Lorentz transformation of a system carrying ''Hidden Momentum''

E. Comay[a)]
School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel

(Received 14 September 1999; accepted 15 March 2000)

The relations between an energy–momentum tensor and its corresponding energy–momentum four-vector are discussed. A particular emphasis is put on conditions guaranteeing that spatial integrals of the energy–momentum densities pertain to a true four-vector. Cases where such integrals are *not* components of a true four-vector are analyzed and the usefulness of the notion of a false four-vector is pointed out. Results are used for explaining Lorentz transformation properties of ''hidden momentum.''

I. INTRODUCTION

As is well known, fields' energy–momentum density and their flux density are represented by $T^{\mu 0}$ and $T^{\mu i}$ entries of the fields' energy–momentum tensor.[1] (Greek indices range from 0 to 3 and Latin ones range from 1 to 3. The diagonal metric $g_{\mu\nu}$ is $(1,-1,-1,-1)$. The symbol $,\mu$ denotes the partial differentiation with respect to x^μ. **i**, **j**, and **k** denote unit vectors in the *x, y*, and *z* directions, respectively.) It follows that the fields' overall energy and momentum are related to the integrals

$$``p^{\mu}" = \frac{1}{c}\int T^{\mu 0}\, d^3x, \tag{1}$$

which is carried out on the entire three-dimensional space.

The four quantities on the left-hand side of (1) are enclosed in quotation marks because it is not evident that they transform as entries of a true four-vector. As a matter of fact, specific examples where the left-hand side of (1) is *not* a four-vector are presented in this work. In such cases, the left-hand side of (1) is called a false four-vector. This issue is the main topic of this work, which discusses sufficient conditions that p^μ is a true four-vector and their implications. It is further explained why this issue is relevant to the Lorentz transformation of ''hidden momentum.''

Section II presents a condition that guarantees that (1) is a true four-vector.[1] Section III includes examples of electromagnetic fields that illustrate this condition. One of these systems contains ''hidden momentum.'' Some mistakes concerning Lorentz transformations of ''hidden momentum'' which have been published recently[2] are explained in Sec. IV. Concluding remarks are presented in Sec. V.

II. SUFFICIENT CONDITIONS FOR ENERGY–MOMENTUM FOUR-VECTOR

As stated above, the four quantities $T^{\mu 0}$ represent energy and momentum densities, respectively. (A division by c is required for the momentum density. In an energy–momentum four-vector, energy is divided by c. It is assumed that a brief terminology like that of the first statement of this section will not be misunderstood.) Hence, if one takes the values of $T^{\mu 0}$ at $t=0$ and carries out an integration on the entire three-dimensional space, the overall energy and momentum associated with $T^{\mu 0}$ is obtained.

Consider two inertial frames, Σ and Σ', and the four quantities $``p^\mu"$ obtained in Σ. The corresponding integration carried out in Σ' uses the tensorial quantities $T'^{\mu 0}$ as found in Σ' at $t'=0$. This point means that the integral carried out in Σ' depends not just on the Lorentz transformation of $T^{\mu\nu}$ from Σ to Σ' but also *on the time adjustment needed for having the simultaneous quantities at* Σ'. This kind of adjustment might affect the integral (1) and is the reason for the quotation marks used on the left-hand side of this expression.

Landau and Lifshitz discuss this issue and prove that the continuity equation for a charge

$$j^{\mu}_{,\mu}=0 \tag{2}$$

is a sufficient condition for obtaining the same amount of charge in any inertial frame,[3] namely, for regarding the electric charge as a Lorentz scalar.

In their discussion, Landau and Lifshitz begin with a proof of charge conservation. The proof uses the four-dimensional Gauss theorem for an integral carried out on the four-volume included between two hyperplanes, S_1 and S_2, defined by $x^0=T_1$ and $x^0=T_2$. Later, they state that ''the proof presented clearly remains valid also for any two integrals $\int j^\mu dS_\mu$, in which the integration is extended over any two infinite hypersurfaces (and not just the hyperplanes $x^0=\text{const}$) which each contain all of three-dimensional space.'' Thus, one concludes that, in particular, the overall charge Q takes the same value for $x^0=T$ at the inertial frame Σ and at $x'^0=T'$ at Σ'. This outcome means that charge transforms as a Lorentz scalar.

The foregoing discussion is extended later to energy–momentum tensors and their corresponding global four-vectors,[4] where the four components of the latter are spatial integrals of the corresponding tensor components. They find that

$$T^{\mu\nu}_{,\nu}=0 \tag{3}$$

guarantees energy–momentum conservation. This relation is calculated and used by Landau and Lifshitz for proving energy–momentum conservation of a system which consists of charged matter and electromagnetic fields.[5] As in the case of charge, (3) can also be used for proving that spatial integrals of $T^{\mu 0}$ (divided by c) are components of the energy–momentum four-vector.

It should be noted that the calculation of Landau and Lifshitz proves that the energy–momentum tensor of electromagnetic fields

$$T_F^{\mu\nu}=\frac{1}{4\pi}\left(F^{\mu\alpha}F^{\beta\nu}g_{\alpha\beta}+\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g^{\mu\nu}\right) \tag{4}$$

does *not* satisfy (3) but

$$T^{\mu\nu}_{,\nu}=-\frac{1}{c}F^{\mu\nu}j_\nu. \tag{5}$$

Evidently, this relation is inconsistent with condition (3), which requires a null four-divergence. Hence, one concludes that, excluding particular cases, *energy and momentum of electromagnetic fields should not be regarded as entries of a true four-vector*.

Examples illustrating this conclusion are presented in the following section. Special attention is devoted to the case of ''hidden momentum.''

