Lagrangian Methods for High Speed Motion. By C. G. Darwin.

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1. In the later developments of Bohr's* spectrum theory, it is necessary to calculate the orbits of electrons moving with such high velocities that there is a sensible increase of mass. The selection of the orbits permitted by the quantum theory almost necessitates the treatment of such problems by Hamiltonian methods. Working on these lines Sommerfeld† and others have calculated with a very high degree of success those spectra which involve the motion of a single electron. But the application of the Hamiltonian function involves a knowledge of the momentum corresponding to any generalized coordinate, and in the formulation of most problems the momenta are not known *a priori* but must be calculated from the corresponding velocities. In other words the formation of the Hamiltonian function must in general be preceded by that of the Lagrangian. An exception occurs in precisely the problems referred to above; for, the electromagnetic theory furnishes directly values for the momentum and kinetic energy of a moving electron in terms of its velocity, and the velocity can be eliminated between them so as to obtain the Hamiltonian function. But in even slightly more complicated cases this simple relation is destroyed—thus the problem of a single electron in a constant magnetic field can only be solved by introducing the artificial conception of rotating axes—and in general it will be necessary to follow the direct course of finding the Lagrangian function in terms of the generalized velocities, and then deducing from it the momenta and the Hamiltonian function in the usual way.

If more than one particle is in motion another difficulty enters. For the interaction of two moving particles depends on a set of retarded potentials and the effect of the retardation is readily seen to be of the same order as the increase of mass with velocity. The calculation of the retardation can only be carried out by expansion and so the results are only approximate. This is not surprising since the methods of conservative dynamics cannot apply to such effects as the dissipation of energy by radiation, effects inevitably required by the electromagnetic theory, though they do not occur in actuality. We can also see from the fact that these radiation terms are of the order of the inverse cube of the velocity of light, that it will be useless to expand beyond the inverse square.

* N. Bohr, *Kgl. Dan. Wet. Selsk.*, 1918.
† A. Sommerfeld, *Ann. Phys.*, vol. 51, p. 1, 1916.

2. We first consider the motion of a single electron in an arbitrary electric and magnetic field varying in any manner with the time and position. If m is the mass for low velocities, the momentum is known to be mv/β, where $\beta = \sqrt{1 - v^2/c^2}$. Starting from this we have quasi-Newtonian equations of motion of the type

$$\frac{d}{dt}\left\{\frac{m}{\beta}\cdot\dot{x}\right\} = F_x \qquad \text{......(2·1).}$$

The force F_x is given from the field **E**, **H** as the vector $e\mathbf{E} + \frac{e}{C}[\mathbf{v}, \mathbf{H}]$, where **v** is the velocity vector of the particle's motion. **E** and **H** can be expressed in terms of the scalar and vector potentials in the form $\mathbf{E} = -\operatorname{grad}\phi - \frac{1}{C}\frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{H} = \operatorname{curl}\mathbf{A}$.

Then if $\mathbf{r}_1$ is the vector x, y, z we have as the vector equation of motion

$$\frac{d}{dt}\left\{\frac{m_1}{\beta_1}\dot{\mathbf{r}}_1\right\} = -e_1\operatorname{grad}\phi - \frac{e_1}{C}\frac{\partial \mathbf{A}}{\partial t} + \frac{e_1}{C}[\dot{\mathbf{r}}_1, \operatorname{curl}\mathbf{A}] \quad \text{...(2·2),}$$

where $\beta_1 = \sqrt{1 - \dot{\mathbf{r}}_1^2/C^2}$.

Let q be any one of three generalized coordinates representing the position of the particle. Take the scalar product of (2·2) by $\frac{\partial \mathbf{r}_1}{\partial q}$. Then since $\frac{\partial \mathbf{r}_1}{\partial q} = \frac{\partial \dot{\mathbf{r}}_1}{\partial \dot{q}}$, we have

$$\left(\frac{\partial \mathbf{r}_1}{\partial q}, \frac{d}{dt}\left\{\frac{m_1}{\beta_1}, \dot{\mathbf{r}}_1\right\}\right) = \frac{d}{dt}\left\{\frac{m_1}{\beta_1}\left(\frac{\partial \mathbf{r}_1}{\partial q}, \dot{\mathbf{r}}_1\right)\right\} - \frac{m_1}{\beta_1}\left(\frac{\partial \dot{\mathbf{r}}_1}{\partial q}, \dot{\mathbf{r}}_1\right)$$

$$= \frac{d}{dt}\left\{\frac{m_1}{\beta_1}\frac{\partial}{\partial \dot{q}}(\tfrac{1}{2}\dot{\mathbf{r}}_1^2)\right\} - \frac{m_1}{\beta_1}\frac{\partial}{\partial q}(\tfrac{1}{2}\dot{\mathbf{r}}_1^2)$$

$$= \mathfrak{D}_q(-m_1 C^2 \beta_1),$$

where $\mathfrak{D}_q = \frac{d}{dt}\frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q}$ the Lagrangian operator.

