LI. *The Dynamical Motions of Charged Particles.* By C. G. Darwin, *M.A., Fellow and Lecturer of Christ's College, Cambridge* *.

1. THE work of Bohr † and of Sommerfeld ‡ and others has given a new importance to problems connected with the orbits of an electron—in particular, to the effect on the orbits of the increase of mass with velocity. The first object of the present paper is to reduce the problem of the motion of any number of charged particles, moving at high velocities in any electric and magnetic field, to a Lagrangian form, so that all the well-known theorems of general dynamics may be made applicable. These principles are then applied to an example, the problem of two bodies; and, finally, as a matter of some theoretical interest (though it was never to be expected that the effect would be perceptible in practice), these results are applied, according to Sommerfeld's quantum principle, to calculate the small influence on the doublets of the hydrogen spectrum, due to the finiteness of mass of the nucleus of the atom.

The application of the methods of general dynamics to such problems is by no means new. Thus Sommerfeld makes much use of the canonical form in the solution of the orbits of a single electron, and much of Bohr's § later work is carried out with the Hamilton-Jacobi partial differential equation. Now the direct application to such problems of the canonical equations of motion implies a knowledge of the momenta corresponding to the various generalized coordinates, whereas in the formulation of any problem it is the velocities which are known and not the momenta. An exception occurs in the case of a single particle in a fixed electric field. Here the linear momentum is known to be $mv/\sqrt{(1-v^2/C^2)}$ ‖, and the momentum corresponding to any other coordinate can be deduced by elementary methods. But even for a single electron a magnetic field upsets this rule, and in the case of several free electrons it is quite impossible to obtain the momenta *à priori*. In other words, for a general method of formulation, the Lagrangian must be found first, before it is possible to proceed to the Hamiltonian, and the Lagrangian, of course,

* Communicated by the Author.

† N. Bohr, Phil. Mag. vol. xxvi. pp. 1, 476, 857 (1913), vol. xxvii. p. 506 (1914), vol. xxix. p. 332 (1915), vol. xxx. p. 394 (1915).

‡ A. Sommerfeld, *Ann. d. Phys.* vol. li. p. 1 (1916).

§ N. Bohr, Kgl. Dan. Vtd. Selsk. 1918.

‖ v the velocity, C the velocity of light, m the mass at low velocities.

will not have the simple connexion with the kinetic and potential energy that it has in ordinary dynamics.

It appears to me most probable that a great part of the theorems of general dynamics here given are already known, for the work of Sommerfeld and the later work of Bohr would be naturally based on them. But in the cases discussed by these writers the Hamiltonian form can be very quickly derived from first principles, and they make no mention of any general method of formulation, so that it seemed to me that it might be worth while to exhibit such a method. In developing this we can start either from Least Action or from modified Newtonian equations of motion. Least Action was shown by Maxwell to be applicable to the æther, and we should therefore only require to prove that the electric and magnetic forces in free space could be made ignorable by the addition of suitable terms for the particles. But this method involves distinctly more advanced dynamical principles, and so, in spite of its superior elegance, I have preferred to proceed by starting from the equations of motion, and have followed methods similar to those by which Lagrange's equations are introduced in dynamical text-books. In this way the problem is kept throughout as a problem of particles, and I hope it will be thereby made more accessible to those unfamiliar with the later developments of dynamics.

2. The problem is really one of relativistic dynamics, but no direct use will be made of the relativity transformations. If the mass of each particle is made the proper function of its velocity and if the electromagnetic equations are used in Lorentz's form *, then the motions described will be invariant for such transformations, and there is no need to go beyond a set of axes fixed in space and a fixed time-scale. This saves us from rather complicated considerations about relative velocity and acceleration.

When several particles are free to move, the difficulty of the problem lies in the fact that the force exerted by one of them on another depends on its position and motion at a certain previous time. In other words, we have to work with *retarded* potentials, and it will be seen that the effect of the retardation is of an order that is not negligible. It can only be calculated by approximation, and so it will be necessary to limit ourselves to motions where the velocities of the

* For the general principles of electromagnetic theory here used reference may be made to H. A. Lorentz, 'The Theory of Electrons,' but I use ordinary electrostatic units and not the Heaviside type.

particles are small, though not negligible, fractions of C, the velocity of light—that is, we shall expand in inverse powers of C. That it should be necessary to approximate is not surprising, as it is well known that according to the classical electromagnetic theory an accelerating electron will radiate, and the consequent dissipation of energy cannot possibly be represented by a Lagrangian form. The radiation of a single electron gives a reactive force on it of amount $\frac{2}{3}\frac{e^2}{C^3}\frac{d^2\mathbf{v}}{dt^2}$, where $\mathbf{v}$ is the velocity vector. Hence we must not expect to be able to find a Lagrangian accurate beyond the terms in C^{-2}. Though Sommerfeld's orbits are worked out without approximation, the neglect of the radiation terms implies that they are really only valid to this degree—that is, if they are regarded as based on the classical electromagnetic theory.

