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Radiation Damping in a Gravitational Field*

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The validity of the principle of equivalence is examined from the point of view of a charged mass point moving in an externally given gravitational field. The procedure is a covariant generalization of Dirac's work on the classical radiating electron. Just as Dirac's calculation was kept Lorentz invariant throughout, so the present calculation is maintained generally covariant throughout. With the aid of *bi-tensors,* which are nonlocal generalizations of ordinary local tensors, the manifest general covariance of each step is achieved in an elegant and useful way. The Green's functions for the scalar and vector wave equations in a curved manifold are obtained and applied to the derivation of the covariant Liénard-Wiechert potentials. The computation of energy-momentum balance across a world tube of infinitesimal radius surrounding the particle world-line then leads to the ponderomotive equations including radiation damping.

Because of the nonlocal electromagnetic field which a charged particle carries with itself, its use as a device to distinguish locally between gravitational and inertial fields is really not allowable. One should be prepared to find an explicit occurrence of the Riemann tensor in the ponderomotive equations, leading to the result that acceleration by a "true" gravitational field can produce bremsstrahlung, thereby causing a reactive force in addition to the force of inertia. It is remarkable, however, that such an explicit occurrence does *not* happen. The particle tries its best to satisfy the equivalence principle in spite of its charge. It is only prevented from doing so (i.e., from following a geodetic path) because of the fact that, contrary to the case of flat space-time, the electromagnetic Green's function in a curved space-time does not generally vanish inside the light cone, but gives rise to a "tail" on any initially sharp pulse of radiation. The ponderomotive equations have exactly the same form as Dirac found for the flat-space-time case except for the addition of an integral over the entire past history of the particle, representing the effect of the "tail."

INTRODUCTION

An important part of the development of any physical theory is the testing of its consequences against its original physical foundations. In the development of

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the general theory of relativity this part plays an anomalously large role for the simple reason that the theory has progressed experimentally so little beyond these foundations. Two experimentally established principles, the principle of equivalence and the principle of the covariance of physical laws, form the basis for the general theory. The concept embodied in the principle of equivalence is a "local" one which, when combined with the covariance concept, leads to the introduction of curvilinear coordinate systems and nonvanishing intrinsic curvature into the description of space-time.

In its simplest form the principle of equivalence states that a gravitational force cannot be distinguished from an inertial force by any experiment which is conducted on a purely local basis. While this principle is certainly valid to a high degree of precision, and may even be valid with absolute precision for neutral matter, there is some question as to whether it can be absolutely valid for matter which carries an electrical charge. To put the question in physical terms, imagine a charged particle located in empty space at a great distance from any gravitating masses. If a force is exerted on the particle it will begin to accelerate, and we know from the laws of classical electrodynamics within the framework of the special theory of relativity, which is valid under the above circumstances, that the particle will radiate, producing a reactive damping force in addition to its mechanical inertial force. Let us next bring the particle to rest in a static gravitational field which exerts a force equal to that to which the particle was previously subjected in empty space. Although the particle experiences the same force in both cases it would be absurd to suppose that it continues to radiate under the latter conditions.

There is, however, a catch here: It is well known that an accelerating charged particle, in fact, does *not* suffer a reactive damping force as long as its absolute acceleration is *uniform*, i.e., constant in magnitude and direction.* It is therefore

* *Note added in proof:* A contribution to the old and much-debated question of the interpretation of this phenomenon has recently been made by T. Fulton and F. Rohrlich (private communication). These authors give a Lorentz invariant asymptotic definition of the rate of radiation from a charged particle which, for arbitrary particle motion in flat space-time, reduces to the usual expression involving the square of the absolute acceleration. They suggest therefore that the absence of the reactive force in the case of uniform acceleration does not imply absence of radiation but merely a special behavior of the internal (nonasymptotic) field energy of the particle which supplies a compensating term in the energy balance equation: the so-called "acceleration energy" which, in the cases of usual experience, averages out to zero.

The problem of giving a similar asymptotic definition of the rate of radiation in a curved space-time is much more difficult, and it seems most unlikely that the total radiation in this case possesses a simple correlation with the absolute acceleration. Nevertheless, only that part of the radiation which is so correlated is considered in the present article, and the reader should be cautioned that the term "bremsstrahlung" as used herein refers only to the radiation which has its immediate local reflection in the departure of the particle motion from geodetic. It is quite possible that the total asymptotic radiation has generally an adbetter to turn the problem around and imagine the particle in an unaccelerated state. If the particle is far from gravitating matter it can be said to be in a state of uniform motion, in which it does not, of course, radiate. If the particle approaches a strong gravitational field, however, the notion of "unaccelerated state" is changed into a different notion, namely, that of a "state of free fall." Does a change of notion now cause the particle to begin to radiate? If a charged particle does not radiate when at rest in a gravitational field, does it also refrain from radiating when falling freely? Or can a charged particle be used, at least in principle, as a local entity which distinguishes between gravitational and inertial forces?

On the basis of physical intuition it would seem reasonable to suppose that a charged particle does, in fact, radiate when deflected by a gravitational field, i.e., that bremsstrahlung can be produced by gravitational as well as electromagnetic forces. This is the problem which will be attacked in the present paper. Before describing the procedure to be used we should point out at once that the idea of using a charged particle to distinguish locally between gravitational and inertial fields is, of course, cheating. A charged particle carries with it an electromagnetic field, which is by no means local. A gravitational field can be readily distinguished from an inertial field by experiments carried out over an extended region, that is, by experiments which measure field *gradients*. The gradient of a field is a second derivative of a potential. In the general theory of relativity the potential is the space-time metric, and second derivatives of the metric are expressed uniquely in a covariant manner by the components of the *Riemann tensor* which describes the intrinsic space-time curvature or, alternatively, the "true" gravitational field. We should therefore not be surprised if, when radiation reaction is included, we find the Riemann tensor entering explicitly into the dynamical equations of a charged particle moving in a gravitational field.

The surprising thing is that the Riemann tensor does *not* so enter, at least to the extent to which the essentially classical calculations of this paper are valid. This does not, however, mean that electro-gravitic bremsstrahlung does not occur. It does. But it has its origin in a more subtle phenomenon having to do with the failure of Huygens' Principle, when taken in the narrowest sense, in a curved space-time. As has been pointed out by Hadamard (*1*), a plane or spherical sharp pulse of light, when propagating in a curved 4-dimensional hyperbolic Riemannian manifold, does not, in general, remain a sharp pulse, but gradually develops a "tail." It is this phenomenon which is responsible for the electro-gravitic bremsstrahlung.

ditional component which can somewhat picturesquely be described as arising from the "static" Coulomb field of the particle, which can be "shaken loose" in bits as it sweeps over the "bumps" in space time. Owing to the difficulty of obtaining asymptotic forms for the Green's functions involved, however, no attempt has so far been made to analyze this component.

The picture, then, is the following: The charged particle tries its best to satisfy the equivalence principle, and on a local basis, in fact, does so. In the absence of an externally applied electromagnetic field the motion of the particle deviates from geodetic motion only because of the unavoidable tail in the propagation function for the electromagnetic field, which enters the picture nonlocally by appearing in an integral over the past history of the particle [see Eq. (5.26)]. Physically, the tail may be pictured as arising from a sort of scatter process, with the "bumps" in space-time playing the role of scatterers, which allows the radiation field originating in the particle, which normally "outruns" the particle, to act directly back on the particle in an anomalous fashion.

In this paper the gravitational field itself is given no dynamical properties; the geometrical structure of space-time is regarded as fixed. It is not supposed that Einstein's empty-space field equations are satisfied; the results hold for a completely arbitrary metric. The calculation is patterned directly on Dirac's famous paper on the classical radiating electron (*2*). Just as Dirac's calculation was kept Lorentz invariant throughout, so the present calculation is maintained generally covariant throughout. The authors believe that this procedure is unique for this type of calculation (i.e., determination of higher order terms in ponderomotive equations) and would like to call attention to some of its features. By being kept *manifestly* covariant at every step the calculation avoids undue complexity in spite of the fact that no special coordinate systems are introduced. The authors believe, in fact, that the complexity is no worse than that involved in similar calculations with special coordinates, and may even be somewhat less. Because of the ability of the covariant procedure to keep separate aspects of the problem always quite distinct, the authors would finally like to suggest that it may be of use in other calculations where the gravitational field itself is dynamically involved.

In order to maintain general covariance in a calculation in which nonlocal questions are involved, it is essential to introduce a generalization of ordinary tensors, which we shall call *n-tensors*. An n-tensor is a set of functions of n space-time points, each member of which is labeled by a set of indices each running from 0 to 3, and which transforms under a coordinate transformation like an ordinary tensor, with the difference that the transformation coefficients do not all refer to the same point, but rather to the n separate points, each point being associated with a subset of the set of all indices on the n-tensor. It is probable that for nearly all practical applications it suffices to consider 2-tensors, or, as we shall call them, *bi-tensors*,[1] associated with only two points. In Section 1 the elementary theory of bi-tensors is outlined and important kinematical examples

[1] This terminology is not to be confused with that which has also been applied in the study of antisymmetric tensors, i.e., use of the term *bi-vector* to mean *6-vector*, with corresponding generalization to tensors of higher order. [See, for example, the work of Schouten (*3*).]

are introduced. In Section 2 the propagation functions and Green's functions for the covariant scalar and vector wave equations are studied and their needed properties derived. In Section 3 the Lagrangian for an electromagnetic field interacting with a charged particle is introduced and a brief resumé is given of the fundamental equations of electrodynamics including the conservation equations. The retarded and advanced fields, as well as the associated fields useful in connection with the scattering problem, are introduced. The generalized covariant Liénard–Wiechert potentials and field strengths are then obtained. Section 4 is devoted to the construction and kinematics of a hyper-tube of infinitesimal radius surrounding the world line of the particle. Finally, with the aid of covariant expansion techniques introduced in preceding sections, the energy-momentum balance of the particle is computed in Section 5 by integrating the stress tensor over the hyper-tube. The ponderomotive equations, including radiation damping, then follow after a classical mass renormalization.

In these calculations the charged particle is assumed to have no spin. The inclusion of spin and magnetic moment is of little interest in an electrodynamic test of the equivalence principle, for it is known (*4*) that spinning neutral particles already deviate from geodetic motion by terms involving the Riemann tensor explicitly, which is an expression of the fact that spin is a nonlocal mechanical phenomenon.

The present calculations, being entirely classical, do not, of course, touch the problem of the influence which quantum phenomena may have on the testability of the equivalence principle. Furthermore, no application is made of the present results to the problem of a charged particle at *rest* in a static gravitational field, nor is there consideration of the problem of damping due to *gravitational radiation* which arises when one studies the truer state of affairs which exists when the gravitational field as well as the electromagnetic field is given dynamical properties and when account is taken of the fact that the metric is actually singular at the location of the particle owing to the particle's gravitational self-field. It is hoped that the two latter problems may be investigated in future publications.

1. BI-TENSORS*

The simplest example of a bi-tensor is the product of two local vectors, $A^\mu(x)$ and $B_\alpha(z)$, for example, taken at different space-time points, x and z:

$$C^\mu{}_\alpha(x,z) = A^\mu(x)B_\alpha(z). \tag{1.1}$$

* After this paper was written the authors learned that bi-tensors had previously been considered by Ruse (*12*) and Synge (*13*). The latter authors have not, however, developed the theory in the detail needed for the purposes of the present work. In particular, the theory seems never previously to have been applied to the study of covariant Green's functions.

We shall here adopt the convention that indices taken from the letters α to κ in the Greek alphabet are always to be associated with the point z, while indices taken from λ to ω are always to be associated with the point x. The coordinates of the points themselves will thus be expressed, for example, as x^μ and z^α.

The coordinate transformation law for the bi-tensor in (1.1) is given by

$$C'^\mu{}_\alpha = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial z^\beta}{\partial z'^\alpha}C^\nu{}_\beta. \tag{1.2}$$

Generalization to bi-tensors with additional indices is obvious. The usual operations, such as contraction and covariant differentiation, may be immediately extended to bi-tensors, with obvious precautions: e.g., contraction may be performed only over indices referring to the same point, and in taking covariant derivatives one should ignore all indices except those which refer to the variable in question. Covariant derivatives may be taken with respect to either variable. Thus, using a dot followed by an index to denote covariant differentiation, we have

$$C^\mu{}_{\alpha\cdot\nu} = C^\mu{}_{\alpha,\nu} + \Gamma_{\sigma\nu}{}^\mu C^\sigma{}_\alpha, \tag{1.3}$$

$$C^\mu{}_{\alpha\cdot\beta} = C^\mu{}_{\alpha,\beta} - \Gamma_{\alpha\beta}{}^\gamma C^\mu{}_\gamma, \tag{1.4}$$

where the comma denotes ordinary differentiation and Γ is the affinity. We do not indicate explicitly the point at which a local quantity such as the affinity is to be taken when the indices themselves suffice for this purpose. Indices generated by covariant differentiation at different points commute, while the usual commutation laws involving the Riemann tensor hold for indices referring to the same point.

One of the points, either x or z, may have no indices associated with it, in which case the bi-tensor in question transforms like the product of an ordinary local tensor at one of the points and a scalar at the other. The bi-tensor may even be an invariant, bearing no indices, in which case we refer to it as a *bi-scalar*. We may also introduce *bi-densities*, of which the most elementary example is the four-dimensional delta function:

$$\delta^{(4)}(x,z) = \delta(x^0 - z^0)\delta(x^1 - z^1)\delta(x^2 - z^2)\delta(x^3 - z^3) = \delta^{(4)}(z,x). \tag{1.5}$$

The delta function may be regarded as a density of weight w at the point x and weight $1 - w$ at the point z, where w is arbitrary. For the sake of symmetry we shall choose $w = \frac{1}{2}$ and write the transformation law of the delta function in the form

$$\delta^{(4)\prime} = \left|\frac{\partial x}{\partial x'}\right|^{1/2}\left|\frac{\partial z}{\partial z'}\right|^{1/2}\delta^{(4)}. \tag{1.6}$$

Of fundamental importance in the study of the nonlocal properties of space-time is the *bi-scalar of geodetic interval*, denoted by $s(x,z)$, which gives the magnitude of the invariant "distance" between x and z as measured along a geodesic joining them. The basic properties of $s(x,z)$ are expressed in the defining equations

$$g^{\mu\nu}s_{\cdot\mu}s_{\cdot\nu} = g^{\alpha\beta}s_{\cdot\alpha}s_{\cdot\beta} = \pm 1, \tag{1.7}$$

$$\lim_{x\to z} s = 0. \tag{1.8}$$

Here $g^{\mu\nu}$ is the contravariant metric, and if its signature is taken as $(- + + +)$ then the interval between x and z is said to be *space-like* when the $+$ sign holds in (1.7) and *time-like* when the $-$ sign holds. The bi-scalar s itself is here taken non-negative. The locus of points x for which $s = 0$ is said to define the *light cone* through z. The relation between the light cone, the surfaces $s =$ constant, and the unit vectors $s_{\cdot\mu}$ is pictured in Fig. 1.

The geodesics emanating from a given point may, at a sufficient distance, begin to cross one another. In this region the bi-scalar of geodetic interval becomes multiple-valued. There will, however, generally be a region close to the given point in which the geodetic interval is single valued. We confine our attention, in the following, to this neighborhood. We note, in passing, the obvious symmetry relation.

$$s(x,z) = s(z,x). \tag{1.9}$$

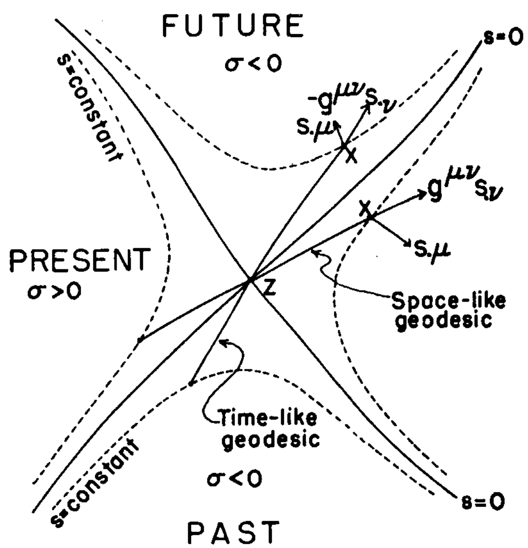

FIG. 1. Geodetic structure of space-time

The geodetic interval, in the single-valued region, can be used as the structural element of a number of covariant expansion techniques which play an important role in the following sections. In order to avoid "branch point" problems it will, however, be more convenient to work with the quantity

$$\sigma \equiv \pm \tfrac{1}{2}s^2, \tag{1.10}$$

which satisfies the equations,

$$\tfrac{1}{2}g^{\mu\nu}\sigma_{\cdot\mu}\sigma_{\cdot\nu} = \tfrac{1}{2}g^{\alpha\beta}\sigma_{\cdot\alpha}\sigma_{\cdot\beta} = \sigma, \tag{1.11}$$

$$\lim_{x\to z}\sigma = 0, \tag{1.12}$$

and which is positive for space-like intervals and negative for time-like ones. Suppose, now, we have a bi-tensor, say $T_{\alpha\beta}$, whose indices all refer to the same point z. If $T_{\alpha\beta}$ is sufficiently differentiable it is possible to expand it about z in the covariant form

$$T_{\alpha\beta} = A_{\alpha\beta} + A_{\alpha\beta}{}^{\gamma}\sigma_{\cdot\gamma} + \tfrac{1}{2}A_{\alpha\beta}{}^{\gamma\delta}\sigma_{\cdot\gamma}\sigma_{\cdot\delta} + O(s^3), \tag{1.13}$$

where the coefficients $A_{\alpha\beta}$, etc., are ordinary local tensors at z. In order to determine these coefficients in terms of the covariant derivatives of $T_{\alpha\beta}$, however, it is necessary to have some information about the covariant derivatives of σ. This can be obtained by repeatedly differentiating Eq. (1.11). We get

$$\sigma_{\cdot\gamma} = g^{\alpha\beta}\sigma_{\cdot\alpha}\sigma_{\cdot\beta\gamma}, \tag{1.14}$$

$$\sigma_{\cdot\gamma\delta} = g^{\alpha\beta}(\sigma_{\cdot\alpha\delta}\sigma_{\cdot\beta\gamma} + \sigma_{\cdot\alpha}\sigma_{\cdot\beta\gamma\delta}), \tag{1.15}$$

$$\sigma_{\cdot\gamma\delta\epsilon} = g^{\alpha\beta}(\sigma_{\cdot\alpha\delta\epsilon}\sigma_{\cdot\beta\gamma} + \sigma_{\cdot\alpha\delta}\sigma_{\cdot\beta\gamma\epsilon} + \sigma_{\cdot\alpha\epsilon}\sigma_{\cdot\beta\gamma\delta} + \sigma_{\cdot\alpha}\alpha_{\cdot\beta\gamma\delta\epsilon}), \tag{1.16}$$

$$\begin{aligned}\sigma_{\cdot\gamma\delta\epsilon\zeta} = g^{\alpha\beta}(&\sigma_{\cdot\alpha\delta\epsilon\zeta}\sigma_{\cdot\beta\gamma} + \sigma_{\cdot\alpha\delta\epsilon}\sigma_{\cdot\beta\gamma\zeta} + \sigma_{\cdot\alpha\delta\zeta}\sigma_{\cdot\beta\gamma\epsilon} + \sigma_{\cdot\alpha\delta}\sigma_{\cdot\beta\gamma\epsilon\zeta} + \sigma_{\cdot\alpha\epsilon\zeta}\sigma_{\cdot\beta\gamma\delta} \\ &+ \sigma_{\cdot\alpha\epsilon}\sigma_{\cdot\beta\gamma\delta\zeta} + \sigma_{\cdot\alpha\zeta}\sigma_{\cdot\beta\gamma\delta\epsilon} + \sigma_{\cdot\alpha}\sigma_{\cdot\beta\gamma\delta\epsilon\zeta}).\end{aligned} \tag{1.17}$$