III. EXAMPLES OF LORENTZ TRANSFORMATION OF ELECTROMAGNETIC FIELDS

In the examples of this section, effects of the following Lorentz ''boost,''

$$L^\mu_\nu=\begin{pmatrix}\gamma & \gamma u/c & 0 & 0\\ \gamma u/c & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{6}$$

are examined. u denotes the three-velocity of the boost which is parallel to the x axis and $\gamma=(1-u^2/c^2)^{-1/2}$. This transformation casts quantities measured in an inertial frame Σ into another frame Σ'.

As is well known,[6] in the case of electromagnetic fields, the energy density is

$$T^{00}=\frac{1}{8\pi}(E^2+B^2) \tag{7}$$

and the momentum density is

$$\frac{1}{c}T^{i0}=\frac{1}{4\pi c}(\mathbf{E}\times\mathbf{B})_i. \tag{8}$$

A. A free electromagnetic wave

Let us consider a monochromatic plane electromagnetic wave[7] traveling in the x direction. In the frame Σ, the fields are

$$\mathbf{E}=E\sin(kx-\omega t)\mathbf{j}, \tag{9}$$

$$\mathbf{B}=B\sin(kx-\omega t)\mathbf{k}, \tag{10}$$

where $B=E$ and $k=\omega/c$. The wavelength of this field is

$$\lambda=2\pi/k. \tag{11}$$

This system does not contain charges and the energy–momentum tensor of the fields satisfies the null four-divergence (3). Hence, one expects that the overall energy and momentum of the fields are components of a true four-vector.

Let us calculate the energy $\mathcal{E}$ and the momentum $\mathbf{p}$ enclosed in a rectangular parallelpiped having a base whose area is unity and its height (which takes the x direction) is λ. It means that this rectangular parallelpiped contains one complete wavelength. Due to the symmetry of the system, the rectangular parallelpiped correctly represents the general problem. The calculation is carried out at $t=0$. Using (7), (9), (10), and (11), one finds

$$\mathcal{E}=\int_0^\lambda\int_0^1\int_0^1\frac{1}{8\pi}(E^2+B^2)dx\,dy\,dz=\frac{\lambda}{8\pi}E^2. \tag{12}$$

In a similar manner, one replaces (7) with (8) and obtains for the x component of the momentum

$$p_x=\frac{1}{c}\int_0^\lambda\int_0^1\int_0^1\frac{1}{4\pi}E_yB_z\,dx\,dy\,dz=\frac{\lambda}{8\pi c}E^2. \tag{13}$$

Other components of the momentum vanish. Now, the energy–momentum four-vector is written by means of the energy and momentum[8]

$$P^\mu=(\mathcal{E}/c,\mathbf{p}). \tag{14}$$

Using these results, one realizes that the expected true four-momentum is

$$P^\mu=\frac{E^2\lambda}{8\pi c}(1,1,0,0). \tag{15}$$

Let us apply the Lorentz transformation (6) and calculate the fields at Σ' for $t'=0$. Using the appropriate formulas,[9] one obtains

$$E'_y=\gamma(1+u/c)E_y, \tag{16}$$

$$B'_z=\gamma(1+u/c)B_z. \tag{17}$$

It means that the electric and magnetic fields increase by the same factor $\gamma(1+u/c)$ and that their product increases by the square of this quantity.

Let us now synchronize the time at Σ'. In Σ, a point on the left-hand side of the rectangular parallelpiped is

$$x_L^\mu=(0,0,y,z) \tag{18}$$

and a corresponding point on its right-hand side is

$$x_R^\mu=(0,\lambda,y,z). \tag{19}$$

Applying the Lorentz transformation (6), one finds that (18) remains unchanged, whereas the transformation of (19) takes the form of

$$x_R'^\mu=(\lambda\gamma u/c,\lambda\gamma,y,z). \tag{20}$$

This result indicates that a time synchronization is required before an integration on $t'=0$ values can take place.

The fields travel at the speed of light c. Hence, at $t'=0$, the right-hand side of the rectangular parallelpiped is at

$$\overline{x_R'}^\mu=(0,\lambda\gamma(1-u/c),y,z). \tag{21}$$

This outcome means that in the case of free electromagnetic waves, the rectangular parallelpiped contracts by the factor $\gamma(1-u/c)$. Combining this result with the factor representing the increase of the fields as given in (16) and (17), one finds

$$P'^\mu=\gamma^3(1+u/c)^2(1-u/c)P^\mu=\gamma(1+u/c)P^\mu. \tag{22}$$

The same result is also obtained from the application of the Lorentz transformation (6) to the four-momentum (15). Therefore, this analysis illustrates the claim that for free electromagnetic fields whose energy–momentum tensor satisfies the null four-divergence (3), the overall energy and momentum are components of a true four-vector.

B. A parallel plate capacitor

A parallel plate capacitor is discussed here. The system contains fields and charges and relation (5) holds for the four-divergence of the energy–momentum tensor of electromagnetic fields. Hence, the null four-divergence (3) does not hold at some points of space. For this reason, one expects that the overall energy and momentum of electromagnetic fields are entries of a false four-vector. The calculation presented below confirms this expectation.

This device has been used in discussions of Lorentz transformations of fields and matter under pressure (Refs. 10 and 11). Relevant calculations required here can be found in these articles. Hence, several points are cited here and the derivation procedure is omitted. The capacitor consists of three parts: its plates, the electric field emerging from the positively charged plate and ending on the negatively charged one, $|\mathbf{E}|=E_x$, and a gas enclosed between the plates. The pressure of this gas balances the electrostatic attraction between the plates.