Now this raises quite unanswered questions of fundamental physics, for there can be no doubt that radiation does not really work in that way at all. But we have no right to claim that the equations with these radiation terms omitted will truly describe the motion simply because the radiation does not in fact occur; for quite apart from radiation there is something wrong with them. This is shown by the spiral orbits * which an electron should describe about a heavy nucleus, when its angular momentum is below a certain value. These orbits involve an ultimate coalescence of the electron with the nucleus, and if the theory were right they should be of fairly frequent occurrence, because, whatever the initial line of motion of the electron, a low angular momentum can always be attained by a sufficiently low initial velocity. This result follows whether the accurate or the approximate formula is used for the variation of mass. As there can be little doubt that coalescence does not in fact occur, it is necessary to invoke some modification of the laws of motion to prevent it, and the quantum naturally suggests itself. The question of these orbits in relation with the quantum has been discussed by Sommerfeld † to a certain extent, though the difficulties are not altogether removed and as in most other applications of the quantum the actual physical motion is quite incomprehensible. But, however that may be, and whatever assumption is made, the type of orbit must be quite altered and the validity of our present methods destroyed. So it is safer to claim validity for our

* C. G. Darwin, Phil. Mag. vol. xxv. p. 201 (1913).

† *Loc. cit.*

work only in cases where the velocities are and remain fairly small fractions of the velocity of light. In this way the spiral orbits will be excluded and the approximation to terms in C^{-2} will represent the facts as closely as is required. This limits our method to the problems of spectroscopy, and cuts out such interesting questions as the collisions of high-speed β particles.

3. The variability of mass of an electron is usually deduced from considerations of electromagnetic momentum. It is found that the linear momentum is $\frac{mv}{\beta}$, where m is the mass for low velocities, v is the constant velocity, and

$$\beta=\sqrt{\left(1-\frac{v^2}{C^2}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By well-known arguments (which, however, cannot quite escape criticism) there follow the equations of motion of the type

$$\frac{d}{dt}\left\{\frac{m}{\beta}\dot{x}\right\}=F_x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where F_x is the total force on the particle in the x direction. The three equations of the type (2) form our starting-point.

The variability of mass is often expressed by considering the quantities m/β and m/β^3 as transverse and longitudinal mass respectively, but these expressions are in fact deduced from (2), and it is useless to put down the more complicated equations in terms of them and then retrace the steps of the argument back to (2). As long as the equations of motion are expressed in terms of rate of change of momentum, instead of mass acceleration, there is no need for the conception of longitudinal mass.

First consider the problem of a single particle of charge e_1 and mass m_1 in any field of electric and magnetic force, variable in time and place. Making use of the vector notation, let $\mathbf{r}_1(=x, y, z)$ be the position of the particle. Let $\mathbf{E}$ and $\mathbf{H}$ be the electric and magnetic forces, ϕ and $\mathbf{A}$ the scalar and vector potentials from which they are derived. Then

$$\mathbf{E}=-\Delta\phi-\frac{1}{C}\frac{\partial\mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{H}=\text{curl}\,\mathbf{A}, \quad . \quad . \quad (3)$$

and (2) becomes

$$\frac{d}{dt}\left\{\frac{m_1}{\beta_1}\dot{\mathbf{r}}_1\right\}=-e_1\Delta\phi-\frac{e_1}{C}\frac{\partial\mathbf{A}}{\partial t}+\frac{e_1}{C}[\dot{\mathbf{r}}_1, \text{curl}\,\mathbf{A}], \quad . \quad (4)$$

where $\beta_1=\sqrt{(1-\dot{\mathbf{r}}_1{}^2/C^2)}$ in the same notation.

Let q_1, q_2, q_3 be three generalized coordinates defining the position of the particle. The components of $\mathbf{r}_1$ are then known functions of the q's and $\frac{\partial\dot{\mathbf{r}}_1}{\partial\dot{q}}=\frac{\partial\mathbf{r}_1}{\partial q}$ for any component of $\mathbf{r}_1$ and any of the q's.

Take the scalar product of (4) by $\frac{\partial\mathbf{r}_1}{\partial q}$.

Then

$$\left(\frac{\partial\mathbf{r}_1}{\partial q}, \frac{d}{dt}\frac{m_1}{\beta_1}\dot{\mathbf{r}}_1\right)=\frac{d}{dt}\left\{\frac{m_1}{\beta_1}\left(\frac{\partial\mathbf{r}_1}{\partial q}, \dot{\mathbf{r}}_1\right)\right\}-\frac{m_1}{\beta_1}\left(\frac{\partial\dot{\mathbf{r}}_1}{\partial q}, \dot{\mathbf{r}}_1\right)$$

$$=\frac{d}{dt}\left\{\frac{m_1}{\beta_1}\frac{\partial}{\partial\dot{q}}\frac{1}{2}\dot{\mathbf{r}}_1{}^2\right\}-\frac{m_1}{\beta_1}\frac{\partial}{\partial q}\frac{1}{2}\dot{\mathbf{r}}_1{}^2$$

and putting in the value of β_1, this reduces to $\mathfrak{D}_q(-mC^2\beta_1)$, where $\mathfrak{D}_q\equiv\frac{d}{dt}\frac{\partial}{\partial\dot{q}}-\frac{\partial}{\partial q}$ the Lagrangian operator. The expression $-mC^2\beta_1$ has an obvious connexion with the "world line" of a particle in relativity theory.