Now Eq. (1.11) itself tells us that

$$\lim_{x\to z}\sigma_{\cdot\alpha} = 0. \tag{1.18}$$

In fact, it is clear that $\sigma_{\cdot\alpha}$ is of the same infinitesimal order as s, as $x \to z$. Setting $x = z$ in Eqs. (1.14) to (1.18), making use of the identities

$$\sigma_{\cdot\delta\gamma\epsilon} + \sigma_{\cdot\epsilon\gamma\delta} = 2\sigma_{\cdot\gamma\delta\epsilon} + R_{\epsilon\delta\gamma}{}^{\zeta}\sigma_{\cdot\zeta}, \tag{1.19}$$

$$\begin{aligned}&\sigma_{\cdot\delta\gamma\epsilon\zeta} + \sigma_{\cdot\epsilon\gamma\delta\zeta} + \sigma_{\cdot\zeta\gamma\delta\epsilon} \\ &\quad = 3\sigma_{\cdot\gamma\delta\epsilon\zeta} + (R_{\epsilon\delta\gamma}{}^{\eta}\sigma_{\cdot\eta})_{\cdot\zeta} + (R_{\zeta\delta\gamma}{}^{\eta}\sigma_{\cdot\eta})_{\cdot\epsilon} + R_{\zeta\epsilon\gamma}{}^{\eta}\sigma_{\cdot\eta\delta} + R_{\zeta\epsilon\delta}{}^{\eta}\sigma_{\cdot\gamma\eta},\end{aligned} \tag{1.20}$$

which follow from repeated use of the law for commuting indices induced by covariant differentiation, and recalling the algebraic identities satisfied by the Riemann tensor, we infer

$$\lim_{x\to z} \sigma_{\cdot\alpha\beta} = g_{\alpha\beta}\,, \tag{1.21}$$

$$\lim_{x\to z} \sigma_{\cdot\alpha\beta\gamma} = 0, \tag{1.22}$$

$$\lim_{x\to z} \sigma_{\cdot\alpha\beta\gamma\delta} = \tfrac{1}{3}(R_{\alpha\gamma\beta\delta} + R_{\alpha\delta\beta\gamma}). \tag{1.23}$$

Our convention for the Riemann tensor is

$$R_{\alpha\beta\gamma}{}^{\delta} = \Gamma_{\gamma\beta}{}^{\delta}{}_{,\alpha} - \Gamma_{\gamma\alpha}{}^{\delta}{}_{,\beta} + \Gamma_{\gamma\beta}{}^{\epsilon}\Gamma_{\epsilon\alpha}{}^{\delta} - \Gamma_{\gamma\alpha}{}^{\epsilon}\Gamma_{\epsilon\beta}{}^{\delta}. \tag{1.24}$$

Returning now to expansions (1.13), we see that

$$A_{\alpha\beta} = \lim_{x\to z} T_{\alpha\beta}\,, \tag{1.25}$$

$$A_{\alpha\beta\gamma} = \lim_{x\to z} T_{\alpha\beta\cdot\gamma} - A_{\alpha\beta\cdot\gamma}\,, \tag{1.26}$$

$$A_{\alpha\beta\gamma\delta} = \lim_{x\to z} T_{\alpha\beta\cdot\gamma\delta} - A_{\alpha\beta\cdot\gamma\delta} - A_{\alpha\beta\gamma\cdot\delta} - A_{\alpha\beta\delta\cdot\gamma}\,. \tag{1.27}$$

Although we shall never need to push our expansions farther than this, it is clear how one would proceed to obtain higher order terms. We note here the particular expansions

$$\sigma_{\cdot\alpha\beta} = g_{\alpha\beta} + \tfrac{1}{3}R_{\alpha}{}^{\gamma}{}_{\beta}{}^{\delta}\,\sigma_{\cdot\gamma}\sigma_{\cdot\delta} + O(s^3), \tag{1.28}$$

$$\sigma_{\cdot\alpha\beta\gamma} = \tfrac{1}{3}(R_{\alpha\gamma\beta}{}^{\delta} + R_{\alpha}{}^{\delta}{}_{\beta\gamma})\sigma_{\cdot\delta} + O(s^2), \tag{1.29}$$

$$\sigma_{\cdot\alpha\beta\gamma\delta} = \tfrac{1}{3}(R_{\alpha\gamma\beta\delta} + R_{\alpha\delta\beta\gamma}) + O(s). \tag{1.30}$$

When we are faced with the problem of expanding a bi-tensor whose indices do not all refer to the same point we cannot, of course, use the above techniques unless we introduce a device which "homogenizes" the indices, i.e., which transforms the given bi-tensor into a new bi-tensor all of whose indices do refer to the same point. The most natural such device is to use what we shall call the *bi-vector of geodetic parallel displacement*, denoted by $\bar{g}_{\mu\alpha}(x,z)$. A quantity of fundamental importance in its own right, this bi-vector is determined by the defining equations

$$\bar{g}_{\mu\alpha\cdot\nu}g^{\nu\sigma}\sigma_{\cdot\sigma} = 0, \qquad \bar{g}_{\mu\alpha\cdot\beta}g^{\beta\gamma}\sigma_{\cdot\gamma} = 0, \tag{1.31}$$

$$\lim_{x\to z} \bar{g}_{\mu}{}^{\alpha} = \delta_{\mu}{}^{\alpha}, \tag{1.32}$$

from which its geometrical significance may be at once inferred. Equations (1.31) state that its covariant derivatives vanish in the directions tangent to the geodesic joining x and z, while Eq. (1.32) states that it reduces to the Kronecker

delta (or the ordinary metric tensor if both indices are in the same position) when the two points coincide. It is therefore evident that the result, for example, of applying $\bar{g}_{\mu}{}^{\alpha}$ to a local vector A_{α} at the point z is to obtain the local vector $\bar{A}_{\mu}$ at the point x which is generated from A_{α} by parallel displacement along the geodesic from z to x. Extension to the geodetic parallel displacement of local tensors of arbitrary order is obvious. In particular, the relations

$$\bar{g}_{\mu}{}^{\alpha}\bar{g}_{\nu}{}^{\beta}g_{\alpha\beta} = g_{\mu\nu}\,, \qquad \bar{g}^{\mu}{}_{\alpha}\bar{g}^{\nu}{}_{\beta}g_{\mu\nu} = g_{\alpha\beta}\,, \tag{1.33}$$

$$\bar{g}_{\mu}{}^{\alpha}\sigma_{\cdot\alpha} = -\sigma_{\cdot\mu}\,, \qquad \bar{g}^{\mu}{}_{\alpha}\sigma_{\cdot\mu} = -\sigma_{\cdot\alpha}\,, \tag{1.34}$$

$$\bar{g}_{\mu\alpha}\bar{g}^{\nu\alpha} = \delta_{\mu}{}^{\nu}, \qquad \bar{g}_{\mu\alpha}\bar{g}^{\mu\beta} = \delta_{\alpha}{}^{\beta}, \tag{1.35}$$

follow immediately from the geometrical interpretation.

The uniqueness of $\bar{g}_{\mu\alpha}$ may be inferred from the fact that for fixed z the first of Eqs. (1.31) may be integrated along each geodesic emanating from z, the initial value of $\bar{g}_{\mu\alpha}$ being set by (1.32). The second of Eqs. (1.31) may alternatively be used, by holding x fixed and integrating to z. This reciprocity is expressed by the symmetry relation

$$\bar{g}_{\mu\alpha}(x,z) = \bar{g}_{\alpha\mu}(z,x). \tag{1.36}$$

Returning now to the covariant expansion problem, suppose we have a bi-tensor, say $T_{\mu\alpha}$, whose indices refer to different points. We first "homogenize" it through application of the parallel displacement bi-vector:

$$\bar{T}_{\alpha\beta} = \bar{g}^{\mu}{}_{\alpha}T_{\mu\beta}\,, \tag{1.37}$$

and then expand the result according to the previously outlined method. Now, however, we should like to express the expansion coefficients in terms of the covariant derivatives of the original bi-tensor $T_{\mu\alpha}$ instead of $\bar{T}_{\alpha\beta}$. Therefore we must study the covariant derivatives of $\bar{g}_{\mu\alpha}$. Repeatedly differentiating Eq. (1.31), we get

$$0 = \bar{g}_{\mu\alpha\cdot\beta\delta}g^{\beta\gamma}\sigma_{\cdot\gamma} + \bar{g}_{\mu\alpha\cdot\beta}g^{\beta\gamma}\sigma_{\cdot\gamma\delta}\,, \tag{1.38}$$

$$0 = \bar{g}_{\mu\alpha\cdot\beta\delta\epsilon}g^{\beta\gamma}\sigma_{\cdot\gamma} + \bar{g}_{\mu\alpha\cdot\beta\delta}g^{\beta\gamma}\sigma_{\cdot\gamma\epsilon} + \bar{g}_{\mu\alpha\cdot\beta\epsilon}g^{\beta\gamma}\sigma_{\cdot\gamma\delta} + \bar{g}_{\mu\alpha\cdot\beta}g^{\beta\gamma}\sigma_{\cdot\gamma\delta\epsilon}\,. \tag{1.39}$$

Setting $x = z$ we then infer

$$\lim_{x\to z} \bar{g}_{\mu\alpha\cdot\beta} = 0, \tag{1.40}$$

$$\lim_{x\to z} \bar{g}_{\mu\alpha\cdot\beta\gamma} = \lim_{x\to z} \tfrac{1}{2}R_{\beta\gamma\alpha}{}^{\delta}\,\bar{g}_{\mu\delta}\,, \tag{1.41}$$

and hence

$$\lim_{x\to z} \bar{T}_{\alpha\beta\cdot\gamma} = \lim_{x\to z} \bar{g}^{\mu}{}_{\alpha}T_{\mu\beta\cdot\gamma}\,, \tag{1.42}$$

$$\lim_{x\to z} \bar{T}_{\alpha\beta\cdot\gamma\delta} = \lim_{x\to z} (\bar{g}^{\mu}{}_{\alpha}T_{\mu\beta\cdot\gamma\delta} + \tfrac{1}{2}R_{\gamma\delta\alpha}{}^{\epsilon}\,\bar{T}_{\epsilon\beta}). \tag{1.43}$$

Tensor *densities* as well as tensors may be subjected to a geodetic parallel displacement by means of the bi-vector $\bar{g}_{\mu}{}^{\alpha}$, provided one also introduces its determinant

$$\bar{\delta} = | \bar{g}_{\mu}{}^{\alpha} |. \tag{1.44}$$

This determinant is a bi-scalar density, having weight 1 at the point x and weight -1 at the point z. It satisfies the equations[2]

$$\bar{\delta}_{.\mu} g^{\mu\nu} \sigma_{.\nu} = 0, \qquad \bar{\delta}_{.\alpha} g^{\alpha\beta} \sigma_{.\beta} = 0, \tag{1.45}$$

$$\lim_{x \to z} \bar{\delta} = 1, \tag{1.46}$$

which are consequences of Eqs. (1.31), (1.32). Equations (1.45) and (1.46) have the unique solution[3]

$$\bar{\delta}(x,z) = g^{1/2}(x) g^{-1/2}(z) = \bar{\delta}^{-1}(z,x), \tag{1.47}$$

where

$$g = -| g_{\mu\nu} |. \tag{1.48}$$

The result, now, of making a parallel displacement of, say, a vector density A_{α} of weight w along the geodesic from z to x is given by

$$\bar{A}_{\mu} = \bar{\delta}^{w} \bar{g}_{\mu}{}^{\alpha} A_{\alpha}. \tag{1.49}$$

Extension to the general case is obvious.

Another determinant of fundamental importance in the theory of geodesics is given by[4]

$$D = -| D_{\mu\alpha} |, \tag{1.50}$$

$$D_{\mu\alpha} = -\sigma_{.\mu\alpha}. \tag{1.51}$$

D is a bi-scalar density of weight 1 at both x and z. It is nonvanishing, at least

[2] Here we make use of the definition $f_{.\mu} = f_{,\mu} - w\,\Gamma_{\nu\mu}{}^{\nu} f$ for the covariant derivative of a density f of weight w.

[3] This solution also follows from an explicit representation of the parallel displacement bi-vector in terms of *vierbeine*, viz.,

$$\bar{g}_{\mu\alpha} = \lambda_{\mu}{}^{(\beta)} \lambda_{\alpha(\beta)}.$$

Here the $\lambda_{\alpha(\beta)}$ are a set of four mutually orthogonal unit vectors at the point z, and the $\lambda_{\mu(\beta)}$ are obtained from these by parallel displacement along the geodesic from z to x. The indices in parentheses are "bein" indices.

[4] This determinant, originally introduced by Lipschitz (*5*), has been generalized by Van Vleck (*6*) for application to Hamilton–Jacobi theory (of which the theory of geodesics is just a special case) and its relation to quantum mechanics and the WKB approximation.

when x and z are sufficiently close together, since, by taking the appropriate covariant derivatives of Eqs. (1.34) and making use of (1.21) and (1.32), one finds

$$\lim_{x \to z} D_{\mu\alpha}(x,z) = g_{\mu\alpha}(z). \tag{1.52}$$

The bi-vector $D_{\mu\alpha}$ therefore has an inverse $D^{-1\,\mu\alpha}$, satisfying

$$D_{\mu\alpha} D^{-1\,\nu\alpha} = \delta_{\mu}{}^{\nu}, \qquad D_{\mu\alpha} D^{-1\,\mu\beta} = \delta_{\alpha}{}^{\beta}. \tag{1.53}$$

It is not difficult to see that D becomes singular at points where the geodesics emanating from z begin to cross one another. It is only necessary to note that D is the Jacobian of the transformation from the variables z^{α}, x^{μ} which specify the geodesic between x and z by means of its end points, to the variables z^{α}, $\sigma_{.\alpha}$ which specify it by means of *one* of its end points and a tangent vector at that point having a length equal to the length of the geodesic. If the tangent vector $\sigma_{.\alpha}$ is varied, the resulting variation in x^{μ} is given by

$$\delta x^{\mu} = -D^{-1\,\mu\alpha} \delta\sigma_{.\alpha}. \tag{1.54}$$

When $D^{-1} = 0$, or $D = \infty$, it is possible to choose a finite variation in $\sigma_{.\alpha}$ which produces no variation in x^{μ}. It is evident that this must occur in regions where the geodesics begin to cross. It can be shown, in fact, (*7*) that the loci of points at which $D^{-1} = 0$ are the *envelopes* of the family of geodesics emanating from z. These envelopes, familiarly known as "caustic surfaces," are generally three dimensional, although degenerate forms having a smaller number of dimensions, including zero (focal points), can occur.

In nonsingular regions the behavior of the determinant D is directly related to the rate at which the geodesics from z are converging or diverging. An important quantitative law can be obtained by repeated differentiation of Eq. (1.11), which yields

$$\sigma_{.\sigma} = g^{\mu\nu} \sigma_{.\mu} \sigma_{.\nu\sigma}, \tag{1.55}$$

$$\sigma_{.\sigma\alpha} = g^{\mu\nu} \sigma_{.\mu\alpha} \sigma_{.\nu\sigma} + g^{\mu\nu} \sigma_{.\mu} \sigma_{.\nu\sigma\alpha}. \tag{1.56}$$

The last equation may be rewritten in the form

$$D_{\sigma\alpha} = g^{\mu\nu} D_{\mu\alpha} \sigma_{.\nu\sigma} + g^{\mu\nu} \sigma_{.\mu} D_{\sigma\alpha.\nu}, \tag{1.57}$$

which, on multiplication by $D^{-1\,\sigma\alpha}$, gives

$$D^{-1}(D\sigma^{.\mu})_{.\mu} = 4. \tag{1.58}$$

This equation may be recast in terms of the arc length s along each geodesic:

$$\sigma^{.\mu}{}_{\mu} = 4 - s\, d(\ln D)/ds, \tag{1.59}$$

from which it may immediately be inferred that D decreases or increases along each geodesic according as the rate of divergence of the neighboring geodesics, which is measured by $\sigma_{.}{}^{\mu}{}_{\mu}$, is greater or less than 4, the rate in flat space-time.

