A new classical theory of electrons. III

By P. A. M. Dirac, F.R.S.
St John's College, University of Cambridge

(*Received* 9 *December* 1953)

The paper deals with several interpenetrating electric streams moving in accordance with the equations of Maxwell and Lorentz. An action principle is set up and then a passage is made to the Hamiltonian form of the equations of motion. The Hamiltonian has considerable analogy to that for point-charge electrons, but there is some discrepancy.

1. Introduction

In two preceding papers (Dirac 1951, 1952) I have proposed that the theory of electrons should be built up from a classical theory of the motion of a continuous stream of electricity rather than the motion of point charges. One then looks upon the existence of discrete electrons as a quantum phenomenon. In this way one has a more reliable foundation for electrodynamic theory, since the difficulties of the infinities inherent in any treatment of point charges no longer occur at the very beginning of one's work.

The preceding papers dealt with the theory of a single electric stream. It is not certain whether such a theory is sufficiently general to serve as a basis for a quantum theory of electrons, as the classical model does not contain anything corresponding to electrons with different velocities in close proximity. In order to broaden the basis, a theory is here built up of several interpenetrating electric streams.

Each stream moves under the influence of the electromagnetic field in accordance with Lorentz's equations of motion, and generates electromagnetic field in accordance with Maxwell's equations. There is no direct interaction between the streams, only an indirect interaction arising from the field generated by one stream influencing the motion of another.

The equations of motion will first be put in the form of an action principle. A passage will then be made to the Hamiltonian form. In order to keep the theory manifestly relativistic, the Hamiltonian will be given for states on general space-like surfaces in space-time.

The theory could be applied to the motion of streams of charged particles in a vacuum tube when collisions can be neglected. For such practical purposes one does not need the complication of the Hamiltonian form. The justification for the calculation of the Hamiltonian form lies in the hope that it may help one to guess the equations for a new quantum theory of electrons, by applying a quantization procedure to the classical equations, bringing in electron spin and Fermi statistics in a suitable way.

One of the main problems of present-day atomic physics is to obtain a theory of electrons which fixes the value of $e^2/\hbar c$, a theory which will only work for one value of this quantity. It seems hopeless to attack this problem from the physical point of view, as one has no clue to what new physical ideas are needed. However, one can be sure that the new theory must incorporate some very pretty mathematics, and by seeking this mathematics one can have some hope of solving the problem. I believe that the required mathematics will be more closely connected with the equations that describe continuous electric streams than those at present used for point charges.

2. The action principle

We suppose we have several streams of electricity, a general one being labelled the ath ($a = 1, 2, \ldots$). The stream lines in space-time of the ath stream are labelled by three parameters η_{ar} ($r = 1, 2, 3$), which have constant values along each stream line. The stream lines are described by our giving these parameters as functions of the four co-ordinates x_μ ($\mu = 0, 1, 2, 3$) of a point in space-time,

$$\eta_{ar} = \eta_{ar}(x).$$

We can, of course, replace the η_{ar} by any three independent functions of them, without altering the stream lines, but only the way they are labelled.

Define the 4-vector $q_{a\mu}$ by

$$q_{a\mu} = \epsilon_{\mu\nu\rho\sigma}\eta_{a1}{}^{\nu}\eta_{a2}{}^{\rho}\eta_{a3}{}^{\sigma}, \tag{1}$$

where $\epsilon_{\mu\nu\rho\sigma} = \pm 1$ when μ, ν, ρ, σ is an even or odd permutation of $0, 1, 2, 3$ and is zero otherwise, and an upper affix ν attached to a variable denotes its derivative with respect to x_ν, A summation is always implied over any repeated suffix in a term, unless it is a suffix a labelling one of the electric streams.

