mathbf{r}\times d\mathbf{M}/dt. \qquad (1)$$

As the numbering of the equation indicates, this is the expression with which we started, the force on the test particle as calculated by Shockley and James. We have shown this explicitly, because had we not done so there would have remained the logical possibility that the Darwin Lagrangian did not give a paradox for the trivial reason that it did not predict the primary effect, the acceleration of the test particle. This possibility has now been eliminated.

We now calculate the force which the test particle exerts on the magnet. If the magnet were an isolated system, we would have

$$Md\mathbf{X}_m/dt=\mathbf{P}_m. \qquad (23)$$

Indeed, Eq. (23) would also hold if the mechanics of the system were Galilean invariant, even if it were not isolated; for in this case, the expression for the c.m. would not involve velocities, and therefore the external forces acting on the system would never enter into the verification of Eq. (23). However, in the actual (relativistic) case, the expression for $\mathbf{X}_m$ *does* involve velocities, through the term

$$\sum \tfrac{1}{2}c^{-2}v_a^2\mathbf{r}_a. \qquad (24)$$

The time derivative of this term will have a contribution from the external forces acting on the magnet (the electric field of the test particle).[9] Thus the correct

[9] A particularly simple illustration of the back-force that counterbalances the force on the test charge is obtained by considering the special case of magnetism arising from the motion of a particle in a fixed orbit or current in a loop. The relativistic mass is different for the sides of the orbit nearest and furthest from the distant test charge, in consequence of the conservation of total energy together with the fact that the electrostatic energy in the field $\mathbf{E}$ exerted by the distant test charge is different on the two sides. This field makes $m\mathbf{v}_a/(1-\beta^2)^{1/2}$ differ from $m\mathbf{v}_a$ by $e_a(\mathbf{E}\cdot\mathbf{r})m\mathbf{v}_a/c^2+O(1/c^4)$. After averaging over an orbital period and some simple vector manipulation, the time derivative of this expression is found to agree with (27), specialized to the case that M is generated by a single particle. Essentially this type of calculation, but in the language of current loops rather than particles, is given by P. Penfield and H. Haus, *The Electrodynamics of Moving Media* (M.I.T. Press, Cambridge, Mass., 1967), p. 215; Proc. IEEE **53**, 442 (1965); Phys. Letters **26A**, 412 (1968). These authors have independently (private communication) noticed that the answer to the Shockley-James paradox is found by including relativity corrections when computing the current flow causing the magnetic moment. That this is the answer is also mentioned by Shockley and James themselves, but not demonstrated in detail. See Ref. 1; also, Science **156**, 542 (1967).

equation is not Eq. (23) but

$$M\frac{d\mathbf{X}_m}{dt}=\mathbf{P}_m-\sum c^{-2}e_a e r^{-3}(\mathbf{r}\cdot\mathbf{v}_a)\mathbf{r}_a. \tag{25}$$

By the same arguments as before, this may be rewritten as

$$M d\mathbf{X}_m/dt=\mathbf{P}_m-er^{-3}c^{-1}\mathbf{r}\times\mathbf{M}. \tag{26}$$

Equation (16) and the conservation of total momentum tell us that P_m is constant; therefore we obtain

$$M d^2\mathbf{X}_m/dt^2=-er^{-3}c^{-1}\mathbf{r}\times d\mathbf{M}/dt, \tag{27}$$

the desired result.

The heart of the matter is the expression (24), the velocity dependence of the longitudinal mass. In mechanics "the total force acting on a system" may be defined in two ways: Either force is the time rate of change of (canonical) momentum, or force is mass times the acceleration of the center of energy. In Newtonian mechanics, these two kinds of force are trivially equivalent. In relativistic mechanics, because of (24), they are different (although, for a closed system, both kinds of force vanish). "What is the force on the magnet?" is a bad question, because it tempts us to confuse the two kinds of force. The force in the first sense does vanish if the magnet is neutral; this is the statement that $\mathbf{P}_m$ is a constant. However, this does not mean that the force in the second sense vanishes; indeed, Eq. (27) shows that it does not.