The energy and momentum enclosed inside a rectangular parallelpiped whose bases lie on the two plates, respectively, are calculated. As in the previous example, this rectangular parallelpiped represents the general problem correctly. Let V_0 denote the volume of this rectangular parallelpiped. The capacitor is motionless in Σ and the energy of the electric field is $V_0E^2/8\pi$. The magnetic field of the system vanishes and, therefore, the electromagnetic momentum vanishes, too. It follows that the electromagnetic energy and momentum can be written as a false four-vector,

$$``P_F^{\mu}"=\frac{V_0E^2}{8\pi c}(1,0,0,0). \tag{23}$$

Let us use the Lorentz transformation (6) and find the respective quantities in Σ'. As is well known, the electric field component which is parallel to the "boost" is unchanged and yields no magnetic field.[9] On the other hand, the self-volume V_0 of the rectangular parallelpiped contracts by a factor γ^{-1}. Therefore, in Σ', the energy and momentum of the electromagnetic fields are

$$``P_F'^{\mu}"=\frac{V_0E^2}{8\pi\gamma}(1,0,0,0). \tag{24}$$

On the other hand, if one treats (23) as a true four-vector and applies the Lorentz transformation (6) to it, one obtains

$$``\overline{P}_F'^{\mu}"=\frac{V_0E^2\gamma}{8\pi}(1,u/c,0,0). \tag{25}$$

Evidently, (24) and (25) are not the same, proving that (23) is a false four-vector.

It is further proved[10,11] that the energy and momentum of the gas enclosed between the plates (or of another material under pressure, used for balancing the electrostatic attraction between the plates) also belongs to this class, namely they are components of a false four-vector. Only the *sum* of these two false four-vectors is a true four-vector, thereby illustrating the self-consistency of special relativity.

C. A device containing "hidden momentum"

Let us examine a device containing "hidden momentum." It is designed so that electric and magnetic fields take simple expressions and integrals and other calculations are straightforward. This feature of the discussion facilitates the presentation of the underlying laws of physics.

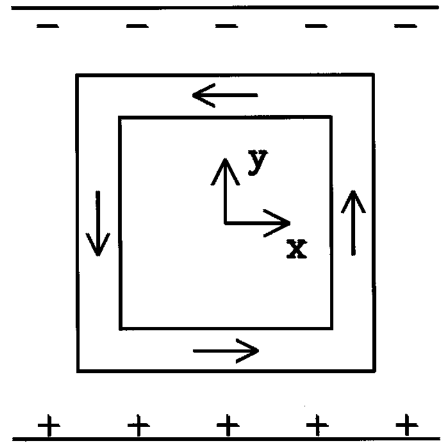

Fig. 1. A solenoid whose cross section is a square, is motionless in a frame Σ, and its axis coincides with the z axis. The solenoid is placed between the plates of a capacitor. The four arrows denote the electric current along the solenoid (see the text).

The device contains a solenoid and a parallel plate capacitor whose magnetic and electric fields are perpendicular to each other. Again, as in the previous example, the system contains charges and currents that yield (5) for the four-divergence of the electromagnetic energy–momentum tensor. Hence, the null four-divergence of the energy–momentum tensor, (3), does not hold. Therefore, one expects that the electromagnetic energy and momentum obtained from the integration of their corresponding densities are components of a false four-vector.

Let us examine a closed pipe that takes the form of a circumference of a square. The pipe is made of an insulating material and contains an incompressible positively charged fluid that flows frictionlessly along it (see Fig. 1). (This kind of uniformly charged fluid is just a hypothetical matter which enables a simple mathematical treatment of the problem.) The pipe is covered with a negative electric charge that screens the electric field of the charged fluid. Hence, in the inertial frame Σ where the closed pipe is motionless, only magnetic field is generated by the closed loop. The cross section of the pipe is small with respect to its length.

The corners of the pipe are placed at four points whose coordinates are $X=\pm 1$ and $Y=\pm 1$, respectively. An infinitely long pile of such pipes makes a solenoid whose axis coincides with the z axis. (As in a standard treatment of solenoids, the insulating material used for building the pipes is thin enough, so that the current can be regarded as uniform on the solenoid's circumference.) In its interior, this solenoid generates a uniform magnetic field in the z direction

$$\mathbf{B}=B\mathbf{k}, \tag{26}$$

whereas the external field vanishes.

Let ρ denote the charge density of the fluid, s the area of the pipe's cross section, and v the fluid's velocity. Thus the electric current along the pipe is $I=\rho vs$. Let N denote the number of closed pipes per unit length in the z direction. Thus the magnetic field (26) is

$$\mathbf{B}=\frac{4\pi}{c}N\rho sv\mathbf{k}. \tag{27}$$

The other component of the device is a parallel plate capacitor whose plates are parallel to the (x, z) plane and are placed at $Y<-1$, $Y>1$, respectively (see Fig. 1). The plates are made of an insulating material, each of which is covered uniformly with a surface charge density $\pm\rho_c$, respectively. In the region between the plates, the capacitor generates a uniform electric field in the y direction,

$$\mathbf{E}=E\mathbf{j}. \tag{28}$$

In a discussion of quantities related to ''hidden momentum,'' one has to consider interaction terms of the solenoid and the capacitor. For this reason, self-interaction terms of the solenoid with itself, as well as those of the capacitor are ignored. Moreover, since the capacitor's plates are made of an insulating material, the self-energy of the capacitor's charges is independent of the solenoid. Similarly, since the hypothetical solenoid's uniformly charged fluid is incompressible, its electric state is assumed here to be unaffected by the electric field of the capacitor.

The electromagnetic interaction dependent momentum density is bilinear in the magnetic field (27) and in the electric field (28). The calculation is restricted to the volume V_0 inside a cube $-1\leqslant x\leqslant 1$, $-1\leqslant y\leqslant 1$, $-1\leqslant z\leqslant 1$ (hence $V_0=8$ is used below). Evidently, due to the symmetry of the device, this cube represents the entire problem correctly.