Equation (1) gives

$$q_{a\mu}\eta_{ar}{}^{\mu} = 0,$$

showing that η_{ar} is constant along a world-line whose direction is in the direction of $q_{a\mu}$. Thus the direction of $q_{a\mu}$ is the direction of the velocity 4-vector $v_{a\mu}$ of the ath stream, so

$$v_{a\mu} = q_{a\mu}/|q_a|, \quad |q_a| = (q_{a\mu}q_a{}^{\mu})^{\frac{1}{2}}.$$

Let us assume the action density

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_a (\alpha_a q_{a\mu}A^{\mu} - \beta_a |q_a|), \tag{2}$$

where α_a and β_a are two new field variables which are constant along the stream lines of the ath stream, so that they are functions only of $\eta_{a1}, \eta_{a2}, \eta_{a3}$. It will be shown that the field equations following from this action density are the correct equations of motion.

The field equations are, for any field quantity K occurring in $\mathscr{L}$, together with its first derivatives K^ν,

$$\left(\frac{\partial\mathscr{L}}{\partial K^\nu}\right)^{\nu} = \frac{\partial\mathscr{L}}{\partial K}. \tag{3}$$

Taking K to be one of the electromagnetic potentials A_μ and remembering $F^{\mu\nu} = A^{\nu\mu} - A^{\mu\nu}$, we get

$$F_{\mu\nu}{}^{\nu} = \sum_a \alpha_a q_{a\mu}. \tag{4}$$

This gives us Maxwell's equations, provided we interpret α_a so that $\alpha_a q_{a0}$ is the electric density of the ath stream.

Taking K to be η_{a1}, we get

$$\{(\alpha_a A^\mu - \beta_a v_a{}^\mu)\,\epsilon_{\mu\nu\rho\sigma}\eta_{a2}{}^\rho \eta_{a3}{}^\sigma\}^\nu = \frac{\partial \alpha_a}{\partial \eta_{a1}} q_{a\mu} A^\mu - \frac{\partial \beta_a}{\partial \eta_{a1}} \,|\, q_a \,|.$$

This reduces to

$$(\alpha A^\mu - \beta v^\mu)^\nu\,\epsilon_{\mu\nu\rho\sigma}\eta_2{}^\rho \eta_3{}^\sigma = \frac{\partial \alpha}{\partial \eta_1} q_\mu A^\mu - \frac{\partial \beta}{\partial \eta_1} \,|\, q \,|,$$

where the suffix a is left understood for brevity. Multiply this equation by $\eta_1{}^\lambda$ and add on the corresponding equations for η_2 and η_3. The result is

$$(\alpha A^\mu - \beta v^\mu)^\nu\,(\delta^\lambda_\kappa \epsilon_{\mu\nu\rho\sigma} + \delta^\lambda_\rho \epsilon_{\mu\nu\sigma\kappa} + \delta^\lambda_\sigma \epsilon_{\mu\nu\kappa\rho})\,\eta_1{}^\kappa \eta_2{}^\rho \eta_3{}^\sigma = \alpha^\lambda q_\mu A^\mu - \beta^\lambda \,|\, q \,|.$$

Applying the general formula

$$\delta^\lambda_\kappa \epsilon_{\mu\nu\rho\sigma} + \delta^\lambda_\mu \epsilon_{\nu\rho\sigma\kappa} + \delta^\lambda_\nu \epsilon_{\rho\sigma\kappa\mu} + \delta^\lambda_\rho \epsilon_{\sigma\kappa\mu\nu} + \delta^\lambda_\sigma \epsilon_{\kappa\mu\nu\rho} = 0, \tag{5}$$

which holds simply because the suffixes can take on only four values, we get

$$(\alpha A^\mu - \beta v^\mu)^\nu\,(\delta^\lambda_\nu v_\mu - \delta^\lambda_\mu v_\nu) = \alpha^\lambda v_\mu A^\mu - \beta^\lambda.$$

Using the equations $\alpha^\nu v_\nu = 0$, $\beta^\nu v_\nu = 0$, which hold because α, β are constant along the stream lines, we get finally

$$\beta v^{\lambda\nu} v_\nu + \alpha(A^{\nu\lambda} - A^{\lambda\nu})\,v_\nu = 0. \tag{6}$$

These are just Lorentz's equations of motion for a stream of charged particles whose charge and rest-mass are in the ratio $\alpha:\beta$.