Inclusion of Spin

We conclude this section with a brief discussion of the effects of spin. We may take account of these by adding to the Darwin Lagrangian (5) an additional term,[10]

$$L'=-\sum_{a\neq b}\tfrac{1}{2}c^{-2}e_a e_b m_b^{-1} \times r_{ab}^{-3}[\mathbf{r}_{ab}\times(g_b\mathbf{v}_a-g_b\mathbf{v}_b+\mathbf{v}_b)]\cdot\mathbf{S}_b+\cdots, \tag{28}$$

where $\mathbf{S}_b$ is the spin of the bth particle, $g_b e_b/2m_b c$ is its ratio of magnetic moment to spin angular momentum, and the triple dots represent direct spin-spin interactions, not involving velocities. Equation (28) is simply the conventional spin-orbit interaction; the strange-looking term not involving the g's is just the effect of Thomas precession. Because of the presence of the Thomas term in (28), one finds that

$$-\frac{d}{dt}\frac{\partial L'}{\partial\mathbf{v}_a}\neq\sum_b\frac{d}{dt}\frac{\partial L'}{\partial\mathbf{v}_b}, \tag{29}$$

where a refers to the test particle and the magnet is represented by the summation over b. If a is devoid of spin and if the magnet's moment arises entirely from spin, then, provided that r_{ab} is large, the left side of (29) is, as expected, identical with the Shockley-James force given in Eq. (1). Since (29) is an inequality, it appears at first sight as though the action-reaction balance expressed in Eq. (15) is spoiled when the spin moment changes with time. However, we shall show that this is not really the case.

The addition (28) to the Lagrangian produces a change in the total momentum,

$$\mathbf{P}\rightarrow\mathbf{P}-\sum_{a\neq b}\tfrac{1}{2}c^{-2}e_a e_b m_b^{-1}r_{ab}^{-3}\mathbf{r}_{ab}\times\mathbf{S}_b. \tag{30}$$

The law of motion of the center of energy may be saved if we make a corresponding addition[11] to the center of energy:

$$c^{-2}E\mathbf{X}\rightarrow c^{-2}E\mathbf{X}-\sum_b\tfrac{1}{2}c^{-2}\mathbf{S}_b\times\mathbf{v}_b. \tag{31}$$

(The time derivative of the spin produces effects of higher order in $1/c^2$ and may be neglected.) All of our previous arguments are clearly unaffected by these additional terms.

The spin factors in (30) and (31) can be regarded as constants, since the wobbles in the orientation of $\mathbf{S}$ which are correlated with the orbital motions are of the order $1/c^2$ and so would only give terms in (30) or (31) of the order $1/c^4$, which can be omitted. Also, any slow precessions of the spin are uncorrelated with the electronic motion and average out. In consequence, $\mathbf{S}_b$ may be replaced in (30) or (31) with its mean value and treated as a constant. Then the sum in (30) vanishes on averaging over the rapid electronic motion, since even when the moment of the magnet changes, the mean acceleration of any particle is zero. (Otherwise the magnet would either decompose or walk off even without any force from the distant test particle.) Correspondingly, the mean velocity of any particle entering in (31) can be considered zero before or after the moment of the magnet is allowed to change. The spin terms then have no effect on the mean position of the center of energy. All this is essentially equivalent to saying that the right and left sides of (29) are equal except for fluctuations which average out and are of no interest to us.

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[10] Actually, what we obtain in this way is not a Lagrangian but a Routhian; it is to be treated as a Lagrangian for the space variables and as a (negative) Hamiltonian for the spin variables. Equation (28) is essentially the generalization to arbitrary spin of an early formula of W. Heisenberg, Z. Physik **39**, 499 (1926).

[11] This is not a surprising result, since this is just the term that must be added to ensure that the Poisson brackets of the components of EX/c with each other should yield the total angular momentum—that is to say, to ensure Lorentz invariance —to the order to which we are working. The Poisson brackets obtained from (9) without the modification (31) yield only the orbital angular momentum.