Instead of working directly with D, which is a bi-density, it will sometimes be more convenient to work with the bi-scalar

$$\Delta = \bar{g}^{-1}D, \tag{1.60}$$

where

$$\bar{g} = -|\bar{g}_{\mu\alpha}|. \tag{1.61}$$

Evidently

$$\bar{g}(x,z) = g^{1/2}(x)g^{1/2}(z) = \bar{g}(z,x), \tag{1.62}$$

and therefore Eq. (1.58) may equally well be written in the form

$$\Delta^{-1}(\Delta\sigma_{.}{}^{\mu})_{.\mu} = 4. \tag{1.63}$$

For purposes of covariant expansion we introduce the bi-tensors

$$\bar{D}_{\alpha\beta} = \bar{g}^{\mu}{}_{\alpha}D_{\mu\beta}, \qquad \bar{D}^{-1\ \alpha\beta} = \bar{g}_{\mu}{}^{\alpha}D^{-1\ \mu\beta}. \tag{1.64}$$

Their expansion coefficients are determined by the limiting behavior of the covariant derivatives of $D_{\mu\alpha}$, which may be obtained by repeatedly differentiating the first of Eqs. (1.34):

$$D_{\mu\beta} = \bar{g}_{\mu}{}^{\alpha}{}_{.\beta}\sigma_{.\alpha} + \bar{g}_{\mu}{}^{\alpha}\sigma_{.\alpha\beta}, \tag{1.65}$$

$$D_{\mu\beta\cdot\gamma} = \bar{g}_{\mu}{}^{\alpha}{}_{.\beta\gamma}\sigma_{.\alpha} + \bar{g}_{\mu}{}^{\alpha}{}_{.\beta}\sigma_{.\alpha\gamma} + \bar{g}_{\mu}{}^{\alpha}{}_{.\gamma}\sigma_{.\alpha\beta} + \bar{g}_{\mu}{}^{\alpha}\sigma_{.\alpha\beta\gamma}, \tag{1.66}$$

$$\begin{aligned} D_{\mu\beta\cdot\gamma\delta} = {} & \bar{g}_{\mu}{}^{\alpha}{}_{.\beta\gamma\delta}\sigma_{.\alpha} + \bar{g}_{\mu}{}^{\alpha}{}_{.\beta\gamma}\sigma_{.\alpha\delta} + \bar{g}_{\mu}{}^{\alpha}{}_{.\beta\delta}\sigma_{.\alpha\gamma} + \bar{g}_{\mu}{}^{\alpha}{}_{.\beta}\sigma_{.\alpha\gamma\delta} + \bar{g}_{\mu}{}^{\alpha}{}_{.\gamma\delta}\sigma_{.\alpha\beta} \\ & + \bar{g}_{\mu}{}^{\alpha}{}_{.\gamma}\sigma_{.\alpha\beta\delta} + \bar{g}_{\mu}{}^{\alpha}{}_{.\delta}\sigma_{.\alpha\beta\gamma} + \bar{g}_{\mu}{}^{\alpha}\sigma_{.\alpha\beta\gamma\delta}. \end{aligned} \tag{1.67}$$

Setting $x = z$, and making use of Eqs. (1.18), (1.21), (1.22), (1.23), (1.40), and (1.41), we well as the algebraic identities satisfied by the Riemann tensor, we infer, in addition to Eq. (1.52), the equations

$$\lim_{x\to z} D_{\mu\beta\cdot\gamma} = 0, \tag{1.68}$$

$$\lim_{x\to z} D_{\mu\beta\cdot\gamma\delta} = \lim_{x\to z} \bar{g}_{\mu}{}^{\alpha}(-\tfrac{2}{3}R_{\alpha\gamma\beta\delta} + \tfrac{1}{3}R_{\alpha\delta\beta\gamma}), \tag{1.69}$$

whence, using (1.42) and (1.43), we get

$$\lim_{x\to z} \bar{D}_{\alpha\beta} = g_{\alpha\beta}, \tag{1.70}$$

$$\lim_{x\to z} \bar{D}_{\alpha\beta\cdot\gamma} = 0, \tag{1.71}$$

$$\lim_{x\to z} \bar{D}_{\alpha\beta\cdot\gamma\delta} = -\tfrac{1}{6}(R_{\alpha\gamma\beta\delta} + R_{\alpha\delta\beta\gamma}), \tag{1.72}$$

and hence

$$\bar{D}_{\alpha\beta} = g_{\alpha\beta} - \tfrac{1}{6}R_{\alpha}{}^{\gamma}{}_{\beta}{}^{\delta}\sigma_{.\gamma}\sigma_{.\delta} + O(s^3), \tag{1.73}$$

$$\bar{D}^{-1\ \alpha\beta} = g^{\alpha\beta} + \tfrac{1}{6}R^{\alpha\gamma\beta\delta}\sigma_{.\gamma}\sigma_{.\delta} + O(s^3). \tag{1.74}$$

Finally, noting that

$$|\bar{D}_{\alpha}{}^{\beta}| = g^{-1}(z)\bar{\delta}^{-1}D = \bar{g}^{-1}D = \Delta, \tag{1.75}$$

we have the expansion

$$\Delta = 1 - \tfrac{1}{6}R^{\alpha\beta}\sigma_{.\alpha}\sigma_{.\beta} + O(s^3), \tag{1.76}$$

our convention for the Ricci tensor being

$$R_{\alpha\beta} = g^{\gamma\delta}R_{\alpha\gamma\beta\delta}, \qquad R = g^{\alpha\beta}R_{\alpha\beta}. \tag{1.77}$$

2. GREEN'S FUNCTIONS

In this section we look for solutions of the covariant scalar and vector wave equations:

$$g^{\mu\nu}\phi_{.\mu\nu} = 0, \tag{2.1}$$

$$g^{\nu\sigma}A_{\mu\cdot\nu\sigma} + R_{\mu}{}^{\nu}A_{\nu} = 0. \tag{2.2}$$

We study the scalar equation first. Following Hadamard (Ref. *1*, p. 100) we try to find a so-called "elementary solution" which, in the case of 4-dimensional space-time, is a bi-scalar having the form

$$G^{(1)} = \frac{1}{(2\pi)^2}\left(\frac{u}{\sigma} + v\ln|\sigma| + w\right), \tag{2.3}$$

where u, v, and w are bi-scalars which are free of singularities and satisfy the normalization condition

$$\lim_{x\to z} u = 1. \tag{2.4}$$

With use of Eqs. (1.11) and (1.63) of the preceding section, a straightforward computation gives

$$\begin{aligned} (2\pi)^2 g^{\mu\nu}G^{(1)}{}_{.\mu\nu} = {} & -\sigma^{-2}g^{\mu\nu}(2u_{.\mu} - u\Delta^{-1}\Delta_{.\mu})\sigma_{.\nu} \\ & + \sigma^{-1}[2v + g^{\mu\nu}(2v_{.\mu} - v\Delta^{-1}\Delta_{.\mu})\sigma_{.\nu} + g^{\mu\nu}u_{.\mu\nu}] \\ & + g^{\mu\nu}v_{.\mu\nu}\ln|\sigma| + g^{\mu\nu}w_{.\mu\nu}. \end{aligned} \tag{2.5}$$

In order for this expression to vanish the coefficient of the logarithmic factor must vanish everywhere, and the coefficients of the singular factors σ^{-2} and σ^{-1} must vanish at least on the light cone, while the term $g^{\mu\nu}w_{.\mu\nu}$ must make up the difference off the light cone. This is achieved most simply by taking the coefficient of σ^{-2} equal to zero everywhere so that we have

$$g^{\mu\nu}(2u_{\cdot\mu} - u\Delta^{-1}\Delta_{\cdot\mu})\sigma_{\cdot\nu} = 0, \tag{2.6}$$

$$g^{\mu\nu}v_{\cdot\mu\nu} = 0, \tag{2.7}$$

and by computing v and w in the form of expansions:

$$v = \sum_{n=0}^{\infty} v_n\sigma^n, \qquad w = \sum_{n=0}^{\infty} w_n\sigma^n. \tag{2.8}$$

Substituting these expansions into (2.7) and the equation

$$2v + g^{\mu\nu}(2v_{\cdot\mu} - v\Delta^{-1}\Delta_{\cdot\mu})\sigma_{\cdot\nu} + g^{\mu\nu}u_{\cdot\mu\nu} + \sigma g^{\mu\nu}w_{\cdot\mu\nu} = 0, \tag{2.9}$$

and making use of (1.11) and (1.63), we find the following recurrence formulae for the coefficients:

$$v_0 + g^{\mu\nu}(v_{0\cdot\mu} - \tfrac{1}{2}v_0\Delta^{-1}\Delta_{\cdot\mu})\sigma_{\cdot\nu} = -\tfrac{1}{2}g^{\mu\nu}u_{\cdot\mu\nu}, \tag{2.10}$$

$$\begin{aligned} v_n + (n+1)^{-1}g^{\mu\nu}(v_{n\cdot\mu} - \tfrac{1}{2}v_n\Delta^{-1}\Delta_{\cdot\mu})\sigma_{\cdot\nu} \\ = -\tfrac{1}{2}n^{-1}(n+1)^{-1}g^{\mu\nu}v_{n-1\cdot\mu\nu}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} w_n + (n+1)^{-1}g^{\mu\nu}(w_{n\cdot\mu} - \tfrac{1}{2}w_n\Delta^{-1}\Delta_{\cdot\mu})\sigma_{\cdot\nu} \\ = -\tfrac{1}{2}n^{-1}(n+1)^{-1}g^{\mu\nu}w_{n-1\cdot\mu\nu} - (n+1)^{-1}v_n \\ + \tfrac{1}{2}n^{-2}(n+1)^{-1}g^{\mu\nu}v_{n-1\cdot\mu\nu}, \end{aligned} \tag{2.12}$$

with $n = 1, 2, 3, \cdots$. Each of these equations may be integrated along each geodesic emanating from the point z, and all the v's are thereby uniquely determined. It is to be noted, however, that the w's are not uniquely fixed, since w_0 remains completely arbitrary. This arbitrariness corresponds to the possibility of adding to $G^{(1)}$ any singularity-free solution of the wave equation. Hadamard (*1*) has shown that the series (2.8) converge uniformly inside the region for which σ is single valued, provided the metric is analytic, and Riesz (*8*) has extended the proof to a broad class of nonanalytic cases.

Equation (2.6) may likewise be integrated along each geodesic emanating from z. Its validity along each geodesic, however, is equivalent to the validity everywhere of the equation

$$u^{-1}u_{\cdot\mu} = \tfrac{1}{2}\Delta^{-1}\Delta_{\cdot\mu}, \tag{2.13}$$

which, with the boundary condition (2.4), has the unique solution

$$u = \Delta^{1/2}. \tag{2.14}$$

When this solution is substituted into (2.10) and use is made of the expansion (1.76), one finds

$$\lim_{x\to z} v = \lim_{x\to z} v_0 = \tfrac{1}{12}R. \tag{2.15}$$

The Green's functions for the wave equation can be obtained from Hadamard's "elementary solution" $G^{(1)}$ by moving into the complex plane. The procedure is familiar from quantum field theory. One introduces the "Feynman propagator"[5]

$$G^F = \frac{1}{(2\pi)^2}\left(\frac{\Delta^{1/2}}{\sigma + i0} + v\ln(\sigma + i0) + w\right) \tag{2.16}$$

and then separates it into real and imaginary parts:

$$G^F = G^{(1)} - 2i\bar{G}. \tag{2.17}$$

Using the well-known formal identities

$$\frac{1}{\sigma + i0} = \mathcal{P}\frac{1}{\sigma} - \pi i\delta(\sigma), \tag{2.18}$$

$$\ln(\sigma + i0) = \ln|\sigma| + \pi i\theta(-\sigma). \tag{2.19}$$

where

$$\theta(\sigma) = \begin{cases} 0 & \text{for } \sigma < 0, \\ 1 & \text{for } \sigma > 0, \end{cases} \tag{2.20}$$

one obtains, for the "symmetric" Green's function $\bar{G}$,

$$\bar{G} = (8\pi)^{-1}[\Delta^{1/2}\delta(\sigma) - v\theta(-\sigma)]. \tag{2.21}$$

Three important properties of this function will immediately be noted. Firstly, it is independent of the bi-scalar w and is hence unique. Secondly, it vanishes for space-like separation of the points x and z, i.e., for $\sigma > 0$. Thirdly, although it has the same delta-function singularity on the light cone as in the case of flat space-time, it does not generally vanish *inside* the light cone. The bi-scalar v

[5] One must exercise caution, in the present context, in calling this function the "vacuum expectation value of a time-ordered product of massless scalar field operators" in a curved space-time with fixed metric. In order to do this one must first define the "vacuum," and this requires a convention about asymptotic boundary conditions, particularly in view of the fact that the given metric is not required to be time independent, and hence can create pairs of scalar photons. An explicit analysis of the propagator into positive and negative frequency components can only be made on the basis of perturbation theory, and this requires the use of some sort of adiabatic switching hypothesis in which space-time is imagined to be flat in the remote past. Only when such boundary requirements are met will the function w be well-defined.

represents the "tail" of the Green's function, which, as was mentioned in the Introduction, is nonvanishing in a curved manifold.

By examining the behavior of the symmetric Green's function in the region where x is close to z, and comparing it with the behavior of the corresponding function for a flat space-time,[6] it is easy to see that it satisfies the covariant generalization of the *inhomogeneous* wave equation satisfied by the latter function, namely

$$g^{\mu\nu}\bar{G}_{\cdot\mu\nu} = -\bar{g}^{-1/2}\delta^{(4)}. \tag{2.22}$$

Moreover, the function $\bar{G}$ can, just as in the flat-space case, be separated into "retarded" and "advanced" parts satisfying the same equation:

$$\bar{G} = \tfrac{1}{2}(G^{\text{ret}} + G^{\text{adv}}), \tag{2.23}$$

$$g^{\mu\nu}G^{\text{ret}}{}_{\cdot\mu\nu} = g^{\mu\nu}G^{\text{adv}}{}_{\cdot\mu\nu} = -\bar{g}^{-1/2}\delta^{(4)}, \tag{2.24}$$

with

$$G^{\text{ret}}(x,z) = 2\theta[\Sigma(x),z]\bar{G}(x,z), \tag{2.25}$$

$$G^{\text{adv}}(x,z) = 2\theta[z,\Sigma(x)]\bar{G}(x,z), \tag{2.26}$$

where $\Sigma(x)$ is an arbitrary space-like hypersurface containing x, and $\theta[\Sigma(x),z] = 1 - \theta[z,\Sigma(x)]$ is equal to 1 when z lies to the past of $\Sigma(x)$ and vanishes when z lies to the future. The fact that the retarded and advanced Green's functions are defined directly in terms of "past" and "future" deserves special emphasis in the present context. When space-time is curved it is generally *not* possible to define these functions in terms of "incoming" and "outgoing" waves, because a wave which starts out, for example, as "outgoing" may find eventually that a portion of itself has become "incoming" owing to "scattering" by the space-time curvature, which is described by the "tail" function v.

The various Green's functions serve to give integral definitions of particular solutions of the general inhomogeneous wave equation

$$g^{1/2}g^{\mu\nu}\phi_{\cdot\mu\nu} = -j. \tag{2.27}$$

Thus

$$\phi^{\text{ret}}(x) = \int G^{\text{ret}}(x,z)j(z)\,d^4z, \tag{2.28}$$

$$\phi^{\text{adv}}(x) = \int G^{\text{adv}}(x,z)j(z)\,d^4z. \tag{2.29}$$

The "source" $j(x)$ is here taken as a scalar density. The Green's functions may also be used to define a solution of the homogeneous wave equation, relative to the source $j(x)$, namely

$$\phi^{\text{rad}}(x) = -\int G(x,z)j(z)\,d^4z, \tag{2.30}$$

where

$$G = G^{\text{adv}} - G^{\text{ret}}, \tag{2.31}$$

$$g^{\mu\nu}G_{\cdot\mu\nu} = 0. \tag{2.32}$$

A very important property of the function G is its ability to express the covariant generalization of Huygens' principle:

$$\phi(z) = \int_\Sigma g^{1/2}(x)[\phi(x)G_{\cdot\mu}(x,z) - \phi_{\cdot\mu}(x)G(x,z)]g^{\mu\nu}(x)\,d\Sigma_\nu. \tag{2.33}$$

Here the value at an arbitrary point z of a function ϕ satisfying the homogeneous wave equation is expressed in terms of Cauchy data (ϕ and $\phi_{\cdot\mu}$) on an arbitrary space-like hypersurface Σ having directed surface element (vector density) $d\Sigma_\mu$. The proof of this relation may be carried out either by choosing Σ so as to pass through z and comparing the singular behavior of $G_{\cdot\mu}$ on Σ with its behavior in the flat-space case, or, formally, by changing the surface integral into a volume integral with the aid of Gauss' theorem. Following the latter procedure we have, for z lying to the past of Σ,

$$\phi(z) = -\int_{\text{past}}^{\Sigma} g^{1/2}(x)g^{\mu\nu}(x)[\phi(x)G^{\text{ret}}{}_{\cdot\mu}(x,z) - \phi_{\cdot\mu}(x)G^{\text{ret}}(x,z)]_{\cdot\nu}\,d^4x, \tag{2.34}$$

and, for z lying to the future of Σ,

$$\phi(z) = -\int_{\Sigma}^{\text{future}} g^{1/2}(x)g^{\mu\nu}(x)[\phi(x)G^{\text{adv}}{}_{\cdot\mu}(x,z) - \phi_{\cdot\mu}(x)G^{\text{adv}}(x,z)]_{\cdot\nu}\,d^4x. \tag{2.35}$$

The validity of these equations follows immediately from (2.24) and the wave equation satisfied by ϕ.