Thus the action density (2) leads to the correct equations of motion. We may use it either with the ratio $\alpha:\beta$ the same for all the streams, so as to describe, for example, the motion of several interpenetrating streams of electrons, or with $\alpha:\beta$ different for the different streams, to describe streams of particles of different nature. It would be mathematically permissible to take $\alpha:\beta$ to vary from one stream line to another in the same stream, but this generalization does not seem to be of any physical importance.

3. Passage to the Hamiltonian

As a preliminary to quantization we must put the equations of motion in the Hamiltonian form. It is desirable that this should be done in a manifestly relativistic manner, and for this purpose one must deal with the state on a general three-dimensional space-like surface in space-time. One must obtain the Hamiltonian which gives the change in such a state as the surface is varied in any way so as to remain always space-like. The action principle gives us immediately the equations for this change in state in the Langrangian form and we can pass over to the Hamiltonian form by standard methods.

With the notation of part II, the Lagrangian is

$$L \equiv \int \mathscr{L} \dot{y}_l \Gamma \,\mathrm{d}^3u.$$

We may write it

$$L \equiv L_F + \sum_a M_a,$$

with
$$L_F \equiv -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\dot{y}_l\Gamma\,\mathrm{d}^3u,$$
$$M_a \equiv \int(\alpha_a q_{a\mu}A^\mu - \beta_a\,|\,q_a\,|)\,\dot{y}_l\Gamma\mathrm{d}^3u.$$

The dynamical co-ordinates are the field quantities η_{ar} and A_μ on the surface. It is preferable to take as basic dynamical co-ordinates, instead of the four quantities A_μ, the tangential and normal components of the 4-vector, A^r and A_l, as was done in part II. This change affects the definition of the conjugate momenta.

To obtain the momenta we must vary the dynamical velocities in L, which involves varying the derivatives of the dynamical co-ordinates in any direction which is not tangential to the surface. To obtain the variation in L_F we may proceed as in part II from equation (28) onwards, but without using equation (30) or (11). Thus instead of (33) we have
$$\dot{y}_\sigma F^{r\sigma} \equiv (\dot{y}_\sigma A^\sigma)^r - A^{r\tau},$$
and instead of the next following equation
$$\dot{y}_l\delta F^r{}_l + F^{r\sigma}\delta\dot{y}_\sigma = (A^\sigma\delta\dot{y}_\sigma)^r - \delta A^{r\tau}.$$
Hence
$$\delta L_F = \int\{F_{rl}(F^{r\sigma}\delta\dot{y}_\sigma + \delta A^{r\tau})\,\Gamma + (F_{rl}\Gamma)^r A^\sigma\delta\dot{y}_\sigma - (\tfrac{1}{4}F_{rs}F^{rs} + \tfrac{1}{2}F_{rl}F^r{}_l)\,\Gamma l^\sigma\delta\dot{y}_\sigma\}\,\mathrm{d}^3u. \quad (7)$$

To obtain δM_a we note that, dropping the suffix a again,
$$\delta q_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon^{rst}\eta_s{}^\rho\eta_t{}^\sigma\delta\eta_r{}^\nu \quad (r,s,t = 1,2,3).$$
Using the kinematic formula
$$\dot{y}_l\delta K^\nu = l^\nu(\delta K^\tau - K^\mu\delta\dot{y}_\mu), \quad (8)$$
which holds for any field variable K, we get
$$\dot{y}_l\delta q_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon^{rst}l^\nu\eta_s{}^\rho\eta_t{}^\sigma(\delta\eta_r{}^\tau - \eta_r{}^\mu\delta\dot{y}_\mu),$$
and so
$$\delta M = \tfrac{1}{2}\int\{(\alpha A^\mu - \beta v^\mu)\,\epsilon_{\mu\nu\rho\sigma}\epsilon^{rst}l^\nu\eta_s{}^\rho\eta_t{}^\sigma(\delta\eta_r{}^\tau - \eta_r{}^\lambda\delta\dot{y}_\lambda) + (\alpha q_\mu A^\mu - \beta\,|\,q\,|)\,l^\sigma\delta\dot{y}_\sigma\}\,\Gamma\mathrm{d}^3u. \quad (9)$$