The interaction dependent momentum of the electromagnetic fields is obtained from the integration of the momentum density on the volume. Only the x component of the momentum is nonzero and the calculation is straightforward,

$$p_{x(\mathrm{elec})}=\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\frac{1}{4\pi c}E_yB_z\,dx\,dy\,dz=\frac{8}{c^2}N\rho s v E_y, \tag{29}$$

where (27) is used.

This electromagnetic momentum is compensated by the mechanical momentum of the system.[11–13] The mechanical momentum is included in the charged fluid that moves along the closed pipes of the solenoid. This quantity is calculated below.[14]

The force exerted by the capacitor's electric field on the moving charges of the fluid is balanced by a mechanical pressure gradient. The fluid's pressure difference, $\Delta\mathcal{P}$, between a point at $Y=1$ and a point at $Y=-1$ renders a force exerted on the portion of the fluid which flows along the pipes' segments at $x=\pm 1$. The force is

$$\mathbf{f}_1=-\Delta\mathcal{P}s\mathbf{j}. \tag{30}$$

(Note that $\mathbf{j}$ is a unit vector and not a current.) This force balances the force exerted by the capacitor's field on this portion of the charged fluid

$$\mathbf{f}_2=2s\rho E\mathbf{j}, \tag{31}$$

where $2s$ is the volume of the charged fluid at each of the $X=\pm 1$ segments of a pipe. In this way one finds an expression for the pressure difference

$$\Delta\mathcal{P}=2\rho E. \tag{32}$$

The energy–momentum tensor of a macroscopic body at rest depends on its energy density ϵ and its pressure $\mathcal{P}$,[14]

$$T^{\mu\nu}=\begin{pmatrix}\epsilon & 0 & 0 & 0\\ 0 & \mathcal{P} & 0 & 0\\ 0 & 0 & \mathcal{P} & 0\\ 0 & 0 & 0 & \mathcal{P}\end{pmatrix}. \tag{33}$$

This tensor is used here for the fluid that moves parallel to the x axis. Performing a Lorentz transformation on (33) for the fluid at the $Y=-1$ segment, one finds

$$T^{\mu\nu}=\begin{pmatrix}\epsilon & (\epsilon+\mathcal{P})v/c & 0 & 0\\ (\epsilon+\mathcal{P})v/c & \mathcal{P} & 0 & 0\\ 0 & 0 & \mathcal{P} & 0\\ 0 & 0 & 0 & \mathcal{P}\end{pmatrix}, \tag{34}$$

where terms proportional to v^2 and higher powers of v are omitted, due to $v\ll c$.

Analogous expressions are obtained for the liquid at the $Y=1$ segment. Here the motion is leftward and the factor v is replaced by $-v$. The pressure at each of the $Y=\pm 1$ segments is uniform. It follows that the integration of the mechanical momentum density is straightforward. Thus one finds that the mechanical momentum enclosed within the cube V_0 is

$$p_{x(\mathrm{mech})}=-4\Delta\mathcal{P}sNv/c^2=-8\rho E_y sNv/c^2, \tag{35}$$

where (32) is used. This result proves that, as expected,[12] the sum of the electromagnetic momentum (29) and the mechanical momentum (35) vanishes for the motionless system discussed.

Now let us turn to the inertial frame Σ' and use the Lorentz transformation (6) for the quantities obtained above. As stated, only the interaction part of components associated with the solenoid with those of the capacitor are treated here. The volume of the cube undergoes a Lorentz contraction by the factor γ,

$$V_0'=V_0/\gamma. \tag{36}$$

The magnetic field of the solenoid increases by a factor γ and also yields an electric field in the y direction,[9]

$$\mathbf{B}'_{(\mathrm{sol})}=\gamma B\mathbf{k}, \tag{37}$$

$$\mathbf{E}'_{(\mathrm{sol})}=\gamma B\frac{u}{c}\mathbf{j}, \tag{38}$$

where B is the quantity used in (26).

Similarly, the fields of the capacitor are

$$\mathbf{E}'_{(\mathrm{cap})}=\gamma E\mathbf{j}, \tag{39}$$

$$\mathbf{B}'_{(\mathrm{cap})}=\gamma E\frac{u}{c}\mathbf{k}, \tag{40}$$

where E is the quantity used in (28).

The foregoing expressions show that in Σ', the interaction part of the electromagnetic momentum density consists of two terms: the product of (39) and (37) and that of (38) and (40). The integration of the momentum density is just a multiplication by the volume (36). Hence, using the abovementioned products and (29), one finds

$$p'_{x(\mathrm{elec})}=\frac{1}{4\pi c}8\gamma BE(1+u^2/c^2)=\gamma(1+u^2/c^2)p_{x(\mathrm{elec})}. \tag{41}$$

This interaction part of the electromagnetic momentum is a counterpart of the ''hidden momentum'' as seen in Σ'.

Let us turn to the mechanical part, namely to the ''hidden momentum.'' The interesting element is the fluid at the $Y=\pm 1$ segments of the solenoid. The $Y=-1$ side is treated first. At Σ, the four-velocity of the liquid is $(1,v/c,0,0)$ (here, as above, powers of v which are greater than 1 are ignored). Thus, using the Lorentz transformation (6), one finds that at Σ', this four-velocity takes the form

$$\nu'^{\mu}_{(y=-1)}=\gamma(1+uv/c^2,(u+v)/c,0,0). \tag{42}$$

Now one has to synchronize the time at Σ'. Assume that the time at Σ is $t=0$. Thus the four-vector of a point on the bottom left part is