Again $$-e_1\left(\frac{\partial \mathbf{r}}{\partial q}, \operatorname{grad}\phi\right) = -e_1\frac{\partial \phi}{\partial q} = e_1\mathfrak{D}\phi.$$

The remainder can be reduced to

$$\frac{e_1}{C}\left(\dot{\mathbf{r}}_1, \frac{\partial \mathbf{A}}{\partial q}\right) - \frac{e_1}{C}\left(\frac{\partial \mathbf{r}_1}{\partial q}, \frac{d\mathbf{A}}{dt}\right) \qquad \text{......(2·3),}$$

where $$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \mathbf{A}}{\partial x}\dot{x} + \frac{\partial \mathbf{A}}{\partial y}\dot{y} + \frac{\partial \mathbf{A}}{\partial z}\dot{z}$$

and so is the total change of $\mathbf{A}$ at the moving particle. (2·3) can be reduced to $-\frac{e_1}{C}\mathfrak{D}_q(\dot{\mathbf{r}}_1, \mathbf{A})$.

Thus the whole equation of motion can be derived from a Lagrangian function

$$L = -m_1 C^2 \beta_1 - e_1\phi + \frac{e_1}{C}(\dot{\mathbf{r}}_1, \mathbf{A}) \qquad \text{......(2·4)}.$$

This is valid for any fields of force including explicit dependence of ϕ and $\mathbf{A}$ on the time. The first term in L, which reduces to the kinetic energy for low velocities, differs from it in general. It is very closely connected with the "world line" of the particle.

3. To treat of the case where several moving particles interact we shall start by supposing that there is a second particle present undergoing a constrained motion so that its coordinates are imagined to be known functions of the time. The same will then be true of the potentials it generates. The motion of e_1 will then be governed by (2·4) if ϕ and $\mathbf{A}$ are expressed in terms of the motion of e_2. These potentials are given by

$$\phi = \frac{e_2}{r + (\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)/C}, \qquad \mathbf{A} = \frac{e_2}{C}\frac{\dot{\mathbf{r}}_2}{r + (\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)/C} \text{...(3·1)}.$$

In these expressions $r^2 = (\mathbf{r}_2 - \mathbf{r}_1)^2$ and the values are to be *retarded* values. If the time of retardation be calculated and the result substituted in (3·1) we obtain

$$\phi = \frac{e_2}{r} + \frac{e_2}{2C^2}\left\{\frac{\dot{\mathbf{r}}_2^{\,2} + (\ddot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r} - \frac{(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)^2}{r^3}\right\}, \ \mathbf{A} = \frac{e_2}{C}\frac{\dot{\mathbf{r}}_2}{r} \text{...(3·2)},$$

where now $\mathbf{r}_1$, $\mathbf{r}_2$ refer to the same instant of time. ϕ is an approximation valid to C^{-2}, but the value of $\mathbf{A}$ has only been found to the degree C^{-1} on account of the further factor C^{-1} in (2·4) which is to multiply it. Then substituting in (2·4) we obtain

$$L = -m_1 C^2\beta_1 - \frac{e_1 e_2}{r} - \frac{e_1 e_2}{2C^2}\left\{\frac{\dot{\mathbf{r}}_2^{\,2} + (\ddot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1) - 2(\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2)}{r} - \frac{(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)^2}{r^3}\right\} \qquad \text{......(3·3)}.$$

The equations of motion are unaffected by adding to L the expression $-m_2 C^2\beta_2 + \frac{d}{dt}\frac{e_1 e_2}{2C^2}\frac{(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r}$. The first is a pure function of the time and so contributes no terms to the equations of motion. The second contributes nothing because for any function f we have

$$\mathfrak{D}\left(\frac{d}{dt}f(q_1, q_2, \ldots, t)\right) \equiv 0.$$

The new form of L then reduces to

$$L = -m_1 C^2\beta_1 - m_2 C^2\beta_2 - \frac{e_1 e_2}{r} + \frac{e_1 e_2}{2C^2}\left\{\frac{(\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2)}{r} + \frac{(\dot{\mathbf{r}}_1, \mathbf{r}_2 - \mathbf{r}_1)(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r^3}\right\} \qquad \text{......(3·4)}.$$

From the complete symmetry of this form the roles of e_1 and e_2 may be interchanged. Further from the covariance of the operator $\mathfrak{D}$ for point transformations, both may be included in the dynamical system, so that if q is any generalized coordinate involving both $\mathbf{r}_1$ and $\mathbf{r}_2$, the equations of motion will be of the form $\mathfrak{D}_q L = 0$.