Because ϕ satisfies the wave equation and because the Cauchy data on Σ may be taken completely arbitrarily, we may infer from Eq. (2.33) that G satisfies not only the equation (2.32) but also the equation

$$g^{\alpha\beta}G_{\cdot\alpha\beta} = 0. \tag{2.36}$$

However, since there is only one unique function, namely $-G(z,x)$, having the properties of $G(x,z)$ and satisfying this equation, we infer from this the symmetry properties

$$G(x,z) = -G(z,x), \tag{2.37}$$

[6] For example, by introducing geodesic normal coordinates at z.

and hence

$$G^{\rm ret}(x,z) = G^{\rm adv}(z,x), \tag{2.38}$$

$$\bar{G}(x,z) = \bar{G}(z,x), \tag{2.39}$$

$$v(x,z) = v(z,x), \tag{2.40}$$

which would be difficult to obtain directly from the defining equations (2.10) and (2.11). These symmetry properties permit one to reexpress the generalized Huygens' principle in the form

$$\phi(x) = \int_\Sigma g^{1/2}(z)[G(x,z)\phi_{\cdot\alpha}(z) - G_{\cdot\alpha}(x,z)\phi(z)]g^{\alpha\beta}(z)\, d\Sigma_\beta. \tag{2.41}$$

We turn now to the vector wave equation. The procedure is entirely analogous to the foregoing. One introduces an "elementary solution" of the form

$$G^{(1)}{}_{\mu\alpha} = \frac{1}{(2\pi)^2}\left(\frac{u_{\mu\alpha}}{\sigma} + v_{\mu\alpha}\ln|\sigma| + w_{\mu\alpha}\right), \tag{2.42}$$

where the functions $u_{\mu\alpha}$, $v_{\mu\alpha}$, $w_{\mu\alpha}$ are now bi-vectors. Expanding $v_{\mu\alpha}$ and $w_{\mu\alpha}$ in series

$$v_{\mu\alpha} = \sum_{n=0}^{\infty} v_{n\,\mu\alpha}\sigma^n, \qquad w_{\mu\alpha} = \sum_{n=0}^{\infty} w_{n\,\mu\alpha}\sigma^n, \tag{2.43}$$

and inserting everything into the equation

$$g^{\nu\sigma}G^{(1)}{}_{\mu\alpha\cdot\nu\sigma} + R_\mu{}^\nu G^{(1)}{}_{\nu\alpha} = 0, \tag{2.44}$$

one obtains the following equations for $u_{\mu\alpha}$ and the v's and w's:

$$g^{\nu\sigma}(2u_{\mu\alpha\cdot\nu} - u_{\mu\alpha}\Delta^{-1}\Delta_{\cdot\nu})\sigma_{\cdot\sigma} = 0, \tag{2.45}$$

$$v_{0\,\mu\alpha} + g^{\nu\sigma}(v_{0\,\mu\alpha\cdot\nu} - \tfrac{1}{2}v_{0\,\mu\alpha}\Delta^{-1}\Delta_{\cdot\nu})\sigma_{\cdot\sigma} = -\tfrac{1}{2}(g^{\nu\sigma}u_{\mu\alpha\cdot\nu\sigma} + R_\mu{}^\nu u_{\nu\alpha}), \tag{2.46}$$

$$v_{n\,\mu\alpha} + (n+1)^{-1}g^{\nu\sigma}(v_{n\,\mu\alpha\cdot\nu} - \tfrac{1}{2}v_{n\,\mu\alpha}\Delta^{-1}\Delta_{\cdot\nu})\sigma_{\cdot\sigma} = -\tfrac{1}{2}n^{-1}(n+1)^{-1}(g^{\nu\sigma}v_{n-1\,\mu\alpha\cdot\nu\sigma} + R_\mu{}^\nu v_{n-1\,\nu\alpha}), \tag{2.47}$$

$$\begin{aligned} w_{n\,\mu\alpha} + (n+1)^{-1}g^{\nu\sigma}(w_{n\,\mu\alpha\cdot\nu} - \tfrac{1}{2}w_{n\,\mu\alpha}\Delta^{-1}\Delta_{\cdot\nu})\sigma_{\cdot\sigma} &= -\tfrac{1}{2}n^{-1}(n+1)^{-1}(g^{\nu\sigma}w_{n-1\,\mu\alpha\cdot\nu\sigma} + R_\mu{}^\nu w_{n-1\,\nu\alpha}) \\ &\quad - (n+1)^{-1}v_{n\,\mu\alpha} + \tfrac{1}{2}n^{-2}(n+1)^{-1}(g^{\nu\sigma}v_{n-1\,\mu\alpha\cdot\nu\sigma} + R_\mu{}^\nu v_{n-1\,\nu\alpha}), \end{aligned} \tag{2.48}$$

with $n = 1, 2, 3, \cdots$. Again $w_{0\,\mu\alpha}$ is arbitrary.

The appropriate normalization for the bi-vector $u_{\mu\alpha}$ is obviously

$$\lim_{x\to z} u_{\mu\alpha}(x,z) = g_{\mu\alpha}(z). \tag{2.49}$$

With this boundary condition we find, upon taking note of (1.31), that Eq. (2.45) has the unique solution

$$u_{\mu\alpha} = \Delta^{1/2}\bar{g}_{\mu\alpha}. \tag{2.50}$$

The Feynman propagator therefore takes the form

$$G^F{}_{\mu\alpha} = \frac{1}{(2\pi)^2}\left(\frac{\Delta^{1/2}\bar{g}_{\mu\alpha}}{\sigma + i0} + v_{\mu\alpha}\ln(\sigma + i0) + w_{\mu\alpha}\right), \tag{2.51}$$

which, upon being separated into real and imaginary parts,

$$G^F{}_{\mu\alpha} = G^{(1)}{}_{\mu\alpha} - 2i\bar{G}_{\mu\alpha}, \tag{2.52}$$

gives, for the symmetric Green's function

$$\bar{G}_{\mu\alpha} = (8\pi)^{-1}[\Delta^{1/2}\bar{g}_{\mu\alpha}\delta(\sigma) - v_{\mu\alpha}\theta(-\sigma)]. \tag{2.53}$$

The appearance of the parallel-displacement bi-vector as a factor in the delta-function term shows that at the front of an initially sharp pulse of electromagnetic radiation the polarization vector is propagated in a parallel manner along the null geodesics.[7] The "twist" in the polarization, produced by the appearance of the Ricci tensor in the vector wave equation as well as by curvature scattering effects, takes place only behind the front, in the "tail" region which is described by the bi-vector $v_{\mu\alpha}$.

Because of the "twist" effects the structure of the bi-vector $v_{\mu\alpha}$ is more complicated than that of the function v in the case of the scalar wave equation. In order to determine its limiting behavior as $x \to z$ we make use of the expansion

$$u_{\mu\alpha} = [1 - \tfrac{1}{12}R^{\beta\gamma}\sigma_{\cdot\beta}\sigma_{\cdot\gamma} + O(s^3)]\bar{g}_{\mu\alpha}. \tag{2.54}$$

From Eq. (1.41) we may infer that

$$\bar{g}_{\mu\alpha\cdot\beta} = \tfrac{1}{2}\bar{g}_{\mu\delta}R_\beta{}^\gamma{}_\alpha{}^\delta\sigma_{\cdot\gamma} + O(s^2), \tag{2.55}$$

$$\bar{g}_{\mu\alpha\cdot\beta\gamma} = \tfrac{1}{2}\bar{g}_{\mu\delta}R_{\beta\gamma\alpha}{}^\delta + O(s), \tag{2.56}$$

and, by symmetry in x and z,

$$\bar{g}_{\mu\alpha\cdot\nu\sigma} = \tfrac{1}{2}\bar{g}_{\tau\alpha}R_{\nu\sigma\mu}{}^\tau + O(s). \tag{2.57}$$

Therefore

$$g^{\nu\sigma}u_{\mu\alpha\cdot\nu\sigma} = -\tfrac{1}{6}\bar{g}_{\mu\alpha}R + O(s), \tag{2.58}$$

[7] The determinant $\Delta^{1/2}$ obviously describes the anomalous "crowding" or "thinning out" of the elementary waves emanating from z, due to curvature-induced deviations of $\sigma_{\cdot}{}^\mu{}_\mu$ from the value 4. The exponent ½ arises from the fact that energy and momentum, which locally are the actually conserved quantities, are quadratic in the field amplitudes.

and, substituting this result into (2.46), we find

$$\lim_{x\to z} v_{\mu\alpha} = \lim_{x\to z} v_{0\,\mu\alpha} = -\tfrac{1}{2}\bar{g}_\mu{}^\beta(R_{\alpha\beta} - \tfrac{1}{6}g_{\alpha\beta}R). \tag{2.59}$$

For later use we record here also the expansions

$$u_{\mu\alpha\cdot\beta} = (-\tfrac{1}{2}\bar{g}_\mu{}^\delta R_{\delta\alpha\beta}{}^\gamma - \tfrac{1}{6}\bar{g}_{\mu\alpha}R_\beta{}^\gamma)\sigma_{\cdot\gamma} + O(s^2), \tag{2.60}$$

$$u_{\mu\alpha\cdot\beta\gamma} = -\tfrac{1}{2}\bar{g}_\mu{}^\delta R_{\delta\alpha\beta\gamma} - \tfrac{1}{6}\bar{g}_{\mu\alpha}R_{\beta\gamma} + O(s), \tag{2.61}$$

which follow from (2.54) and (2.55).

We define, as before, the various Green's functions

$$G^{\text{ret}}{}_{\mu\alpha}(x,z) = 2\theta[\Sigma(x),z]\bar{G}_{\mu\alpha}(x,z), \tag{2.62}$$

$$G^{\text{adv}}{}_{\mu\alpha}(x,z) = 2\theta[z,\Sigma(x)]\bar{G}_{\mu\alpha}(x,z), \tag{2.63}$$

$$G_{\mu\alpha} = G^{\text{adv}}{}_{\mu\alpha} - G^{\text{ret}}{}_{\mu\alpha}, \tag{2.64}$$

satisfying the equations

$$\bar{G}_{\mu\alpha} = \tfrac{1}{2}(G^{\text{ret}}{}_{\mu\alpha} + G^{\text{adv}}{}_{\mu\alpha}), \tag{2.65}$$

$$\begin{aligned} g^{\nu\sigma}\bar{G}_{\mu\alpha\cdot\nu\sigma} + R_\mu{}^\nu\bar{G}_{\nu\alpha} &= g^{\nu\sigma}G^{\text{ret}}{}_{\mu\alpha\cdot\nu\sigma} + R_\mu{}^\nu G^{\text{ret}}{}_{\nu\alpha} \\ &= g^{\nu\sigma}G^{\text{adv}}{}_{\mu\alpha\cdot\nu\sigma} + R_\mu{}^\nu G^{\text{adv}}{}_{\nu\alpha} = -\bar{g}^{-1/2}\bar{g}_{\mu\alpha}\delta^{(4)}, \end{aligned} \tag{2.66}$$

$$g^{\nu\sigma}G_{\mu\alpha\cdot\nu\sigma} + R_\mu{}^\nu G_{\nu\alpha} = 0. \tag{2.67}$$

Integral solutions of the general inhomogeneous vector wave equation, as well as the generalized Huygen's principle for the homogeneous equation, will be given in terms of these functions in the next section. By proceeding exactly as in the scalar case one can derive the symmetry properties

$$G_{\mu\alpha}(x,z) = -G_{\alpha\mu}(z,x), \tag{2.68}$$

$$G^{\text{ret}}{}_{\mu\alpha}(x,z) = G^{\text{adv}}{}_{\alpha\mu}(z,x), \tag{2.69}$$

$$\bar{G}_{\mu\alpha}(x,z) = \bar{G}_{\alpha\mu}(z,x), \tag{2.70}$$

$$v_{\mu\alpha}(x,z) = v_{\alpha\mu}(z,x). \tag{2.71}$$

Finally, we may establish an important relation between the bi-scalar and bi-vector Green's functions. For this we need the easily verified identity

$$(\bar{g}^{-1/2}\bar{g}^\mu{}_\alpha\delta^{(4)})_{\cdot\mu} = -(\bar{g}^{-1/2}\delta^{(4)})_{\cdot\alpha}. \tag{2.72}$$

Taking the covariant divergence of the first of Eqs. (2.66), we find, after commuting indices and making use of the algebraic properties of the Riemann tensor,

$$\begin{aligned}(\bar{g}^{-1/2}\delta^{(4)})_{\cdot\alpha} &= g^{\nu\sigma}\bar{G}^\mu{}_{\alpha\cdot\nu\sigma\mu} + (R^\mu{}_\nu\bar{G}^\nu{}_\alpha)_{\cdot\mu} \\ &= g^{\nu\sigma}\bar{G}^\mu{}_{\alpha\cdot\mu\nu\sigma}.\end{aligned} \tag{2.73}$$

From the properties of $\bar{G}^\mu{}_{\alpha\cdot\mu}$ and arguments of uniqueness we then infer

$$\bar{G}^\mu{}_{\alpha\cdot\mu} = -\bar{G}_{\cdot\alpha}, \tag{2.74}$$

and hence

$$G^{\text{ret}\,\mu}{}_{\alpha\cdot\mu} = -G^{\text{ret}}{}_{\cdot\alpha}, \qquad G^{\text{adv}\,\mu}{}_{\alpha\cdot\mu} = -G^{\text{adv}}{}_{\cdot\alpha}, \qquad G^\mu{}_{\alpha\cdot\mu} = -G_{\cdot\alpha}, \tag{2.75}$$

which follow from (2.74) because the derivative of the step function θ which intervenes in these latter equations can make a contribution only when $x = z$ at which point $\bar{g}^\mu{}_\alpha\,\theta_{\cdot\mu} = -\theta_{\cdot\alpha}$.

3. EQUATIONS OF CLASSICAL ELECTRODYNAMICS

The Lagrangian density for a structureless point particle of charge e and "bare" mass m_0, interacting with an electromagnetic field $F_{\mu\nu}$ in a space-time of arbitrary fixed metric, is given in nonrationalized units by

$$\begin{aligned}\mathfrak{L} &= -m_0c^2\int(-g_{\alpha\beta}\dot{z}^\alpha\dot{z}^\beta)^{1/2}\delta^{(4)}\,d\tau + e\int A_\alpha\dot{z}^\alpha\delta^{(4)}\,d\tau - (16\pi)^{-1}g^{1/2}F_{\mu\nu}F^{\mu\nu}, \\ &= c\int L_0\delta^{(4)}\,d\tau + c^{-1}A_\mu j^\mu - (16\pi)^{-1}g^{1/2}F_{\mu\nu}F^{\mu\nu},\end{aligned} \tag{3.1}$$

where c is the velocity of light and

$$F_{\mu\nu} = A_{\nu\cdot\mu} - A_{\mu\cdot\nu}, \tag{3.2}$$

$$L_0 = -m_0c(-g_{\alpha\beta}\dot{z}^\alpha\dot{z}^\beta)^{1/2}, \tag{3.3}$$

$$j^\mu = ec\int\bar{\delta}^{1/2}\bar{g}^\mu{}_\alpha\dot{z}^\alpha\delta^{(4)}\,d\tau. \tag{3.4}$$

The particle traces out a world-line in space-time given by a set of functions $z^\alpha(\tau)$, where τ is an arbitrary parameter which increases monotonically as the particle goes to the future. Dots over the z's denote differentiation with respect to this parameter. Multiple dots will be used to denote repeated *absolute covariant differentiation* with respect to τ. Thus

$$\dot{z}^\alpha = dz^\alpha/d\tau, \tag{3.5}$$

$$\ddot{z}^\alpha = d\dot{z}^\alpha/d\tau + \Gamma_{\beta\gamma}{}^\alpha\dot{z}^\beta\dot{z}^\gamma, \tag{3.6}$$

$$\dddot{z}^\alpha = d\ddot{z}^\alpha/d\tau + \Gamma_{\beta\gamma}{}^\alpha\ddot{z}^\beta\dot{z}^\gamma, \text{ etc.} \tag{3.7}$$

These structures all transform like contravariant vectors. Use of absolute covariant derivatives allows one to write the τ-derivatives of any scalar function f of the z's in the manifestly covariant forms

$$\dot{f} = f_{\cdot\alpha}\, \dot{z}^{\alpha}, \tag{3.8}$$

$$\ddot{f} = f_{\cdot\alpha\beta}\, \dot{z}^{\alpha} \dot{z}^{\beta} + f_{\cdot\alpha}\, \ddot{z}^{\alpha}, \tag{3.9}$$

$$\dddot{f} = f_{\cdot\alpha\beta\gamma}\, \dot{z}^{\alpha} \dot{z}^{\beta} \dot{z}^{\gamma} + 3 f_{\cdot\alpha\beta}\, \ddot{z}^{\alpha} \dot{z}^{\beta} + f_{\cdot\alpha}\, \dddot{z}^{\alpha}, \tag{3.10}$$

$$\ddddot{f} = f_{\cdot\alpha\beta\gamma\delta}\, \dot{z}^{\alpha} \dot{z}^{\beta} \dot{z}^{\gamma} \dot{z}^{\delta} + 5 f_{\cdot\alpha\beta\gamma}\, \ddot{z}^{\alpha} \dot{z}^{\beta} \dot{z}^{\gamma} + f_{\cdot\alpha\beta\gamma}\, \dot{z}^{\alpha} \dot{z}^{\beta} \ddot{z}^{\gamma} + 4 f_{\cdot\alpha\beta}\, \dddot{z}^{\alpha} \dot{z}^{\beta} + 3 f_{\cdot\alpha\beta}\, \ddot{z}^{\alpha} \ddot{z}^{\beta} + f_{\cdot\alpha}\, \ddddot{z}^{\alpha}, \text{ etc.} \tag{3.11}$$