$$X_L^{\mu}=(0,-1,-1,z) \tag{43}$$

and at the right end of the $Y=-1$ segment

$$X_R^{\mu}=(0,1,-1,z). \tag{44}$$

Applying the Lorentz transformation (6), one finds that at Σ', these points are

$$X_L'^{\mu}=(-\gamma u/c,-\gamma,-1,z), \tag{45}$$

$$X_R'^{\mu}=(\gamma u/c,\gamma,-1,z). \tag{46}$$

The time synchronization is done so that all events at Σ' are examined at $t'=0$. It follows that the four-vector (46) must be shifted by $\Delta t=-\gamma u/c^2$. Using (42), one finds that in Σ', the liquid's three-velocity is

$$v_x=\frac{u+v}{1+uv/c^2}. \tag{47}$$

Thus, the fluid element, which in Σ was at (46), is seen in Σ' at $t'=0$ at

$$X_R''^{\mu}=\left(0,\frac{1}{\gamma(1+uv/c^2)},-1,z\right). \tag{48}$$

A similar calculation yields for point (45) the $t'=0$ values

$$X_R''^{\mu}=\left(0,\frac{-1}{\gamma(1+uv/c^2)},-1,z\right). \tag{49}$$

This calculation shows that the charged liquid at the $Y=-1$ side undergoes a Lorentz contraction by a factor

$$a=\frac{1}{\gamma(1+uv/c^2)}. \tag{50}$$

On the other hand, the charge which is distributed uniformly on the insulating material of the pipes is motionless in Σ (and there screens the electric field of the charged fluid). The static charge undergoes the ordinary Lorentz contraction of $1/\gamma$. It means that in Σ', the complete screening does not hold any more and the net charge density per unit area at $Y=-1$ is

$$\rho'_{(\mathrm{area})}=\rho_{(\mathrm{area})}[\gamma(1+uv/c^2)-\gamma]=\rho_{(\mathrm{area})}\gamma uv/c^2, \tag{51}$$

where

$$\rho_{\mathrm{area}}=\rho sN \tag{52}$$

is the density of the positive charge in Σ. This outcome must be consistent with the solenoid's electric field (38), as seen in Σ'.

Indeed, in Σ', the nonvanishing electric field emanating from the $Y=-1$ side of the solenoid indicates that the charge density at this part is nonzero, too. The following calculation shows that the charge density obtained above is the precise quantity.

Taking the uniform electric field at the inner part of the moving solenoid (38), the value of the magnetic field (27) and relation (51) for the charge density at Σ', one obtains

$$\mathbf{E}'_{(\mathrm{sol})}=\frac{\gamma u}{c^2}4\pi N\rho sv\,\mathbf{j}=\frac{\gamma uv}{c^2}4\pi\rho_{(\mathrm{area})}\mathbf{j}=4\pi\rho'_{(\mathrm{area})}\mathbf{j}. \tag{53}$$

Thus one finds that (51) and (38) are consistent with the Maxwell equation $\operatorname{div}\mathbf{E}=4\pi\rho$. This is an example of the self-consistency of relativistic electrodynamics.

The nonzero charge density (53) is a relativistic effect which emphasizes the claims presented above. Although the mean charge density vanishes in Σ, the current j^{μ} is nonzero there. Hence a nonzero charge arises in Σ' and yields a nonzero three-force.

The mechanical ''hidden momentum'' is obtained from the pressure-related terms of the Lorentz transformation of the tensor (34) and from the corresponding tensor which pertains to the $Y=1$ part of the solenoid, where $-v$ replaces v of (34). Performing the calculations for the terms which are proportional to the pressure $\mathcal{P}$, one finds the required tensor component for $Y=-1$,

$$T'^{10}_{(\mathrm{press})}=\gamma^2\frac{v}{c}\left(1+\frac{u^2}{c^2}\right)\mathcal{P}_{Y=-1}. \tag{54}$$

For the $Y=1$ part of the solenoid, one replaces v by $-v$ in (54) and obtains an analogous expression.

As in the previous cases, the integration of each part of $Y=\pm 1$ reduces to a multiplication by the volume of the fluid (which contracts by a factor γ^{-1}). Thus one adds the contribution of the two sides and finds

$$p_{x(\mathrm{press})}=-4\gamma\left(1+\frac{u^2}{c^2}\right)v\Delta\mathcal{P}Ns/c^2=\gamma\left(1+\frac{u^2}{c^2}\right)p_{x(\mathrm{press})}, \tag{55}$$

where (35) is used. Hence, as in Σ, one finds that in Σ', too, the mechanical part (55) of the ''hidden momentum'' balances its electromagnetic counterpart (41).

IV. A PREVIOUS DISCUSSION OF LORENTZ TRANSFORMATION OF ''HIDDEN MOMENTUM''

The problem of Lorentz transformation of ''hidden momentum'' has been discussed recently in the literature.[2] Section III of Ref. 2 contains a general discussion of this problem, where the following Lorentz transformation formulas are postulated:

$$U_{\mathrm{elm}}=\gamma(U_0+\mathbf{v}\cdot\mathbf{P}_0), \tag{70}$$

$$\mathbf{P}_{\mathrm{elm}}=\gamma(\mathbf{P}_0+U_0\mathbf{v}/c^2), \tag{71}$$

$$U_{\mathrm{mech}}=\gamma(m_0c^2+\mathbf{v}\cdot\mathbf{P}_h), \tag{73}$$

$$\mathbf{P}_{\mathrm{mec}}=\gamma(\mathbf{P}_h+m_0\mathbf{v}). \tag{74}$$

Here U_0 and $\mathbf{P}_0$ are the system's rest frame electromagnetic energy and momentum, respectively. m_0 is the mechanical rest mass, $\mathbf{P}_h$ is the rest frame mechanical momentum, and U_{elm} and $\mathbf{P}_{\text{elm}}$ (U_{mec} and $\mathbf{P}_{\text{mec}}$) are the system's electromagnetic (mechanical) energy and momentum, respectively. (The numbering of the quoted equations is as in Ref. 2.)