The momenta $B_r, B_l, \xi_a{}^r, w^\sigma$ conjugate to A^r, A_ρ, η_{ar}, y_σ respectively are equal to coefficients in the variation of L, thus,
$$\delta L = \int\{B_r\delta A^{r\tau} + B_l\delta A_l{}^\tau + \sum_a \xi_a{}^r\delta\eta_{ar}{}^\tau + w^\sigma\delta\dot{y}_\sigma\}\,\mathrm{d}^3u. \quad (10)$$

Comparing coefficients in (7), (9) and (10), we get
$$B_r = F_{rl}\Gamma, \quad (11)$$
$$B_l = 0, \quad (12)$$
$$\xi^r = \tfrac{1}{2}(\alpha A^\mu - \beta v^\mu)\,\epsilon_{\mu\nu\rho\sigma}\epsilon^{rst}l^\nu\eta_s{}^\rho\eta_t{}^\sigma\Gamma, \quad (13)$$
$$w^\sigma = F_{rl}F^{r\sigma}\Gamma + (F_{rl}\Gamma)^r A^\sigma - (\tfrac{1}{4}F_{rs}F^{rs} + \tfrac{1}{2}F_{rl}F^r{}_l)\,\Gamma l^\sigma$$
$$- \Sigma\{\tfrac{1}{2}(\alpha A^\mu - \beta v^\mu)\,\epsilon_{\mu\nu\rho\lambda}\epsilon^{rst}l^\nu\eta_s{}^\rho\eta_t{}^\lambda\eta_r{}^\sigma - (\alpha q_\mu A^\mu - \beta\,|\,q\,|)\,l^\sigma\}\,\Gamma,$$

the Σ referring to a sum over all the streams. Simplifying the last equation with the help of the preceding ones and picking out its tangential and normal components, we get

$$w^s = B_r F^{rs} + B_r{}^r A^s - \Sigma \xi^r \eta_r{}^s, \tag{14}$$

$$w_l = B_r{}^r A_l - \tfrac{1}{4} F_{rs} F^{rs} \Gamma + \tfrac{1}{2} B_r B^r \Gamma^{-1} + \Sigma(-\xi^r \eta_{rl} + \alpha q_\mu A^\mu \Gamma - \beta \mid q \mid \Gamma). \tag{15}$$

We must now obtain equations involving only the dynamical co-ordinates and momenta, i.e. equations independent of the dynamical velocities. Equations (12) and (14) as they stand are of this type. Another can be deduced from (13) and (15), and one more can be deduced from the field equations (4), as shown below.

Equation (4) multiplied by l_μ gives

$$\Sigma \alpha q_l = l_\mu F^{\mu\nu}{}_\nu = l_\mu (F^{\mu\nu s} y_{\nu s} + F^{\mu\nu}{}_l l_\nu).$$

Owing to the antisymmetry of $F^{\mu\nu}$ and the symmetry of the curvature tensor $l_\mu{}^s y_{\nu s}$ in μ and ν this becomes

$$\Sigma \alpha q_l = (l_\mu F^{\mu\nu})^s y_{\nu s} = (F_{lr} y^{\nu r})^s y_{\nu s}.$$