The action functional for the system is given by

$$S = c^{-1} \int \mathfrak{L}\, d^4x, \tag{3.12}$$

where the integration is to be extended over the region between any two spacelike hypersurfaces. If variations in the dynamical variables z^{α} and A_{μ} are taken which vanish on these hypersurfaces then the action suffers the variation

$$\delta S = \int (-m_0 g_{\alpha\beta}\, \ddot{z}^{\beta} + e c^{-1} F_{\alpha\beta}\, \dot{z}^{\beta}) \delta z^{\alpha}\, d\tau + c^{-1} \int [-(4\pi)^{-1} g^{1/2} F^{\mu\nu}{}_{\cdot\nu} + c^{-1} j^{\mu}] \delta A_{\mu}\, d^4x, \tag{3.13}$$

provided, as will henceforth be assumed, τ is taken to be the proper time of the particle, so that

$$g_{\alpha\beta}\, \dot{z}^{\alpha} \dot{z}^{\beta} = -c^2, \tag{3.14}$$

$$g_{\alpha\beta}\, \dot{z}^{\alpha} \ddot{z}^{\beta} = 0, \tag{3.15}$$

$$g_{\alpha\beta}\, \dot{z}^{\alpha} \dddot{z}^{\beta} = -g_{\alpha\beta}\, \ddot{z}^{\alpha} \ddot{z}^{\beta} \equiv -\ddot{z}^2. \tag{3.16}$$

Application of the stationary action principle yields the dynamical equations

$$m_0\, \ddot{z}^{\alpha} = e c^{-1} F^{\alpha}{}_{\beta}\, \dot{z}^{\beta}, \tag{3.17}$$

$$g^{1/2} F^{\mu\nu}{}_{\cdot\nu} = 4\pi c^{-1} j^{\mu}. \tag{3.18}$$

Although Eq. (3.17) is the correct equation to give the motion of a charged particle in a given electromagnetic field which has no dynamical properties, and Eq. (3.18) is the correct equation to give the production of an electromagnetic field by a given current density j^{μ} which has no dynamical properties, together they lead to the well-known difficulty that the quantity $F^{\alpha}{}_{\beta}$ appearing on the right of (3.17) is divergent when given by (3.18). For the moment, however, we pretend ignorance of this fact and proceed in an entirely formal way. Firstly it will be noted that (3.17) is consistent with (3.15) owing to the antisymmetry of $F_{\alpha\beta}$. Secondly, we note that since

$$F^{\mu\nu}{}_{\cdot\nu\mu} = \tfrac{1}{2}(R_{\nu\mu}{}^{\mu}{}_{\sigma} F^{\sigma\nu} + R_{\nu\mu}{}^{\nu}{}_{\sigma} F^{\mu\sigma}) = 0, \tag{3.19}$$

the current density j^{μ} must be conserved. This, however, is automatically guaranteed by the form of expression (3.4); for, using the identity

$$(\bar{\delta}^{1/2}\, \bar{g}^{\mu}{}_{\alpha}\, \delta^{(4)})_{\cdot\mu} = -(\bar{\delta}^{1/2}\, \delta^{(4)})_{\cdot\alpha}, \tag{3.20}$$

which is an alternative form of (2.72), we have

$$j^{\mu}{}_{\cdot\mu} = -ec \int \dot{z}^{\alpha} (\bar{\delta}^{1/2} \delta^{(4)})_{\cdot\alpha}\, d\tau = -ec \int d(\bar{\delta}^{1/2} \delta^{(4)}) = 0. \tag{3.21}$$

The only assumption which has been made in setting the final expression equal to zero is that space-time is open in the time direction so that the world point of the particle becomes arbitrarily remote from any given space-time point provided one goes sufficiently far into the past or future, i.e., toward the limiting values of the parameter τ. For an arbitrary parameter these limiting values may be arbitrary, but for the proper time they are, of course, $\pm\infty$, and will frequently be so indicated.

The stress density of the system is given by

$$T^{\mu\nu} = T_P{}^{\mu\nu} + T_F{}^{\mu\nu}, \tag{3.22}$$

$$T_P{}^{\mu\nu} = m_0 c \int \bar{\delta}^{1/2} \bar{g}^{\mu}{}_{\alpha}\, \bar{g}^{\nu}{}_{\beta}\, \dot{z}^{\alpha} \dot{z}^{\beta} \delta^{(4)}\, d\tau, \tag{3.23}$$

$$T_F{}^{\mu\nu} = (4\pi)^{-1} g^{1/2} (F^{\mu}{}_{\sigma} F^{\nu\sigma} - \tfrac{1}{4} g^{\mu\nu} F_{\sigma\tau} F^{\sigma\tau}). \tag{3.24}$$

Using Eqs. (1.40) and (3.20) and the identity

$$F_{\mu\nu\cdot\sigma} + F_{\nu\sigma\cdot\mu} + F_{\sigma\mu\cdot\nu} = (R_{\mu\sigma\nu}{}^{\tau} + R_{\nu\mu\sigma}{}^{\tau} + R_{\sigma\nu\mu}{}^{\tau}) A_{\tau} = 0, \tag{3.25}$$

we find, in virtue of the dynamical equations,

$$T_P{}^{\mu\nu}{}_{\cdot\nu} = -m_0 c \int \bar{g}^{\mu}{}_{\alpha}\, \dot{z}^{\alpha} \dot{z}^{\beta} (\bar{\delta}^{1/2} \delta^{(4)})_{\cdot\beta}\, d\tau = m_0 c \int \bar{\delta}^{1/2} \bar{g}^{\mu}{}_{\alpha}\, \ddot{z}^{\alpha} \delta^{(4)}\, d\tau = e \int \bar{\delta}^{1/2}\, \bar{g}^{\mu}{}_{\alpha}\, F^{\alpha}{}_{\beta}\, \dot{z}^{\beta} \delta^{(4)}\, d\tau = c^{-1} F^{\mu}{}_{\nu}\, j^{\nu}, \tag{3.26}$$

$$T_F{}^{\mu\nu}{}_{\cdot\nu} = (4\pi)^{-1} g^{1/2} [F^{\mu}{}_{\sigma} F^{\nu\sigma}{}_{\cdot\nu} - \tfrac{1}{2} g^{\mu\nu} (F_{\sigma\tau\cdot\nu} + F_{\tau\nu\cdot\sigma} + F_{\nu\sigma\cdot\tau}) F^{\sigma\tau}] = -c^{-1} F^{\mu}{}_{\sigma}\, j^{\sigma}, \tag{3.27}$$

from which we obtain the conservation law

$$T^{\mu\nu}{}_{\cdot\nu} = 0. \tag{3.28}$$

In order to make use of this conservation law to obtain the true ponderomotive equations, including radiation reaction, we must obtain explicit expressions for the electromagnetic field by solving Eq. (3.18). In order to solve Eq. (3.18), we must choose a special gauge. If, in the gauge transformation

$$A^*_\mu = A_\mu + \Lambda_{\cdot\mu}, \tag{3.29}$$

we choose the gauge parameter to satisfy

$$g^{\mu\nu}\Lambda_{\cdot\mu\nu} = -A^\mu{}_{\cdot\mu}, \tag{3.30}$$

i.e., if we take, for example,

$$\Lambda(x) = \int \bar{G}(x,x')A^{\mu'}{}_{\cdot\mu'}(x')g^{1/2}(x')\,d^4x', \tag{3.31}$$

then A^*_μ will satisfy the covariant Lorentz conditions

$$A^{*\mu}{}_{\cdot\mu} = 0. \tag{3.32}$$

We shall assume such a gauge transformation already to have been carried out. Then we may rewrite Eq. (3.18) in the form

$$\begin{aligned}-4\pi c^{-1} j^\mu &= -g^{1/2} g^{\mu\sigma} g^{\nu\tau}(A_{\tau\cdot\sigma\nu} - A_{\sigma\cdot\tau\nu}) \\ &= g^{1/2}(g^{\nu\sigma} A^\mu{}_{\cdot\nu\sigma} + R^\mu{}_\nu A^\nu),\end{aligned} \tag{3.33}$$

of which particular solutions are given by

$$A^{\rm ret}{}_\mu(x) = 4\pi c^{-1}\int G^{\rm ret}{}_{\mu\nu'}(x,x')j^{\nu'}(x')\,d^4x', \tag{3.34}$$

$$A^{\rm adv}{}_\mu(x) = 4\pi c^{-1}\int G^{\rm adv}{}_{\mu\nu'}(x,x')j^{\nu'}(x')\,d^4x', \tag{3.35}$$

yielding the retarded and advanced proper fields of the particle:

$$F^{\rm ret}{}_{\mu\nu} = A^{\rm ret}{}_{\nu\cdot\mu} - A^{\rm ret}{}_{\mu\cdot\nu}, \tag{3.36}$$

$$F^{\rm adv}{}_{\mu\nu} = A^{\rm adv}{}_{\nu\cdot\mu} - A^{\rm adv}{}_{\mu\cdot\nu}. \tag{3.37}$$

It is easy to see that the solutions (3.34), (3.35) satisfy the Lorentz condition (3.32). One simply makes use of the identities (2.75) and (3.21), and performs an integration by parts.

The total field may be expressed in the alternative forms

$$F_{\mu\nu} = F^{\rm in}{}_{\mu\nu} + F^{\rm ret}{}_{\mu\nu} = F^{\rm out}{}_{\mu\nu} + F^{\rm adv}{}_{\mu\nu}, \tag{3.38}$$

which serve as definitions for the fields $F^{\rm in}{}_{\mu\nu}$ and $F^{\rm out}{}_{\mu\nu}$. Another useful form is

$$F_{\mu\nu} = \bar{F}^{\rm free}{}_{\mu\nu} + \bar{F}_{\mu\nu}, \tag{3.39}$$

where

$$\bar{F}_{\mu\nu} = \tfrac{1}{2}(F^{\rm ret}{}_{\mu\nu} + F^{\rm adv}{}_{\mu\nu}), \tag{3.40}$$

$$\begin{aligned}\bar{F}^{\rm free}{}_{\mu\nu} &= \tfrac{1}{2}(F^{\rm in}{}_{\mu\nu} + F^{\rm out}{}_{\mu\nu}) \\ &= F^{\rm in}{}_{\mu\nu} + \tfrac{1}{2}F^{\rm rad}{}_{\mu\nu} = F^{\rm out}{}_{\mu\nu} - \tfrac{1}{2}F^{\rm rad}{}_{\mu\nu},\end{aligned} \tag{3.41}$$

$$F^{\rm rad}{}_{\mu\nu} = F^{\rm ret}{}_{\mu\nu} - F^{\rm adv}{}_{\mu\nu}. \tag{3.42}$$

The fields $\bar{F}_{\mu\nu}$ and $F^{\rm rad}{}_{\mu\nu}$ may be expressed in terms of potentials $\bar{A}_\mu$ and $A^{\rm rad}{}_\mu$ which are defined by integral expressions of the form (3.34) (3.35), involving the functions $\bar{G}_{\mu\nu'}$ and $G_{\mu\nu'}$, respectively. The various fields thus defined satisfy the equations

$$g^{1/2}F^{\rm ret\,\mu\nu}{}_{\cdot\nu} = g^{1/2}F^{\rm adv\,\mu\nu}{}_{\cdot\nu} = g^{1/2}\bar{F}^{\mu\nu}{}_{\cdot\nu} = 4\pi c^{-1}j^\mu, \tag{3.43}$$

$$F^{\rm in\,\mu\nu}{}_{\cdot\nu} = F^{\rm out\,\mu\nu}{}_{\cdot\nu} = \bar{F}^{\rm free\,\mu\nu}{}_{\cdot\nu} = F^{\rm rad\,\mu\nu}{}_{\cdot\nu} = 0. \tag{3.44}$$

Substituting the explicit forms (2.53), (2.62), (2.63), and (3.4) into Eqs. (3.34), (3.35), and, for the sake of compactness, replacing the designations "ret" and "adv" by "−" and "+", respectively, we get.

$$\begin{aligned}A^\pm{}_\mu &= 4\pi e\int_{-\infty}^{\infty} G^\pm{}_{\mu\alpha}\dot{z}^\alpha\,d\tau \\ &= \pm e\int_{\tau_\Sigma}^{\pm\infty}[u_{\mu\alpha}\,\delta(\sigma) - v_{\mu\alpha}\,\theta(-\sigma)]\dot{z}^\alpha\,d\tau,\end{aligned} \tag{3.45}$$

where τ_Σ is the value of the proper time at the point of intersection of the world line of the particle with an arbitrary space-like hypersurface $\Sigma(x)$ containing x (see Fig. 2). Changing the variable of integration from τ to σ, noting that

$$\sigma_\Sigma = \sigma(x,z(\tau_\Sigma)) > 0, \tag{3.46}$$

$$\sigma(x,z(\pm\infty)) = -\infty \quad \text{for non-"runaway" trajectories}, \tag{3.47}$$

$$d\sigma = \sigma_{\cdot\alpha}\dot{z}^\alpha\,d\tau, \tag{3.48}$$

and defining the advanced and retarded proper times, $\tau_\pm$, of the particle relative to the point x by

$$\sigma(x,z(\tau_\pm)) = 0, \qquad \tau_+ > \tau_\Sigma, \quad \tau_- < \tau_\Sigma \tag{3.49}$$

we find

$$A^\pm{}_\mu = \mp e[u_{\mu\alpha}\dot{z}^\alpha(\sigma_{\cdot\beta}\dot{z}^\beta)^{-1}]_{\tau=\tau_\pm} \mp e\int_{\tau_\pm}^{\pm\infty} v_{\mu\alpha}\dot{z}^\alpha\,d\tau. \tag{3.50}$$

These are the covariant *Liénard–Wiechert potentials.* The corresponding field strengths may be obtained by straightforward differentiation, with use of the relation

$$(\sigma_{\cdot\mu} + \sigma_{\cdot\alpha}\,\dot{z}^{\alpha}\,\tau_{\pm\,\cdot\mu})_{\tau=\tau_\pm} = 0, \tag{3.51}$$

which expresses the x-dependence of $\tau_\pm$. We find

$$\begin{aligned} F^{\pm}{}_{\mu\nu} = \mp e\{&(u_{\nu\alpha}\,\sigma_{\cdot\mu} - u_{\mu\alpha}\,\sigma_{\cdot\nu})\dot{z}^{\alpha}(\sigma_{\cdot\beta\gamma}\,\dot{z}^{\beta}\,\dot{z}^{\gamma} + \sigma_{\cdot\beta}\,\ddot{z}^{\beta})(\sigma_{\cdot\delta}\,\dot{z}^{\delta})^{-3} \\ &- [(u_{\nu\alpha}\,\sigma_{\cdot\mu} - u_{\mu\alpha}\,\sigma_{\cdot\nu})_{\cdot\beta}\,\dot{z}^{\alpha}\,\dot{z}^{\beta} + (u_{\nu\alpha}\,\sigma_{\cdot\mu} - u_{\mu\alpha}\,\sigma_{\cdot\nu})\ddot{z}^{\alpha}](\sigma_{\cdot\gamma}\,\dot{z}^{\gamma})^{-2} \\ &+ (u_{\nu\alpha\cdot\mu} - u_{\mu\alpha\cdot\nu} + v_{\nu\alpha}\,\sigma_{\cdot\mu} - v_{\mu\alpha}\,\sigma_{\cdot\nu})\dot{z}^{\alpha}(\sigma_{\cdot\beta}\,\dot{z}^{\beta})^{-1}\}_{\tau=\tau_\pm} \\ &\mp e\int_{\tau_\pm}^{\pm\infty}(v_{\nu\alpha\cdot\mu} - v_{\mu\alpha\cdot\nu})\dot{z}^{\alpha}\,d\tau. \end{aligned} \tag{3.52}$$

Equations (3.50) and (3.52) were derived under the assumption that σ is single valued. Even though this may be the case for the leading terms, which involve the behavior of the particle only at the retarded or advanced proper times, it will not generally be true for the "tail" terms involving integrations over the whole past or furture history of the particle. However, since the wave equation is linear, the superposition principle holds, and it is clear that the appropriate bi-vector $v_{\mu\alpha}$ to use in the "tail" term is the sum of the v's for all the different geodesics between x and z, each term in the sum representing the contribution of

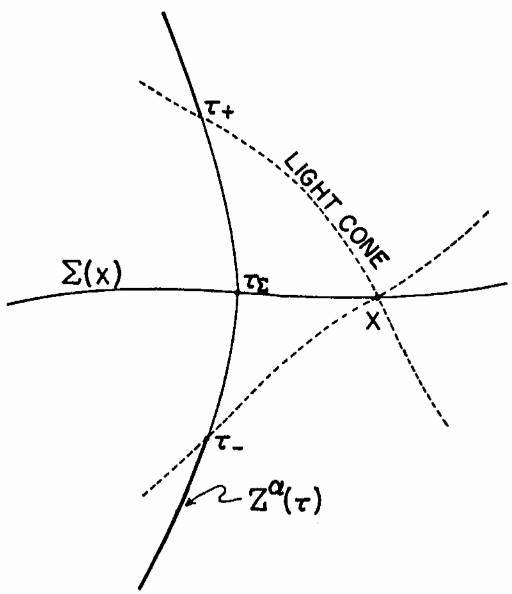

FIG. 2. Advanced and retarded proper times

an elementary wave.[8] Similarly, if the point x is taken in a region sufficiently far from the world line of the particle, where the null geodesics (so-called "characteristics") emanating from the world line begin to cross, then each leading term must be replaced by a sum of terms, one for each geodesic. These modifications are perhaps most clearly indicated by the generalized Huygens' principle for the homogeneous vector wave equation:[9]

$$\begin{aligned} A_\mu(x) = \int_{\Sigma'} g^{1/2}(x')[&G_{\mu\nu'}(x,x')A^{\nu'}{}_{\cdot\sigma'}(x') \\ &- G_{\mu\nu'\cdot\sigma'}(x,x')A^{\nu'}(x')]g^{\sigma'\tau'}(x')\,d\Sigma'_{\tau'}. \end{aligned} \tag{3.53}$$

Because of the superposition of elementary waves propagating from the hypersurface Σ' it is obvious that the correct propagation function to use here as well as everywhere else [e.g., in Eq. (2.41)] is the sum, over all distinct geodesics between x and x', of the elementary Green's function (2.53) [or, in the scalar case, (2.21)]. It is evident that discontinuities in the field may occur on account of the crossing of null geodesics.