The analysis carried out in this work clearly proves that these equations are incorrect. The capacitor of Sec. III B can be used as a counterexample to (70)–(74). Evidently, in the rest frame of the capacitor there is no electromagnetic momentum and no mechanical (''hidden'') one. Thus one substitutes $\mathbf{P}_0 = \mathbf{P}_h = 0$ in (70)–(74) and examines the outcome in the frame Σ'. Equation (24) clearly shows that in Σ' there is no electromagnetic momentum, contrary to (71) which is $\mathbf{P}_{\text{elm}} = \gamma U_0 \mathbf{v}/c^2$ (note that the electromagnetic energy $U_0 \neq 0$). Similarly, as (24) shows, in Σ', the electromagnetic energy *reduces* by the factor γ, unlike (70).

Moreover, the mechanical quantities do not transform like (73) and (74). This is proved in detail in Refs. 10 and 11. Only the *sum* of the mechanical and electromagnetic energy and momentum false four-vectors transforms as a true four-vector.

A system containing ''hidden momentum'' is discussed in Sec. III C. The results found there can also be used for disproving (70)–(74). Thus, in Σ', the ''hidden momentum'' part of the mechanical momentum is given in (55). As seen, it is $\gamma(1+u^2/c^2)$ times the mechanical ''hidden momentum'' of the rest frame. This outcome negates (74). Moreover, (41) above shows that the interaction part of the electromagnetic momentum is increased by the same factor $\gamma(1+u^2/c^2)$, contrary to (71).

Obviously, the postulated equations (70)–(74) are inconsistent with the null four-divergence condition of (3). This point casts a new light on the significance of this condition in the case of a relativistic treatment of energy and momentum of classical systems.

V. CONCLUDING REMARKS

The usefulness of the notion of false four-vectors is explained. These objects are associated with spatial integrals of energy and momentum density, as given by an energy–momentum tensor which does not satisfy the null four-divergence (3). Following a discussion of Landau and Lifshitz, it is proved here that a null four-divergence is a sufficient condition for having a true energy—momentum four-vector whose components are obtained from spatial integrals of corresponding densities. Examples illustrating this subject are presented in Sec. III. It is shown there that in the case of a free electromagnetic wave, whose energy–momentum tensor has a null four-divergence, the integrals of the appropriate densities are related to a true four-vector. On the other hand, in examples of systems of fields and charges (or currents), integrals of energy and momentum densities of electromagnetic fields are related to a false four-vector. This property also holds for the mechanical sector of the system. Only the sum of the mechanical and electromagnetic false four-vectors is a true four-vector.

Note added (Received 6 March 2000): In his Response to this work,[15] Hnizdo claims that the problem discussed above can be treated in two alternative (and mutually contradictory) ways. One method of calculation treats quantities in the standard way and shows that only the overall energy and momentum of the system transform like components of a four-vector, whereas energy–momentum of fields as well as those of matter, transform like a false four-vector. The other method *postulates* that the electromagnetic and the mechanical parts do transform like four-vectors. The second approach is called the covariant method. The domain of validity of each method is the main topic of this note. (For a discussion of the notion of the domain of validity of a theory, the reader is referred to Rohrlich's book.[16])

The starting point of this note is the validity of Maxwellian electrodynamics for a system whose charge density is bounded. This matter is denoted below by the term standard Maxwellian basis. Moreover, even in cases which do not belong to the domain of validity of classical physics, this theory is assumed to be *mathematically* correct. In such cases, only mathematical aspects of the theory are considered.

The phenomenon of charge quantization motivates the introduction of particles carrying a quantized quantity of charge into the theory. Two kinds of charged particles are discussed here. Particles of the first kind are tiny objects whose volume is small (say, a sphere whose $r > 0$). The other kind is an elementary classical point charge. Particles of these kinds are called hereinafter extended charges and point charges, respectively. Four cases are discussed below.

A. A single extended charge

This system falls within the domain of validity of the standard Maxwellian basis. The particle is stabilized by means of a Poincaré force. The energy—momentum of the entire system transforms like a four-vector and the electromagnetic and the mechanical parts of the energy–momentum transform like false four-vectors. (References and discussions of the Poincaré force can be found on the appropriate pages of Refs. 16 and 17.)

B. A single point charge

In classical physics, an elementary particle is pointlike (see Ref. 1, pp. 43 and 44). Hence, in this case, no Poincaré forces can exist. If one applies the laws of the standard maxwellian basis to a point charge, then very serious problems arise. Two of these problems are the infinite energy and the 4/3 factor obtained for the momentum components, if a Lorentz transformation is applied to a motionless charge. The covariant method solves the latter problem. If this approach is augmented by a mass renormalization procedure which removes the infinite energy of the fields, then results agree with those obtained from a different analysis.[18] Here, the interaction of fields of a single particle is removed from the system's energy–momentum tensor.

C. A system which consists of more than one extended charge

This system, like that of case *A*, is explained perfectly by the standard Maxwellian basis, because its charge density is bounded.

D. A system which consists of more than one point charge

A self-consistent solution of this problem can be achieved if its two-particle interaction is the same as the limit of an analogous system of extended charges whose radius tends to zero. Thus, as in case C, the energy of the interaction fields transforms as shown by the standard Maxwellian basis. An attempt to do it differently is inconsistent with special relativity. Indeed, the kinematics of particles entails a unique Lorentz transformation of their position and velocity. Hence, the mechanical energy and momentum of the system transform like those of an analogous system belonging to case *C*. It follows that there is no room for ambiguity of the laws of transformation of mechanical energy and momentum. For this reason, the postulate used for (73) and (74) of Ref. 2 is wrong. Illustrations of this matter are given in Refs. 10 and 11. In particular, see Ref. 11, Sec. II B, pp. 1030–1032.