The next term in the equation is

$$-e_1\left(\frac{\partial\mathbf{r}_1}{\partial q}, \Delta\phi\right)=-e_1\frac{\partial\phi}{\partial q}=e_1\mathfrak{D}_q\phi.$$

For the remainder we simplify by writing out one component of the vector product,

$$[\dot{\mathbf{r}}_1, \text{curl}\,\mathbf{A}]_x=\left(\dot{x}\frac{\partial\mathbf{A}_x}{\partial x}+\dot{y}\frac{\partial\mathbf{A}_y}{\partial x}+\dot{z}\frac{\partial\mathbf{A}_z}{\partial x}\right)$$

$$-\left(\dot{x}\frac{\partial\mathbf{A}_x}{\partial x}+\dot{y}\frac{\partial\mathbf{A}_x}{\partial y}+\dot{z}\frac{\partial\mathbf{A}_x}{\partial z}\right),$$

and the second factor is $\frac{d\mathbf{A}_x}{dt}-\frac{\partial\mathbf{A}_x}{\partial t}$, where $\frac{d\mathbf{A}}{dt}$ denotes the total rate of change of $\mathbf{A}$ at the moving particle. Thus

$$\frac{e_1}{C}\left(\frac{\partial\mathbf{r}_1}{\partial q}, -\frac{\partial\mathbf{A}}{\partial t}+[\dot{\mathbf{r}}_1, \text{curl}\,\mathbf{A}]\right)=\frac{e_1}{C}\left(\dot{\mathbf{r}}_1, \frac{\partial\mathbf{A}}{\partial q}\right)-\frac{e_1}{C}\left(\frac{\partial\mathbf{r}_1}{\partial q}, \frac{d\mathbf{A}}{dt}\right)$$

$$=-\frac{e_1}{C}\mathfrak{D}_q(\dot{\mathbf{r}}_1, \mathbf{A}).$$

So the equations of motion can be derived from a Lagrangian

$$L=-m_1C^2\beta_1-e_1\phi+\frac{e_1}{C}(\dot{\mathbf{r}}_1, \mathbf{A}). \quad . \quad . \quad . \quad (5)$$

This expression is valid for any fields, including explicit dependence of ϕ and $\mathbf{A}$ on the time. In the case of a constant magnetic field it is a matter of indifference what particular integral is taken in finding $\mathbf{A}$ from $\mathbf{H}$. For the general value of $\mathbf{A}$ is given by the addition of a term $\Delta\Omega$ to any particular value, where Ω is a function of x, y, z. This adds on to L a term $(\dot{\mathbf{r}}_1, \Delta\Omega) = \frac{d\Omega}{dt}$, and if Ω is any function of x, y, z and t whatever

$$\mathfrak{D}_q \frac{d\Omega}{dt} = \frac{d}{dt}\frac{\partial\Omega}{\partial q} - \frac{\partial}{\partial q}\frac{d\Omega}{dt} = 0, \quad . \quad . \quad . \quad (6)$$

so that the extra terms will be without effect on the equations of motion.

4. We next find the Lagrangian for a number of freely moving interacting particles. Suppose there is a second one of charge e_2 and mass m_2 at $\mathbf{r}_2$. This particle is in motion, but at first we imagine it outside the dynamical system, that is we suppose $\mathbf{r}_2$ to be known in terms of the time. Then the motion of e_1 is governed by (5), where ϕ and $\mathbf{A}$ are to be calculated from the position and motion of e_2. The potentials are given by

$$\phi = \frac{e_2}{r + (\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)/\mathrm{C}}\bigg|_{\text{ret.}}, \quad \mathbf{A} = \frac{e_2}{\mathrm{C}}\frac{\dot{\mathbf{r}}_2}{r + (\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)/\mathrm{C}}\bigg|_{\text{ret.}}. \quad (7)$$

In these expressions $r^2 = (\mathbf{r}_2 - \mathbf{r}_1)^2$ and $(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)/r$ is the component of velocity of e_2 away from e_1. The quantities are all to have *retarded* values. If the effect reaching e_1 at time t, left e_2 at time $t - \tau$, we have

$$\begin{aligned}\mathrm{C}^2\tau^2 &= (-\mathbf{r}_1 + \mathbf{r}_2 - \dot{\mathbf{r}}_2\tau + \tfrac{1}{2}\ddot{\mathbf{r}}_2\tau^2 - \dots)^2 \\ &= r^2 - 2\tau(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1) + \tau^2\{\mathbf{r}_2{}^2 + (\ddot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)\} - \dots .\end{aligned}$$

Solving by approximation we find

$$\tau = \frac{r}{\mathrm{C}} - \frac{(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{\mathrm{C}^2} + \frac{r}{2\mathrm{C}^3}\{\dot{\mathbf{r}}_2{}^2 + (\ddot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1) + (\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)^2/r^2\}. \quad . \quad . \quad . \quad (8)$$