We finally point out that there may be *no* null geodesics connecting the point x to the particle world line. This can happen, for example, if x is located at a sufficient distance from a "runaway" trajectory, i.e., one in which the particle accelerates asymptotically to the velocity of light. In this case the Liénard–Wiechert potentials vanish at x. Such cases are, however, of no interest to us here.

4. CONSTRUCTION OF THE WORLD TUBE

In order to determine the effect of radiation reaction upon the particle we must keep a record of the energy-momentum balance between it and the field. Since the ponderomotive equations describe the local behavior of the particle they can be obtained only if we keep an instantaneous record in the immediate vicinity of the particle. To do this we shall construct a small sphere about the particle, across the surface of which the energy-momentum flow will be determined. In the course of time such a sphere generates a hypersurface in the space-time manifold. It is the precise construction of this hypersurface, or *world tube*, to which we now turn our attention.

We begin by introducing, at a point z on the world line of the particle, three unit vectors $n_i{}^{\alpha}$ which are orthogonal to each other and to the world line itself:[10]

$$n_i{}^{\alpha}\,n_{j\alpha} = \delta_{ij}, \qquad n_{i\alpha}\,\dot{z}^{\alpha} = 0. \tag{4.1}$$

[8] Since the series (2.8) and (2.43) no longer generally converge in the multiple-valued region it is necessary to define the individual v's by a continuation process.

[9] Through use of the last of the identities (2.75), Gauss' theorem, an integration by parts, and the homogeneous vector wave equation, it is easy to show that this Huygens' principle maintains the Lorentz condition.

[10] Latin indices range over the values 1, 2, 3.

We next introduce a set of direction cosines Ω_i, satisfying,

$$\Omega_i \Omega_i = 1, \tag{4.2}$$

in terms of which we can specify the direction relative to the $n_i{}^\alpha$ of an arbitrary unit vector perpendicular to the world line at z. Then, starting in the direction of this arbitrary vector we construct a geodesic from z extending out a fixed distance ϵ to a point x. The coordinates of the point z will depend on the direction cosines Ω_i and on the proper time τ which identifies the point z. This dependence may be indicated explicitly by expressing these coordinates in the form $x^\mu(\Omega,\tau)$, although the arguments Ω,τ will in practice be suppressed. The locus of all points generated in this way for various values of Ω_i is the "sphere" with which we surround the particle.

From Eqs. (1.10), (1.11), (4.1), and (4.2) we may infer that the bi-scalar σ which describes the geodesic between x and z satisfies the equations

$$\sigma = \tfrac{1}{2}\epsilon^2, \tag{4.3}$$

$$\sigma_{\cdot\alpha} = -\epsilon\, n_{i\alpha}\Omega_i, \tag{4.4}$$

$$\sigma_{\cdot\alpha}\dot{z}^\alpha = 0. \tag{4.5}$$

A variation $\delta\Omega_i$ in the direction cosines produces a variation in the point x which is given by

$$\sigma_{\cdot\mu\alpha}\,\delta x^\mu = -\epsilon\, n_{i\alpha}\,\delta\Omega_i, \tag{4.6}$$

or, in virtue of (1.51),

$$\delta x^\mu = \epsilon\, D^{-1\,\mu}{}_\alpha\, n_i{}^\alpha\,\delta\Omega_i. \tag{4.7}$$

A pair of independent variations $\delta_1\Omega_i$, $\delta_2\Omega_i$ in the direction cosines defines an element $d\Omega$ of solid angle by the relation

$$\Omega_i d\Omega = \epsilon_{ijk}\,\delta_1\Omega_j\,\delta_2\Omega_k, \tag{4.8}$$

where ϵ_{ijk} is the 3-dimensional antisymmetric permutation symbol. This solid angle defines an element of 2-dimensional area on the surface of the sphere, enclosed by the parallelogram formed from the corresponding displacements $\delta_1 x^\mu$, $\delta_2 x^\mu$. It is not, however, this 2-dimensional surface element which is of prime interest to us, but rather a 3-dimensional element of the world tube generated by the sphere as the proper time τ varies.

A general displacement of the point x on the tube, produced by independent variations of τ and the Ω_i, may be expressed as a linear combination of $\delta_1 x^\mu$, $\delta_2 x^\mu$ and a third displacement $\delta_3 x^\mu$ orthogonal to these two (see Figure 3):

$$g_{\mu\nu}\delta_1 x^\mu\,\delta_3 x^\nu = 0, \qquad g_{\mu\nu}\delta_2 x^\mu\,\delta_3 x^\nu = 0. \tag{4.9}$$

Denoting such a general displacement by δx^μ, we have, on varying Eqs. (4.3) and (4.5) which define the tube, and holding ϵ constant,

$$0 = \sigma_{\cdot\mu}\,\delta x^\mu + \sigma_{\cdot\alpha}\dot{z}^\alpha\,d\tau = \sigma_{\cdot\mu}\,\delta x^\mu, \tag{4.10}$$

$$0 = -D_{\mu\alpha}\dot{z}^\alpha\,\delta x^\mu - \kappa^2\,d\tau, \tag{4.11}$$

where

$$\kappa = (-\sigma_{\cdot\alpha\beta}\dot{z}^\alpha\dot{z}^\beta - \sigma_{\cdot\alpha}\ddot{z}^\alpha)^{1/2}. \tag{4.12}$$

Equation (4.10) states that geodesics normal to the world line are also normal to the tube. As a check, we may verify Eq. (4.10) directly for the dipslacements $\delta_1 x^\mu$ and $\delta_2 x^\mu$. From Eq. (1.11) we have

$$\sigma_{\cdot\mu} = g^{\alpha\beta}\sigma_{\cdot\mu\alpha}\sigma_{\cdot\beta}, \tag{4.13}$$

or

$$\sigma_{\cdot\mu} D^{-1\,\mu}{}_\alpha = -\sigma_{\cdot\alpha}. \tag{4.14}$$

Substituting (4.7) into (4.10), we therefore have

$$0 = -\epsilon\,\sigma_{\cdot\alpha}n_i{}^\alpha\,\delta\Omega_i = \epsilon^2 n_{i\alpha}\Omega_i n_j{}^\alpha\,\delta\Omega_j = \epsilon^2\Omega_i\,\delta\Omega_i, \tag{4.15}$$

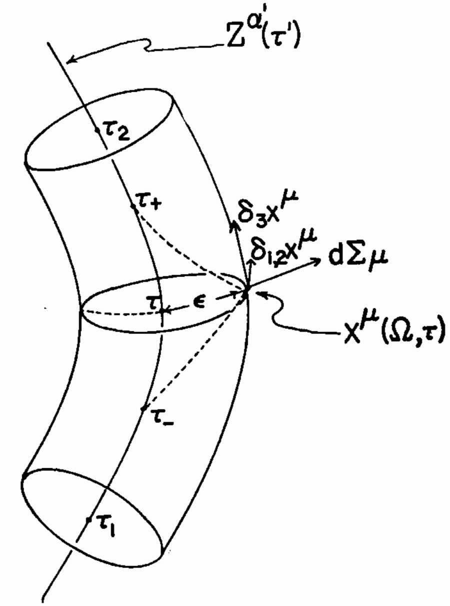

FIG. 3. The world tube

which must be satisfied in virtue of Eq. (4.2). A similar verification of Eq. (4.11) for $\delta_1 x^\mu$ and $\delta_2 x^\mu$ ($d\tau = 0$) follows immediately from (4.7) and the second orthogonality relation (4.1).

Before proceeding to the explicit construction of the displacement $\delta_3 x^\mu$ and thence to the surface element of the tube, it will be convenient to record a few more of the properties of the unit vectors $n_i^{\ \alpha}$, which follow from Eqs. (4.1), when combined with Eq. (3.14), namely,

$$n_i^{\ \alpha}\, n_i^{\ \beta} = g^{\alpha\beta} + c^{-2}\, \dot z^\alpha\, \dot z^\beta, \tag{4.16}$$

$$\epsilon_{\alpha\beta\gamma\delta}\, \dot z^\delta = c\, g^{-1/2}(z)\, \epsilon_{ijk}\, n_{i\alpha}\, n_{j\beta}\, n_{k\gamma}\,, \tag{4.17}$$

$$\begin{aligned}\epsilon_{\alpha\beta\gamma\delta}\, n_i^{\ \delta} = {} & c^{-1}\, g^{-1/2}(z)\, \epsilon_{ijk}(n_{j\alpha}\, n_{k\beta}\, g_{\gamma\delta} \\ & + n_{j\beta}\, n_{k\gamma}\, g_{\alpha\delta} + n_{j\gamma}\, n_{k\alpha}\, g_{\beta\delta})\dot z^\delta,\end{aligned} \tag{4.18}$$

$$\epsilon_{\alpha\beta\gamma\delta}\, n_i^{\ \gamma}\, n_j^{\ \delta} = c^{-1}\, g^{-1/2}(z)\, \epsilon_{ijk}(n_{k\beta}\, g_{\alpha\delta} - n_{k\alpha}\, g_{\beta\delta})\dot z^\delta. \tag{4.19}$$

The latter three equations, involving the 4-dimensional permutation symbol $\epsilon_{\alpha\beta\gamma\delta}$, are obtained by considering minors of the matrix formed by the four vectors $n_i^{\ \alpha}$, $\dot z^\alpha$ and observing that the determinant of the matrix itself is equal to $cg^{-1/2}(z)$. It is not necessary for future purposes to specify anthing beyond what has already been recorded about the behavior of the unit vectors $n_i^{\ \alpha}$ as τ varies. One can, however, imagine them to be "rotationless." This condition is expressed by the equation

$$n_i^{\ \alpha}\, \dot n_{j\alpha} = 0, \tag{4.20}$$

which, together with the equation

$$\dot n_{i\alpha}\, \dot z^\alpha + n_{i\alpha}\, \ddot z^\alpha = 0, \tag{4.21}$$

implies

$$\dot n_i^{\ \alpha} = c^{-2}\, n_{i\beta}\, \ddot z^\beta\, \dot z^\alpha. \tag{4.22}$$

The construction of $\delta_3 x^\mu$ now depends on the observation that (4.9), (4.10) and (4.11) constitute a set of four simultaneous equations which serve to fix it completely. It is not difficult to see that the solution of these equations is given by

$$\delta_3 x^\mu = -\kappa^2\, M^{-1}\, \epsilon^{\nu\sigma\tau\mu}\, \delta_1 x_\nu\, \delta_2 x_\sigma\, \sigma_{\cdot\tau}\, d\tau, \tag{4.23}$$

where

$$M = \epsilon^{\nu\sigma\tau\mu}\, \delta_1 x_\nu\, \delta_2 x_\sigma\, \sigma_{\cdot\tau}\, D_{\mu\alpha}\, \dot z^\alpha. \tag{4.24}$$

These equations may be reexpressed in terms of the element of solid angle $d\Omega$ by noting, in virtue of (4.7) and (4.8), that

$$\delta_1 x_\nu\, \delta_2 x_\sigma - \delta_1 x_\sigma\, \delta_2 x_\nu = \epsilon^2\, D^{-1}{}_{\nu\alpha}\, D^{-1}{}_{\sigma\beta}\, n_i^{\ \alpha}\, n_j^{\ \beta}\, \epsilon_{ijk}\, \Omega_k\, d\Omega. \tag{4.25}$$

Introducing the bi-vector of geodetic parallel displacement, and making use of the bi-tensor defined in (1.64) as well as Eqs. (1.34), (1.76), (4.4), and (4.19), we may now write

$$\begin{aligned}\delta_3 x^\mu &= \tfrac12\, \epsilon^2 \kappa^2 M^{-1}\, \epsilon^{\mu\nu\sigma\tau}\, \sigma_{\cdot\tau}\, D^{-1}{}_{\nu\alpha}\, D^{-1}{}_{\sigma\beta}\, n_i^{\ \alpha}\, n_j^{\ \beta}\, \epsilon_{ijk}\, \Omega_k\, d\Omega\, d\tau \\ &= -\tfrac12\, \epsilon^2 \kappa^2 M^{-1}\, \bar\delta\, \bar g^\mu{}_\gamma\, \epsilon^{\gamma\delta\epsilon\zeta}\, \sigma_{\cdot\zeta}\, \bar D^{-1}{}_{\delta\alpha}\, \bar D^{-1}{}_{\epsilon\beta}\, n_i^{\ \alpha}\, n_j^{\ \beta}\, \epsilon_{ijk}\, \Omega_k\, d\Omega\, d\tau \\ &= \tfrac12\, \epsilon^2 \kappa^2 M^{-1}\, \bar g\, \Delta^{-1}\, \bar g^\mu{}_\gamma\, \epsilon_{\alpha\beta\delta\epsilon}\, \bar D^{\gamma\delta}\, \bar D^{\zeta\epsilon}\, \sigma_{\cdot\zeta}\, n_i^{\ \alpha}\, n_j^{\ \beta}\, \epsilon_{ijk}\, \Omega_k\, d\Omega\, d\tau \\ &= -c^{-1}\, \epsilon^3\, \kappa^2 M^{-1}\, g^{1/2}(x)\, \Delta^{-1}\, \bar g^{\mu\gamma}(\bar D_{\gamma\alpha}\, \bar D_{\delta\beta} - \bar D_{\gamma\beta}\, \bar D_{\delta\alpha})\dot z^\alpha\, \Omega^\beta\, \Omega^\delta\, d\Omega\, d\tau,\end{aligned} \tag{4.26}$$

in which we have used the abbreviation

$$\Omega^\alpha = n_i^{\ \alpha}\, \Omega_i\,. \tag{4.27}$$

Similarly,

$$M = c^{-1}\, \epsilon^3\, g^{1/2}(x)\, \Delta^{-1}(\bar D_{\gamma\alpha}\, \bar D_{\delta\beta} - \bar D_{\gamma\beta}\, \bar D_{\delta\alpha})\bar D^\gamma{}_\epsilon\, \dot z^\alpha\, \dot z^\epsilon\, \Omega^\beta\, \Omega^\delta\, d\Omega. \tag{4.28}$$

The directed (vector density) surface element defined by the independent displacements $\delta_1 x^\mu$, $\delta_2 x^\mu$, $\delta_3 x^\mu$, is given by

$$\begin{aligned}d\Sigma_\mu &= \epsilon_{\mu\nu\sigma\tau}\, \delta_1 x^\nu\, \delta_2 x^\sigma\, \delta_3 x^\tau \\ &= \epsilon^2\, c^{-1}\, g^{-1/2}(x)\, \Delta^{-1}\, \bar g_\mu{}^\gamma\, \bar g_\tau{}^\delta(\bar D_{\gamma\alpha} \bar D_{\delta\beta} - \bar D_{\gamma\beta}\, \bar D_{\delta\alpha})\dot z^\alpha\, \Omega^\beta\, d\Omega\, \delta_3 x^\tau.\end{aligned} \tag{4.29}$$