Now
$$\Gamma^r \Gamma^{-1} = \tfrac{1}{2} \gamma_{st} \gamma^{str} = \gamma_{st} y^{\nu sr} y_\nu{}^t$$
$$= y^{\nu rs} y_{\nu s} = -y^{\nu r} y_{\nu s}{}^s,$$

so
$$y^{\nu r} (\Gamma y_{\nu s})^s = 0.$$

Hence
$$\Gamma \Sigma \alpha q_l = (F_{lr} y^{\nu r} \Gamma y_{\nu s})^s = -B_s{}^s \tag{16}$$

from (11). Now from (1)

$$q_l = \epsilon^{\mu\nu\rho\sigma} l_\mu (\eta_{1l} l_\nu + \eta_1{}^r y_{\nu r})(\eta_{2l} l_\rho + \eta_2{}^s y_{\rho s})(\eta_{3l} l_\sigma + \eta_3{}^t y_{\sigma t})$$
$$= \epsilon^{\mu\nu\rho\sigma} l_\mu \eta_1{}^r \eta_2{}^s \eta_3{}^t y_{\nu r} y_{\rho s} y_{\sigma t}, \tag{17}$$

which involves only tangential derivatives of the η's and is thus a function of the dynamical co-ordinates only. Hence equation (16) involves only dynamical co-ordinates and momenta.

Formula (5) multiplied by $\eta_1{}^\kappa \eta_2{}^\rho \eta_3{}^\sigma$ gives

$$-(\delta_\nu^\lambda \epsilon_{\rho\sigma\kappa\mu} + \delta_\mu^\lambda \epsilon_{\nu\rho\sigma\kappa}) \eta_1{}^\kappa \eta_2{}^\rho \eta_3{}^\sigma = \tfrac{1}{2} \epsilon_{\mu\nu\rho\sigma} \epsilon^{rst} \eta_r{}^\lambda \eta_s{}^\rho \eta_t{}^\sigma.$$

Multiplying this by $(\alpha A^\mu - \beta v^\mu) l^\nu \Gamma$ and using (1) and (13), we get

$$(\alpha q_\mu A^\mu - \beta \mid q \mid) l^\lambda \Gamma - (\alpha A^\lambda - \beta v^\lambda) q_l \Gamma = \xi^r \eta_r{}^\lambda.$$

The tangential and normal parts of this equation are

$$\xi^r \eta_r{}^s = -(\alpha A^s - \beta v^s) q_l \Gamma, \tag{18}$$

$$\xi^r \eta_{rl} = (\alpha q_\mu A^\mu - \beta \mid q \mid) \Gamma - (\alpha A_l - \beta v_l) q_l \Gamma. \tag{19}$$

Substituting (19) in (15) and using (16), we get

$$w_l = -\tfrac{1}{4} F_{rs} F^{rs} \Gamma + \tfrac{1}{2} B_r B^r \Gamma^{-1} - \Gamma \Sigma \beta v_l q_l, \tag{20}$$

which involves dynamical velocities only through v_l. From (18)

$$\beta^2 v_l^2 = \beta^2 (1 - v^s v_s)$$
$$= \beta^2 - (\xi^r \eta_r{}^s q_l^{-1} \Gamma^{-1} + \alpha A^s)(\xi^t \eta_{ts} q_l^{-1} \Gamma^{-1} + a A_s). \tag{21}$$

On substituting the square root of (21) for βv_l in (20), we get an equation involving only dynamical co-ordinates and momenta.

Equations (12), (14), (16) and (20) with (21) are all the independent equations involving only dynamical co-ordinates and momenta. They serve as the Hamiltonians, i.e. if any of them is written as $H = 0$, then H can be used to give rise to a continuous change in each dynamical variable X according to the standard formula

$$\mathrm{d}X/\mathrm{d}\tau = [X, H].$$

The Hamiltonian (12) gives rise to a change in A_l with all the other dynamical variables remaining constant, and is thus trivial. The Hamiltonian (14) gives rise to a change in the parametrization of the surface by the u's, which is just a mathematical change with no physical significance. The Hamiltonian (16) gives rise to a change of gauge in the tangential part of the potentials A^s. The Hamiltonian (20) with (21) gives rise to the change in all the dynamical variables accompanying the motion of the surface normal to itself, and is thus the important Hamiltonian giving the change in the physical state with time.