Substituting in (7)

$$\phi = \frac{e_2}{r} + \frac{e_2}{2\mathrm{C}^2}\left\{\frac{\dot{\mathbf{r}}_2{}^2 + (\ddot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r} - \frac{(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)^2}{r^3}\right\}, \quad \mathbf{A} = \frac{e_2}{\mathrm{C}}\frac{\dot{\mathbf{r}}_2}{r}. \quad . \quad . \quad . \quad (9)$$

The solution of $\mathbf{A}$ is only carried to this degree, because of the further factor C^{-1} in L which multiplies it. Substituting in (5) we have

$$\mathrm{L} = -m_1\mathrm{C}^2\beta_1 - \frac{e_1e_2}{r} - \frac{e_1e_2}{2\mathrm{C}_2}\left\{\frac{\dot{\mathbf{r}}_2{}^2 + (\ddot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1) - 2(\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2)}{r} - \frac{(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)^2}{r^3}\right\}.$$

Add to this the expression $-m_2\mathrm{C}^2\beta_2 + \frac{d}{dt}\frac{e_1e_2}{2\mathrm{C}^2}\frac{(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r}$. The first term is without effect because it is a pure function of the time, the second by (6). The result is

$$\mathrm{L} = -m_1\mathrm{C}^2\beta_1 - m_2\mathrm{C}^2\beta_2 - \frac{e_1e_2}{r} + \frac{e_1e_2}{2\mathrm{C}^2}\left\{\frac{(\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2)}{r} + \frac{(\dot{\mathbf{r}}_1, \mathbf{r}_2 - \mathbf{r}_1)(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r^3}\right\}. \quad (10)$$

From its complete symmetry (10) will also give the motion of e_2 when $\mathbf{r}_1$ is regarded as known in terms of the time. Thus the equations of motion of e_1 are $\mathfrak{D}_{\mathbf{r}_1}\mathrm{L} = 0$ and of e_2 are $\mathfrak{D}_{\mathbf{r}_2}\mathrm{L} = 0$. If q be any generalized coordinate involving both $\mathbf{r}_1$ and $\mathbf{r}_2$ we have

$$\mathfrak{D}_q\mathrm{L} = \left(\frac{\partial\mathbf{r}_1}{\partial q}, \mathfrak{D}_{\mathbf{r}_1}\mathrm{L}\right) + \left(\frac{\partial\mathbf{r}_2}{\partial q}, \mathfrak{D}_{\mathbf{r}_2}\mathrm{L}\right) = 0,$$

as may be seen by writing out the values of $\mathfrak{D}_{\mathbf{r}_1}$ and $\mathfrak{D}_{\mathbf{r}_2}$ or directly from the covariance of the operator $\mathfrak{D}$ for point transformations. Thus (10) is the Lagrangian for the simultaneous motion of the two particles, which can now be regarded as both belonging to the dynamical system.

The last term in (10) is only accurate to the terms in C^{-2}, so for the sake of consistency the first two should only be expanded to this degree. They are then of the form

$$-m_1\mathrm{C}^2 + \tfrac{1}{2}m_1\dot{\mathbf{r}}_1{}^2 + \frac{1}{8\mathrm{C}^2}m_1\dot{\mathbf{r}}_1{}^4.$$

Thus we have the complete Lagrangian for any number of charged particles in any field in the form :

$$\mathrm{L} = \Sigma\tfrac{1}{2}m_1\dot{\mathbf{r}}_1{}^2 + \Sigma\frac{1}{8\mathrm{C}^2}m_1\dot{\mathbf{r}}_1{}^4 - \Sigma e_1\phi + \Sigma\frac{e_1}{\mathrm{C}}(\dot{\mathbf{r}}_1, \mathbf{A}) - \Sigma\Sigma\frac{e_1e_2}{r_{12}} + \Sigma\Sigma\frac{e_1e_2}{2\mathrm{C}^2}\left\{\frac{(\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2)}{r_{12}} + \frac{(\dot{\mathbf{r}}_1, \mathbf{r}_2 - \mathbf{r}_1)(\dot{\mathbf{r}}_2, \mathbf{r}_2 - \mathbf{r}_1)}{r_{12}{}^3}\right\}. \quad . \quad . \quad . \quad (11)$$

The double summations are for each pair of particles counted once only.

Finally, re-writing (11) without the vector notation

$$L=\Sigma\tfrac{1}{2}m_1v_1^2+\Sigma\frac{1}{8C^2}m_1v_1^4-\Sigma e_1\phi_1+\Sigma\frac{e_1}{C}v_1A_1\cos\chi_1-\Sigma\Sigma\frac{e_1e_2}{r_{12}}$$

$$+\Sigma\Sigma\frac{e_1e_2}{2C^2}\frac{v_1v_2}{r_{12}}(\cos\psi_{12}-\cos\theta_1^2\cos\theta_2'), \quad . \quad (12)$$

where e_1, m_1, v_1 are the charge, mass, and velocity of the first particle ;

ϕ_1 and A_1 are the scalar and vector potentials at e_1 due to external fields ;

χ_1 is the angle between the line of motion of e_1 and the vector potential ;

r_{12} is the distance between e_1 and e_2 ;

ψ_{12} is the angle between their lines of motion ;

θ_1^2 is the angle between the line of motion of e_1 and the line joining it to e_2.