By substituting (4.26) and (4.26) into (4.29) we can now use the proper time τ and solid angle Ω as integration variables in evaluating integrals over the world tube. We shall be interested in the case in which the tube radius ϵ becomes infinitesimally small, and therefore, it will suffice to use expansions in powers of ϵ in evaluating expressions (4.26), (4.27), and (4.29). From (1.74) we have

$$\bar D_{\alpha\beta} = g_{\alpha\beta} - \tfrac16\, \epsilon^2\, R_{\alpha\gamma\beta\delta}\, \Omega^\gamma\, \Omega^\delta + O(\epsilon^3), \tag{4.30}$$

$$\bar D_{\alpha\beta}\, \Omega^\alpha = \Omega_\beta + O(\epsilon^3), \tag{4.31}$$

$$\bar D_{\alpha\beta}\, \dot z^\delta\, \Omega^\alpha = O(\epsilon^3), \tag{4.32}$$

and hence

$$M = -c^{-1}\, \epsilon^3\, g^{1/2}(x)\, \Delta^{-1}[c^2 + \tfrac13\, \epsilon^2 R_{\alpha\gamma\beta\delta}\, \dot z^\alpha\, \dot z^\beta\, \Omega^\gamma\, \Omega^\delta + O(\epsilon^3)]\, d\Omega,$$

$$\begin{aligned}\delta_3 x^\mu = {} & -c^{-1}\, \epsilon^3\, \kappa^2\, M^{-1}\, g^{1/2}(x)\, \Delta^{-1}\, \bar g^{\mu\alpha}[g_{\alpha\beta} - \tfrac16\, \epsilon^2\, R_{\alpha\gamma\beta\delta}\, \Omega^\gamma\, \Omega^\delta \\ & + O(\epsilon^3)]\dot z^\beta\, d\Omega\, d\tau\end{aligned} \tag{4.33}$$

$$= c^{-2}\kappa^2\, \bar g^\mu{}_\alpha[\dot z^\alpha - \tfrac16\, \epsilon^2(g^{\alpha\beta} + 2c^{-2}\, \dot z^\alpha\, \dot z^\beta)R_{\beta\gamma\epsilon\delta}\, \dot z^\epsilon\, \Omega^\gamma\, \Omega^\delta + O(\epsilon^3)]\, d\tau, \tag{4.34}$$

$$\begin{aligned}d\Sigma_\mu = {} & \epsilon^2 c^{-1}\, g^{-1/2}(x)\Delta^{-1}\, \bar g_{\mu\alpha}\, \bar g_{\tau\beta}[\dot z^\alpha\, \Omega^\beta - \dot z^\beta\, \Omega^\alpha \\ & - \tfrac16\, \epsilon^2(R^\alpha{}_{\gamma\epsilon\delta}\, \Omega^\beta - R^\beta{}_{\gamma\epsilon\delta}\, \Omega^\alpha)\dot z^\epsilon\, \Omega^\gamma\, \Omega^\delta + O(\epsilon^3)]\, d\Omega\, \delta_3 x^\tau \\ = {} & \epsilon^2\, c^{-1}\, \kappa^2\, g^{-1/2}(x)\bar g_{\mu\alpha}\, \Omega^\alpha(1 + \tfrac16\, \epsilon^2\, R_{\beta\gamma}\, \Omega^\beta\Omega^\gamma)\, d\Omega\, d\tau + O(\epsilon^5).\end{aligned} \tag{4.35}$$

In addition to these expansions we shall also need expansions for the retarded and advanced field strengths (3.52). As a first step we obtain expansions for the retarded and advanced proper times at which the quantities appearing in (3.52) are to be evaluated. Introducing

$$\delta_\pm = \tau_\pm - \tau = O(\epsilon), \tag{4.36}$$

and recalling the defining equation (3.49), we may write

$$0 = \sigma + \delta_\pm \dot\sigma + \tfrac{1}{2}\delta_\pm^2 \ddot\sigma + \tfrac{1}{6}\delta_\pm^3 \dddot\sigma + \tfrac{1}{24}\delta_\pm^4 \ddddot\sigma + O(\epsilon^5), \tag{4.37}$$

where σ and its derivatives are here to be evaluated at the points x and z. Then making use of Eqs. (3.8), (3.9), (3.10), (3.11), and the expansions (1.28), (1.29), (1.30), and taking note of (3.15), (3.16), (4.3), (4.5), (4.12), and the symmetry properties of the Riemann tensor, we find that Eq. (4.37) becomes

$$0 = \tfrac{1}{2}\epsilon^2 - \tfrac{1}{2}\delta_\pm^2 \kappa^2 + \tfrac{1}{6}\delta_\pm^3 \sigma_{\cdot\alpha}\dddot z^\alpha - \tfrac{1}{24}\delta_\pm^4 \ddot z^2 + O(\epsilon^5), \tag{4.38}$$

from which we obtain, on inverting the series and making use of (4.4),

$$\delta_\pm^2 = \epsilon^2\kappa^{-2}(1 \pm \tfrac{1}{3}\epsilon^2\kappa^{-3}\dddot z^\alpha \Omega_\alpha - \tfrac{1}{12}\epsilon^2\kappa^{-4}\ddot z^2) + O(\epsilon^5), \tag{4.39}$$

$$\delta_\pm = \pm\epsilon\kappa^{-1}(1 \mp \tfrac{1}{6}\epsilon^2\kappa^{-3}\dddot z^\alpha\Omega_\alpha - \tfrac{1}{24}\epsilon^2\kappa^{-4}\ddot z^2) + O(\epsilon^4). \tag{4.40}$$

It is noteworthy that the Riemann tensor makes no explicit appearance to this order.

We also need the expansions

$$\begin{aligned}(\sigma_{\cdot\alpha}\dot z^\alpha)_{\tau=\tau_\pm} &= (\dot\sigma)_{\tau=\tau_\pm}\\ &= \dot\sigma + \delta_\pm\ddot\sigma + \tfrac{1}{2}\delta_\pm^2\dddot\sigma + \tfrac{1}{6}\delta_\pm^3\ddddot\sigma + O(\epsilon^4)\\ &= -\delta_\pm\kappa^2 + \tfrac{1}{2}\delta_\pm^2\sigma_{\cdot\alpha}\dddot z^\alpha - \tfrac{1}{6}\delta_\pm^3\ddot z^2 + O(\epsilon^4)\\ &= \mp\epsilon\kappa(1 \pm \tfrac{1}{3}\epsilon^2\kappa^{-3}\dddot z^\alpha\Omega_\alpha + \tfrac{1}{8}\epsilon^2\kappa^{-4}\ddot z^2) + O(\epsilon^4),\end{aligned} \tag{4.41}$$

$$(\sigma_{\cdot\alpha}\dot z^\alpha)^{-1}_{\tau=\tau_\pm} = \mp\epsilon^{-1}\kappa^{-1}(1 \mp \tfrac{1}{3}\epsilon^2\kappa^{-3}\dddot z^\alpha\Omega_\alpha - \tfrac{1}{8}\epsilon^2\kappa^{-4}\ddot z^2) + O(\epsilon^2), \tag{4.42}$$

$$(\sigma_{\cdot\alpha}\dot z^\alpha)^{-3}_{\tau=\tau_\pm} = \mp\epsilon^{-3}\kappa^{-3}(1 \mp \epsilon^2\kappa^{-3}\dddot z^\alpha\Omega_\alpha - \tfrac{3}{8}\epsilon^2\kappa^{-4}\ddot z^2) + O(\epsilon^0). \tag{4.43}$$

In the calculations of the next section we shall not actually integrate over the entire world tube. In fact, we shall integrate only over an infinitesimal portion of it, *plus* the "caps" at the ends given by geodetic cross sections τ = constant. We shall therefore need also an expression for the surface elements of the caps. Here we may take Ω and ϵ as integration variables, and the displacement $\delta_3 x^\mu$ will now correspond to a variation in ϵ. It is to be noted, however, that in order to preserve continuity of orientation of the surface element relative to the interior of the tube, $\delta_3 x^\mu$ must be directed inward toward the world line on the positive (future) end of the tube and outward on the negative (past) end. From (4.4) we therefore have, on the positive end,

$$\sigma_{\cdot\mu\alpha}\delta_3 x^\mu = \Omega_\alpha\, d\epsilon \quad\text{or}\quad \delta_3 x^\mu = -D^{-1\,\mu\alpha}\,\Omega_\alpha\, d\epsilon, \tag{4.44}$$

which, when substituted into (4.29), gives

$$d\Sigma_\mu = -c^{-1} g^{-1/2}(x)\Delta^{-1} D_{\mu\alpha}\dot z^\alpha \epsilon^2\, d\epsilon\, d\Omega, \tag{4.45}$$

and, on the negative end, the same expressions with opposite signs.

5. THE PONDEROMOTIVE EQUATIONS

The energy balance between field and particle is expressed in differential form by the conservation law (3.28) which states that the covariant divergence of the total stress density vanishes. For practical application this differential characterization must be replaced by an integral one. In the case of flat space-time, as is well known, one integrates the divergence over a space-time volume and then uses Gauss' theorem to replace the volume integral by an integral over a hypersurface. In the presence case, however, one cannot do this since the integral $\int T^{\mu\nu}{}_{\cdot\nu}\, d^4x$ is not an invariant, nor even a vector.

There is, nevertheless, a natural procedure to overcome this difficulty which suggests itself, namely, to consider the integral $\int \bar g_\mu{}^\alpha T^{\mu\nu}{}_{\cdot\nu}\, d^4x$, in which the bivector of geodetic parallel displacement is introduced in order to refer contributions to the integral at the variable point x back to some fixed point z. The latter integral *is* a local contravariant vector at z, and Gauss' theorem can be used. If we now let Σ denote the surface of the world tube between two proper times τ_1 and τ_2, and denote by Σ_1 and Σ_2 the corresponding end "caps," and by V the interior of the tube, enclosed by Σ, Σ_1, and Σ_2, we may write

$$\begin{aligned}0 &= c^{-1}\int_V \bar g_\mu{}^\alpha T^{\mu\nu}{}_{\cdot\nu}\, d^4x\\ &= c^{-1}\left(\int_\Sigma + \int_{\Sigma_1} + \int_{\Sigma_2}\right)\bar g_\mu{}^\alpha T^{\mu\nu}\, d\Sigma_\nu - c^{-1}\int_V \bar g_\mu{}^\alpha{}_{\cdot\nu} T^{\mu\nu}\, d^4x.\end{aligned} \tag{5.1}$$

Let us next take the limit $\epsilon \to 0$. The integrals over Σ_1, Σ_2, and V will then retain contributions only from the particle stress density; and if, furthermore, we take the fixed point z to lie on the particle world line at a proper time τ between τ_1 and τ_2, we then have, assuming $\tau_1 < \tau_2$, making use of (3.23) and (4.45), and remembering the sign conditions attached to the latter equation,

$$\begin{aligned}0 = \lim_{\epsilon\to 0}\frac{1}{c}\int_{\tau_1}^{\tau_2}\int_{4\pi} \bar g_\mu{}^\alpha T^{\mu\nu}\, d\Sigma_\nu + m_0[\bar g_{\beta'}{}^\alpha(z(\tau'),z(\tau))\dot z^{\beta'}(\tau')]_{\tau'=\tau_1}^{\tau'=\tau_2}\\ - m_0\int_{\tau_1}^{\tau_2} \bar g_{\beta'}{}^\alpha{}_{\cdot\gamma'}(z(\tau'),z(\tau))\dot z^{\beta'}(\tau')\dot z^{\gamma'}(\tau')\, d\tau'.\end{aligned} \tag{5.2}$$

Here we have made the replacement

$$\int_{\Sigma} \rightarrow \int_{\tau_1}^{\tau_2} \int_{4\pi} \tag{5.3}$$

to emphasize that the integral over the world tube will be computed explicitly in terms of an integral over proper time and in integral over solid angle. In passing to the limit $\epsilon = 0$ we have, of course, rendered the value of this integral divergent. However, since all calculations have been carried out in a covariant manner we can isolate the divergent part in an invariant fashion and eventually absorb it in a mass renormalization.

The third step is to let τ_1 and τ_2 both approach τ. Denoting their infinitesimal separation in the limit by $d\tau$ we see that (5.2) becomes

$$0 = m_0 \ddot{z}^\alpha \, d\tau + \lim_{\epsilon \to 0} \frac{1}{c} \int_{4\pi} \bar{g}_\mu{}^\alpha \, T^{\mu\nu} \, d\Sigma_\nu \, . \tag{5.4}$$

The remainder of this section will be devoted to computing the second term of this equation.

We must first get the retarded and advanced proper fields (3.52) in the form of expansions. With the understanding that all quantities are now to refer to the world tube, we begin by computing

$$\begin{aligned}
&\{\mu_{\mu\alpha}\, \sigma_{\cdot\nu}\, \dot{z}^\alpha(\sigma_{\cdot\beta\gamma}\, \dot{z}^\beta \dot{z}^\gamma + \sigma_{\cdot\beta}\, \ddot{z}^\beta) - [(u_{\mu\alpha}\, \sigma_{\cdot\nu})_{\cdot\beta}\, \dot{z}^\alpha \dot{z}^\beta + u_{\mu\alpha}\, \sigma_{\cdot\nu}\, \ddot{z}^\alpha](\sigma_{\cdot\gamma}\, \dot{z}^\gamma)\}_{\tau=\tau\pm} \\
&= \{u_{\mu\alpha}\, \sigma_{\cdot\nu}\, \dot{z}^\alpha\, \ddot{\sigma} - [(u_{\mu\alpha}\, \sigma_{\cdot\nu})_{\cdot\beta}\, \dot{z}^\alpha \dot{z}^\beta + u_{\mu\alpha}\, \sigma_{\cdot\nu}\, \ddot{z}^\alpha]\, \dot{\sigma}\}_{\tau=\tau\pm} \\
&= u_{\mu\alpha}\, \sigma_{\cdot\nu}\, \dot{z}^\alpha\, \ddot{\sigma} + \delta_\pm\, u_{\mu\alpha}\, \sigma_{\cdot\nu}\, \dot{z}^\alpha\, \dddot{\sigma} \\
&\quad + \tfrac{1}{2}\, \delta_\pm{}^2[(u_{\mu\alpha}\, \sigma_{\cdot\nu})_{\cdot\beta}\, \dot{z}^\alpha \dot{z}^\beta\, \dddot{\sigma} + u_{\mu\alpha}\, \sigma_{\cdot\nu}\, \dot{z}^\alpha\, \ddddot{\sigma} - (u_{\mu\alpha}\, \sigma_{\cdot\nu})_{\cdot\beta\gamma}\, \dot{z}^\alpha \dot{z}^\beta \dot{z}^\gamma\, \ddot{\sigma} \\
&\qquad\qquad - (u_{\mu\alpha}\, \sigma_{\cdot\nu})_{\cdot\beta}(2\ddot{z}^\alpha \dot{z}^\beta + \dot{z}^\alpha \ddot{z}^\beta)\ddot{\sigma} - u_{\mu\alpha}\, \sigma_{\cdot\nu}\, \dddot{z}^\alpha\, \ddot{\sigma}] \\
&\quad + \tfrac{1}{6}\, \delta_\pm{}^3[2(u_{\mu\alpha}\, \sigma_{\cdot\nu})_{\cdot\beta}\, \dot{z}^\alpha \dot{z}^\beta\, \ddddot{\sigma} - 2(u_{\mu\alpha}\, \sigma_{\cdot\nu})_{\cdot\beta\gamma\delta}\, \dot{z}^\alpha \dot{z}^\beta \dot{z}^\gamma \dot{z}^\delta\, \ddot{\sigma} \\
&\qquad\qquad - (u_{\mu\alpha}\, \sigma_{\cdot\nu})_{\cdot\beta}(6\, \dddot{z}^\alpha \dot{z}^\beta + 6\, \ddot{z}^\alpha \ddot{z}^\beta + 2\dot{z}^\alpha \dddot{z}^\beta)\ddot{\sigma}] \\
&\quad + O(\epsilon^4).
\end{aligned} \tag{5.5}$$

Here we have simply carried out repeated differentiations with respect to τ and have dropped certain terms by using the facts that $\dot{\sigma} = \sigma_{\cdot\alpha}\, \dot{z}^\alpha = 0$, and that $\sigma_{\cdot\nu}$ and $(u_{\mu\alpha}\, \sigma_{\cdot\nu})_{\cdot\beta\gamma}$ are of order ϵ. We now use the relations

$$\ddot{\sigma} = -\kappa^2, \tag{5.6}$$

$$\dddot{\sigma} = -\epsilon\Omega_\alpha\, \ddot{z}^\alpha + O(\epsilon^2), \tag{5.7}$$

$$\ddddot{\sigma} = -\ddot{z}^2 + O(\epsilon), \tag{5.8}$$

together with Eqs. (1.34), (1.51), (1.52), (1.64), (2.50), (4.4) and the expan-

sions (1.74), (2.60), (2.61), (4.39), (4.40), as well as the expansion

$$\sigma_{\cdot\nu\beta\gamma} = -D_{\nu\beta\cdot\gamma} = \bar{g}_\nu{}^\alpha(\tfrac{2}{3}\, R_{\alpha\gamma\beta}{}^\delta - \tfrac{1}{3}\, R_\alpha{}^\delta{}_{\beta\gamma})\sigma_{\cdot\delta} + O(\epsilon^2), \tag{5.9}$$

which follows from Eqs. (1.40), (1.69), and (1.70), to rewrite expression (5.5) in the form

$$\begin{aligned}
&\bar{g}_{\mu\alpha}\, \bar{g}_{\nu\beta}(-\epsilon\kappa^2\, \Delta^{1/2}\, \dot{z}^\alpha\, \Omega^\beta \mp \epsilon^3\, \kappa^{-1}\, \dot{z}^\alpha\, \Omega^\beta\, \Omega_\gamma\, \ddot{z}^\gamma + \tfrac{1}{2}\, \epsilon^3\, \kappa^{-2}\, \dot{z}^\alpha\, \dot{z}^\beta\, \Omega_\gamma\, \ddot{z}^\gamma \\
&\quad - \tfrac{1}{2}\, \epsilon^3\, \kappa^{-2}\, \dot{z}^\alpha\, \Omega^\beta\, \ddot{z}^2 - \tfrac{1}{12}\, \epsilon^3\, \dot{z}^\alpha\, \Omega^\beta\, R_{\gamma\delta}\, \dot{z}^\gamma\, \dot{z}^\delta - \tfrac{1}{2}\, \epsilon^3\, \dot{z}^\beta\, R^\alpha{}_{\gamma\delta\epsilon}\, \dot{z}^\gamma\, \dot{z}^\delta\, \Omega^\epsilon \\
&\quad - \tfrac{1}{6}\, \epsilon^3\, \dot{z}^\alpha\, \dot{z}^\beta\, R_{\gamma\delta}\, \dot{z}^\gamma\, \Omega^\delta - \tfrac{1}{3}\, \epsilon^3\, \dot{z}^\alpha\, R^\beta{}_{\gamma\delta\epsilon}\, \dot{z}^\gamma\, \dot{z}^\delta\, \Omega^\epsilon - \epsilon^2\, \ddot{z}^\alpha\, \dot{z}^\beta - \tfrac{1}{2}\, \epsilon^2\, \dot{z}^\alpha\, \ddot{z}^\beta \\
&\quad + \tfrac{1}{2}\, \epsilon^3\, \dddot{z}^\alpha\Omega^\beta \pm \tfrac{1}{3}\, \epsilon^3\, \kappa^{-3}\, \dot{z}^\alpha\, \dot{z}^\beta\, \ddot{z}^2 \pm \tfrac{1}{6}\, \epsilon^3\, \kappa^{-1}\, \dot{z}^\alpha\, \dot{z}^\beta\, R_{\gamma\delta}\, \dot{z}^\gamma\, \dot{z}^\delta \\
&\quad \mp \epsilon^3\, \kappa^{-1}\, \dddot{z}^\alpha\, \dot{z}^\beta \mp \epsilon^3\, \kappa^{-1}\, \ddot{z}^\alpha\, \ddot{z}^\beta \mp \tfrac{1}{3}\, \epsilon^3\, \kappa^{-1}\, \dot{z}^\alpha\, \dddot{z}^\beta) + O(\epsilon^4).
\end{aligned} \tag{5.10}$$