The foregoing discussion shows that the covariant method applies to case *B* and to the *self-interaction* of particles belonging to case *D*. (A self-consistent presentation of a theory where self-interactions of point charges are removed from the energy–momentum tensor can be found in Ref. 18.) Unlike (73) and (74) of Ref. 2, the mechanical part does not transform covariantly. In his Response,[15] Hnizdo does not even try to settle this contradiction.

Now, the overall energy–momentum (which is the sum of the mechanical and the electromagnetic parts) transforms covariantly. Hence, since the mechanical part does not transform covariantly, the electromagnetic part must follow suit, in order to compensate for noncovariant effects of the mechanical part.

In Ref. 4 of his Response, Hnizdo discusses results obtained from an analysis of infinitely long devices. His claims do not affect the validity of this work, because, in special relativity, energy density is defined locally and one may examine appropriate finite volumes, which are parts of the entire device. This approach is analogous to a standard textbook discussion of a parallel-plate capacitor. This issue can be explained briefly as follows. ∞ is not a number but a limit of a sequence. Let the device be enclosed within a cube whose linear size is L. The integration of field quantities is carried out within another cube, which is concentric with the former, and whose linear size is $10^{20}L$. Now consider a sequence of such devices where $L \rightarrow \infty$. Everything is OK for every element of the sequence and the results are valid.

[a)]Electronic mail: eli@tauphy.tau.ac.il

[1]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975), pp. 77–80.
[2]V. Hnizdo, ''Hidden momentum and the electromagnetic mass of a charge and current carrying body,'' Am. J. Phys. **65**, 55–65 (1997).
[3]See Ref. 1, pp. 71–73.
[4]See Ref. 1, p. 78.
[5]See Ref. 1, pp. 80–83.
[6]See Ref. 1, p. 81.
[7]See Ref. 1, pp. 110–117.
[8]See Ref. 1, p. 28.
[9]See Ref. 1, p. 62.
[10]W. Rindler and J. Denur, ''A simple relativistic paradox about electrostatic momentum,'' Am. J. Phys. **56**, 795 (1988).
[11]E. Comay, ''Exposing 'hidden momentum,' '' Am. J. Phys. **64**, 1028 (1996).
[12]S. Coleman and J. H. Van Vleck, ''Origin of 'Hidden Momentum Forces on Magnets,'' Phys. Rev. **171** (5), 1370–1375 (1968), and references therein.
[13]L. Vaidman, ''Torque and force on a magnetic dipole,'' Am. J. Phys. **58**, 978–983 (1990), case III.
[14]See Ref. 1, p. 85.
[15]V. Hnizdo, ''Response to 'Lorentz transformation of a system carrying 'Hidden Momentum,' ' '' Am. J. Phys. **68**, 1014 (2000).
[16]F. Rohrlich, *Classical Charged Particles* (Addison–Wesley, Reading, MA, 1965), pp. 3–6.
[17]E. Comay, ''Lorentz Transformation of Electromagnetic Systems and the 4/3 Problem,'' Z. Naturforsch., A: Phys. Sci. **46**, 377–383 (1991).
[18]E. Comay, ''Decomposition of Electromagnetic Fields into Radiation and Bound Components,'' Am. J. Phys. **65**, 862–867 (1997).

Response to "Lorentz transformation of a system carrying 'Hidden Momentum,'" by E. Comay [Am. J. Phys. 68, 1007 (2000)]

V. Hnizdo
National Institute for Occupational Safety and Health, 1095 Willowdale Road, Morgantown, West Virginia 26505

(Received 31 January 2000; accepted 15 March 2000)

We respond to Comay's criticism of the use of covariant definitions of the electromagnetic and mechanical energy–momenta in an analysis of the role of hidden momentum in the total energy–momentum four vector of a macroscopic body.

In his paper,[1] Comay calls the use in Ref. 2 of covariant definitions of the "electromagnetic" and "mechanical" contributions to the total energy and momentum of a macroscopic body "mistakes." He illustrates in detail, using the examples of a free electromagnetic wave, a charged capacitor, and a current-carrying solenoid inside a charged capacitor, the well-known fact that the energy and momentum of the electromagnetic field in a system (i.e., the "electromagnetic" energy–momentum) form a covariant four vector only when the four divergence of the field's energy–momentum tensor vanishes. Similarly, he shows that the hidden mechanical momentum in his third example, obtained as the integral of the simultaneous values of the momentum-density component of the energy–momentum tensor of the charged fluid that is the current carrier in the system, does not transform as the momentum part of an energy–momentum four vector. Here again, the reason for this is that the four divergence of the fluid's energy–momentum tensor can be shown not to vanish. Only the total energy–momentum of the system, i.e., the sum of the electromagnetic and mechanical contributions, is a covariant four vector, as the four divergence of the total energy–momentum tensor of a closed system vanishes.[3]

This is shown also in Ref. 2, where detailed calculations on the examples of finite exactly solvable systems[4] illustrate the fact that only the total energy–momentum is a covariant four vector when the standard definitions are used for the electromagnetic and mechanical contributions. And it is shown in Ref. 5, without relying on any specific example, how the noncovariant "electromagnetic" and "mechanical" energy–momenta calculated using the standard energy–momentum tensors combine to form a covariant four vector of the total energy–momentum of a macroscopic body that carries general stationary macroscopic charge and/or current distributions.