There is a certain interest in knowing how far wrong the approximation (11) will be according to the classical theory. This is done by calculating the next term for τ in (8) and evaluating the corresponding terms in (9). These are then substituted in (5). The force on e_1 from e_2 is found to be $\frac{2}{3}\frac{e_1e_2}{C^3}\dddot{\mathbf{r}}_2$. Thus the total force on e_1 is $\frac{2}{3}\frac{e_1}{C^3}\Sigma e_s\dddot{\mathbf{r}}_s$. The summation will include e_1, as well as the rest, as this term is the reactive force of an electron's radiation on itself. From the point of view of generalized coordinates we have $\frac{\partial\dddot{\mathbf{r}}}{\partial\dddot{q}}=\frac{\partial\mathbf{r}}{\partial q}$, so that the equations of motion can be put in the form $\mathfrak{D}_qL=\frac{\partial F}{\partial\ddot{q}}$, where $F=\frac{1}{3C^3}(\Sigma e_1\ddot{\mathbf{r}}_1)^2$. If F is neglected altogether, it is easy to see that the ratio of terms omitted to those included is of the order v/C.

5. When the Lagrangian for any problem has been found, the transition to the Hamiltonian follows in the usual way. We find the momenta $p=\frac{\partial L}{\partial\dot{q}}$ and solve for the $\dot{q}$'s in terms of them. The Hamiltonian is then $H=\Sigma p\dot{q}-L$ expressed in q's and p's, and the equations of motion have the canonical form $\dot{p}=-\frac{\partial H}{\partial q}$, $\dot{q}=\frac{\partial H}{\partial p}$. Thus if $\mathbf{p}_1$ be the momentum corresponding to each component of $\mathbf{r}_1$, it is easy to see that, extending the use of the vector notation, we have

$$H=\Sigma\frac{\mathbf{p}_1^2}{2m_1}-\Sigma\frac{\mathbf{p}_1^4}{8C^2m_1^3}+\Sigma e_1\phi-\Sigma\frac{e_1}{Cm_1}(\mathbf{p}_1,\ \mathbf{A})+\Sigma\Sigma\frac{e_1e_2}{r_{12}}$$

$$-\Sigma\Sigma\frac{e_1e_2}{2C^2m_1m_2}\left\{\frac{(\mathbf{p}_1,\ \mathbf{p}_2)}{r_{12}}+\frac{(\mathbf{p}_1,\ \mathbf{r}_2-\mathbf{r}_1)(\mathbf{p}_2,\ \mathbf{r}_2-\mathbf{r}_1)}{r_{12}^3}\right\},$$

All the developments of general dynamics (such as the Hamilton-Jacobi partial differential equation etc.) follow at once, with the exception of such theorems as depend on the kinetic energy having a quadratic form.

For many problems it will be quicker to work in the Lagrangian form direct. When q_s does not occur explicitly in L, we have an integral $\frac{\partial L}{\partial\dot{q}_s}=p_s$ a constant, and when this coordinate is "ignored" the modified Lagrangian is $L'=L-p_s\dot{q}_s$. The energy integral will exist when the external fields ϕ and $\mathbf{A}$ do not contain the time explicitly and is then of the usual form $\Sigma\dot{q}_s\frac{\partial L}{\partial\dot{q}_s}-L=\text{const.}$ Applying this to (11) we have the integral

$$\Sigma\tfrac{1}{2}m_1\dot{\mathbf{r}}_1^2+\Sigma\tfrac{3}{8}m_1\dot{\mathbf{r}}_1^4+\Sigma e_1\phi+\Sigma\Sigma\frac{e_1e_2}{r}$$

$$+\Sigma\Sigma\frac{e_1e_2}{2C^2}\left\{\frac{(\dot{\mathbf{r}}_1,\ \dot{\mathbf{r}}_2)}{r}+\frac{(\dot{\mathbf{r}}_1,\ \mathbf{r}_2-\mathbf{r}_1)(\dot{\mathbf{r}}_2,\ \mathbf{r}_2-\mathbf{r}_1)}{r^3}\right\}$$

$$=\text{const.} \quad . \quad . \quad . \quad (13)$$

The first two terms can be obtained either direct in the expanded form, or else from the fact that

$$\Sigma\dot{q}_s\frac{\partial}{\partial\dot{q}_s}(-\beta_1)+\beta_1=\frac{1}{\beta_1}=\left(1-\frac{\dot{\mathbf{r}}_1^2}{C^2}\right)^{-1/2},$$

agreeing with the known fact that the kinetic energy of an electron is mC^2/β.