4. Simplification of the Hamiltonian

The preceding work involves a parametrization of the surface, given by u_r, and also a parametrization of the stream lines of each stream, given by η_{ar}. It is possible to eliminate the latter parametrization by a change of variables and so effect a simplification of the equations. The former cannot be eliminated so long as one deals with states on a general curved surface.

A density ρ referred to unit variation of the parameters η_1, η_2, η_3 is connected with the corresponding density ρ^* referred to unit variation of the parameters u_1, u_2, u_3 by

$$\rho^* = \rho\partial(\eta_1, \eta_2, \eta_3)/\partial(u_1, u_2, u_3). \tag{22}$$

From (17) we can express q_l as the product of two determinants

$$q_l = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \eta_1{}^1 & \eta_1{}^2 & \eta_1{}^3 \\ 0 & \eta_2{}^1 & \eta_2{}^2 & \eta_2{}^3 \\ 0 & \eta_3{}^1 & \eta_3{}^2 & \eta_3{}^3 \end{vmatrix} \begin{vmatrix} l_0 & l_1 & l_2 & l_3 \\ y_{01} & y_{11} & y_{21} & y_{31} \\ y_{02} & y_{12} & y_{22} & y_{32} \\ y_{03} & y_{13} & y_{23} & y_{33} \end{vmatrix}.$$

The first of these determinants is just $\partial(\eta_1, \eta_2, \eta_3)/\partial(u_1, u_2, u_3)$. The second, if multiplied by itself transposed, with a minus sign inserted in the second, third and fourth rows, gives

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ 0 & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ 0 & \gamma_{31} & \gamma_{32} & \gamma_{33} \end{vmatrix} = -\Gamma^{-2},$$

so it is just Γ^{-1}. Hence

$$q_l = \Gamma^{-1}\partial(\eta_1, \eta_2, \eta_3)/\partial(u_1, u_2, u_3), \tag{23}$$

and (22) becomes

$$\rho^* = \rho q_l \Gamma. \tag{24}$$

From the interpretation of equation (4) we saw that αq_μ is the electric density and current of one of the streams in the x_μ frame of reference, so $\alpha q_l \Gamma$ is the electric density in the space of the surface, referred to unit variation of the parameters u_r. Hence α is the electric density referred to unit variation of the parameters η_r. Thus α is a variable that has to be eliminated when we eliminate the η-parametrization. Instead of α we must use the electric density $\alpha^* = \alpha q_l \Gamma$ referring to the parameters u_r. Similarly, β is the rest-mass density referred to the parameters η_r and has to be eliminated. Instead we must use $\beta^* = \beta q_l \Gamma$, the rest-mass density referred to the parameters u_r.

The variables η_{ar} and their conjugates $\xi_a{}^r$ occur in the Hamiltonians only in the combination $\xi_a{}^r \eta_{ar}{}^s$. Let us put

$$P_a{}^s = \xi_a{}^r \eta_{ar}{}^s, \tag{25}$$

and look upon the $P_a{}^s$ as new dynamical variables. In terms of them the Hamiltonian equations (14), (16) and (20) with (21) may be written

$$w^s - B_r F^{rs} - B_r{}^r A^s + \Sigma P^s = 0, \tag{26}$$

$$B_s{}^s + \Sigma \alpha^* = 0, \tag{27}$$

$$w_l + \tfrac{1}{4} F_{rs} F^{rs} \Gamma - \tfrac{1}{2} B_r B^r \Gamma^{-1} - \Sigma\{\beta^{*2} - (P^s + \alpha^* A^s)(P_s + \alpha^* A_s)\}^{\frac{1}{2}} = 0. \tag{28}$$

All reference to the η-parametrization has now disappeared.

We now have as dynamical variables describing the ath stream and occurring in the Hamiltonians only $P_a{}^s$, $\alpha_a{}^*$ and $\beta_a{}^*$. In order to see that these variables form a self-contained set, we must verify that the Poisson bracket of any two of them is a function of them.