The employment of definitions that separately impose the relativistic four-vector covariance on the electromagnetic and nonelectromagnetic ("mechanical") energy–momenta has been pioneered by Rohrlich[6] as a procedure that needs no explicit consideration of nonelectromagnetic forces to deal with the problem of the noncovariance of the energy–momentum of the electron in classical electron theory. The use of this procedure has been criticized on numerous occasions by proponents of the standard definitions, as these are sufficient to produce a covariant total energy–momentum when the contribution of nonelectromagnetic forces, necessary for the stability of an extended particle, to the energy–momentum of the classical electron is included. (That has been shown already by Poincaré,[7] and the nonelectromagnetic stresses needed for the stability of an extended charged particle are usually called Poincaré stresses.) Comay's criticism is a relapse into yet another round of that old debate,[8] in regard to which it has now been recognized by several authors[9] that the point here is not that only one of the procedures of Rohrlich's covariant definitions and Poincaré stresses is a "correct" one while the other is "wrong," but that either procedure can be used for the purpose of constructing a covariant *total* energy–momentum of a system that has electromagnetic and nonelectromagnetic components. The recent third edition of Jackson's classic text on classical electrodynamics[10] has a thorough discussion of the problem of the electromagnetic mass, Poincaré stresses, and covariant definitions without expressing any preference for one of the two procedures over the other.

We reiterate here the point emphasized in Refs. 2 and 5, namely that the covariant definitions have a formal character akin to the procedure of the renormalization of mass in quantum electrodynamics, which is carried out in a covariant fashion separately from any nonelectromagnetic contribution to the rest mass of a charged particle. (In fact, it can be argued that a mass renormalization with a negative nonelectromagnetic mass is implied also in classical electrodynamics whenever a charged body is assigned a rest mass that is smaller than the electromagnetic mass due to the body's charge distribution.) In principle, the electromagnetic energy–momentum arising from a macroscopic distribution of charge and current is measurable separately from the body's "mechanical" energy–momentum (unlike in the electron or any other "elementary" particle), and the standard electromagnetic and mechanical energy–momenta would agree observationally with the covariantly defined quantities only in one inertial frame of reference, namely the reference frame in which the standard and covariant definitions coincide.

The purpose of a covariant-definition procedure for a macroscopic system is that of the construction of a covariant *total* energy–momentum, and the separately covariant "electromagnetic" and "mechanical" energy–momenta obtained in such a procedure serve only that purpose. The covariant definitions of the electromagnetic and mechanical energy–momenta were used in Ref. 2 not because they are the only "correct" definitions of such quantities for the systems in question, but to show how the covariant-definition procedure, along with the Poincaré stresses procedure of the standard definitions, would consistently take into account the existence of hidden mechanical momentum.

[1]E. Comay, "Lorentz transformation of a system carrying 'Hidden Momentum,'" Am. J. Phys. **68**, 1007 (2000).
[2]V. Hnizdo, "Hidden momentum and the electromagnetic mass of a charge

and current carrying body,'' Am. J. Phys. **65**, 55–65 (1997).

[3]Comay cites in detail from the proof of this statement in L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975). There are several such proofs in the literature, the oldest going back some 80 years: H. Weyl, *Space-Time-Matter* (Dover, New York, 1950), first American printing of the 4th edition of 1922, Sec. 33; W. Pauli, *Theory of Relativity* (Pergamon, London, 1958), Sec. 21; C. Møller, *The Theory of Relativity* (Clarendon, Oxford, 1972), 2nd ed., Sec. 6.2; C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Sec. 5.8, case (c).

[4]Comay's examples are systems of infinite extension, and also of infinite energy and, except in his second example, infinite momentum. While the use of such systems simplifies the requisite integrations, infinite systems are, strictly speaking, unphysical; moreover, the general theorem that guarantees that the total momentum of a stationary macroscopic system vanishes [see, e.g., L. Vaidman, ''Torque and force on a magnetic dipole,'' Am. J. Phys. **58**, 978–983 (1990); V. Hnizdo, ''Hidden mechanical momentum and the field momentum in stationary electromagnetic and gravitational systems,'' *ibid.* **65**, 515–518 (1997)] cannot be applied to an infinite system. Presumably, the assumption here is that when the systems Comay considers are finite, the relative contribution of the fringing fields (and of the wave-packet ''tails'' in the case of a free electromagnetic wave) to the quantities of interest can be shown to be arbitrarily small when suitable dimensions of the systems are sufficiently large.

[5]V. Hnizdo, ''Covariance of the total energy–momentum four vector of a charge and current carrying macroscopic body,'' Am. J. Phys. **66**, 414–418 (1998).

[6]F. Rohrlich, ''Self-energy and stability of the classical electron,'' Am. J. Phys. **28**, 639–643 (1960); ''Electromagnetic momentum, energy, and mass,'' *ibid.* **38**, 1310–1316 (1970); *Classical Charged Particles* (Addison–Wesley, Reading, MA, 1965 and 1990).

[7]There is an English, modernized presentation of Poincaré's 1906 paper on the electron by H. M. Schwartz, ''Poincaré's Rendincoti paper on relativity. I,'' Am. J. Phys. **39**, 1287–1294 (1971); ''II,'' **40**, 862–872 (1972); ''III,'' **40**, 1282–1287 (1972).

[8]Reference 5 gives several references to the debate, both in the context of classical electron theory and the Trouton–Noble experiment (on the latter, see S. A. Teukolsky, Ref. 9).

[9]D. J. Griffiths and R. E. Owen, ''Mass renormalization in classical electrodynamics,'' Am. J. Phys. **51**, 1120–1126 (1983); S. A. Teukolsky, ''The explanation of the Trouton–Noble experiment revisited,'' *ibid.* **64**, 1104–1109 (1996); F. Rohrlich, ''The dynamics of a charged sphere and the electron,'' *ibid.* **65**, 1051–1056 (1997).

[10]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1999), 3rd ed., Secs. 16.4–16.6.