6. We now apply these results to the "Problem of Two Bodies." Take $e_1=-e$, $m_1=m$ and $e_2=E$, $m_2=M$. The motion is supposed to take place in a plane and the particles

are at (x_1, y_1) (x_2, y_2) at any time. Then from (11) we have

$$L=\tfrac{1}{2}m(\dot{x}_1^2+\dot{y}_1^2)+\tfrac{1}{2}M(\dot{x}_2^2+\dot{y}_2^2)+\frac{1}{8C^2}m(\dot{x}_1^2+\dot{y}_1^2)^2$$
$$+\frac{1}{8C^2}M(\dot{x}_2^2+\dot{y}_2^2)^2+\frac{Ee}{r}-\frac{Ee}{2C^2r^3}\Big\{r^2(\dot{x}_1\dot{x}_2+\dot{y}_1\dot{y}_2)$$
$$+[\dot{x}_1(x_1-x_2)+\dot{y}_1(y_1-y_2)]\,[\dot{x}_2(x_1-x_2)+\dot{y}_2(y_1-y_2)]\Big\}. \quad . \quad . \quad . \quad (14)$$

The first transformation is

$$x_1=X+M\xi/(M+m), \quad x_2=X-m\xi/(M+m),$$

with similar expressions for y_1, y_2. Then X, Y may conveniently be called the centroid, though except for low velocities it has none of the properties ordinarily associated with the name. Then

$$L=\tfrac{1}{2}(M+m)(\dot{X}^2+\dot{Y}^2)+\frac{1}{2}\frac{Mm}{M+m}(\dot{\xi}^2+\dot{\eta}^2)$$
$$+\frac{1}{8C^2}(M+m)(\dot{X}^2+\dot{Y}^2)^2+\frac{1}{2C^2}\frac{Mm}{M+m}(\dot{\xi}\dot{X}+\dot{\eta}\dot{Y})^2$$
$$+\frac{1}{4C^2}\frac{Mm}{M+m}(\dot{X}^2+\dot{Y}^2)(\dot{\xi}^2+\dot{\eta}^2)$$
$$+\frac{1}{2C^2}\frac{Mm(M-m)}{(M+m)^2}(\dot{\xi}\dot{X}+\dot{\eta}\dot{Y})(\dot{\xi}^2+\dot{\eta}^2)$$
$$+\frac{1}{8C^2}\frac{Mm(M^2-Mm+m^2)}{(M+m)^3}(\dot{\xi}^2+\dot{\eta}^2)^2+\frac{Ee}{r}$$
$$-\frac{Ee}{2C^2r^3}\Big\{r^2(\dot{X}^2+\dot{Y}^2)+(\xi\dot{X}+\eta\dot{Y})^2$$
$$+\frac{M-m}{M+m}[r^2(\dot{\xi}\dot{X}+\dot{\eta}\dot{Y})+(\xi\dot{X}+\eta\dot{Y})(\xi\dot{\xi}+\eta\dot{\eta})]$$
$$-\frac{Mm}{(M+m)^2}[r^2(\dot{\xi}^2+\dot{\eta}^2)+(\xi\dot{\xi}+\eta\dot{\eta})]\Big\}. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

As $r^2=\xi^2+\eta^2$, X and Y do not occur explicitly in (15), and so we have integrals

$$\frac{\partial L}{\partial \dot{X}}=p_x \qquad \frac{\partial L}{\partial \dot{Y}}=p_y. \quad . \quad . \quad . \quad . \quad (16)$$

For considering quasi-elliptic orbits we naturally take $p_x=p_y=0$ for the integration constants. If this is not done it will be found that the modified Lagrangian deduced from (15), if expressed in polar coordinates, contains θ explicitly. Under these conditions the ordinary integral of angular momentum does not exist. But any such case could be worked out easily by taking $p_x=p_y=0$ and when the complete solution has been found, applying a linear relativity transformation to give the system the proper motion of translation. Such a transformation would be expected to introduce the time explicitly into the formulæ. So it appears that the angular momentum integral would be replaced by a complicated integral involving both θ and t. The study of such an integral might have an analytical interest, but it would appear that in any specified case where p_x does not vanish (and the same applies to motions of the particles which are not in a plane) the required results could be quickest attained by relativity transformations. Thus, in studying the collision of a moving electron with a stationary, we should work out the orbit with both moving in such a way that $p_x=p_y=0$, and afterwards apply the transformation which would reduce one of them initially to rest.

Taking $p_x=p_y=0$ in (16) we have equations of which the solution is

$$\dot{X}=-\frac{1}{2C^2}\frac{Mm(M-m)}{(M+m)^3}\dot{\xi}(\dot{\xi}^2+\dot{\eta}^2)$$
$$+\frac{Ee}{2C^2r^3}\frac{M-m}{(M+m)^2}[r^2\dot{\xi}+\xi(\xi\dot{\xi}+\eta\dot{\eta})]. \quad . \quad . \quad (17)$$