Working from the fundamental Poisson bracket relation

$$[\eta_s, \xi'^r] = \delta(u-u')\,\delta_s^r,$$

where the $'$ attached to a variable denotes its value at the point u', we find

$$\begin{aligned} [P^r, P'^s] &= [\xi^p \eta_p{}^r, \xi'^q \eta_q'{}^{s'}] \\ &= \delta^r(u-u')\,\delta_q^p\,\xi^p \eta_q'{}^{s'} + \delta^s(u-u')\,\delta_q^p\,\xi'^q \eta_p{}^r \\ &= \delta^r(u-u')\,P^s + \delta^s(u-u')\,P'^r. \end{aligned} \tag{29}$$

Again

$$\begin{aligned} [\alpha, P'^s] &= [\eta_r, P'^s]\,\partial\alpha/\partial\eta_r \\ &= \delta(u-u')\,\eta_r{}^s \partial\alpha/\partial\eta_r = \delta(u-u')\,\alpha^s. \end{aligned}$$

After some reduction one finds

$$[\partial(\eta_1, \eta_2, \eta_3)/\partial(u_1, u_2, u_3), P'^s] = \{\delta(u-u')\,\partial(\eta_1, \eta_2, \eta_3)/\partial(u_1, u_2, u_3)\}^s, \tag{30}$$

so that

$$[\alpha^*, P'^s] = \{\delta(u-u')\,\alpha^*\}^s. \tag{31}$$

Similarly

$$[\beta^*, P'^s] = \{\delta(u-u')\,\beta^*\}^s. \tag{32}$$

Finally

$$[\alpha^*, \alpha'^*] = 0, \quad [\beta^*, \beta'^*] = 0, \quad [\alpha^*, \beta'^*] = 0.$$

Thus the Poisson brackets of P^s, α^*, β^* are all expressible as functions of themselves and these variables form a self-contained set. They are adequate to describe one of the electric streams, and are the most convenient variables for this purpose in a Hamiltonian theory.

5. Discussion

The Hamiltonians are now the left-hand sides of (12), (26), (27) and (28). It should be noted that they do not involve the variables A_l, B_l except for (12), and the only effect of this Hamiltonian is to give rise to arbitrary changes in A_l. Thus we can omit the dynamical variables A_l, B_l from the theory and work only with the Hamiltonians (26), (27) and (28). The disappearance of the normal component of the potentials and its conjugate momentum seems to be a general feature of any theory of electrodynamics expressed in the Hamiltonian form.

The important Hamiltonian, giving rise to a motion of the surface normal to itself, is (28). This Hamiltonian, with the term w_l omitted, equals the total energy density in the surface. The various terms here are what one would expect. The first two are the magnetic and electric parts of the electromagnetic energy density. The remaining terms give the energy densities of the individual streams. Each of them is a square root, of a form analogous to the kinetic energy of a point-charge electron, with β^* taking the place of the rest-mass, α^* taking the place of the charge and P^s taking the place of the momentum. Thus it appears that P^s should be interpreted as the momentum density of a stream.

Equation (18) gives

$$P^s = \beta^* v^s - \alpha^* A^s. \tag{33}$$

This is to be compared with the equation

$$p^s = mv^s - eA^s$$

for a point electron, and shows that the usual relation holds between the momentum density and the velocity of a stream.

It is rather surprising that the variables P^s do not have zero Poisson brackets with one another. The analogy with a point electron breaks down at this stage. One gets somewhat simpler Poisson bracket relations if one works with the momentum per unit rest-mass, i.e. the quantity P^s/β^*, instead of the momentum density P^s. One finds after some reduction

$$[P^r/\beta^*, P'^s/\beta'^*] = \delta(u-u')\{(P^r/\beta^*)^s - (P^s/\beta^*)^r\}/\beta^*,$$

which is simpler than the Poisson bracket relations (29) for P^s because it does not involve derivatives of $\delta(u-u')$. But there is still no analogy with the point electron.

References

Dirac, P. A. M. 1951 *Proc. Roy. Soc.* A, **209**, 291 (part I).
Dirac, P. A. M. 1952 *Proc. Roy. Soc.* A, **212**, 330 (part II).