Combining this with expansions (1.77), (4.42), (4.43), and with

$$\begin{aligned}
(u_{\mu\alpha\cdot\nu} + v_{\mu\alpha}\, \sigma_{\cdot\nu})\dot{z}^\alpha &= \epsilon\, \bar{g}_{\mu\alpha}\, \bar{g}_{\nu\beta}(-\tfrac{1}{2}\, \dot{z}^\delta\, R_\delta{}^{\alpha\beta\gamma}\, \Omega_\gamma \\
&\quad - \tfrac{1}{6}\, \dot{z}^\alpha\, R^{\beta\gamma}\, \Omega_\gamma - \tfrac{1}{2}\, \Omega^\beta\, R^\alpha{}_\gamma\, \dot{z}^\gamma + \tfrac{1}{12}\, \dot{z}^\alpha\, \Omega^\beta\, R) + O(\epsilon^2),
\end{aligned} \tag{5.11}$$

which follows from (2.59), (2.60) and the symmetry of $u_{\mu\alpha}$ under interchange of μ and α, and x and z, we find for the retarded and advanced field strengths

$$\begin{aligned}
F^\pm{}_{\mu\nu} &= e(\bar{g}_{\mu\alpha}\, \bar{g}_{\nu\beta} - \bar{g}_{\nu\alpha}\, \bar{g}_{\mu\beta})[\epsilon^{-2}\, \kappa^{-1}\, \dot{z}^\alpha\, \Omega^\beta + \tfrac{1}{2}\, \epsilon^{-1}\, \kappa^{-3}\, \ddot{z}^\alpha\, \dot{z}^\beta + \tfrac{1}{8}\, \kappa^{-5}\, \dot{z}^\alpha\, \Omega^\beta\, \ddot{z}^2 \\
&\quad - \tfrac{1}{2}\, \kappa^{-3}\, \ddot{z}^\alpha\, \Omega^\beta \pm \tfrac{2}{3}\, \kappa^{-4}\, \dddot{z}^\alpha\, \dot{z}^\beta - \tfrac{1}{12}\, \kappa^{-1}\, \dot{z}^\alpha\, \Omega^\beta\, R + \tfrac{1}{6}\, \kappa^{-1}\, \dot{z}^\alpha\, R^\beta{}_\gamma\, \Omega^\gamma \\
&\quad - \tfrac{1}{2}\, \kappa^{-1}\, \Omega^\alpha R^\beta{}_\gamma\, \dot{z}^\gamma - \tfrac{1}{12}\, \kappa^{-1}\, \dot{z}^\alpha\, \Omega^\beta\, R_{\gamma\delta}\, \Omega^\gamma\, \Omega^\delta - \tfrac{1}{2}\, \kappa^{-1}\, R^\alpha{}_\gamma{}^\beta{}_\delta\, \dot{z}^\gamma\, \Omega^\delta \\
&\quad + \tfrac{1}{12}\, \kappa^{-3}\, \dot{z}^\alpha\, \Omega^\beta\, R_{\gamma\delta}\, \dot{z}^\gamma\, \dot{z}^\delta - \tfrac{1}{6}\, \kappa^{-3}\, \dot{z}^\alpha\, R^\beta{}_{\gamma\delta\epsilon}\, \dot{z}^\gamma\, \dot{z}^\delta\, \Omega^\epsilon \\
&\quad \pm \frac{1}{2} \int_\tau^{\pm\infty} f^{\alpha\beta}{}_{\gamma'}(z(\tau), z(\tau'))\dot{z}^{\gamma'}(\tau')\, d\tau'] \\
&\quad + O(\epsilon),
\end{aligned} \tag{5.12}$$

where

$$f_{\mu\nu\alpha} = v_{\mu\alpha\cdot\nu} - v_{\nu\alpha\cdot\mu} \, . \tag{5.13}$$

We note that, since σ certainly becomes single valued in the limit $\epsilon \to 0$, direct use of Eq. (3.52) is valid here; difficulties with multiple-valuedness can occur only in the "tail" term.

From Eq. (5.12) it follows at once that the field $F^{\mathrm{rad}}{}_{\mu\nu}$ is everywhere finite. At the location of the particle itself we have, in fact,

$$\begin{aligned}
F^{\mathrm{rad}\ \alpha\beta} &= F^{-\alpha\beta} - F^{+\alpha\beta} \\
&= \tfrac{4}{3}\, e\, c^{-4}(\dot{z}^\alpha \dddot{z}^\beta - \dot{z}^\beta\, \dddot{z}^\alpha) + e \int_{-\infty}^{\infty} \epsilon(\tau - \tau') f^{\alpha\beta}{}_{\gamma'}\, \dot{z}^{\gamma'}(\tau')\, d\tau',
\end{aligned} \tag{5.14}$$

where

$$\epsilon(\tau) = \tau/|\tau| = \theta(\tau) - \theta(-\tau). \tag{5.15}$$

On the other hand, for the average of the retarded and advanced fields we have

$$\begin{aligned}\bar F_{\mu\nu} = e(\bar g_{\mu\alpha}\bar g_{\nu\beta} - \bar g_{\nu\alpha}\bar g_{\mu\beta})&\Big[\epsilon^{-2}\kappa^{-1}\dot z^\alpha\Omega^\beta + \tfrac12\epsilon^{-1}\kappa^{-3}\ddot z^\alpha\dot z^\beta + \tfrac18\kappa^{-5}\dot z^\alpha\Omega^\beta\ddot z^2 - \tfrac12\kappa^{-3}\ddot z^\alpha\Omega^\beta \\ &+ \text{terms linear and cubic in the } \Omega\text{'s involving the Riemann tensor} \\ &+ \tfrac14\int_{-\infty}^{\infty} f^{\alpha\beta}{}_{\gamma'}\dot z^{\gamma'}(\tau')\,d\tau'\Big] + O(\epsilon).\end{aligned} \tag{5.16}$$

The first term inside the square brackets, representing the Coulomb field, diverges quadratically as $\epsilon\to 0$.

By breaking the total electromagnetic field up in the manner of Eq. (3.39), we may now use Eq. (5.16) to compute the stress density on the world tube. Noting that the field $\bar F^{\text{free}}{}_{\mu\nu}$ is singularity-free, or at any rate has no singularities arising from the particle itself, we may write

$$\begin{aligned}c^{-1}\bar g_\mu{}^\alpha T^{\mu\nu}\,d\Sigma_\nu &= (4\pi c)^{-1} g^{1/2}[\bar g_\mu{}^\alpha(\bar F^\mu{}_\sigma \bar F^{\nu\sigma} + \bar F^{\text{free}\,\mu}{}_\sigma \bar F^{\nu\sigma} + \bar F^\mu{}_\sigma \bar F^{\text{free}\,\nu\sigma})\,d\Sigma_\nu \\ &\quad - (\tfrac14\bar F_{\sigma\tau}\bar F^{\sigma\tau} + \tfrac12\bar F^{\text{free}}{}_{\sigma\tau}\bar F^{\sigma\tau})\bar g^{\mu\alpha}\,d\Sigma_\mu] + O(\epsilon).\end{aligned} \tag{5.17}$$

Using (3.14), (3.15), (3.16), (4.35) and the expansions

$$\kappa^2 = -\sigma_{\cdot\alpha\beta}\dot z^\alpha\dot z^\beta - \sigma_{\cdot\alpha}\ddot z^\alpha = c^2 + \epsilon\Omega_\alpha\ddot z^\alpha + O(\epsilon^2), \tag{5.18}$$

$$\kappa^{-2} = c^{-2} - \epsilon c^{-4}\Omega_\alpha\ddot z^\alpha + O(\epsilon^2), \tag{5.19}$$

we find, by straightforward computation,

$$\begin{aligned}c^{-1}g^{1/2}(\bar g_\mu{}^\alpha\bar F^\mu{}_\sigma\bar F^{\nu\sigma}\,d\Sigma_\nu - \tfrac14\bar F_{\sigma\tau}\bar F^{\sigma\tau}\bar g^{\mu\alpha}\,d\Sigma_\mu) &= e^2\Big[-\tfrac12\epsilon^{-2}\Omega^\alpha + \tfrac12\epsilon^{-1}c^{-2}\ddot z^\alpha - \tfrac34 c^{-4}\ddot z^\alpha\ddot z^\beta\Omega_\beta + \tfrac12 c^{-4}\Omega^\alpha\ddot z^2 \\ &+ \text{terms of odd degree in the } \Omega\text{'s involving the Riemann tensor} \\ &- \tfrac12 c^{-1}\dot z^\beta\int_{-\infty}^{\infty} f^\alpha{}_{\beta\gamma'}\dot z^{\gamma'}(\tau')\,d\tau'\Big]\,d\Omega\,d\tau + O(\epsilon),\end{aligned} \tag{5.20}$$

$$c^{-1}g^{1/2}\bar g_\mu{}^\alpha\bar F^{\text{free}\,\mu}{}_\sigma\bar F^{\nu\sigma}\,d\Sigma_\nu = -e c^{-1}\bar F^{\text{free}\,\alpha}{}_\beta\dot z^\beta\,d\Omega\,d\tau + O(\epsilon), \tag{5.21}$$

$$c^{-1}g^{1/2}(\bar g_\mu{}^\alpha\bar F^\mu{}_\sigma\bar F^{\text{free}\,\nu\sigma}\,d\Sigma_\nu - \tfrac12 F^{\text{free}}{}_{\sigma\tau}\bar F^{\sigma\tau}\bar g^{\mu\alpha}\,d\Sigma_\mu) = O(\epsilon). \tag{5.22}$$

We now draw attention to the remarkable fact that, in the expansion of $\bar g_\mu{}^\alpha T^{\mu\nu}\,d\Sigma_\nu$, all terms which involve the Riemann tensor and which do not vanish as $\epsilon\to 0$ are, with the exception of the "tail" term, of odd degree in the direction cosines. Such terms will all be eliminated when the integration over solid angle is performed, and the Riemann tensor will therefore enter into the ponderomotive equations only implicitly through the "tail" term. Carrying out the integration, we get, in fact,

$$\begin{aligned}\frac1c\int_{4\pi}\bar g_\mu{}^\alpha T^{\mu\nu}\,d\Sigma_\nu &= \Big[\frac{e^2}{2\epsilon c^2}\ddot z^\alpha - \frac{e^2}{2c}\dot z^\beta\int_{-\infty}^{\infty} f^\alpha{}_{\beta\gamma'}\dot z^{\gamma'}(\tau')\,d\tau' - \frac{e}{c}\bar F^{\text{free}\,\alpha}{}_\beta\dot z^\beta\Big]\,d\tau + O(\epsilon).\end{aligned} \tag{5.23}$$

The divergent term in (5.23) has the same kinematical structure as the mass term in Eq. (5.4). It therefore has the effect of an unobservable mass renormalization, and with the introduction of the "observed" mass

$$m = m_0 + \lim_{\epsilon\to 0}\tfrac12 e^2\epsilon^{-1}c^{-2}, \tag{5.24}$$

Eq. (5.4) takes the form[11]

$$m\ddot z^\alpha = ec^{-1}\bar F^{\text{free}\,\alpha}{}_\beta\dot z^\beta + \tfrac12 e^2c^{-1}\dot z^\beta\int_{-\infty}^{\infty} f^\alpha{}_{\beta\gamma'}\dot z^{\gamma'}(\tau')\,d\tau'. \tag{5.25}$$

For purposes of application to physically set boundary conditions in the remote past it is more appropriate to work with the field $F^{\text{in}}{}_{\alpha\beta}$. Referring to Eqs. (3.41) and (5.14), we see that Eq. (5.25) then becomes

$$\begin{aligned}m\ddot z^\alpha &= ec^{-1}F^{\text{in}\,\alpha}{}_\beta\dot z^\beta + \tfrac23 e^2c^{-3}(\dddot z^\alpha - c^{-2}\dot z^\alpha\ddot z^2) \\ &\quad + e^2c^{-1}\dot z^\beta\int_{-\infty}^{\tau} f^\alpha{}_{\beta\gamma'}\dot z^{\gamma'}(\tau')\,d\tau'.\end{aligned} \tag{5.26}$$

The second term on the right is the familiar classical radiation damping term. When $F^{\text{in}\,\alpha}{}_\beta = 0$ and space-time is flat, the physical solution of (5.26) is $\ddot z^\alpha = 0$, i.e., geodesic motion.[12] When space-time is curved, however, the presence of a non-

[11] Equation (5.25) may be regarded as the *definition* of the result of substituting (5.16) in Eq. (3.17) and taking the limit $\epsilon\to 0$. It is only by the roundabout procedure of computing momentum-energy balance and performing a mass renormalization that a meaning can be given to the indeterminate expression $F^\alpha{}_\beta\dot z^\beta$ which appears in (3.17) and again in (3.26) and (3.27).

[12] There are, of course, also the inadmissible "runaway" solutions. Such solutions presumably exist as well in curved space-times. In the case of flat space-time it is known that there are circumstances under which *all* solutions are inadmissible, notably when $F^{\text{in}}{}_{\mu\nu}$ is simply the Coulomb field of a static external point charge. [See Eliezer (*9*).] This represents

vanishing "tail" prevents this from generally being a solution, and, as stated in the Introduction, radiation damping then occurs even when $F^{\text{in}\,\alpha}{}_\beta$ vanishes. No attempt at a detailed analysis of the "tail" term will be made in the present article. It is hoped that an investigation of some of its effects in the case of a static metric can be carried out in a future paper.

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a serious defect of the classical theory, and it persists, as Eliezer (*9*) has shown, even if the point charge is allowed to move under its own dynamics. That is to say, the equations of motion for an assembly of two or more point charges mutually interacting through their electromagnetic fields, have no physical solutions unless $F^{\text{in}}{}_{\mu\nu}$ itself is chosen in a rather unphysical way. [See, however, Wheeler and Feynman (*10*).] A very interesting although probably difficult problem would be to see if this situation continues to persist for the equations of a charged particle which moves in the curved space-time produced, according to Einstein's equations, by a charged mass point (*11*). This suggests the importance, in the n-particle problem, of taking into account the dynamical properties of the gravitational field and the fact that the metric is actually singular at the location of each particle.

In the case of a fixed metric the results of the present paper can easily be extended to the n-particle problem. Each particle (we may label them by indices A or B running from 1 to n) will have its own retarded and advanced fields, $F_A{}^{\text{ret}}{}_{\mu\nu}$ and $F_A{}^{\text{adv}}{}_{\mu\nu}$, and Eq. (5.26) will hold separately for each particle in the form

$$m_A \ddot z_A{}^\alpha = e_A c^{-1} F_A{}^{\text{in}\,\alpha}{}_\beta\, \dot z_A{}^\beta + \frac{2}{3} e_A{}^2 c^{-3}(\dddot z_A{}^\alpha - c^{-2} \dot z_A{}^\alpha \ddot z_A{}^2)$$
$$+ e_A{}^2 c^{-1} \dot z_A{}^\beta \int_{-\infty}^{\tau_A} f^\alpha{}_{\beta\gamma'}(z_A(\tau_A), z_A(\tau'))\, \dot z_A{}^{\gamma'}(\tau')\, d\tau',$$

where

$$F_A{}^{\text{in}}{}_{\mu\nu} = F_{\mu\nu} - F_A{}^{\text{ret}}{}_{\mu\nu}\,.$$

(Attention should be called to the fact that each particle will have its own proper time, and the dots above denote absolute covariant differentiation with respect to the proper time of the particle in question.) In practice the physical boundary conditions do not specify the fields $F_A{}^{\text{in}}{}_{\mu\nu}$, but rather the field

$$F^{\text{in}}{}_{\mu\nu} = F_{\mu\nu} - \sum_B F_B{}^{\text{ret}}{}_{\mu\nu}$$
$$= F_A{}^{\text{in}}{}_{\mu\nu} - \sum_{B \neq A} F_B{}^{\text{ret}}{}_{\mu\nu}\,.$$

In terms of this field the ponderomotive equations become

$$m_A\, \ddot z_A{}^\alpha = e_A c^{-1}\, F^{\text{in}\,\alpha}{}_\beta\, \dot z_A{}^\beta + e c^{-1} \sum_{B \neq A} F_B{}^{\text{ret}\,\alpha}{}_\beta\, \dot z_A{}^\beta$$
$$+ \tfrac{2}{3}\, e_A{}^2 c^{-3} (\dddot z_A{}^\alpha - c^{-2}\, \dot z_A{}^\alpha\, \ddot z_A{}^2)$$
$$+ e_A{}^2 c^{-1} \dot z_A{}^\beta \int_{-\infty}^{\tau_A} f^\alpha{}_{\beta\gamma'}(z_A(\tau_A), z_A(\tau'))\, \dot z_A{}^{\gamma'}(\tau')\, d\tau'.$$

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