APPENDIX: CLASSICAL COHESIVE FORCES

We observed in the Introduction that it is impossible, by virtue of Earnshaw's theorem, for there to exist a classical static array of point particles held together by electromagnetic forces alone. Therefore, in the context of classical physics, the analysis of the main body of this paper, which assumes that the only forces acting between the particles of the magnet are electromagnetic, is incomplete. The purpose of this Appendix is to fill the gap in our argument by showing how to construct, by adding terms of order $1/c^2$ to a Newtonian Lagrangian of the form

$$L=-\sum_a \tfrac{1}{2}m_a v_a^2+\tfrac{1}{2}\sum_{a\neq b} V_{ab}\,, \tag{A1}$$

where V_{ab} depends only on r_{ab}^2, a Lagrangian that is like Darwin's in that it is "relativistic neglecting terms of order $1/c^3$."

A theory is relativistic if we can find 10 observables (energy, momentum, angular momentum, and center of energy) such that their Poisson brackets are the appropriate ones for the generators of the inhomogeneous Lorentz group. It is approximately relativistic in the sense which we have in mind if these Poisson brackets are the correct ones, neglecting terms of order $1/c^3$. Any time-independent Lagrangian possessing manifest translational and rotational invariance will automatically have the right Poisson brackets for energy, momentum, and angular momentum. Therefore the only Poisson brackets which we have to check are those involving the center of energy.

To the desired order, the center of energy is determined just by the Newtonian terms in the Lagrangian:

$$c^{-2}E\mathbf{X}=\sum_a m_a(1+\tfrac{1}{2}c^{-2}v_a^2)\mathbf{r}_a+\tfrac{1}{4}\sum_{a\neq b} c^{-2}V_{ab}(\mathbf{r}_a+\mathbf{r}_b)\,. \tag{A2}$$

We note that the Poisson bracket of this object with the momentum is automatically correct to the desired order, since a space translation adds to (A2) a term proportional to the energy. The same is true of the Poisson bracket with the angular momentum, since (A2) obviously transforms like a vector under rotations. As for the Poisson brackets of the components of (A2) with each other, the only nonvanishing contributions are the brackets of the first and second terms which trivially give the angular momentum, as they should.

Thus the only Poisson bracket that depends on the (as yet undetermined) terms in the Lagrangian of order $1/c^2$ is that of the center of energy with the energy. That is to say, the terms of order $1/c^2$ in the Lagrangian must be chosen such that the time derivative of $c^{-2}EX$ is the total momentum.

In the order to which we are working,

$$c^{-2}\frac{d(E\mathbf{X})}{dt}=\sum_a m_a(1+\tfrac{1}{2}c^{-2}v_a^2)\mathbf{v}_a+\sum_{a\neq b}\tfrac{1}{2}c^{-2}V_{ab}\mathbf{v}_a -\sum_{a\neq b} c^{-2}V_{ab}'(\mathbf{v}_a\cdot\mathbf{r}_{ab})\mathbf{r}_{ab}\,, \tag{A3}$$

where the prime indicates differentiation of V with respect to its argument r_{ab}^2. The right side of Eq. (A3) must be the momentum. This is guaranteed if we choose our Lagrangian to be

$$L=\tfrac{1}{2}\sum_a m_a v_a^2+\tfrac{1}{8}\sum_a m_a c^{-2}v_a^4 -\tfrac{1}{2}\sum_{a\neq b} V_{ab}+\tfrac{1}{4}c^{-2}\sum_{a\neq b} V_{ab}\mathbf{v}_a\cdot\mathbf{v}_b -\tfrac{1}{2}c^{-2}\sum_{a\neq b} V_{ab}'(\mathbf{v}_a\cdot\mathbf{r}_{ab})(\mathbf{v}_b\cdot\mathbf{r}_{ab})\,. \tag{A4}$$

(Note that this reduces to the Darwin Lagrangian if V_{ab} is the Coulomb potential.)

All of the arguments constructed in Sec. 3 for the Darwin Lagrangian hold for the more general Lagrangian described by Eq. (A4): The law of motion of the center of energy is valid, by construction. If we introduce nonelectromagnetic forces only for the particles of the magnet, the decomposition of the center of energy, Eq. (14), is still true, as is the consequent equality of Newtonian forces, Eq. (15). Under the same restriction, all the subsequent equations of Sec. 3 are valid without alteration. In short, nonelectromagnetic cohesive forces within the magnet, provided that they do not violate the principle of relativity, do not affect any of our conclusions.