For the sake of consistency, as the last term in (3·4) is only an approximation valid to C^{-2}, the first two should be expanded only to this power. The first term will give

$$-m_1 C^2 + \tfrac{1}{2}m_1\dot{\mathbf{r}}_1^{\,2} + \frac{1}{8C^2}m_1\dot{\mathbf{r}}_1^{\,4}.$$

Generalizing our result to the case of any number of particles in any external field we have

$$L = \Sigma\tfrac{1}{2}m_1\dot{\mathbf{r}}_1^{\,2} + \Sigma\frac{1}{8C^2}m_1\dot{\mathbf{r}}_1^{\,4} - \Sigma e_1\phi + \Sigma\frac{e_1}{C}(\dot{\mathbf{r}}_1\mathbf{A}) - \Sigma\Sigma\frac{e_1 e_2}{r_{12}} + \Sigma\Sigma\frac{e_1 e_2}{2C^2}\left\{\frac{(\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2)}{r_{12}} + \frac{(\dot{\mathbf{r}}_1, \mathbf{r}_2 - \mathbf{r}_1)(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r_{12}^{\,3}}\right\} \text{...(3·5)}.$$

The double summations are taken counting each pair once only.

4. The transition to the Hamiltonian now follows the ordinary rules. We find momenta $p = \frac{\partial L}{\partial \dot{q}}$ and solve for the $\dot{q}$'s in terms of the p's. This can be done in spite of the cubic form of the equations in the q's by use of the approximation in powers of C. The Hamiltonian function will then be $H = \Sigma p\dot{q} - L$ and the equations of motion will be the canonical equations $\dot{q} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q}$. If $\mathbf{p}_1$ be the momentum corresponding to $\mathbf{r}_1$, the Hamiltonian in these coordinates will be

$$H = \Sigma\frac{\mathbf{p}_1^{\,2}}{2m_1} - \Sigma\Sigma\frac{\mathbf{p}_1^{\,4}}{8C^2 m_1^{\,3}} + \Sigma e_1\phi - \Sigma\frac{e_1}{Cm_1}(\mathbf{p}_1, \mathbf{A}) + \Sigma\Sigma\frac{e_1 e_2}{r_{12}} - \Sigma\Sigma\frac{e_1 e_2}{2C^2 m_1 m_2}\left\{\frac{(\mathbf{p}_1, \mathbf{p}_2)}{r_{12}} + \frac{(\mathbf{p}_1, \mathbf{r}_2 - \mathbf{r}_1)(\mathbf{p}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r_{12}^{\,3}}\right\} \text{...(4·1)}.$$

All the applications of general dynamics, such as the Hamilton Jacobi partial differential equation, follow from this. As in ordinary dynamics, many problems can be conveniently solved in the Lagrangian form. The solution will usually depend on finding integrals corresponding to coordinates which do not occur explicitly in L and if ϕ and $\mathbf{A}$ do not involve the time explicitly there is also the energy integral. This has the form

$$\Sigma \tfrac{1}{2} m \dot{\mathbf{r}}_1{}^2 + \Sigma \tfrac{3}{8} \frac{m_1}{C^2} \dot{\mathbf{r}}_1{}^4 + \Sigma e_1 \phi + \Sigma\Sigma \frac{e_1 e_2}{r_{12}}$$

$$+ \Sigma\Sigma \frac{e_1 e_2}{2C^2} \left\{ \frac{(\dot{\mathbf{r}}_1 \dot{\mathbf{r}}_2)}{r} + \frac{(\dot{\mathbf{r}}_1, \mathbf{r}_2 - \mathbf{r}_1)(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r^3} \right\} = \text{const.} \ldots (4 \cdot 2).$$

This completes the development of the method. Its direct applications are naturally somewhat limited, since, even with the large order terms only, there are comparatively few problems that are soluble. A problem of some interest that can be solved completely is the motion of two attracting particles, where their masses have a finite ratio*.

* A discussion of this problem by the present writer will be found in *Phil. Mag.*, Vol. 39, p. 537 (1920), together with a somewhat fuller account of the general theory.