Next form $L'=L-p_x\dot{X}-p_y\dot{Y}$. This is given by simply omitting $\dot{X}$, $\dot{Y}$ from (15), since $\dot{X}$, itself of the order C^{-2}, occurs everywhere either squared or else multiplied by C^{-2}. In polar coordinates we then have

$$L'=\frac{1}{2}\frac{Mm}{M+m}w^2+\frac{1}{8C^2}\frac{Mm(M^2-Mm+m^2)}{(M+m)^3}w^4$$
$$+\frac{Ee}{r}+\frac{Ee}{2C^2}\frac{Mm}{(M+m)^2}\frac{2\dot{r}^2+r^2\dot{\theta}^2}{r}, \quad . \quad (18)$$

where $w^2=\dot{r}^2+r^2\dot{\theta}^2$. The integral of angular momentum is

$$\frac{Mm}{M+m}r^2\dot{\theta}\left\{1+\frac{1}{2C^2}\frac{M^2-Mm+m^2}{(M+m)^2}w^2+\frac{Ee}{C^2(M+m)}\frac{1}{r}\right\}=p$$
$$. \quad . \quad . \quad (19)$$

and of energy is

$$\frac{1}{2}\frac{Mm}{M+m}w^2+\frac{3}{8C^2}\frac{Mm(M^2-Mm+m^2)}{(M+m)^3}w^4-\frac{Ee}{r}$$
$$+\frac{Ee}{2C^2}\frac{Mm}{(M+m)^2}\frac{2\dot{r}^2+r^2\dot{\theta}^2}{r}=-W. \quad . \quad . \quad (20)$$

The integration constant is taken as $-W$, so that W may be positive for elliptic orbits. Following the usual procedure we eliminate the time between (19) and (20), and express the orbit in terms of θ and u, where $u=1/r$. The result is an equation of the form

$$\left(\frac{du}{d\theta}\right)^2=\alpha u^3-(1-\beta)u^2+2gu-k, \quad . \quad . \quad (21)$$

where $\alpha=\frac{Ee}{C^2(M+m)}$, $\beta=\frac{E^2e^2}{C^2p^2}\frac{M^2-Mm+m^2}{(M+m)^2}$,

$$g=\frac{Ee}{p^2}\frac{Mm}{M+m}\left(1-\frac{W}{C^2}\frac{M^2-Mm+m^2}{Mm(M+m)}\right),$$
$$k=\frac{2W}{p^2}\frac{Mm}{M+m}\left(1-\frac{W}{2C^2}\frac{M^2-Mm+m^2}{Mm(M+m)}\right).$$

The solution of this equation to the same order of approximation as before is

$$u=q+s\cos\lambda\theta+l\cos 2\lambda\theta, \quad . \quad . \quad . \quad . \quad (22)$$

where
$$\lambda=1-\tfrac{3}{2}\alpha g-\tfrac{1}{2}\beta,$$
$$q=g+\tfrac{3}{4}\alpha(3g^2-k)+\beta g,$$
$$s^2=g^2-k+\alpha g(4g^2-3k)+\beta(2g^2-k),$$
$$l=-\tfrac{1}{4}\alpha(g^2-k).$$

The last three expressions cannot be much simplified, but

$$\lambda=1-\frac{E^2e^2}{2C^2p^2},$$

which is independent of the masses and depends only on the angular momentum. It is the same as Sommerfeld's result and implies an advance of perigee by $\pi E^2e^2/C^2p^2$ each revolution. The term in l makes a slight increase in the radius at the apses, and a decrease at the ends of the latus rectum. The solution of the relative orbit is completed by finding the time from (19). The formula is complicated and of no special interest.

We next solve for the motion of the centroid. As X and Y are of the order C^{-2}, it will be sufficient to use large order values for ξ, etc. Then changing the independent variable to θ and making use of the known value of w^2, (17) gives

$$\frac{dX}{d\theta}=-\frac{1}{2C^2}\frac{M-m}{(M+m)^2}\left\{2(Eeu-W)\frac{d\xi}{d\theta}-Eeu\left(\frac{d\xi}{d\theta}+\frac{\xi}{r}\frac{dr}{d\theta}\right)\right\},$$

and this is directly integrable in the form

$$X=-\frac{1}{2C^2}\frac{M-m}{(M+m)^2}(Eeu-2W)\xi.$$

If a is the half major axis $a=q/(q^2-s^2)$ and $W=Ee/2a$ and we have

$$X=-\frac{1}{2C^2}\frac{M-m}{(M+m)^2}\frac{Ee}{a}(r-a)\cos\theta. \quad . \quad . \quad (23)$$

If the motion of the centroid is to be valid for many revolutions, we must replace $\cos\theta$ by $\cos\lambda\theta$. Both are the same to the degree of approximation considered, but $\cos\lambda\theta$ will enable the centroid to keep pace with the motion of the apse.

From (23) we see that both particles and their centroid always remain collinear with the origin, which is the invariable point of the system. Observe that this invariable point cannot be calculated by taking the centre of mass of the particles as though each had its mass increased separately by the effect of velocity. Such a process would give X proportional to $w^2\xi$ or $(r-2a)\cos\theta$. There is in fact no simple definition for the invariable point.

Expressed in polar coordinates the centroid describes the curve

$$R=-\frac{Ee}{2C^2}\frac{M-m}{(M+m)^2}\epsilon\cdot\frac{\epsilon+\cos\theta}{1+\epsilon\cos\theta}, \quad . \quad . \quad (24)$$

where $\epsilon=\sqrt{(1-s^2/q^2)}$ is the eccentricity of the relative orbit of the particles. If $M>m$, R is negative at perigee, that is to say, the centroid is towards M. At apogee it is an equal distance towards m. If the time average of R be taken it is found to be

$$+\frac{Ee}{2C^2}\frac{M-m}{(M+m)^2}\tfrac{1}{2}\epsilon^2, \quad . \quad . \quad . \quad . \quad . \quad (25)$$

that is, on the average it is towards m. As the velocity of the lighter particle is the higher, so its mass is the more increased by the motion, and so (25) is directly contrary to what would be expected at first sight. Observe that (24) shows that the centroid is at rest at the invariable point in the case of any circular orbit, as well as for the obvious cases $M=m$ and M infinite.

7. Finally we apply these results to Bohr's theory of spectra. To do so we use Sommerfeld's* quantum relations, so as to determine the integration constants p and W. These relations are

$$nh=\int p_\theta d\theta, \qquad n'h=\int p_r dr,$$

where the integrations are carried round a complete period of the variable in each case. Then

$$nh=\int_0^{2\pi} p\,d\theta=2\pi p \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and

$$n'h=\int_0^{2\pi/\lambda}\frac{\partial L'}{\partial r}\frac{dr}{d\theta}d\theta.$$

If the values be taken from (18) and (22), the last gives after some partial integration

$$n'h=p\lambda\int_0^{2\pi}-\frac{s\cos\phi+4l\cos 2\phi}{q+s\cos\phi+l\cos 2\phi}+\alpha\frac{s^2\sin^2\phi}{q+s\cos\phi}d\phi.$$

The evaluation of the integral is rather long, but by taking advantage of the smallness of l and α it can be reduced to

$$n'h=2\pi p\lambda\left\{\frac{q}{(q^2-s^2)^{1/2}}-1+4l\frac{q}{s}\right.$$
$$\left.-\frac{l}{s^2(q^2-s^2)^{3/2}}(4q^4-6q^2s^2+3s^4)+\alpha\left[q-(q^2-s^2)^{1/2}\right]\right\}.$$

Putting in the values from (22) we have

$$n'h=2\pi p\left\{\frac{g}{\sqrt{k}}-1+\tfrac{3}{2}\alpha g+\tfrac{1}{2}\beta-\alpha\sqrt{k}\right\}.$$

This is to be solved for W by using the values given in (21) and (26). The result is

$$W=\frac{2\pi^2E^2e^2}{h^2}\frac{Mm}{(M+m)}\frac{1}{(n+n')^2}$$
$$\left\{1+\frac{\rho^2}{(n+n')^2}\left[\frac{1}{4}+\frac{1}{4}\frac{Mm}{(M+m)^2}+\frac{n'}{n}\right]\right\}, \quad . \quad . \quad (27)$$

where $\rho=2\pi Ee/Ch$. The spectrum lines are given by

$$\nu=\{W(n_2, n_2')-W(n_1, n_1')\}/h.$$

* *Loc. cit.*

It was not of course to be anticipated that our work should give any effect perceptible experimentally for the distribution of lines in the hydrogen spectrum, but it is interesting to observe what extremely little difference the finite mass of the hydrogen nucleus does make. In the first place there is the factor $M/(M+m)$ in the large terms, corresponding to a slight alteration in Balmer's constant. This comes out of ordinary dynamics and was given by Sommerfeld. In addition, we have a minute shift of the whole position of the composite lines, represented by the term in $\frac{1}{4}\frac{Mm}{(M+m)^2}$. But the fine structure of each line, which is given by the term in n'/n, remains absolutely unaffected by the mass of the nucleus.

LII. *The Specific Heat of Carbon Dioxide and Steam.*

By W. T. David, *M.A.**

1. THE specific heat of many gases, notably carbon dioxide and steam, increases very considerably with temperature. In this paper the suggestion is put forward that the specific heat of these gases depends to an appreciable extent upon volume and density as well as temperature.

2. Some of my experiments upon the emission of radiation in gaseous explosions indicate that the intrinsic radiance from thicknesses of gas containing the same number and kind of radiating molecules does not depend upon the temperature alone, even after correcting the radiation for absorption. This implies that the vibratory energy of the radiating molecules is not solely dependent upon the gas temperature. It depends upon the volume and the density of the gas as well †. I have suggested ‡ an explanation of this in terms of the kinetic theory of gases which it will be convenient to repeat here briefly. A radiating molecule as it describes its free-path loses energy owing to the emission of radiation and gains energy owing to the absorption of energy from the æther. Its vibratory energy will thus increase or decrease according as the absorption is greater or less than the emission. During collision with another molecule there will be a transference of energy between the vibratory and the rotational and translational energies, which, as Mr. Jeans

* Communicated by the Author.
† Phil. Trans. A. vol. ccxi. (1911) pp. 402 & 406.
‡ Phil. Mag. Feb. 1913, p. 267.